JEE MAIN 2021

18 March shift II

SECTION – A

1. Let the system of linear equations

$4x+\lambda y+2z=0$

$2x-y+z=0$

$\mu x+2y+3z=0, \lambda, \mu \in R$

Has a non-trivial solution. Then which of the following is true?

(1) $\mu=6, \quad \lambda \in R$

(2) $\lambda = 2, \mu \in R$

(3) $\lambda = 3, \mu \in R$

(4) $\mu = -6, \quad \lambda \in R$

Ans. **(1)**

Sol. For non trivial solution

$\Delta = 0$

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow 4(-3-2)-\lambda(6-\mu)+2(4+\mu)=0$

$\Rightarrow -20-6\lambda+\lambda\mu+8+2\mu=0$

$\Rightarrow -12-6\lambda+\lambda\mu+2\mu=0$

$\Rightarrow -6(\lambda+2)+\mu(\lambda+2)=0$

$\Rightarrow \quad (\lambda+2)(\mu-6)=0$

$\mu=6, \quad \lambda \in R$

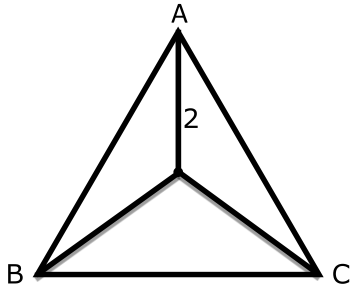
2. A pole stands vertically inside a triangular park ABC. Let the angle of elevation of the top of the pole from each corner of the park be $\frac{\pi}{3}$. If the radius of the circumcircle of ΔABC is 2, then the height of the pole is equal to:

(1) $\frac{1}{\sqrt{3}}$ (2) $\sqrt{3}$

(3) $2\sqrt{3}$ (4) $\frac{2\sqrt{3}}{3}$

Ans. **(3)**

Sol.



$\tan 60° = \frac{h}{2} \Rightarrow h = 2\sqrt{3}$

3. Let in a series of 2n observations, half of them are equal to a and remaining half are equal to –a. Also by adding a constant b in each of these observations, the mean and standard deviation of new set become 5 and 20, respectively. Then the value of $a^2 + b^2$ is equal to:

(1) 250 (2) 925

(3) 650 (4) 425

Ans. **(4)**

Sol. Given series

(a,a,a…….n times), (–a, –a, –a,…… n times)

Now $\bar{x} = \frac{\sum x_i}{2n} = 0$

as $x_i \to x_i + b$

then $\bar{x} \to \bar{x} + b$

So , $\bar{x} + b = 5 \Rightarrow b = 5$

No change in S.D. due to change in origin

$\sigma = \frac{\sum x_i^2}{2n} - (\bar{x})^2 = \sqrt{\frac{2na^2}{2n} - 0}$

$20 = \sqrt{a^2} \Rightarrow a = 20$

$a^2 + b^2 = 425$

4. Let $g(x) = \int_0^x f(t)\,dt$, where f is continuous function in [0, 3] such that $\frac{1}{3} \le f(t) \le 1$ for all $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$ for all $t \in (1,3]$. The largest possible interval in which g(3) lies is:

(1) [1,3] (2) $\left[-1, -\frac{1}{2}\right]$ (3) $\left[-\frac{3}{2}, -1\right]$ (4) $\left[\frac{1}{3}, 2\right]$

Ans. **(4)**

Sol. $\int_0^1 \frac{1}{3}dt + \int_1^3 0.dt \le g(3) \le \int_0^1 1.dt + \int_1^3 \frac{1}{2}dt$

$\frac{1}{3} \le g(3) \le 2$

5. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha \in R$, then the value of $27\sec^6\alpha + 8\operatorname{cosec}^6\alpha$ is equal to:

(1) 250 (2) 500

(3) 400 (4) 350

Ans. **(1)**

Sol. $15\sin^4\theta + 10\cos^4\theta = 6$

$\Rightarrow 15\sin^4\theta + 10(1-\sin^2\theta)^2 = 6$

$\Rightarrow 25\sin^4\theta - 20\sin^2\theta + 4 = 0$

$\Rightarrow (5\sin^2\theta - 2)^2 = 0 \Rightarrow \sin^2\theta = \frac{2}{5}, \cos^2\theta = \frac{3}{5}$

Now $27\operatorname{cosec}^6\theta + 8\sec^6\theta = 27\left(\frac{125}{27}\right) + 8\left(\frac{125}{8}\right) = 250$

6. Let $f : R - \{3\} \to R - \{1\}$ be defind by $f(x) = \frac{x-2}{x-3}$.

Let $g : R - R$ be given as $g(x) = 2x - 3$. Then, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$ is equal to

(1) 7 (2) 5

(3) 2 (4) 3

Ans. **(2)**

Sol. $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$

$\Rightarrow \frac{3x-2}{x-1} + \frac{x+3}{2} = \frac{13}{2}$

$\Rightarrow 2(3x-2) + (x-1)(x+3) = 13(x-1)$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x = 2$ or 3

7. Let S_1 be the sum of frist 2n terms of an arithmetic progression. Let S_2 be the sum of first 4n terms of the same arithmetic progression. If $(S_2–S_1)$ is 1000, then the sum of the first 6n terms of the arithmetic progression is equal to :

(1) 3000 (2) 7000 (3) 5000 (4) 1000

Ans **(1)**

Sol. $S_{4n} – S_{2n} = 1000$

$\Rightarrow \frac{4n}{2}(2a + (4n–1)d) – \frac{2n}{2}(2a+(2n–1)d)=1000$

$\Rightarrow 2an + 6n^2d–nd = 1000$

$\Rightarrow \frac{6n}{2}(2a + (6n–1)d) = 3000$

$\therefore S_{6n} = 3000$

8. Let $S_1 : x^2 + y^2 = 9$ and $S_2 : (x-2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points:

(1) $\left(\frac{1}{2}, \pm\frac{\sqrt{5}}{2}\right)$ (2) $\left(2, \pm\frac{3}{2}\right)$

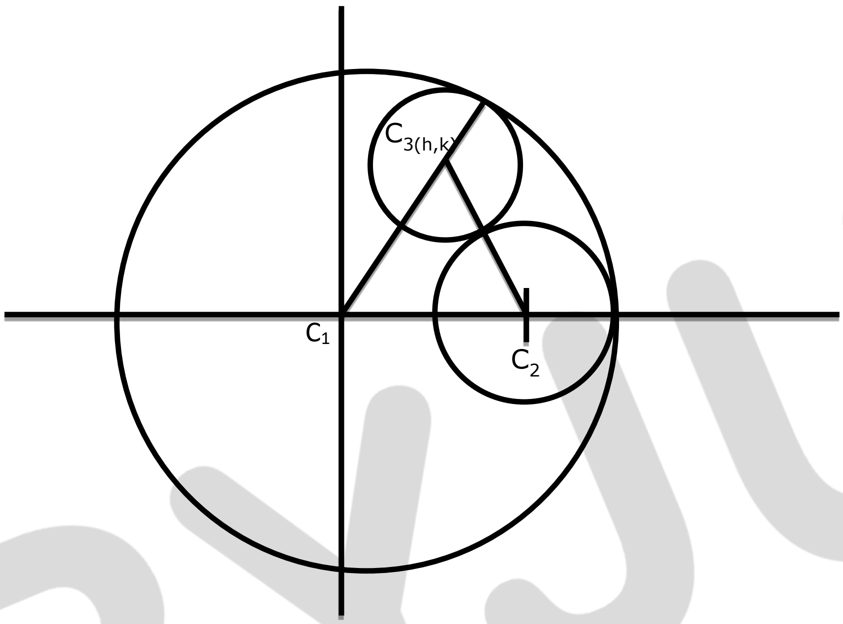
(3) $(1, \pm 2)$ (4) $(0, \pm\sqrt{3})$

Ans. **(2)**

Sol. $C_1 : (0,0)$, $r_1 = 3$

$C_2 : (2, 0)$, $r_2 = 1$

Let centre of variable circle be $C_3(h,k)$ and radius be r.



$C_3C_1 = 3 - r$

$C_2C_3 = 1 + r$

$C_3C_1 + C_2C_3 = 4$

So locus is ellipse whose focii are C_1 & C_2

And major axis is $2a = 4$ and $2ae = C_1C_2 = 2$

$\Rightarrow e = \frac{1}{2}$

$\Rightarrow b^2 = 4\left(1 - \frac{1}{4}\right) = 3$

Centre of ellipse is midpoint of C_1 & C_2 is (1,0)

Equation of ellipse is $\frac{(x-1)^2}{2^2}+\frac{y^2}{(\sqrt{3})^2}=1$

Now by cross checking the option $\left(2,\pm\frac{3}{2}\right)$ satisfied it.

9. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line x + y = 3. If R and r be the radius of circumcircle and incircle respectively of ΔABC, then (R+ r) is equal to

(1) $2\sqrt{2}$

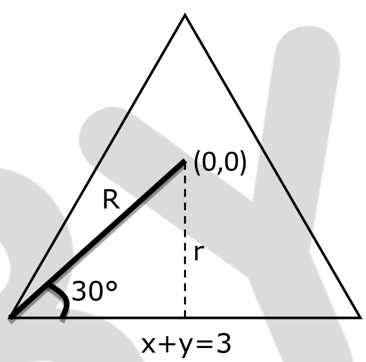
(2) $3\sqrt{2}$

(3) $7\sqrt{2}$

(4) $\frac{9}{\sqrt{2}}$

Ans. **(4)**

Sol.



r = $\left|\frac{0+0-3}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}$

sin30° = $\frac{r}{R}=\frac{1}{2}$

R = 2r

So, r + R = 3r = $3\times\left(\frac{3}{\sqrt{2}}\right)=\frac{9}{\sqrt{2}}$

10. In a triangle ABC, if $|\vec{BC}|$ = 8, $|\vec{CA}|$ = 7, $|\vec{AB}|$ = 10, then the projection of the vector $\vec{AB}$ on $\vec{AC}$ is equal to:

(1) $\frac{25}{4}$

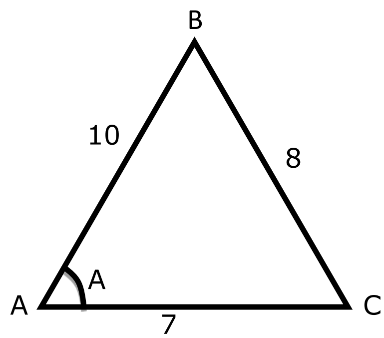
(2) $\frac{85}{14}$

(3) $\frac{127}{20}$

(4) $\frac{115}{16}$

Ans. **(2)**

Sol.



Projection of AB on AC is = AB cos A

= 10 cos A

By cosine rule

$$\cos A = \frac{10^2 + 7^2 - 8^2}{2.10.7}$$

$$= \frac{85}{140}$$

$$\Rightarrow \quad 10 \cos A = 10\left(\frac{85}{140}\right) = \frac{85}{14}$$

11. Let in a Binomial distribution, consisting of 5 independent trials, probabilities of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to:

(1) $\frac{80}{243}$

(2) $\frac{32}{625}$

(3) $\frac{128}{625}$

(4) $\frac{40}{243}$

Ans. **(2)**

Sol. $^5C_1p^1q^4 = 0.4096$ ...(1)

$^5C_2p^2q^3 = 0.2048$ ...(2)

$\frac{(1)}{(2)} \Rightarrow \frac{q}{2p} = 2 \Rightarrow q = 4p$

$p + q = 1 \Rightarrow P = \frac{1}{5}, q = \frac{4}{5}$

P (exactly 3) = $^5C_3(p)^3(q)^2 = {}^5C_3\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2$

$= 10 \times \frac{1}{125} \times \frac{16}{25} = \frac{32}{625}$

12. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors perpendicular to each other and $|\vec{a}| = |\vec{b}|$. If $|\vec{a}\times\vec{b}| = |\vec{a}|$, then the angle between the vectors $\left(\vec{a}+\vec{b}+\left(\vec{a}\times\vec{b}\right)\right)$ and $\vec{a}$ is equal to :

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(2) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(3) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$

(4) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Ans. **(2)**

Sol. Given $|\vec{a}\times\vec{b}| = |\vec{a}| = |\vec{b}|$

$$\cos\theta = \frac{\vec{a}\left(\vec{a}+\vec{b}+\left(\vec{a}\times\vec{b}\right)\right)}{|\vec{a}|.|\vec{a}+\vec{b}+\vec{a}\times\vec{b}|}$$

Let $|\vec{a}| = a$

$$\cos\theta = \frac{a^2+0+0}{a\times\sqrt{a^2+a^2+a^2}} = \frac{a^2}{a^2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

13. Let a complex number be w = $1-\sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and arg(z)–arg(w) = $\frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to:

(1) $\frac{1}{2}$

(2) 4

(3) 2

(4) $\frac{1}{4}$

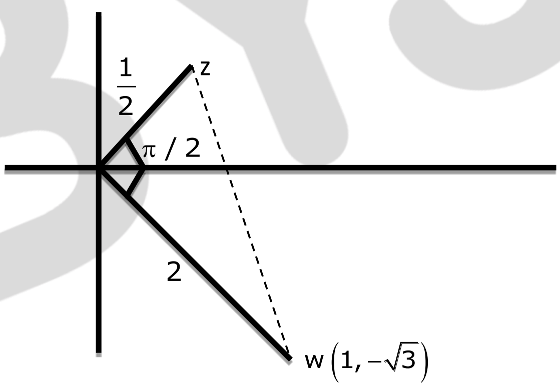
Ans. **(1)**

Sol. $w = 1-\sqrt{3}i$

$|w| = 2$

$|zw| = 1 \Rightarrow |z| = \frac{1}{|w|} = \frac{1}{2}$

$\arg(z) - \arg(w) = \pi/2$



Area of $\Delta = \frac{1}{2}.\frac{1}{2}.2 = \frac{1}{2}$

14. The area bounded by the curve $4y^2 = x^2(4-x)(x-2)$ is equal to:

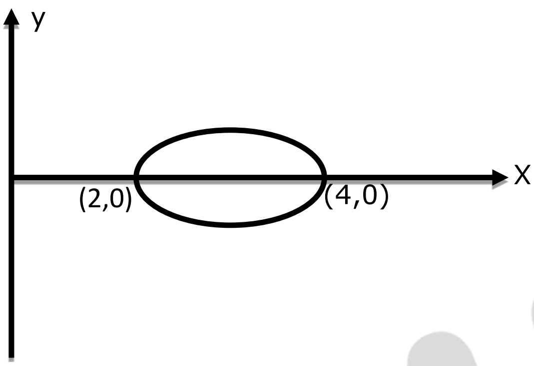
(1) $\frac{3\pi}{2}$

(2) $\frac{\pi}{16}$

(3) $\frac{\pi}{8}$ (4) $\frac{3\pi}{8}$

Ans. **(1)**

Sol. domain of $4y^2 = x^2(4-x)(x-2)$ is $[2,4] \cup \{0\}$



Area of loop = $2 \times \frac{1}{2} \times \int_2^4 x\sqrt{(4-x)(x-2)}dx$

Put , $x = 4\sin^2\theta + 2\cos^2\theta$

$dx = (8\sin\theta\cos\theta - 4\cos\theta\sin\theta)d\theta$

$= 4\sin\theta\cos\theta d\theta$

= $\int_0^{\pi/2} (4\sin^2\theta + 2\cos^2\theta)\sqrt{(2\cos^2\theta)(2\sin^2\theta)}(4\sin\theta\cos\theta)d\theta$

= $\int_0^{\pi/2} (4\sin^2\theta + 2\cos^2\theta)8(\cos\theta\sin\theta)^2 d\theta$

= $\int_0^{\pi/2} 32\sin^4\theta\cos^2\theta d\theta + \int_0^{\pi/2} 16\sin^2\theta\cos^4\theta d\theta$

Using wallis theorm

= $32.\frac{3.1.1}{6.4.2}\frac{\pi}{2} + 16.\frac{3.1.1.}{6.4.2.}\frac{\pi}{2}$

$= \pi + \pi / 2 = 3\pi / 2$

15. Define a relation R over a class of n × n real matrices A and B as "ARB iff there exists a non-singular matrix P such that $PAP^{-1} = B$"

The which of the following is true?

(1) R is reflexive, symmetric but not transitive

(2) R is symmetric, transitive but not reflexive,

(3) R is an equivalence relation

(4) R is reflexive, transitive but not symmetic

Ans. **(3)**

Sol. For reflexive

$(B, B) \in R \;\Rightarrow\; B = PBP^{-1}$

Which is true for P = I

$\therefore$ R is Reflexive

For symmetry

As $(B, A) \in R$ for matrix P

$B = PAP^{-1} \;\Rightarrow\; P^{-1}B = P^{-1}PAP^{-1}$

$\Rightarrow$ $P^{-1}BP = IAP^{-1}P = IAI$

$P^{-1}BP = A \Rightarrow A = P^{-1}BP$

$\therefore$ $(A, B) \in R$ for matrix P^{-1}

$\therefore$ R is symmetric

For transitivity

$B = PAP^{-1}$ and $A = PCP^{-1}$

$\Rightarrow$ $B = P(PCP^{-1})P^{-1}$

$\Rightarrow$ $B = P^2C(P^{-1})^2 \Rightarrow B = P^2C(P^2)^{-1}$

$\therefore$ $(B, C) \in R$ for matrix P^2

$\therefore$ R is transitive

So R is equivalence

16. If P and Q are two statements, then which of the following compound statement is a tautology?

(1) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$

(2) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$

(3) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

(4) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$

Ans. **(2)**

Sol. $(P \Rightarrow Q) \wedge \sim Q$

$\equiv (\sim P \vee Q) \wedge \sim Q$

$\equiv \sim P \wedge \sim Q) \vee (Q \wedge \sim Q)$

$\equiv (\sim P \vee \sim Q) \vee (Q \wedge \sim Q)$

$\equiv \sim (P \vee Q)$

Now,

(1) $\sim (P \vee Q) \Rightarrow P$

$\equiv (P \vee Q) \vee P$

$\equiv P \vee Q$

(2) $\sim (P \vee Q) \Rightarrow \sim P$

$\equiv (P \vee Q) \vee \sim P$

$\equiv T$

(3) $\sim (P \vee Q) \Rightarrow (P \wedge Q)$

$\equiv (P \vee Q) \vee (P \wedge Q)$

$\equiv P \vee Q$

(4) $\sim(P \vee Q) \Rightarrow Q$

$\equiv (P \vee Q) \vee Q$

$\equiv P \vee Q$

17. Consider a hyperbola $H: x^2 - 2y^2 = 4$. Let the tangent at a point $P\left(4, \sqrt{6}\right)$ meet the x-axis at Q and latus rectum at $R\left(x_1, y_1\right), x_1 > 0$. If F is a focus of H which is nearer to the point P, then the are of ΔQFR is equal to:

(1) $\sqrt{6} - 1$

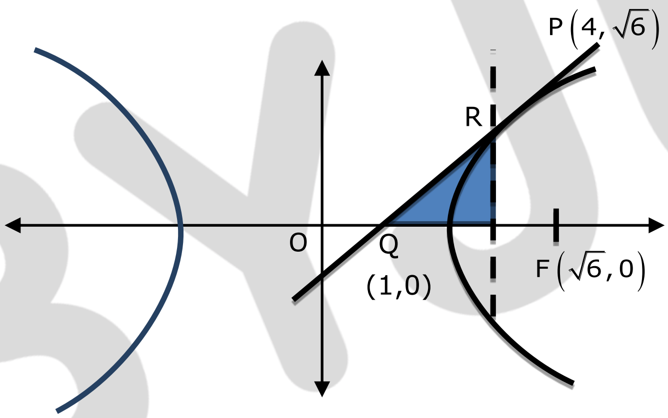
(2) $4\sqrt{6} - 1$

(3) $4\sqrt{6}$

(4) $\frac{7}{\sqrt{6}} - 2$

Ans. **(4)**

Sol.



Tangent at $P\left(4, \sqrt{6}\right)$

$4(x) - 2 \cdot \sqrt{6}(y) = 4$

$\Rightarrow 2x - \sqrt{6}(y) = 2$ ...(1)

For Q, put y = 0

Q(1,0)

Equation of Latus rectum:

$X = ae = 2\sqrt{\frac{3}{2}} = \sqrt{6}$ ...(2)

Solving (1) & (2), we get

$R\left(\sqrt{6}, 2-\frac{2}{\sqrt{6}}\right)$

Area of $\Delta QFR = \frac{1}{2} \times QF \times FR$

$= \frac{1}{2}\left(\sqrt{6}-1\right)\left(2-\frac{2}{\sqrt{6}}\right)$

$= \frac{7}{\sqrt{6}} - 2$

18. Let $f : R \to R$ be a function defined as

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin 2x}{2X} & , \text{If } x < 0 \\ b & , \text{If } x = 0 \\ \dfrac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & , \text{If } x > 0 \end{cases}$$

If f is continuous at x = 0, then the value of a + b is equal

(1) –2

(2) $-\frac{2}{5}$

(3) $-\frac{3}{2}$

(4) –3

Ans. **(3)**

Sol. 'f' is continuous at x = 0

$\Rightarrow f(0^-) = f(0) = f(0^+)$

$f(0^-) = \lim_{x \to 0^-} \frac{\sin(a+1)x + \sin 2x}{2x}$

$\lim_{x \to 0^-} \left\{ \frac{\sin(a+1)x}{(a+1)x} \cdot \frac{(a+1)}{2} + \frac{\sin(2x)}{2x} \right\}$

$= \frac{a+1}{2} + 1$ ...(1)

$f(0^+) = \lim_{x \to 0^+} \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}$

$$= \lim_{x\to 0^+} \frac{bx^3}{b.x^{5/2}.\left(\sqrt{x+bx^3}+\sqrt{x}\right)}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{1+bx^2}+1}$$

$$= \frac{1}{2} \qquad ...(2)$$

f(0) = b ...(3)

From (1),(2) and (3)

$$\therefore \frac{a+1}{2}+1 = \frac{1}{2} = b$$

$\Rightarrow$ a = –2 & b = ½

Thus, a + b = –3/2

19. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (y+1)\left((y+1)e^{x^2/2} - x\right)$, $0 < x < 2.1$, with y(2) = 0. Then the value of $\frac{dy}{dx}$ at x = 1 is equal to:

(1) $\frac{e^{5/2}}{\left(1+e^2\right)^2}$

(2) $\frac{5e^{1/2}}{\left(e^2+1\right)^2}$

(3) $-\frac{2e^2}{\left(1+e^2\right)^2}$

(4) $\frac{-e^{3/2}}{\left(e^2+1\right)^2}$

Ans. **(4)**

Sol. $\frac{dy}{dx} = (y+1)\left((y+1)e^{\frac{x^2}{2}} - x\right)$

$$\Rightarrow \frac{-1}{(y+1)^2}\frac{dy}{dx} - x\left(\frac{1}{y+1}\right) = -e^{\frac{x^2}{2}}$$

Put, $\frac{1}{y+1} = z$

$$-\frac{1}{(y+1)^2}.\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore \dfrac{dz}{dx} + z(-x) = -e^{\frac{x^2}{2}}$

I.F = $e^{\int -x dx} = e^{\frac{-x^2}{2}}$

$z.\left(e^{-\frac{x^2}{2}}\right) = -\int e^{-\frac{x^2}{2}}.e^{\frac{x^2}{2}} dx = -\int 1.dx = -x + C$

$\Rightarrow \dfrac{e^{-\frac{x^2}{2}}}{y+1} = -x + C \quad ...(1)$

Given y = 0 at x = 2

Put in (1)

$\dfrac{e^{-2}}{0+1} = -2 + C$

C = $e^{-2} + 2 \quad ...(2)$

From (1) and (2)

$y + 1 = \dfrac{e^{-x^2/2}}{e^{-2} + 2 - x}$

Again, at x = 1

$\Rightarrow \ y + 1 = \dfrac{e^{3/2}}{e^2 + 1}$

$\Rightarrow \ y + 1 = \dfrac{e^{3/2}}{e^2 + 1}$

$\therefore \left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{e^{3/2}}{e^2+1}\left(\dfrac{e^{3/2}}{e^2+1} \times e^{1/2} - 1\right)$

$= -\dfrac{e^{3/2}}{(e^2+1)^2}$

20. Let a tangent be drawn to the ellipse $\dfrac{x^2}{27} + y^2 = 1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$ where $\theta \in \left(0, \dfrac{\pi}{2}\right)$. Then the value of θ such that the sum of intercepts on axes made by tangent is minimum is equal to :

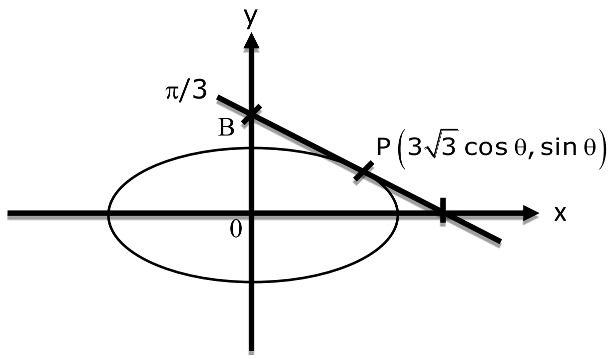
(1) $\dfrac{\pi}{8}$

(2) $\dfrac{\pi}{6}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

Ans. **(2)**

Sol.



Equation of tangent

$$\frac{x}{3\sqrt{3}}\cos\theta + y\sin\theta = 1$$

$$A\left(\frac{3\sqrt{3}}{\cos\theta}, 0\right), B\left(0, \frac{1}{\sin\theta}\right)$$

Now sum of intercept $= \frac{3\sqrt{3}}{\cos\theta} + \frac{1}{\sin\theta}$

Let $y = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$$y' = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$$

$$y' = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

SECTION – B

1. Let P be a plane containing the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+5}{2}$ and parallel to the line $\frac{x-3}{4} = \frac{y-2}{-3} = \frac{z+5}{7}$. If the point $(1, -1, \alpha)$ lies on the plane P, then the value of $|5\alpha|$ is equal to __________ .

Ans. **(38)**

Sol. DR's of normal $\vec{n} \equiv \vec{b}_1 \times \vec{b}_2$

$$\vec{n} = \begin{vmatrix} i & j & i \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix}$$

(34, −13, −25)

$P \equiv 34(x-1) - 13(y+6) - 25(z+5) = 0$

$Q(1, -1, \alpha)$ lies on P.

$\Rightarrow$ 3(1–1) -13(–1+6) -25(α+5) = 0

$\Rightarrow$ –25(α+5) =65

$\Rightarrow$ +5α = –38

$\Rightarrow$ |5α| = 38

2. $\sum_{r=1}^{10} r!\left(r^3 + 6r^2 + 2r + 5\right) = \alpha(11!)$

Then the value of α is equal to _______ .

Ans. **(160)**

Sol. $T_r = r!\ \left((r+1)(r+2)(r+3) - 9r - 1\right)$

$= (r+3)! - 9r.r! - r!$

$= (r+3)! - 9(r+1-1))r! - r!$

$= (r+3)! - 9(r+1)! + 8r!$

$= \{(r+3)! - (r+1)!\} - 8\{(r+1)! - r!\}$

Now, $\sum_{r=1}^{10} T_r = \{13! + 12! - 3! - 2!\} - 8\ \{11! - 1!\}$

$= 13! + 12! - 8(11!)$

$= (13 \times 12 + 12 - 8)11!$

$= 160 \times 11!$

Thus, $\alpha = 160$

3. The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}, x \neq 1,$ is equal to ________.

Ans. **(210)**

Sol. Given, $\left(\left(x^{1/3}+1\right)-\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10} = \left(x^{1/3}-x^{-1/2}\right)^{10}$

General term, $T_{r+1} = {}^{10}C_r \left(x^{1/3}\right)^{10-r}\left(-x^{-1/2}\right)^r$

For term independent of x

$\frac{10-r}{3}-\frac{r}{2}=0 \Rightarrow 20-2r-3r=0$

$\Rightarrow r = 4$

Therefore required term, $T_5 = {}^{10}C_4 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$

4. Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$. If $\sum_{k=0}^{10}(2^2+3k)\ {}^nC_k = \alpha.3^{10}+\beta.2^{10}, \alpha, \beta \in R,$ then $\alpha+\beta$ is equal to _______.

Bonus

Sol. n must be equal to 10

$\sum_{k=0}^{10}(2^2+3k)\ {}^nC_k$

$= \sum_{k=0}^{10}(4+3k)\ {}^nC_k$

$= 4\sum_{k=0}^{10} {}^nC_k + 3\sum_{k=0}^{10} k\,{}^nC_k$

$= 4(2^{10}) + 3\times 10\times 2^9$

$= 19 \times 2^{10}$

$\therefore \alpha = 0$ and $\beta = 19$

Thus, $\alpha + \beta = 19$

5. Let P(x) be a real polynomial of degree 3 which vanishes at x = –3. Let P(x) have local minima at x = 1, local maxima at x = –1 and $\int_{-1}^{1} P(x)dx = 18$, then the sum of all the coefficients of the polynomial P(x) is equal to _______________ .

Ans. **(8)**

Sol. $P'(x) = a(x + 1)(x - 1)$

$\therefore P(x) = \frac{ax^3}{3} - ax + C$

P(–3) = 0 (given)

$\Rightarrow a(-9 + 3) + C = 0$

$\Rightarrow 6a = C$ ...(i)

Also, $\int_{-1}^{1} P(x)dx = 18 \Rightarrow \int_{-1}^{1} \left(a\left(\frac{x^3}{3} - x\right) + C \right) dx = 18$

$\Rightarrow 0 + 2C = 18 \Rightarrow C = 9$

from(i)

$a = \frac{3}{2}$

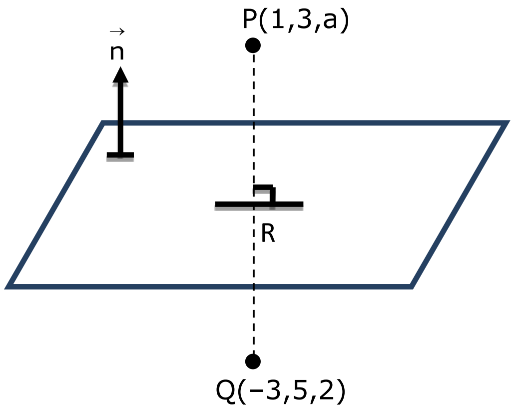
$\therefore P(x) = \frac{x^3}{2} - \frac{3}{2}x + 9$

Sum of co-efficient = –1 + 9 = 8

6. Let the mirror image of the point (1, 3, a) with respect to the plane $\vec{r}.\left(2\hat{i} - \hat{j} + \hat{k}\right) - b = 0$ be (–3, 5, 2). Then, the value of |a+b| is equal to ______.

Ans. **(1)**

Sol.



Plane : 2x –y + z = b

$R \equiv \left(-1, 4, \frac{a+2}{2}\right) \rightarrow$ on plane

$\therefore -2-4+\frac{a+2}{2}=b$

$\Rightarrow$ a + 2= 2b + 12 $\Rightarrow$ a = 2b + 10 ...(1)

PQ <4, –2, a –2>

$\therefore \frac{2}{4}=\frac{-1}{-2}=\frac{1}{a-2} \Rightarrow a-2=2 \Rightarrow a=4, b=-3$

$\therefore |a+b|=1$

7. If f(x) and g(x) are two polynomials such that the polynomial $P(x) = f(x^3) + x\,g(x^3)$ is divisible by $x^2 + x + 1$, then P(1) is equal to _______.

Ans. **0**

Sol. roots of x^2+x+1 are ω and ω^2 now

$Q(\omega)=f(1)+\omega g(1)=0$ ...(1)

$Q(\omega^2)=f(1)+\omega^2 g(1)=0$ ...(2)

Adding (1) and (2)

$\Rightarrow$ 2f(1) – g(1) = 0

$\Rightarrow$ g(1) = 2f (1)

$\Rightarrow$ f(1) = g(1) = 0

Therefore , Q(1) = f(1) + g(1) = 0 + 0 = 0

8. Let I be an identity matrix of order 2 × 2 and $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$. Then the value of $n \in N$ for which $P^n = 5I - 8P$ is equal to ________.

Ans. **(6)**

Sol. $P^2 = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$

$P^4 = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -15 & 11 \end{bmatrix}$

$P^6 = \begin{bmatrix} -4 & 3 \\ -15 & 11 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$

and $5I - 8P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 8\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$

$\Rightarrow P^6 = 5I - 8P$

Thus, n = 6

9. Let $f : R \to R$ satisfy the equation f(x + y) = f(x). f(y) for all x, $y \in R$ and f(x) $\neq 0$ for any $x \in R$. If the function f is differentiable at x = 0 and f'(0) = 3, then $\lim_{h\to 0}\frac{1}{h}(f(h) - 1)$ is equal to _______.

Ans. **(3)**

Sol. f(x + y) = f(x) .f(y) then

$\Rightarrow$ $f(x) = a^{kx}$

$\Rightarrow$ $f'(x) = (a^{kx})k\,\ell n\, a$

$\Rightarrow$ $f'(0) = k\,\ell n\, a = 3$ (given f'(0) = 3)

$\Rightarrow$ $a = e^{3/k}$

$\therefore$ $f(x) = (e^{3/k})^{kx} = e^{3x}$

Now, $\lim_{h\to 0}\frac{f(h) - 1}{h} = \lim_{h\to 0}\left(\frac{e^{3h} - 1}{3h} \times 3\right) = 1 \times 3 = 3$

10. Let y = y(x) be the solution of the differential equation $xdy - ydx = \sqrt{(x^2 - y^2)}dx,\ x \geq 1$, with $y(1) = 0$. If the area bounded by the line x = 1, x = e^{π}, y = 0 and y = y(x) is $\alpha e^{2\pi} + b$, then the value of $10(\alpha + \beta)$ is equal to ________.

Ans. **(4)**

Sol. $xdy - ydx = \sqrt{x^2 - y^2}dx \Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \frac{y^2}{x^2}}dx \Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$

$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ell n|x| + c$

At $x = 1, y = 0 \Rightarrow c = 0$

$y = x \sin(\ell nx)$

$$A = \int_1^{e^{\pi}} x \sin(\ell nx)\,dx$$

$$x = e^t, dx = e^t dt = \int_0^{\pi} e^{2t} \sin(t)dt$$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5}(2\sin t - \cos t)\right)_0^{\pi} = \frac{1 + e^{2\pi}}{5}$$

$$\alpha = \frac{1}{5},\ \beta = \frac{1}{5}$$

Thus, $10(\alpha + \beta) = 4$
