JEE MAIN – 2021

16th Mar Shift – II Maths Question Paper

Section A

Multiple Choice Question:

1. The least value of |z| where z is complex number which satisfies the inequality $\exp\left(\frac{(|z|+3)\,(|z|-1)}{||z|+1|}\log_e 2\right) \geq \log_{\sqrt{2}}|5\sqrt{7}+9i|, i=\sqrt{-1}$, is equal to :

(1) 2

(2) 3

(3) 8

(4) $\sqrt{5}$

Ans. **(2)**

Sol. $2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3 \Rightarrow \frac{\left(|z|+3\right)\left(|z|-1\right)}{|z|+1} \geq 3$

$\Rightarrow |z|^2 + 2|z| - 3 \geq 3|z| + 3$

$\Rightarrow |z|^2 - |z| - 6 \geq 0$

$\left(|z|-3\right)\left(|z|+2\right) \geq 0$

$|z|_{min} = 3$

2. Let $f : S \to S$ where $S = (0, \infty)$ be a twice differentiable function such that $f(x+1) = xf(x)$. If $g : S \to \mathbf{R}$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

(1) $\dfrac{197}{144}$

(2) $\dfrac{187}{144}$

(3) $\dfrac{205}{144}$

(4) 1

Ans. **(3)**

Sol. f(x+1) = xf(x)

g(x+1) = log$_e$(f(x+1))

g(x+1) = log$_e$ x + log$_e$ f(x)

g(x+1) − g(x) = log$_e$ x

$g''(x+1) - g''(x) = -\dfrac{1}{x^2}$

g''(2) − g''(1) = −1

$g''(3) - g''(2) = -\dfrac{1}{4}$

$$g''(4) - g''(3) = -\frac{1}{9}$$

$$g''(5) - g''(4) = -\frac{1}{16}$$

$$g''(5) - g''(1) = -\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right]$$

$$\therefore \left|g''(5) - g''(1)\right| = \left[\frac{144 + 36 + 16 + 9}{16 \times 9}\right] = \frac{205}{144}$$

3. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x, 0 \le x \le \frac{\pi}{3}$,

with $y(0) = 0$, then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\log_e 2$

(2) $\frac{1}{2}\log_e 2$

(3) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$

(4) $\frac{1}{4}\log_e 2$

Ans. (3)

Sol. I.F. $= e^{\int \tan x\, dx}$

$= e^{\ln \sec x}$

$= \sec x$

Solution of the equation:

$y(\sec x) = \int (\sin x)(\sec x)\, dx$

$\Rightarrow \frac{y}{\cos x} = \ell n(\sec x) + c$

Put $x = 0$, $c = 0$

$\therefore y = \cos x\, \ell n(\sec x)$

Put $x = \pi/4$

$y = \frac{1}{\sqrt{2}}\ln\sqrt{2} = \frac{1}{2\sqrt{2}}\ell n2$

$y = \frac{\ln 2}{2\sqrt{2}}$

4. If the foot of the perpendicular from point (4, 3, 8) on the line

$L_1 : \dfrac{x-a}{\ell} = \dfrac{y-2}{3} = \dfrac{z-b}{4}$, $\ell \neq 0$ is (3, 5, 7), then the shortest distance between the line

L_1 and line $L_2 : \dfrac{x-2}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{5}$ is equal to :

(1) $\sqrt{\dfrac{2}{3}}$

(3) $\dfrac{1}{2}$

(2) $\dfrac{1}{\sqrt{3}}$

(4) $\dfrac{1}{\sqrt{6}}$

Ans. **(4)**

Sol. (3, 5, 7) lies on given line L_1

$\dfrac{3-a}{\ell} = \dfrac{3}{3} = \dfrac{7-b}{4}$

$\dfrac{7-b}{4} = 1 \Rightarrow b = 3$

$\dfrac{3-a}{\ell} = 1 \Rightarrow 3 - a = \ell$

A (4, 3, 8)

B (3, 5, 7)

DR's of AB = (1, −2, 1)

AB $\perp$ line L_1

$(1)(\ell) + (-2)(3) + 4(1) = 0$

$\Rightarrow \ell = 2$

a = 1

a = 1, b = 3, ℓ = 2

$\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$

$\dfrac{x-2}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{5}$

S.D. = $\dfrac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix} \right\|} = \dfrac{1}{\sqrt{6}}$

5. If (x, y, z) be an arbitrary point lying on a plane P which passes through the points (42, 0, 0), (0, 42, 0) and (0, 0, 42), then the value of the expression

$$3 + \frac{x-11}{(y-19)^2(z-12)^2} + \frac{y-19}{(x-11)^2(z-12)^2} + \frac{z-12}{(z-11)^2(y-19)^2} - \frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

is equal to :

(1) 3   (3) 39

(2) 0   (4) −45

Ans. **(1)**

Sol. Equation of plane is x + y + z = 42

or, $(x-11) + (y-19) + (z-12) = 0$

Now, $\frac{x-11}{(y-19)^2(z-12)^2} + \frac{y-19}{(x-11)^2(z-12)^2} + \frac{z-12}{(z-11)^2(y-19)^2}$

$= \frac{(x-11)^3 + (y-19)^3 + (z-12)^3}{(x-11)^2(y-19)^2(z-12)^2}$

$= \frac{3(x-11)(y-19)(z-12)}{(x-11)^2(y-19)^2(z-12)^2}$ [If $a+b+c = 0$, then $a^3 + b^3 + c^3 = 3abc$]

$= \frac{3}{(x-11)(y-19)(z-12)}$

∴ The given expression is equal to

$3 + \frac{3}{(x-11)(y-19)(z-12)} - \frac{3}{(x-11)(y-19)(z-12)} = 3$

6. Consider the integral $I = \int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}} dx$, where [x] denotes the greatest integer less than

or equal to x. Then the value of I is equal to :

(1) 45 (e − 1)   (3) 9(e − 1)

(2) 45 (e + 1)   (4) 9(e + 1)

Ans. **(1)**

Sol. $I = \int_0^{10} [x] \cdot e^{[x]+1-x} dx$

$= \int_1^2 e^{2-x} dx + \int_2^3 2 \cdot e^{3-x} dx + \int_3^4 3 \cdot e^{4-x} dx + \dots\dots + \int_9^{10} 9e^{10-x} dx$

$= - \{(1 - e) + 2(1 - e) + 3(1 - e) + \dots\dots + 9(1 - e)\}$
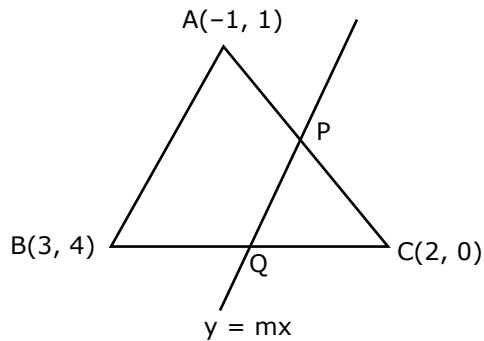
$= 45(e - 1)$

7. Let A (−1, 1), B (3, 4) and C(2, 0) be given three points. A line y = mx, m > 0, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of △ABC and △PQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

(1) $\frac{4}{15}$ (3) 2

(2) 1   (4) 3

Ans. **(2)**

Sol.

$A_1 = $ Area of $\Delta ABC = \dfrac{1}{2} \left\| \begin{matrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 4 & 1 \end{matrix} \right\|$

$\Rightarrow A_1 = \dfrac{13}{2}$

Equation of line AC is $y - 1 = -\dfrac{1}{3}(x + 1)$

Solving it with line $y = mx$, we get $P\left(\dfrac{2}{3m+1}, \dfrac{2m}{3m+1}\right)$

Equation of line BC is $y - 0 = 4(x - 2)$

Solving it with line $y = mx$, we get $Q\left(\dfrac{-8}{m-4}, \dfrac{-8m}{m-4}\right)$

$A_2 = $ Area of $\Delta PQC = \dfrac{1}{2} \left\| \begin{matrix} 2 & 0 & 1 \\ \dfrac{2}{3m+1} & \dfrac{2m}{3m+1} & 1 \\ \dfrac{-8}{m-4} & \dfrac{-8m}{m-4} & 1 \end{matrix} \right\| = \dfrac{A_1}{3} = \dfrac{13}{6}$

$\Rightarrow \dfrac{26m^2}{3m^2 - 11m - 4} = \pm\dfrac{13}{6}$

$\Rightarrow 12m^2 = \pm\left(3m^2 - 11m - 4\right)$

Taking +ve sign,

$9m^2 + 11m + 4 = 0$ (Rejected $\because$ m is imaginary)

Taking –ve sign,

$15m^2 - 11m - 4 = 0$

$m = 1, \ -\dfrac{4}{15}$

$\Rightarrow m = 1$ as $m > 0$

8. Let *f* be a real valued function, defined on **R** − {−1, 1} and given by

$f(x) = 3\log_e \left| \dfrac{x-1}{x+1} \right| - \dfrac{2}{x-1}$. Then in which of the following intervals, function *f*(x) is

increasing?

(1) $\left(-\infty, -1 \right) \cup \left(\left[\dfrac{1}{2}, \infty \right) - \{1\} \right)$

(2) $\left(-1, \dfrac{1}{2} \right]$

(3) $(-\infty, \infty) - \{-1, 1\}$

(4) $\left(-\infty, \dfrac{1}{2} \right] - \{-1\}$

Ans. **(1)**

Sol. $f'(x) = \left(\dfrac{x+1}{x-1} \times \dfrac{x+1-(x-1)}{(x+1)^2} \right) 3 + \dfrac{2}{(x-1)^2} = \dfrac{6}{(x-1)(x+1)} + \dfrac{2}{(x-1)^2}$

$= \dfrac{4(2x-1)}{(x+1)(x-1)^2}$

$\therefore\ x \in \left(-\infty, -1 \right) \cup \left[\dfrac{1}{2}, \infty \right) - \{1\}$

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, (a<0) be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line x + 2y = 0, is equal to :

(1) $\sqrt{10}$ (3) $\sqrt{11}$

(2) $\sqrt{6}$ (4) $\sqrt{7}$

Ans. **(2)**

Sol. $2\sqrt{a^2/4 - c} = 2\sqrt{2}$

$\sqrt{a^2 - 4c} = 2\sqrt{2}$

$a^2 - 4c = 8$ … (1)

$2\sqrt{a^2 - c} = 2\sqrt{5}$

$a^2 - c = 5$ …(2)

(2) – (1), we get

$3c = -3 \Rightarrow c = -1$

$a^2 = 4 \Rightarrow a = -2$

$x^2 + y^2 - 2x - 4y - 1 = 0$

Equation of tangent $2x - y + \lambda = 0$

$\therefore p = r$

$\left| \frac{0 - 0 + \lambda}{\sqrt{5}} \right| = \sqrt{6}$

$\Rightarrow \lambda = \pm\sqrt{30}$

$\therefore$ tangents are $2x - y \pm \sqrt{30} = 0$

Distance from origin $= \dfrac{\sqrt{30}}{\sqrt{5}} = \sqrt{6}$

10. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

(1) $\dfrac{4}{9}$　　　　　　　　　　　　　　　　(3) $\dfrac{3}{7}$

(2) $\dfrac{9}{56}$　　　　　　　　　　　　　　　(4) $\dfrac{11}{27}$

Ans. **(1)**

Sol. Total case $= 6\lfloor 6$

Fav. case = (0, 1, 2, 3, 4, 5) + (0, 1, 2, 4, 5, 6) + (1, 2, 3, 4, 5, 6)

$= 5\lfloor 5 + 5\lfloor 5 + \lfloor 6$

$= 1920$

Probability $= \dfrac{1920}{6\lfloor 6} = \dfrac{4}{9}$

11. Let $\alpha \in \mathbf{R}$ be such that the function $f(x) = \begin{cases} \dfrac{\cos^{-1}\left(1 - \{x\}^2\right)\sin^{-1}\left(1 - \{x\}\right)}{\{x\} - \{x\}^3} & x \neq 0 \\ \alpha & x = 0 \end{cases}$ is

continuous at x = 0, where $\{x\} = x-[x]$, [x] is the greatest integer less than or equal to x. Then :

(1) $\alpha = \dfrac{\pi}{4}$　　　　　　　　　　　　　(3) $\alpha = 0$

(2) No such α exists　　　　　　　　　(4) $\alpha = \dfrac{\pi}{\sqrt{2}}$

Ans. **(2)**

Sol.

$$RHL = \lim_{x \to 0^+} \frac{\cos^{-1}(1-x^2)\sin^{-1}(1-x)}{x(1-x^2)} = \frac{\pi}{2} \lim_{x \to 0^+} \frac{\cos^{-1}(1-x^2)}{x}$$

$$= \frac{\pi}{2} \lim_{x \to 0^+} \frac{-1}{\sqrt{1-(1-x^2)^2}}(-2x) \qquad \text{(L' Hospital Rule)}$$

$$= \pi \lim_{x \to 0^+} \frac{x}{\sqrt{2x^2 - x^4}} = \pi \lim_{x \to 0^+} \frac{1}{\sqrt{2-x^2}} = \frac{\pi}{\sqrt{2}}$$

$$LHL = \lim_{x \to 0^-} \frac{\cos^{-1}(1-(1+x)^2)\sin^{-1}(-x)}{(1+x)-(1+x)^3} = \frac{\pi}{2}\lim_{x \to 0^-} \frac{\sin^{-1}x}{(1+x)[(1+x)^2-1]} = \frac{\pi}{2}\lim_{x \to 0^-} \frac{\sin^{-1}x}{x^2+2x}$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

As LHL ≠ RHL, so $f(x)$ is not continuous at $x = 0$

12. The maximum value of $f(x) = \begin{vmatrix} \sin^2 x & 1+\cos^2 x & \cos 2x \\ 1+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}$, $x \in \mathbf{R}$ is :

(1) $\sqrt{7}$　　　　　　　　　　　　　　　(2) $\sqrt{5}$

(3) 5　　　　　　　　　　　　　　　　　　(4) $\dfrac{3}{4}$

Ans. **(2)**

Sol. $C_1 \to C_1 + C_2$

$$\begin{vmatrix} 2 & 1+\cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$= (-1)[2\sin 2x - \cos 2x] = \cos 2x - 2\sin 2x$

Maximum value = $\sqrt{5}$

13. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these

points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to:

(1) 1890

(2) 795

(3) 717

(4) 1173

Ans. **(3)**

Sol. $\alpha = {}^6C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^7C_1 = 378 + 315 + 270 + 210 = 1173$
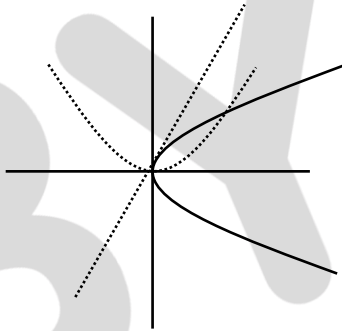
$\beta = {}^5C_1\,{}^6C_1\,{}^7C_1\,{}^9C_1 = 1890$

$\Rightarrow \beta - \alpha = 1890 - 1173 = 717$

14. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2, 1) is :

(1) $2x + y = 5$ (3) $x + 3y = 5$

(2) $x + 2y = 4$ (4) $x - y = 1$

Ans. **(4)**

Sol. Image of $y^2 = 4x$ w.r.t. $y = x$ is $x^2 = 4y$

Tangent from (2, 1)

$xx_1 = 2(y + y_1)$

$2x = 2(y + 1)$

$x = y + 1$

15. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\dfrac{3x}{5}\right) + \sin^{-1}\left(\dfrac{4x}{5}\right) = \sin^{-1}x$ is equal to :

(1) 1   (3) 3

(2) 2   (4) 0

Ans. **(3)**

Sol.   Taking sine on both sides

$$\frac{3x}{5}\sqrt{1 - \frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1 - \frac{9x^2}{25}} = x$$

$$\Rightarrow 3x\sqrt{25 - 16x^2} = 25x - 4x\sqrt{25 - 9x^2}$$

$$\Rightarrow x = 0 \text{ or } 3\sqrt{25 - 16x^2} = 25 - 4\sqrt{25 - 9x^2}$$

$$\Rightarrow 9\left(25 - 16x^2\right) = 625 - 200\sqrt{25 - 9x^2} + 16\left(25 - 9x^2\right)$$

$$\Rightarrow 200\sqrt{25 - 9x^2} = 800$$

$$\Rightarrow \sqrt{25 - 9x^2} = 4$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

$\therefore$ Total number of solutions = 3

16.   Let C_1 be the curve obtained by solution of differential equation

$2xy\dfrac{dy}{dx} = y^2 - x^2, x > 0$. Let the curve C_2 be the solution of $\dfrac{2xy}{x^2 - y^2} = \dfrac{dy}{dx}$. If both the

curves pass through (1, 1), then the area enclosed by the curves C_1 and C_2 is equal to :

(1) $\dfrac{\pi}{2} - 1$

(3) $\pi - 1$

(2) $\dfrac{\pi}{4} + 1$

(4) $\pi + 1$

Ans.   **(1)**

Sol. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

Put y = vx

$$v + x\frac{dv}{dx} = \frac{v^2 x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{(v^2 + 1)}{2v}$$

$$\Rightarrow \frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

$$\ell n(v^2 + 1) = -\ell nx + \ell nc \Rightarrow v^2 + 1 = \frac{c}{x}$$

$$\Rightarrow \frac{y^2}{x^2} + 1 = \frac{c}{x} \Rightarrow x^2 + y^2 = cx$$
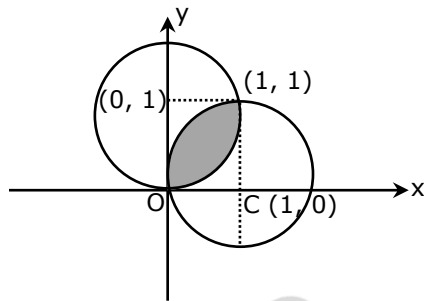
It passes through (1, 1)

$$\therefore x^2 + y^2 - 2x = 0$$

Similarly, for second differential equation $\frac{dx}{dy} = \frac{x^2 - y^2}{2xy}$

equation of curve is $x^2 + y^2 - 2y = 0$

Now required area is

$$= \left(\frac{1}{4} \times \pi \times 1^2 - \frac{1}{2} \times 1 \times 1 \right) \times 2$$

$$= \left(\frac{\pi}{2} - 1 \right) \text{ sq. units}$$

17. Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$. If $\vec{r} \times \vec{a} = \vec{b} \times \vec{r}, \vec{r} \cdot \left(\alpha\hat{i} + 2\hat{j} + \hat{k} \right) = 3$ and

$\vec{r} \cdot \left(2\hat{i} + 5\hat{j} - \alpha\hat{k} \right) = -1, \alpha \in R$, then the value of $\alpha + |\vec{r}|^2$ is equal to :

(1) 11   (3) 9

(2) 15   (4) 13

Ans. **(2)**

Sol. $\vec{r} \times \vec{a} = -\vec{r} \times \vec{b}$

$\vec{r} \times \left(\vec{a} + \vec{b} \right) = 0$ $\qquad \left(\vec{a} + \vec{b} = 3\hat{i} - \hat{j} + 2\hat{k} \right)$

$\vec{r} \parallel \left(\vec{a} + \vec{b} \right)$

$\vec{r} = \lambda \left(\vec{a} + \vec{b} \right)$

$\because \vec{r} \cdot \left(2\hat{i} + 5\hat{j} - \alpha\hat{k} \right) = -1$

$\lambda \left[3\hat{i} - \hat{j} + 2\hat{k} \right] \cdot \left[2\hat{i} + 5\hat{j} - \alpha\hat{k} \right] = -1$

$\Rightarrow \lambda(6 - 5 - 2\alpha) = -1$

$\lambda(1 - 2\alpha) = -1$ ...(1)

$\vec{r} \cdot \left(\alpha\hat{i} + 2\hat{j} + \hat{k}\right) = 3$

$\lambda\left(3\hat{i} - \hat{j} + 2\hat{k}\right) \cdot \left(\alpha\hat{i} + 2\hat{j} + \hat{k}\right) = 3$

$\Rightarrow \lambda[3\alpha - 2 + 2] = 3 \Rightarrow \lambda\alpha = 1$ ...(2)

Solving (1) & (2)

$\lambda\left[1 - \dfrac{2}{\lambda}\right] = -1$

$\lambda - 2 = -1 \Rightarrow \lambda = 1$ and $\alpha = 1$

$\vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\alpha + |\vec{r}|^2 \Rightarrow 1 + 14 = 15$

18. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)\,dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b+c)$ is equal to :

(1) 7   (3) 15

(2) 11   (4) 9

Ans. **(1)**

Sol. $(x - 2)Q(x) + 5 = x^2 + bx + c$

Put $x = 2$

$5 = 2b + c + 4$ ...(1)

$\int_0^1 \left(x^2 + bx + c\right)dx = 1$

$\Rightarrow \dfrac{1}{3} + \dfrac{b}{2} + c = 1$

$\dfrac{b}{2} + c = \dfrac{2}{3}$ ...(2)

Solving (1) & (2)

$b = \dfrac{2}{9}$

$c = \dfrac{5}{9}$

$9(b+c) = 7$

19. If the points of intersections of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = 4b$,

b > 4 lie on the curve $y^2 = 3x^2$, then b is equal to :

(1) 5   (3) 12

(2) 6   (4) 10

Ans. **(3)**

Sol. $\dfrac{x^2}{16} + \dfrac{y^2}{b^2} = 1$ …(1)

$x^2 + y^2 = 4b$ …(2)

$y^2 = 3x^2$ …(3)

From eqns. (2) and (3), $x^2 = b$ and $y^2 = 3b$

From eqn. (1), $\dfrac{b}{16} + \dfrac{3b}{b^2} = 1$

$\Rightarrow b^2 + 48 = 16b$

$\Rightarrow b = 12$

20. Let A = {2, 3, 4, 5, ……, 30} and '$\cong$' be an equivalence relation on A×A, defined by $(a, b) \cong (c, d)$, if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

(1) 7   (3) 6

(2) 5   (4) 8

Ans. **(1)**

Sol. ad = bc

(a, b) R (4, 3) $\Rightarrow$ 3a = 4b

$a = \dfrac{4}{3}b$

b must be multiple of 3

b can be 3, 6, 9, ……, 30

Also, a must be less than or equal to 30.

(a, b) = (4, 3), (8, 6), (12, 9) (16, 12), (20, 15), (24, 18), (28, 21)

$\Rightarrow$ 7 ordered pairs

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Section B

Integer Type:

1. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. If $\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = 8$, then the value of $\vec{c} \cdot (\vec{a} \times \vec{b})$ is equal to _____

Ans. (28)

Sol. $\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c}\ \vec{a}\ \vec{b}]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (3, -2, 1)$$

$\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \Rightarrow C \parallel \vec{a} \times \vec{b}$

$\vec{c} = \lambda(\vec{a} \times \vec{b})$

$\Rightarrow \vec{c} = \lambda(3\hat{i} - 2\hat{j} + \hat{k})$

$\vec{c}(\hat{i} + \hat{j} + 3\hat{k}) = 8$

$\Rightarrow 3\lambda - 2\lambda + 3\lambda = 8$

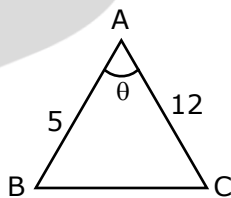$\Rightarrow 4\lambda = 8 \Rightarrow \lambda = 2$

$\vec{c} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c}\ \vec{a}\ \vec{b}] = \begin{vmatrix} 6 & -4 & 2 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 18 + 8 + 2 = 28$

2. In $\triangle ABC$, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of $\triangle ABC$ is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of $\triangle ABC$, then the value of 2R + r (in cm) is equal to _____
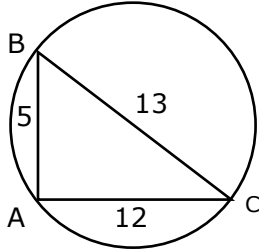
Ans. (15)

Sol.

Area $= \frac{1}{2}(5)(12)\sin\theta = 30$

$\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

ABC is right angled at A.

$$r = (s-a) \tan \frac{A}{2}$$

$$r = (s-a)$$

$$2R + r = s \quad (\because a = 2R)$$

$$2R + r = \frac{30}{2} = 15$$

3. Consider the statistics of two sets of observations as follows:

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value

of n is equal to _____

Ans. **(5)**

Sol. For group-1 : $\dfrac{\sum x_i}{10} = 2 \Rightarrow \sum x_i = 20$

$\dfrac{\sum x_i^2}{10} - (2)^2 = 2 \Rightarrow \sum x_i^2 = 60$

For group-2 : $\dfrac{\sum y_i}{n} = 3 \Rightarrow \sum y_i = 3n$

$\dfrac{\sum y_i^2}{n} - 3^2 = 1 \Rightarrow \sum y_i^2 = 10n$

Now, combined variance

$$\sigma^2 = \frac{\sum\left(x_i^2 + y_i^2\right)}{10+n} - \left(\frac{\sum(x_i + y_i)}{10+n}\right)^2$$

$$\Rightarrow \frac{17}{9} = \frac{60 + 10n}{10+n} - \frac{(20 + 3n)^2}{(10+n)^2}$$

$$\Rightarrow 17\left(n^2 + 20n + 100\right) = 9\left(n^2 + 40n + 200\right)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$\Rightarrow 2n^2 - 5n - 25 = 0 \Rightarrow n = 5$$

4. Let
$S_n(x) = \log_{a^{\frac{1}{2}}} x + \log_{a^{\frac{1}{3}}} x + \log_{a^{\frac{1}{6}}} x + \log_{a^{\frac{1}{11}}} x + \log_{a^{\frac{1}{18}}} x + \log_{a^{\frac{1}{27}}} x + \ldots$ up to n-terms,

where a >1. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, then value of a is equal to_____

Ans. **(16)**

Sol. $S_n(x) = 2\log_a x + 3\log_a x + 6\log_a x + 11\log_a x + ...$

$S_n(x) = \log_a x(2 + 3 + 6 + 11 + ...)$

$S_r = 2 + 3 + 6 + 11$

General term, $T_r = r^2 - 2r + 3$

$S_n(x) = \sum_{r=1}^{n} \log_a x \,(r^2 - 2r + 3)$

$\Rightarrow S_{24}(x) = \log_a x \sum_{r=1}^{24}(r^2 - 2r + 3)$

$\Rightarrow 1093 = 4372 \log_a x$

$\Rightarrow \log_a x = \dfrac{1}{4}$

$\Rightarrow x = a^{1/4}$

$S_{12}(2x) = \log_a(2x) \sum_{r=1}^{12}\left(r^2 - 2r + 3\right)$

$\Rightarrow 265 = 530 \log_a(2x)$

$\Rightarrow \log_a(2x) = \dfrac{1}{2}$

$\Rightarrow 2x = a^{1/2}$

After solving (i) and (ii), we get

$a^{1/4} = 2$

$\Rightarrow a = 16$

5. Let n be a positive integer. Let $A = \sum_{k=0}^{n}(-1)^k \,{}^nC_k\left[\left(\dfrac{1}{2}\right)^k + \left(\dfrac{3}{4}\right)^k + \left(\dfrac{7}{8}\right)^k + \left(\dfrac{15}{16}\right)^k + \left(\dfrac{31}{32}\right)^k\right]$

If $63A = 1 - \dfrac{1}{2^{30}}$, then n is equal to _____

Ans. **(6)**

Sol. $A = \left(\dfrac{1}{2}\right)^n + \left(\dfrac{1}{4}\right)^n + \left(\dfrac{1}{8}\right)^n + \left(\dfrac{1}{16}\right)^n + \left(\dfrac{1}{32}\right)^n$

$= \dfrac{1}{2^n} + \dfrac{1}{2^{2n}} + \dfrac{1}{2^{3n}} + \dfrac{1}{2^{4n}} + \dfrac{1}{2^{5n}}$

$= \dfrac{1}{2^n}\left[\dfrac{1 - \left(\dfrac{1}{2^n}\right)^5}{1 - \dfrac{1}{2^n}}\right]$

$A = \dfrac{2^{5n} - 1}{2^{5n}(2^n - 1)}$

$$63A = \frac{63(2^{5n} - 1)}{2^{5n}(2^n - 1)}$$

$$= \frac{63}{2^n - 1}\left(1 - \frac{1}{2^{5n}}\right) = 1 - \frac{1}{2^{30}}$$

$$\Rightarrow n = 6$$

6. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined as

$$f(x) = \begin{cases} x + a, & x < 0 \\ |x - 1|, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x + 1, & x < 0 \\ (x - 1)^2 + b, & x \geq 0 \end{cases}$$

where a, b are non-negative real numbers. If (gof) (x) is continuous for all x ∈ **R**, then a + b is equal to _____

Ans. **(1)**

Sol.
$$g\big[f(x)\big] = \begin{bmatrix} f(x) + 1 & f(x) < 0 \\ (f(x) - 1)^2 + b & f(x) \geq 0 \end{bmatrix}$$

$$g\big[f(x)\big] = \begin{bmatrix} x + a + 1 & x + a < 0 \ \& \ x < 0 \\ |x - 1| + 1 & |x - 1| < 0 \ \& \ x \geq 0 \\ (x + a - 1)^2 + b & x + a \geq 0 \ \& \ x < 0 \\ (|x - 1| - 1)^2 + b & |x - 1| \geq 0 \ \& \ x \geq 0 \end{bmatrix}$$

$$g\big[f(x)\big] = \begin{bmatrix} x + a + 1 & x \in (-\infty, -a) \ \& \ x \in (-\infty, 0) \\ |x - 1| + 1 & x \in \phi \\ (x + a - 1)^2 + b & x \in [-a, \infty) \ \& \ x \in (-\infty, 0) \\ (|x - 1| - 1)^2 + b & x \in R \ \& \ x \in [0, \infty) \end{bmatrix}$$

$$g\big[f(x)\big] = \begin{bmatrix} x + a + 1 & x \in (-\infty, -a) \\ (x + a - 1)^2 + b & x \in [-a, 0) \\ (|x - 1| - 1)^2 + b & x \in [0, \infty) \end{bmatrix}$$

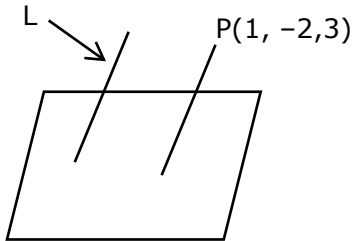Given g(f(x)) is continuous.

So, at x = −a & at x = 0

1 = b + 1 & (a−1)^2 + b = b

$\Rightarrow$ b = 0 & a = 1

$\Rightarrow$ a + b = 1

7. If the distance of the point (1, −2, 3) from the plane x + 2y − 3z + 10 = 0 measured parallel to the line, $\dfrac{x-1}{3} = \dfrac{2-y}{m} = \dfrac{z+3}{1}$ is $\sqrt{\dfrac{7}{2}}$, then the value of |m| is equal to_____

Ans. (2)

Sol.

$\dfrac{x-1}{3} = \dfrac{y+2}{-m} = \dfrac{z-3}{1} = \lambda$

Point $Q(3\lambda + 1, -m\lambda - 2, \lambda + 3)$ lies on plane.

$(3\lambda + 1) + 2(-m\lambda - 2) - 3(\lambda + 3) + 10 = 0$

$\Rightarrow 3\lambda - 2m\lambda - 3\lambda + 1 - 4 - 9 + 10 = 0$

$\Rightarrow -2m\lambda = 2$

$m\lambda = -1 \Rightarrow \lambda = -\dfrac{1}{m}$

$Q\left[-\dfrac{3}{m} + 1, -1, -\dfrac{1}{m} + 3\right]$

$PQ = \sqrt{\dfrac{7}{2}}$

$\sqrt{\left(-\dfrac{3}{m}\right)^2 + 1 + \left(-\dfrac{1}{m}\right)^2} = \sqrt{\dfrac{7}{2}}$

$\Rightarrow \dfrac{10 + m^2}{m^2} = \dfrac{7}{2}$

$\Rightarrow 20 + 2m^2 = 7m^2$

$m^2 = 4 \quad \Rightarrow |m| = 2$

8. Let $\dfrac{1}{16}$, a and b be in G.P. and $\dfrac{1}{a}$, $\dfrac{1}{b}$, 6 be in A.P., where a, b, > 0. Then 72(a+b) is equal to _____

Ans. (14)

Sol. $a^2 = \dfrac{b}{16}$ and $\dfrac{2}{b} = \dfrac{1}{a} + 6$

Solving, we get a = $\dfrac{1}{12}$ or a = $-\dfrac{1}{4}$ [rejected]

If a = $\dfrac{1}{12} \Rightarrow b = \dfrac{1}{9}$

$\therefore 72(a+b) = 72\left(\dfrac{1}{12} + \dfrac{1}{9}\right) = 14$

9. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2×1 matrices with real entries such that $A = XB$,

where $X = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and $k \in \mathbf{R}$. If $a_1^2 + a_2^2 = \dfrac{2}{3}\left(b_1^2 + b_2^2\right)$ and $\left(k^2 + 1\right)b_2^2 \neq -2b_1 b_2$, then

the value of k is _____

Ans. (1)

Sol.

$XB = A$

$\Rightarrow \dfrac{1}{\sqrt{3}} \begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$b_1 - b_2 = \sqrt{3}a_1 \Rightarrow 3a_1^2 = b_1^2 + b_2^2 - 2b_1 b_2$

$b_1 + kb_2 = \sqrt{3}a_2 \Rightarrow 3a_2^2 = b_1^2 + k^2 b_2^2 + 2kb_1 b_2$

Adding both equations, we get

$3(a_1^2 + a^2) = 2b_2^2 + (k^2 + 1)b_2^2 + 2b_1 b_2(k - 1)$

$\Rightarrow 2b_1^2 + 2b_2^2 = 2b_1^2 + (k^2 + 1)b_2^2 + 2b_1 b_2(k - 1)$

$\Rightarrow (k^2 - 1)b_2^2 + 2b_1 b_2(k - 1) = 0$

$\Rightarrow (k - 1)\left((k + 1)b_2^2 + 2b_1 b_2\right) = 0$

$\Rightarrow k = 1$

10. For real numbers α, β, γ and δ, if

$$\int \dfrac{\left(x^2 - 1\right) + \tan^{-1}\left(\dfrac{x^2 + 1}{x}\right)}{\left(x^4 + 3x^2 + 1\right)\tan^{-1}\left(\dfrac{x^2 + 1}{x}\right)}\, dx$$

$$= \alpha \log_e\left(\tan^{-1}\left(\dfrac{x^2 + 1}{x}\right)\right) + \beta \tan^{-1}\left(\dfrac{\gamma\left(x^2 - 1\right)}{x}\right) + \delta \tan^{-1}\left(\dfrac{x^2 + 1}{x}\right) + C$$

where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to_____

Ans. (6)

Sol. $\displaystyle\int \dfrac{x^2 - 1}{\left(x^4 + 3x^2 + 1\right)\tan^{-1}\left(\dfrac{x^2 + 1}{x}\right)}\, dx + \int \dfrac{1}{x^4 + 3x^2 + 1}\, dx$

$$\int \dfrac{1 - \dfrac{1}{x^2}}{\left[\left(x + \dfrac{1}{x}\right)^2 + 1\right]\tan^{-1}\left(x + \dfrac{1}{x}\right)}\, dx + \int \dfrac{dx}{x^4 + 3x^2 + 1}$$

$$\qquad\qquad\quad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\qquad\qquad\quad I_1 \qquad\qquad\qquad\qquad\qquad I_2$$

$\tan^{-1}\left(x + \dfrac{1}{x}\right) = t$

$$I_1 = \int \frac{dt}{t}$$

$$I_1 = \ln(t) = \ln\left|\tan^{-1}\left(x + \frac{1}{x}\right)\right|$$

Now,

$$I_2 = \int \frac{dx}{x^4 + 3x^2 + 1}$$

$$= \frac{1}{2}\int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 3x^2 + 1}dx$$

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}}dx - \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + 3 + \frac{1}{x^2}}dx\right]$$

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 5} - \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 + 1}dx\right]$$

$$\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\qquad\quad x - \frac{1}{x} = u \qquad\qquad x + \frac{1}{x} = v$$

$$= \frac{1}{2}\left[\int \frac{du}{u^2 + \left(\sqrt{5}\right)^2} - \int \frac{dv}{v^2 + 1}\right]$$

$$I_2 = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2}\tan^{-1}\left(x + \frac{1}{x}\right)$$

$$I = I_1 + I_2 = \ln\left|\tan^{-1}\left(x + \frac{1}{x}\right)\right| + \frac{1}{2\sqrt{5}}\ln\left(\frac{x^2 - 1}{\sqrt{5}x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C$$

$$\therefore \alpha = 1,\ \beta = \frac{1}{2\sqrt{5}},\ \gamma = \frac{1}{\sqrt{5}},\ \delta = -\frac{1}{2}$$

$$10(\alpha + \beta\lambda + \delta) = 10\left[1 + \frac{1}{10} - \frac{1}{2}\right]$$

$$= 10\left(\frac{1}{10} + \frac{1}{2}\right)$$

$$= 1 + 5 = 6$$