JEE MAIN – 2021

16th Mar Shift – I Maths Question Paper

Section A

Multiple Choice Question:

1. Consider three observations a, b and c such that b = a+c.If the standard deviation of a+2, b+2, c+2 is d, then which of the following is true?

(1) $b^2 = a^2 + c^2 + 3d^2$

(2) $b^2 = 3(a^2 + c^2) - 9d^2$

(3) $b^2 = 3(a^2 + c^2) + 9d^2$

(4) $b^2 = 3(a^2 + c^2 + d^2)$

Ans. (2)

Sol. for a, b, c

mean = $\bar{x} = \frac{a+b+c}{3}$

$\bar{x} = \frac{2b}{3}$

S.D. of a, b, c = d

$d^2 = \frac{a^2+b^2+c^2}{3} - \frac{4b^2}{9}$

$b^2 = 3a^2 + 3c^2 - 9d^2$

2. Let a vector $\alpha\hat{i} + \beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{\imath} + \hat{\jmath}$ by an angle 45° about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices (α, β), $(0, \beta)$ and (0, 0) is equal to:

(1) 1

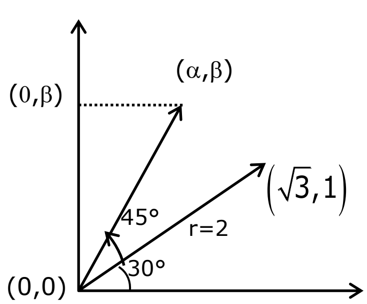
(2) $\frac{1}{2}$

(3) $\frac{1}{\sqrt{2}}$

(4) $2\sqrt{2}$

Ans. (2)

Sol.



$(\alpha, \beta) \equiv (2\cos 75°, 2\sin 75°)$

Area = $\frac{1}{2}$ (2 cos75°)(2 sin 75°)

= sin(150°) = $\frac{1}{2}$ square unit

3. If for a>0, the feet of perpendiculars from the points A(a, –2a, 3) and B(0, 4, 5) on the plane $lx + my + nz = 0$ are points C(0, –a, –1) and D respectively, then the length of line segment CD is equal to :

(1) $\sqrt{41}$

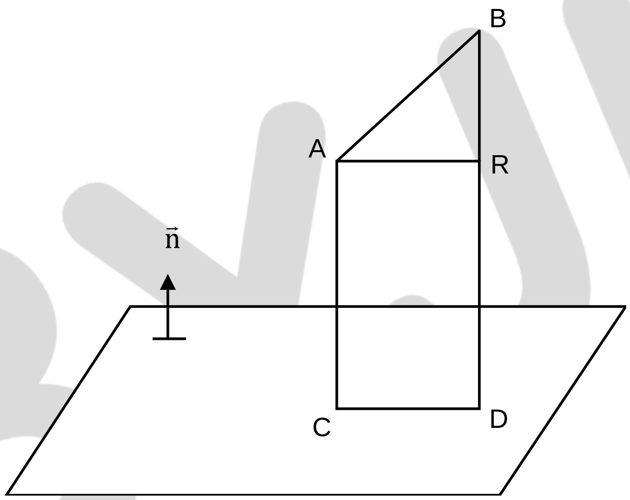
(2) $\sqrt{55}$

(3) $\sqrt{31}$

(4) $\sqrt{66}$

Ans. **(4)**

Sol.



Direction cosines of plane =λ(direction cosines of line AC)

$\therefore$ direction cosines of plane = $\lambda a, -\lambda a, 4\lambda$

Hence equation plane is: $ax - ay + 4z = 0$

$\because$ point C lies on plane

$\therefore a(0) - a(-a) + 4(-1) = 0 \Rightarrow a = 2 \quad (\because a > 0)$

So plane is $2x - 2y + 4z = 0$, $C \equiv (0, -2, -1)$

So for coordinates of D,

$\frac{x-0}{2} = \frac{y-4}{-2} = \frac{z-5}{4} = -\left(\frac{2(0)-2(4)+4(5)}{2^2+2^2+4^2}\right)$

$D \equiv (-1,5,3)$

$\therefore CD = \sqrt{66}$ unit

4. The range of $a \in R$ for which the function

$f(x) = (4a-3)(x + \log_e 5) + 2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right),\ x \neq 2n\pi, n \in N$ has critical points, is :

(1) $\left[-\frac{4}{3}, 2\right]$

(2) $[1, \infty)$

(3) $(-\infty, -1]$

(4) $(-3, 1)$

Ans. **(1)**

Sol. $f(x) = (4a-3)(x + \ln 5) + 2(a-7)\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \cdot \sin^2\frac{x}{2}\right)$

$f(x) = (4a-3)(x + \log_e 5) + (a-7)\sin x$

$\Rightarrow f'(x) = (4a-3) + (a-7)\cos x = 0$

$\Rightarrow \cos x = \frac{-(4a-3)}{a-7}$

$\Rightarrow\ -1 \leq -\frac{(4a-3)}{a-7} < 1 \qquad (\because -1 \leq \cos x \leq 1)$

$-1 < \frac{4a-3}{a-7} \leq 1$

$\frac{4a-3}{a-7} - 1 \leq 0$ and $\frac{4a-3}{a-7} + 1 > 0$

$\Rightarrow a \in \left[\frac{4}{3}, 7\right)$ and $a \in (-\infty, 2) \cup (7, \infty)$

$\Rightarrow \frac{-4}{3} \leq a < 2$

5. Let the functions f: R→R and g: R →R be defined as :

$f(x) = \begin{cases} x+2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 3x-2, & x \geq 1 \end{cases}$

Then, the number of points in R where (fog)(x) is NOT differentiable is equal to :

(1) 1

(2) 2

(3) 3

(4) 0

Ans. **(1)**

Sol. $fog(x)=\begin{cases} x^3+2, & x<0 \\ x^6, & 0\le x<1 \\ (3x-2)^2, & x\ge 1 \end{cases}$

Clearly fog(x) is discontinuous at x = 0 then non-differentiable at x = 0

Now,

at x = 1

$\text{RHD} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(3(1+h)-2)^2-1}{h} = 6$

$\text{LHD} = \lim_{h\to 0^-} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^-} \frac{(1-h)^6-1}{-h} = 6$

Number of points of non-differentiability = 1

6. Let a complex number z, $|z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \le 2$. Then, the largest value of |z| is equal to _____

(1) 5
(2) 8
(3) 6
(4) 7

Ans. **(4)**

Sol. $\frac{|z|+11}{(|z|-1)^2} \ge \frac{1}{2}$

$2|z| + 22 \ge (|z| - 1)^2$

$2|z| + 22 \ge |z|^2 - 2|z| + 1$

$|z|^2 - 4|z| - 21 \le 0$

$(|z| - 7)(|z| + 3) \le 0$

$\Rightarrow |z| \le 7$

$\therefore |z|_{max} = 7$

7. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

(1) $\frac{3}{4}$
(2) $\frac{52}{867}$
(3) $\frac{39}{50}$
(4) $\frac{22}{425}$

Ans. **(3)**

Sol. $P\left(\overline{S}_{\text{missing}} \mid \text{both found spade}\right) = \dfrac{P\left(\overline{S_m} \cap BFS\right)}{P(BFS)}$

$$= \frac{\left(1-\frac{13}{52}\right)\times\frac{13}{51}\times\frac{12}{50}}{\left(1-\frac{13}{52}\right)\times\frac{13}{51}\times\frac{12}{50}+\frac{13}{52}\times\frac{12}{51}\times\frac{11}{50}}$$

$$= \frac{39}{50}$$

8. If n is the number of irrational terms in the expansion of $\left(3^{\frac{1}{4}}+5^{\frac{1}{8}}\right)^{60}$, then (n−1) is divisible by :

(1) 8
(2) 26
(3) 7
(4) 30

Ans. **(2)**

Sol. $T_{r+1} = {}^{60}C_r\left(3^{1/4}\right)^{60-r}\left(5^{1/8}\right)^r$

rational if $\dfrac{60-r}{4}, \dfrac{r}{8}$, both are whole numbers, $r \in \{0,1,2,....60\}$

$\dfrac{60-r}{4} \in W \Rightarrow r \in \{0,4,8,....60\}$

and $\dfrac{r}{8} \in W \Rightarrow r \in \{0,8,16,...56\}$

$\therefore$ Common terms $r \in \{0,8,16,....56\}$
So 8 terms are rational
Then irrational terms = 61 – 8 = 53 = n
$\therefore n - 1 = 52 = 13 \times 2^2$
factors 1,2,4,13,26,52

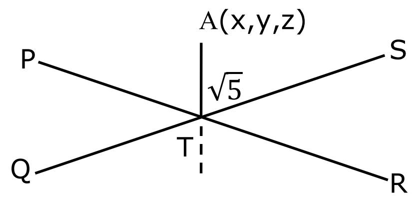
9. Let the position vectors of two points P and Q be $3\hat{i}-\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}-4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are (4, −1, 2) and (−2,1,−2) respectively. Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overrightarrow{PR}$ and $\overrightarrow{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is :

(1) $\sqrt{5}$
(2) $\sqrt{171}$
(3) $\sqrt{227}$
(4) $\sqrt{482}$

Ans. (2)

Sol. $\vec{p} = 3\hat{i} - \hat{j} + 2\hat{k}$ & $\vec{q} = \hat{i} + 2\hat{j} - 4\hat{k}$

$\overrightarrow{v_{PR}} = (4, -1, 2)$ & $\overrightarrow{v_{QS}} = (-2, 1, -2)$



$L_{PR}: \vec{r} = (3\hat{i} - \hat{j} + 2\hat{k}) + \lambda(4\hat{i} - 1\hat{j} + 2\hat{k})$

$L_{QS}: \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \mu(-2\hat{i} + 1\hat{j} - 2\hat{k})$

Now T on PR $= (3 + 4\lambda, -1 - \lambda, 2 + 2\lambda)$

Similarly T on QS $= (1 - 2\mu, 2 + \mu, -4 - 2\mu)$

For λ & μ: $\left.\begin{array}{l} 3 + 4\lambda = 1 - 2\mu \Rightarrow \mu + 2\lambda = -1 \\ -1 - \lambda = 2 + \mu \Rightarrow \mu + \lambda = -3 \end{array}\right\} \begin{array}{l} \lambda = 2 \\ \mu = -5 \end{array}$

And $2 + 2\lambda = -4 - 2\mu$

$\Rightarrow T: (11, -3, 6)$

D.R. of TA $= \overrightarrow{v_{QS}} \times \overrightarrow{v_{PR}}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -2 \\ 4 & -1 & 2 \end{vmatrix} = 0\hat{i} - 4\hat{j} - 2\hat{k}$$

$L_{TA}: \vec{r} = (11\hat{i} - 3\hat{j} + 6\hat{k}) + \lambda(-4\hat{j} - 2\hat{k})$

Now $A = (11, -3 - 4\lambda, 6 - 2\lambda)$

$TA = \sqrt{5}$

$\Rightarrow (4\lambda)^2 + (2\lambda)^2 = 5$

$\Rightarrow 16\lambda^2 + 4\lambda^2 = 5 \Rightarrow \lambda = \pm\frac{1}{2}$

A: (11, –5, 5) or A: (11, –1, 7)

$|A| = \sqrt{121 + 25 + 25}$ or $|A| = \sqrt{121 + 1 + 49}$

$= \sqrt{171}$ or $\sqrt{171}$

10. If the three normals drawn to the parabola, $y^2=2x$ pass through the point (a, 0) $a \neq 0$, then 'a' must be greater than:

(1) 1

(2) $\frac{1}{2}$

(3) $-\frac{1}{2}$

(4) −1

Ans. (1)

Sol. Let the equation of the normal is

$y = mx - 2am - am^3$

here $4a = 2 \Rightarrow a = \frac{1}{2}$

$y = mx - m - \frac{1}{2}m^3$

It passes through A(a, 0) then

$0 = am - m - \frac{1}{2}m^3$

$m = 0, m^2 - 2(a-1) = 0$

For real values of m

$2(a - 1) > 0$

$\therefore a > 1$

11. Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1} + 3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to :

(1) $\tan^{-1}\left(\frac{3}{2}\right)$

(2) $\cot^{-1}\left(\frac{3}{2}\right)$

(3) $\frac{\pi}{2}$

(4) $\tan^{-1}(3)$

Ans. (2)

Sol. $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{6^r(3-2)}{\left(1+\left(\frac{3}{2}\right)^{2r+1}\right)2^{2r+1}}\right)$

$$\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2^r\cdot 3^{r+1}-3^r2^{r+1}}{\left(1+\left(\frac{3}{2}\right)^{2r+1}\right)2^{2r+1}}\right)$$

$$\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{\left(\frac{3}{2}\right)^{r+1}-\left(\frac{3}{2}\right)^{r}}{1+\left(\frac{3}{2}\right)^{r+1}\left(\frac{3}{2}\right)^{r}}\right)=\sum_{r=1}^{\infty}\left[\tan^{-1}\left(\frac{3}{2}\right)^{r+1}-\tan^{-1}\left(\frac{3}{2}\right)^{r}\right]=\frac{\pi}{2}-\tan^{-1}\frac{3}{2}=\cot^{-1}\frac{3}{2}$$

12. The number of roots of the equation, $(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$ in the interval $[0, \pi]$ is equal to :

(1) 3 (3) 4

(2) 2 (4) 8

Ans. (3)

Sol. $(81)^{\sin^2 x}+(81)^{1-\sin^2 x}=30$

$$(81)^{\sin^2 x}+\frac{81}{(81)^{\sin^2 x}}=30$$

Let $(81)^{\sin^2 x}=t$

$t+\frac{81}{t}=30 \quad \Rightarrow t^2 + 81 = 30t$

$\Rightarrow t^2 - 30t + 81 = 0$
$\Rightarrow t^2 - 27t - 3t + 81 = 0$
$\Rightarrow (t - 3)(t - 27) = 0$
$\Rightarrow t = 3, 27$

$\Rightarrow (81)^{\sin^2 x}=3, 3^3$

$\Rightarrow 3^{4\sin^2 x}=3^1, 3^3$

$\Rightarrow 4\sin^2 x=1, 3$

$\Rightarrow \sin^2 x=\frac{1}{4}, \frac{3}{4}$

in $[0, \pi]$ $\sin x \geq 0$

$\sin x=\frac{1}{2}, \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$

Number of solutions = 4

13. If y=y(x) is the solution of the differential equation, $\frac{dy}{dx} + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function y(x) over **R** is equal to :

(1) 8

(3) $-\frac{15}{4}$

(2) $\frac{1}{2}$

(4) $\frac{1}{8}$

Ans. **(4)**

Sol. $\frac{dy}{dx} + 2\tan x \cdot y = \sin x$

I.F. $= e^{\ln(\sec^2 x)} = \sec^2 x$

$\Rightarrow y\sec^2 x = \int \tan x \sec x\, dx = \sec x + c$

Now $x = \frac{\pi}{3}, y = 0$

$c = -2$

$\therefore y = \cos x - 2\cos^2 x$

$y = -2\left(\cos^2 x - \frac{1}{2}\cos x\right) = -2\left(\left(\cos x - \frac{1}{4}\right)^2 - \frac{1}{16}\right)$

$y = \frac{1}{8} - 2\left(\cos x - \frac{1}{4}\right)^2$

$\therefore \quad y_{max} = \frac{1}{8}$

14. Which of the following Boolean expression is a tautology?

(1) $(p \wedge q) \wedge (p \to q)$

(3) $(p \wedge q) \vee (p \to q)$

(2) $(p \wedge q) \vee (p \vee q)$

(4) $(p \wedge q) \to (p \to q)$

Ans. **(4)**

Sol.

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	T	T	T	T
F	T	F	T	T	T
T	F	F	T	F	T
F	F	F	F	T	T

15. Let $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}, i = \sqrt{-1}$. Then, the system of linear equations $A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has :

(1) No solution

(2) Exactly two solutions

(3) A unique solution

(4) Infinitely many solutions

Ans. **(1)**

Sol. $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}\begin{bmatrix} i & -i \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^4 = 4\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 = 64\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 64\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$128\begin{bmatrix} x-y \\ -x+y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix} \Rightarrow 128(x-y) = 8$$

$\Rightarrow x - y = \frac{1}{16}$ ....(1) and $128(-x + y) = 64 \Rightarrow x - y = \frac{-1}{2}$ ...(2)

$\Rightarrow$ no solution (from eq. (1) & (2))

16. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x+\cos x) = \frac{1}{2}(\log_{10} n-1)$, $n>0$,

then the value of n is equal to :

(1) 16

(2) 20

(3) 12

(4) 9

Ans. **(3)**

Sol. $\log_{10}(\sin x) + \log_{10}(\cos x) = -1$

$\sin x \cdot \cos x = \frac{1}{10}$ ....(1)

and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$

$\Rightarrow \sin x + \cos x = \left(\frac{n}{10}\right)^{\frac{1}{2}}$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{n}{10}$ (squaring)

$\Rightarrow 1 + 2\left(\frac{1}{10}\right) = \frac{n}{10}$ (using equation(1))

$\Rightarrow \frac{n}{10} = \frac{12}{10} \Rightarrow n = 12$

17. The locus of the midpoints of the chord of the circle, $x^2+y^2=25$ which is tangent to the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is :

(1) $(x^2+y^2)^2 - 16x^2 + 9y^2 = 0$

(2) $(x^2+y^2)^2 - 9x^2 + 144y^2 = 0$

(3) $(x^2+y^2)^2 - 9x^2 - 16y^2 = 0$

(4) $(x^2+y^2)^2 - 9x^2 + 16y^2 = 0$

Ans. (4)

Sol. tangent of hyperbola

$y = mx \pm \sqrt{9m^2 - 16}$ ...(i)

which is a chord of circle with mid-point (h, k)
so equation of chord $T = S_1$
$hx + ky = h^2 + k^2$

$y = -\frac{hx}{k} + \frac{h^2+k^2}{k}$ ...(ii)

by (i) and (ii)

$m = -\frac{h}{k}$ and $\sqrt{9m^2 - 16} = \frac{h^2+k^2}{k}$

$$9\frac{h^2}{k^2}-16=\frac{\left(h^2+k^2\right)^2}{k^2}$$

locus $9x^2 - 16y^2 = (x^2 + y^2)^2$

18. Let [x] denote greatest integer less than or equal to x. If for $n \in N$,

$\left(1-x+x^3\right)^n=\sum_{j=0}^{3n} a_j x^j$, then $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$ is equal to :

(1) 1
(2) n
(3) 2^{n-1}
(4) 2

Ans. **(1)**

Sol. $\left(1-x+x^3\right)^n=\sum_{j=0}^{3n} a_j x^j$

$(1-x + x^3)^n = a_0 + a_1x + a_2x^2 + ... + a_{3n} x^{3n}$

Put x = 1

$1 = a_0 + a_1 + a_2 + a_3 + a_4 + ... + a_{3n}$ ...(1)

Put x = –1

$1 = a_0 - a_1 + a_2 - a_3 + a_4$ ...... $(-1)^{3n}a_{3n}$ ...(2)

Add (1) + (2)

$\Rightarrow a_0 + a_2 + a_4 + a_6 + ... = 1$

Sub (1) – (2)

$\Rightarrow a_1 + a_3 + a_5 + a_7 + ... = 0$

Now $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$

$= (a_0 + a_2 + a_4 + ...) + 4(a_1 + a_3 + ...)$

$= 1 + 4\times0$

$= 1$

19. Let P be a plane lx+my+nz=0 containing the line, $\frac{1-x}{1}=\frac{y+4}{2}=\frac{z+2}{3}$. If plane P divides the line segment AB joining points A(−3, −6, 1) and B(2, 4, −3) in ratio k : 1 then the value of k is equal to :

(1) 1.5

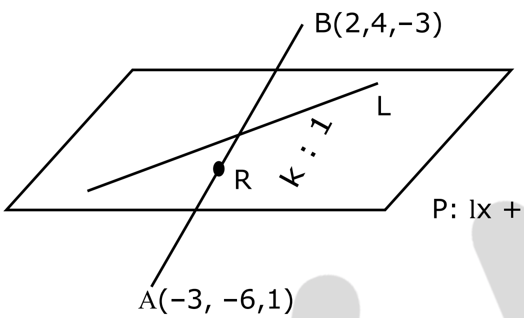
(2) 2

(3) 4

(4) 3

Ans. **(2)**

Sol.



$L=\frac{x-1}{-1}=\frac{y+4}{2}=\frac{z+2}{3}$

P: lx + my + nz = 0

Line lies on plane

$-\ell + 2m + 3n = 0$ ...(1)

Point on line (1,−4,−2) lies on plane

$\ell - 4m - 2n = 0$ ...(2)

from (1) & (2)

$-2m + n = 0 \Rightarrow 2m = n$

$\ell = 3n + 2m \Rightarrow \ell = 4n$

$\ell : m : n :: 4n : \frac{n}{2} : n$

$\ell : m : n :: 8n : n : 2n$

$\ell : m : n :: 8 : 1 : 2$

Now equation of plane is 8x + y + 2z = 0

R divide AB is ratio k : 1

$R:\left(\frac{-3+2k}{k+1},\frac{-6+4k}{k+1},\frac{1-3k}{k+1}\right)$ lies on plane

$8\left(\frac{-3+2k}{k+1}\right)+\left(\frac{-6+4k}{k+1}\right)+2\left(\frac{1-3k}{k+1}\right)=0$

$-24 + 16k - 6 + 4k + 2 - 6k = 0$

$-28 + 14k = 0$

$k = 2$

20. The number of elements in the set $\{x \in R : (|x| - 3)|x + 4| = 6\}$ is equal to :

(1) 2

(2) 1

(3) 3

(4) 4

Ans. **(1)**

Sol. **Case-1** $x \le -4$

$(-x - 3)(-x - 4) = 6$

$\Rightarrow (x + 3)(x + 4) = 6$

$\Rightarrow x^2 + 7x + 6 = 0$

$\Rightarrow x = -1$ or -6

but $x \le -4$

$x = -6$

Case-2 $x \in (-4, 0)$

$(-x - 3)(x + 4) = 6$

$\Rightarrow -x^2 - 7x - 12 - 6 = 0$

$\Rightarrow x^2 + 7x + 18 = 0$

$D < 0$ No solution

Case-3 $x \ge 0$

$(x - 3)(x + 4) = 6$

$\Rightarrow x^2 + x - 12 - 6 = 0$

$\Rightarrow x^2 + x - 18 = 0$

$x = \frac{-1 \pm \sqrt{1+72}}{2}$

$\therefore x = \frac{\sqrt{73}-1}{2}$ only

Hence 2 elements only

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Section B

Integer Type:

1. Let f: (0, 2) → R be defined as $f(x) = \log_2\left(1+\tan\left(\frac{\pi x}{4}\right)\right)$. Then,

$\lim_{n\to\infty}\frac{2}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f(1)\right)$ is equal to _______

Ans. (1)

Sol.

$$E = 2\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$E = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\tan\frac{\pi x}{4}\right)dx \quad \ldots(i)$$

replacing x →1 – x

$$E = \frac{2}{\ln 2}\int_0^1 \ln\left(1+tan\frac{\pi}{4}(1-x)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\tan\left(\frac{\pi}{4}-\frac{\pi}{4}x\right)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\frac{1-\tan\frac{\pi}{4}x}{1+\tan\frac{\pi}{4}x}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \left(\ell n2-\ell n\left(1+\tan\frac{\pi x}{4}\right)\right)dx \quad \ldots.(ii)$$

equation (i) + (ii)
E = 1

2. The total number of 3×3 matrices A having entries from the set {0, 1, 2, 3} such that the sum of all the diagonal entries of AA^T is 9, is equal to ____

Ans. **(766)**

Sol. $AA^T = \begin{bmatrix} x & y & z \\ a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} x & a & d \\ y & b & e \\ z & c & f \end{bmatrix}$

$$= \begin{bmatrix} x^2+y^2+z^2 & ax+by+cz & dx+ey+fz \\ ax+by+cz & a^2+b^2+c^2 & ad+be+cf \\ dx+ey+fz & ad+be+cf & d^2+e^2+f^2 \end{bmatrix}$$

$Tr(AA^T) = x^2 + y^2 + z^2 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 9$

all→1	=1
one 3, rest = 0	$\frac{9!}{8!} = 9$
two 2 , one 1 & rest 0	$\frac{9!}{2!6!} = 63 \times 4 = 252$
one 2 , five 1, rest 0	$\frac{9!}{5!3!} = 63 \times 8 = 504$
Total	= 766

3. Let $f:R\to R$ be a continuous function such that $f(x)+f(x+1)=2$, for all $x \in R$. If $I_1 = \int_0^8 f(x)\,dx$ and $I_2 = \int_{-1}^3 f(x)\,dx$, then the value of $I_1 + 2I_2$ is equal to ____

Ans. **(16)**

Sol. $f(x) + f(x + 1) = 2$ …..(i)
$x \to (x + 1)$
$f(x + 1) + f(x + 2) = 2$ …..(ii)
by (i) & (ii)
$f(x) - f(x + 2) = 0$
$f(x + 2) = f(x)$
f(x) is periodic with T = 2

$$I_1 = \int_0^{2\times 4} f(x)dx = 4\int_0^2 f(x)dx$$

$$I_2 = \int_{-1}^3 f(x)dx = \int_0^4 f(x+1)dx = \int_0^4 (2-f(x))dx$$

$I_2 = 8 - 2\int_0^2 f(x)dx$

$I_1 + 2I_2 = 16$

4. Consider an arithmetic series and a geometric series having four initial terms from the set {11,8,21,16,26,32,4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to ____

Ans. **(3)**

Sol. By observation
A.P : 11, 16, 21, 26 …….
G.P : 4, 8, 16, 32 …….
So common terms are 16, 256, 4096

5. If the normal to the curve $y(x) = \int_0^x (2t^2 - 15t + 10)dt$ at a point (a, b) is parallel to the line x+3y = −5, a>1, then the value of |a+6b| is equal to ____

Ans. **(406)**

Sol. $y'(x) = (2x^2 - 15x + 10)$
at point (a, b) normal is
$3 = (2a^2 - 15a + 10)$
$\Rightarrow 2a^2 - 15a + 7 = 0$
$\Rightarrow 2a^2 - 14a - a + 7 = 0$
$\Rightarrow 2a(a - 7) - 1(a - 7) = 0$

$a = \frac{1}{2}$ or 7,

given a > 1 $\therefore a = 7$
also P lies on curve

$\therefore b = \int_0^a (2t^2 - 15t + 10)dt$

$b = \int_0^7 (2t^2 - 15t + 10)dt$

$6b = -413$
$\therefore |a + 6b| = 406$

6. If $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then a+b+c is equal to ______

Ans. **(4)**

Sol. $$\lim_{x\to 0}\frac{\left\{a\left(1+x+\frac{x^2}{2!}+\ldots\right)-b\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)+c\left(1-x+\frac{x^2}{2!}\ldots\right)\right\}}{x\left(x-\frac{x^3}{3!}+\ldots\right)}=2$$

$$\therefore \lim_{x\to 0}\frac{(a-b+c)+x(a-c)+x^2\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}\right)+\ldots.}{x^2\left(1-\frac{x^2}{6}\ldots\right)}=2$$

$\therefore a - b + c = 0$

$\& a - c = 0$

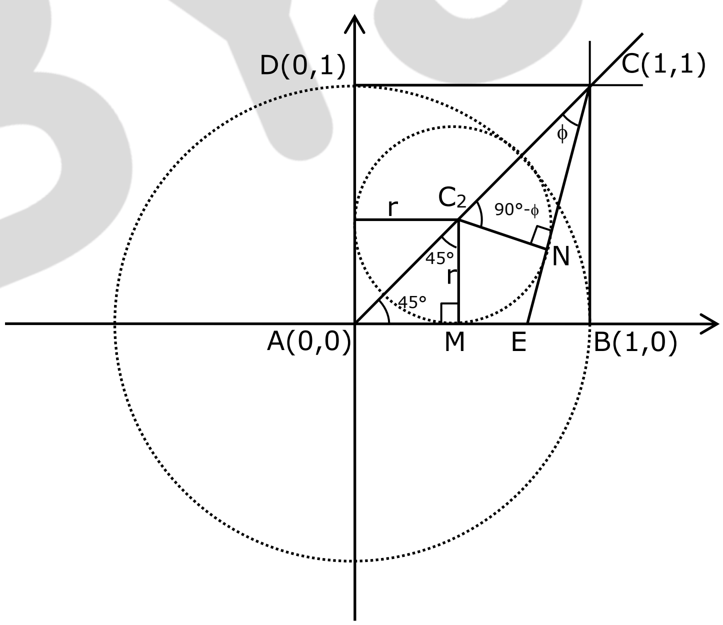
$\& \frac{a}{2}+\frac{b}{2}+\frac{c}{2}=2$

$\Rightarrow a + b + c = 4$

7. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha+\sqrt{3}\beta$, where α, β are integers, then $\alpha+\beta$ is equal to ___

Ans. **(1)**

Sol.



(i) $\sqrt{2}r+r=1$

$r = \frac{1}{\sqrt{2}+1}$

$r = \sqrt{2}-1$

(ii) $CC_2 = 2\sqrt{2}-2 = 2\left(\sqrt{2}-1\right)$

From ΔCC_2N = $\sin\phi = \frac{\sqrt{2}-1}{2\left(\sqrt{2}-1\right)}$

$\phi = 30°$

(iii) In ΔACE are sine law

$\frac{AE}{\sin\phi} = \frac{AC}{\sin 105°}$

$AE = \frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3}+1}.2\sqrt{2}$

$AE = \frac{2}{\sqrt{3}+1} = \sqrt{3}-1$

$\therefore EB = 1-\left(\sqrt{3}-1\right)$

$2-\sqrt{3}$

$\alpha = 2, \beta = -1 \Rightarrow \alpha+\beta = 1$

8. Let z and w be two complex numbers such that $w = z\bar{z} - 2z + 2, \left|\frac{z+i}{z-3i}\right| = 1$and Re (w) has

minimum value. Then, the minimum value of $n \in N$ for Which w^n is real, is equal to ____

Ans. **(4)**

Sol. Let z = x + iy

$|z + i| = |z-3i|$

$\Rightarrow y = 1$

Now w= $x^2 + y^2 - 2x - 2iy + 2$

$w = x^2+1 - 2x - 2i + 2$

$Re(w) = x^2 - 2x + 3$

$\text{Re}(w) = (x-1)^2 + 2$

$\text{Re}(w)_{min}$ at $x = 1 \Rightarrow z = 1 + i$

Now $w = 1 + 1 - 2 - 2i + 2$

$w = 2(1-i) = 2\sqrt{2}e^{i\left(\frac{-\pi}{4}\right)}$

$w^n = 2\sqrt{2}e^{i\left(\frac{-n\pi}{4}\right)}$

If w^n is real $\Rightarrow n = 4$

9. Let $P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega+1 \end{bmatrix}$ where $\omega = \frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $\left(P^{-1}AP - I_3\right)^2$ is $\alpha\omega^2$, then the value of α is equal to ______

Ans. **(36)**

Sol. $\left|P^{-1}AP - I\right|^2$

$= |(P^{-1}AP - I)(P^{-1}AP - I)|$

$= \left|P^{-1}APP^{-1}AP - 2P^{-1}AP + I\right|$

$= \left|P^{-1}A^2P - 2P^{-1}AP + P^{-1}IP\right|$

$= \left|P^{-1}\left(A^2 - 2A + I\right)P\right|$

$= \left|P^{-1}\left(A - I\right)^2 P\right|$

$= \left|P^{-1}\right|\left|A - I\right|^2\left|P\right|$

$= \left|A - I\right|^2$

$$= \begin{vmatrix} 1 & 7 & \omega^2 \\ -1 & -\omega-1 & 1 \\ 0 & -\omega & -\omega \end{vmatrix}^2$$

$=\left(1\left(\omega(\omega+1)+\omega\right)-7\omega+\omega^2.\omega\right)^2$

$=\left(\omega^2+2\omega-7\omega+1\right)^2$

$=\left(\omega^2-5\omega+1\right)^2$

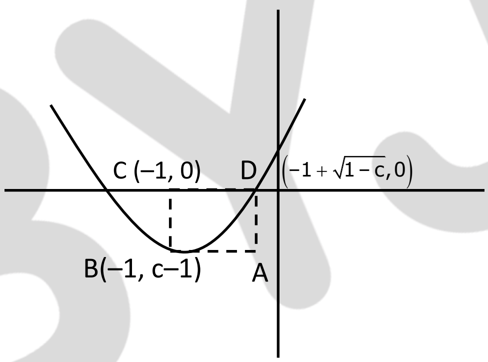
$=\left(-6\omega\right)^2$

$=36\omega^2 \Rightarrow \alpha=36$

10. Let the curve y=y(x) be the solution of the differential equation, $\frac{dy}{dx}=2(x+1)$. If the numerical value of area bounded by the curve y=y(x) and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of y(1) is equal to ___

Ans. **(2)**

Sol. $y = x^2 + 2x + c$



Area of rectangle $(ABCD)=|(c-1)(\sqrt{1-c})|$

Area of parabola and x-axis = $2\left(\frac{2}{3}\left((1-c)^{3/2}\right)\right)=\frac{4\sqrt{8}}{3}$

$1 - c = 2 \Rightarrow c = -1$

Equation of $f(x) = x^2 + 2x - 1$

$f(1) = 1 + 2 - 1 = 2$