Q1. Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

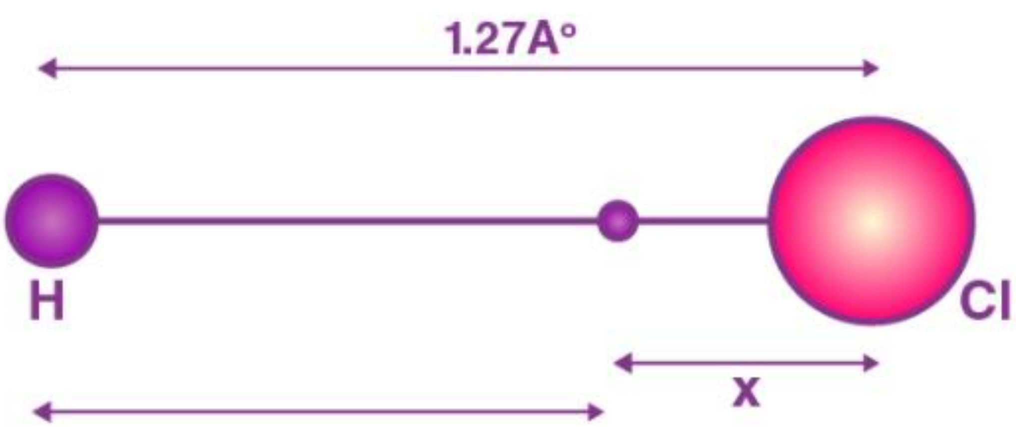
Solution:

All the given structures are symmetric bodies having a very uniform mass density. Thus, for all the above bodies their center of mass will lie in their geometric centres.

It is not always necessary for a body's center of mass to lie inside it, for example, the center of mass of a circular ring is at its center.

Q2. In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Solution:



Given,

mass of hydrogen atom = 1 unit

mass of chlorine atom = 35.5 unit (As a chlorine atom is 35.5 times the size)

Let the center of mass lie at a distance x from the chlorine atom

Thus, the distance of center of mass from the hydrogen atom = 1.27 – x

Assuming that the center of mass of HCL lies at the origin,

Hydrogen will lie on the left side of the origin and chlorine lie on the right side of the origin

$x = (-m (1.27 - x) + 35.5mx)/(m + 35.5m) = 0$

$-m (1.27 - x) + 35.5mx = 0$

-1.27+x+35.5x = 0
36.5x = 1.27

Therefore, x = (1.27)/36.5

= 0.035 Å

The center of mass lies at 0.035 Å from the chlorine atom.

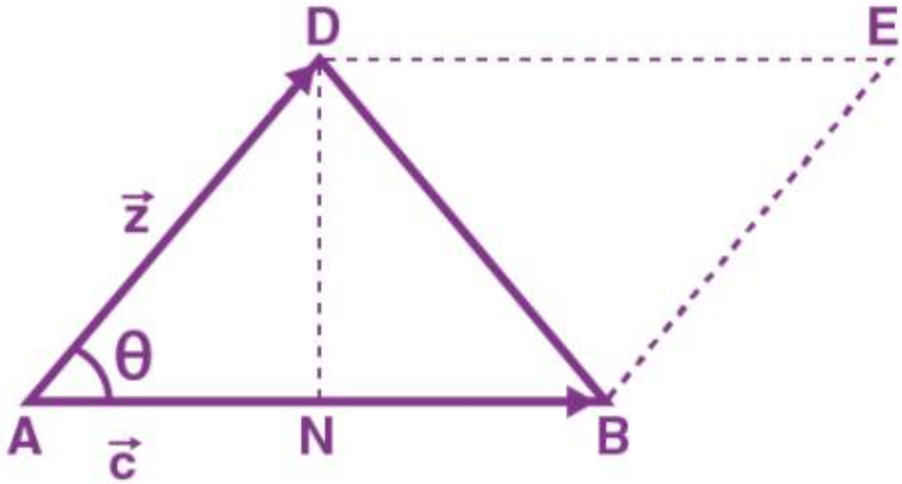
Q3. A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Solution:

The child and the trolley constitute a single system and the child moving inside the trolley is a purely internal motion. Since there is no external force on the system the velocity of the center of mass of the system will not change.

Q4. Show that the area of the triangle contained between the vectors a and b is one half of the magnitude of a × b

Solution:



The side AD is equal to $\vec{a}$ and AB is equal to $\vec{b}$

Considering ΔADN :

sinθ = DN/AD = DN / $\vec{a}$

DN = $\vec{a}$ sinθ

Now, by definition $\left|\vec{a} \times \vec{b}\right|$ = $|\vec{a}|\,|\vec{b}|$ sinθ

= AB x DN

= (1/2) x [2 x AB x DN] (since area of the triangle ADB = 2 x AB x DN]

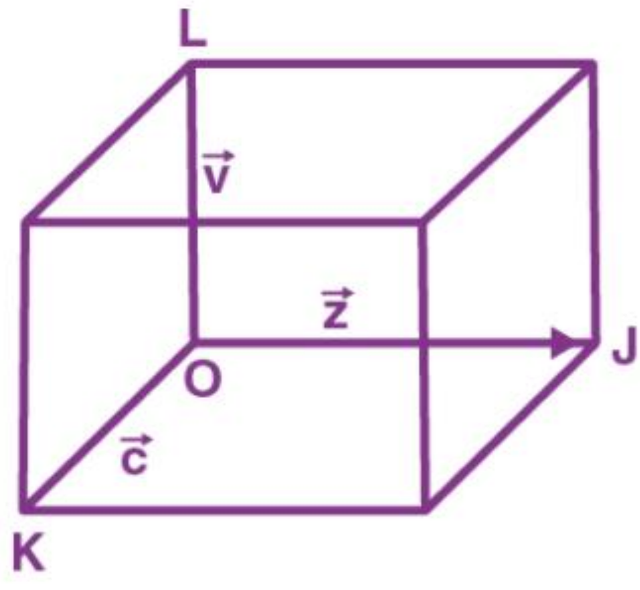
= (1/2) x $\left|\vec{a} \times \vec{b}\right|$

Thus, area of ΔADB = ½ × $\left|\vec{a} \times \vec{b}\right|$

Q5. Show that a.(b × c) is equal in magnitude to the volume of the parallelepiped formed on the three vectors, a, b and c.

Solution:

Let the parallelepiped formed be:



Here,

$$\vec{OJ} = \vec{a}\ ,\ \vec{OL} = \vec{b}\ and\ \vec{OK} = \vec{c}$$

$\hat{n}$ is a unit vector along OJ perpendicular to the plane containing $\vec{b}\ and\ \vec{c}$

Now, $\vec{b} \times \vec{c}$ = bcsin90^0 $\hat{n}$

= bc $\hat{n}$

$\vec{a}.(\vec{b} \times \vec{c}) = a.(bc\hat{n})$

= abc cosθ $\hat{n}$

= abc cos 0^0

= abc

Volume of the parallelepiped.

Q6. Find the components along the x, y, z axes of the angular momentum l of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

Solution:

Linear momentum , $\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

Position vector of the body , $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Angular momentum , $\vec{I} = \vec{r} \times \vec{p}$

= $(x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$

= $\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{bmatrix}$ = $\hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$

$l_x\hat{i} + l_y\hat{j} + l_z\hat{k} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$

From this we can conclude;

$l_X = yp_Z - zp_{y_y}$, $l_Y = zp_X - xp_Z$ and $l_Z = xp_Y - yp_X$

If the body only moves in the x-y plane then $z = p_z = 0$. Which means :

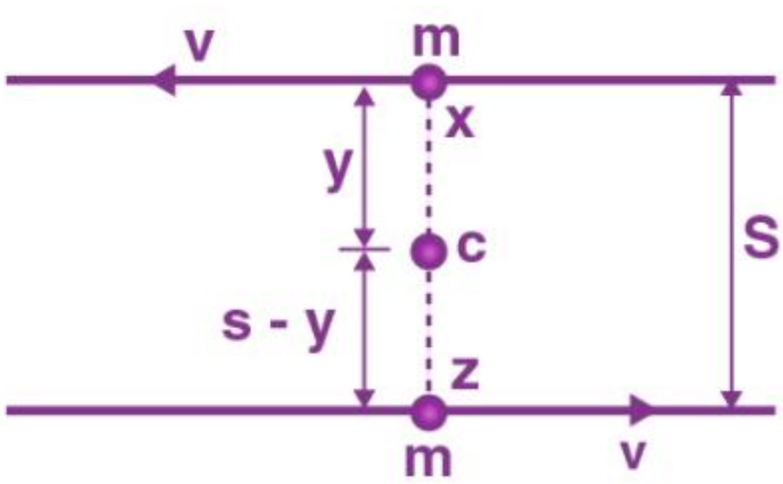
$l_X = l_Y = 0$

And hence only $l_Z = xp_Y - yp_X$, which is just the z component of angular momentum.

Q7. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the angular momentum vector of the two-particle system is the same whatever be the point about which the angular momentum is taken

Solution:

Let us consider three points be Z, C and X:



Angular momentum at Z,

L_Z =mv x 0 + mv x d

=mvd——(1)

Angular momentum about x,

L_X = mv x d + mv x 0

= mvd——(2)

Angular momentum about C,

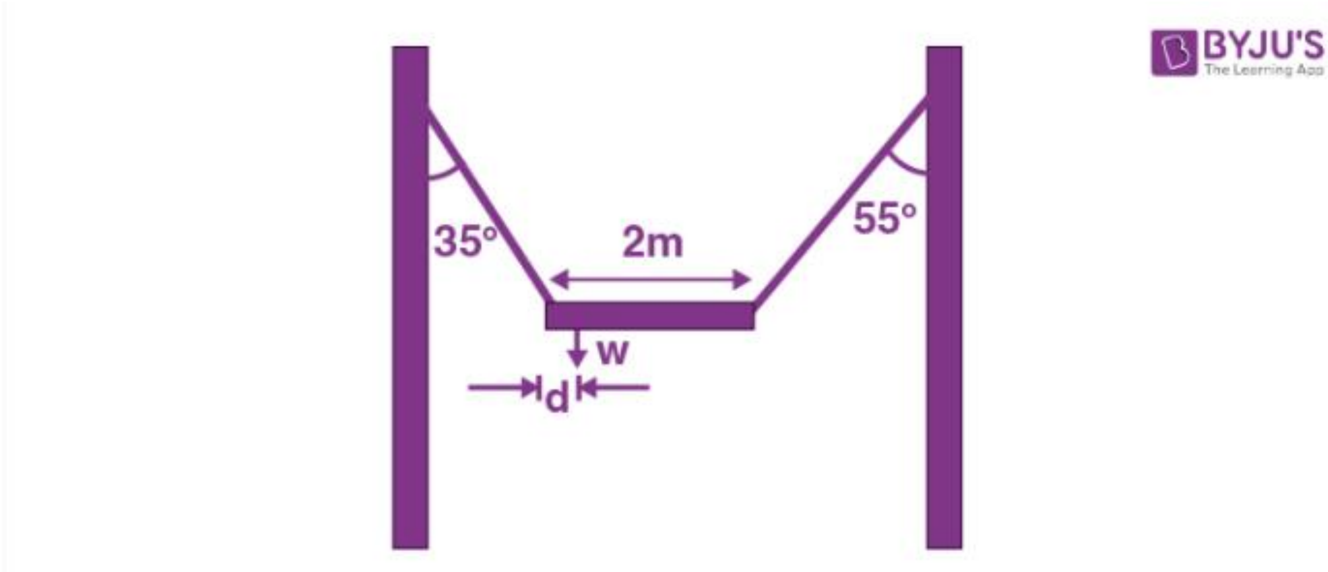
L_C = mv x y + mv x (d – y) = mvd——(3)

Thus we can see that ;

$L_Z = L_X = L_C$

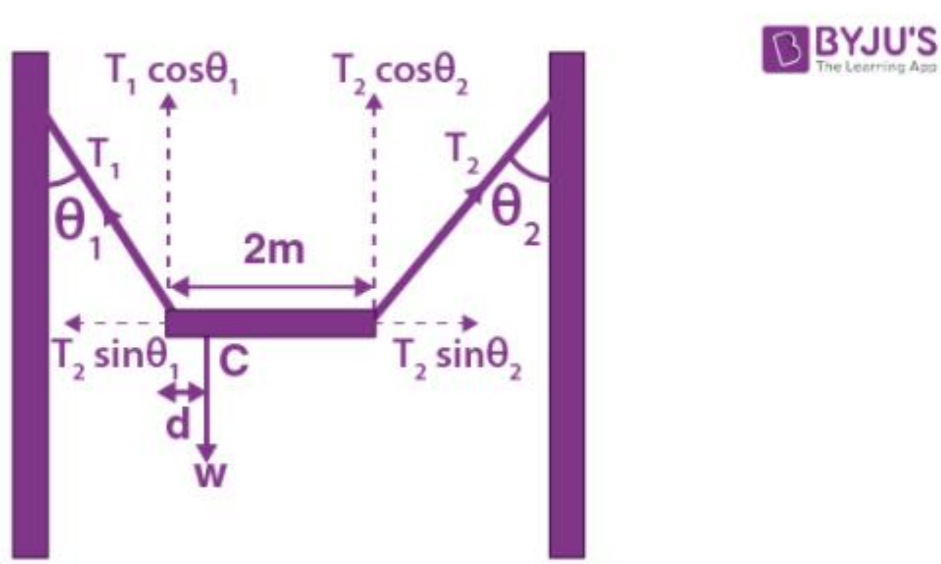
This proves that the angular momentum of a system does not depend on the point about which its taken.

Q8. A 2m irregular plank weighing W kg is suspended in the manner shown below, by strings of negligible weight. If the strings make an angle of 35^0 and 55^0 respectively with the vertical, find the location of center of gravity of the plank from the left end.



Solution:

The free body diagram of the above figure is:



Given,

Length of the plank, l = 2 m

$\theta_1 = 35^0$ and $\theta_2 = 55^0$

Let T_1 and T_2 be the tensions produced in the left and right strings respectively.

So at translational equilibrium we have;

$T_1 \sin\theta_1 = T_2 \sin\theta_2$

$T_1 / T_2 = \sin\theta_2 / \sin\theta_1 = \sin 55 / \sin 35$

$T_1 / T_2 = 0.819 / 0.573 = 1.42$

$T_1 = 1.42T_2$

Let 'd' be the distance of the center of gravity of the plank from the left.

For rotational equilibrium about the centre of gravity:

$T_1 \cos 35 \times d = T_2 \cos 55 \ (2 – d)$

$(T_1 / T_2) \times 0.82d = (2 \times 0.57 – 0.57d)$

Substituting $T_1 = 1.42T_2$

$1.42 \times 0.82\ d + 0.57\ d = 1.14$

$1.73d = 1.14$

Therefore $d = 0.65$m

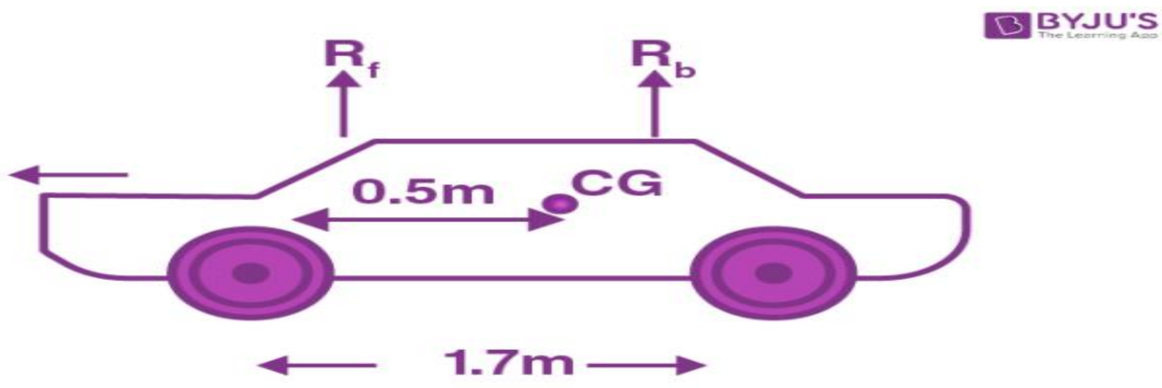
Q9. A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Solution:

Given,
Mass of the car, m= 1800 kg

Distance between the two axles, d = 1.8 m
Distance of the C.G. (centre of gravity) from the front axle = 1.05 m
The free body diagram of the car can be drawn as:

Let R_f and R_b be are the forces exerted by the level ground on the front and back wheels, respectively.



At translational equilibrium:
$R_f + R_b = mg$
$= 1800 \times 9.8$
$= 14700$ N . . . . (1)

For rotational equilibrium about the C.G., we have:
$R_f (1.05) = R_b (1.8 - 1.05)$
$R_f / R_b = 0.75/1.05 = 5/7$

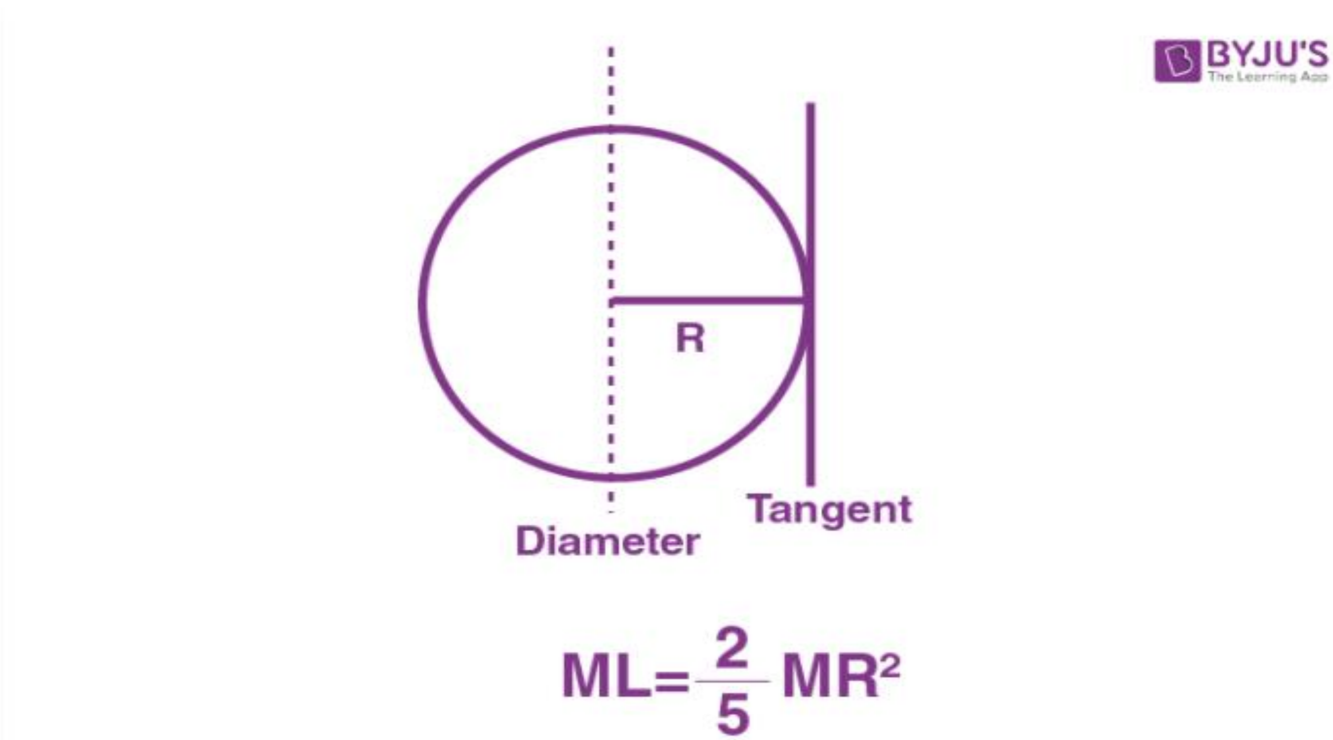
$R_f = (5/7) R_b$ . . . . (2)
Using value of equation (2) in equation (1), we get:
$(5/7) R_b + R_b = 14700$
$R_b = 10290$ N
$\therefore R_f = (5/7) R_b = (5/7)10290 = 7350$ N

Q10. (a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.
(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

Solution:

Given,

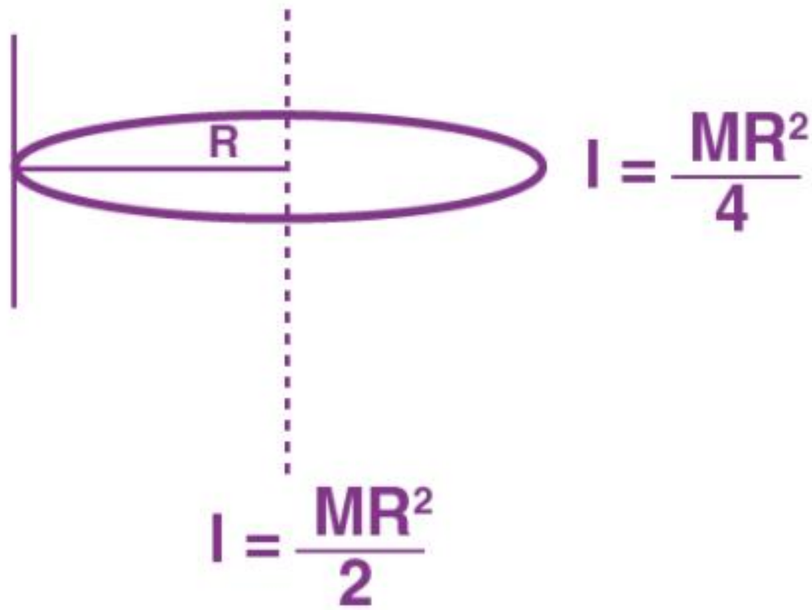
(a) The moment of inertia (M.I.) of a sphere about its diameter = $2MR^2/5$



According to the theorem of parallel axes, M.I of a sphere about a tangent to the sphere = $2MR^2/5 + MR^2 = (7MR^2)/5$

(b) Given, moment of inertia of a disc about its diameter = $(MR^2)1/4$

(i)According to the theorem of perpendicular axis, the moment of inertia of a planar body (lamina) about an axis passing through its center and perpendicular to the disc = $2 \times (1/4)MR^2 = MR^2/2$



The situation is shown in the given figure.

(ii) Using the theorem of parallel axes:

Moment of inertia about an axis normal to the disc and going through a point on its circumference

$= MR^2/2 + MR^2$

$= (3MR^2)/2$

Q11. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Solution:

Let m be the mass and r be the radius of the solid sphere and also the hollow cylinder.

The moment of inertia of the hollow cylinder about its standard axis, $I_1 = MR^2$

Moment of inertia of the solid sphere about an axis passing through its centre,

$I_2 = (2MR^2)/5$

Let T be the magnitude of the torque being exerted on the two structures, producing angular accelerations of α_2 and α_1 in the sphere and the cylinder, respectively.

Thus we have, $T = I_1\alpha_1 = I_2\alpha_2$

$$\therefore \alpha_2 / \alpha_1 = I_1 / I_2 = \frac{MR^2}{\frac{2}{5}MR^2} = 5/2$$

$\alpha_2 > \alpha_1$ . . . . . .(1)
Now, using the relation :
$\omega = \omega_0 + \alpha t$
Where,
ω_0 = Initial angular velocity
t = Time of rotation
ω = Final angular velocity
For equal ω_0 and t, we have:
$\omega \propto \alpha$ . . . . (2)
From equations (1) and (2), we can write:
$\omega_2 > \omega_1$
Thus, from the above relation, it is clear that the angular velocity of the solid sphere will be greater than that of the hollow cylinder.

Q12. A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Solution:

Given,

Mass of the cylinder, m = 20 kg
Angular speed, ω = 100 rad s^{-1}
Radius of the cylinder, r = 0.25 m
The moment of inertia of the solid cylinder:
$I = mr^2 / 2$
$= (1/2) \times 20 \times (0.25)^2$
$= 0.625$ kg m^2
(a) $\therefore$ Kinetic energy = $(1/2)\, I\, \omega^2$
$= (1/2) \times 0.625 \times (100)^2 = 3125$ J
(b) $\therefore$ Angular momentum, $L = I\omega$
$= 0.625 \times 100$
= 62. 5 Js

Q13. (i) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction

(ii) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Solution:

(a) Given,

Initial angular velocity, ω_1= 40 rev/min
let the final angular velocity = ω_2
Let the boy's moment of inertia with hands stretched out = I_1
Let the boy's moment of inertia with hands folded in = I_2
We know:
$I_2 = (2/5)\ I_1$
As no external forces are acting on the boy, the angular momentum will be constant.
Thus, we can write:
$I_2\,\omega_2 = I_1\,\omega_1$
$\omega_2 = (I_1/I_2)\ \omega_1$
$= [\ I_1 / (2/5)I_1\ ] \times 40 = (\ 5/2\ ) \times 40\ =\ 100$ rev/min

(b) Final kinetic energy of rotation, $E_F = (1/2)\ I_2\ \omega_2^2$
Initial kinetic energy of rotation, $E_I = (1/2)\ I_1\ \omega_1^2$
$E_F / E_I = (1/2)\ I_2\ \omega_2^2 / (1/2)\ I_1\ \omega_1^2$
$= (2/5)\ I_1\ (100)^2 / I_1\ (40)^2$
$= 2.5$
$\therefore E_F = 2.5\ E_I$
It is clear that there is an increase in the kinetic energy of rotation and it can be attributed to the internal energy used by the boy to fold his hands.

Q14. A rope of negligible mass is wound around a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Solution:

Given,

Mass of the hollow cylinder, m = 3 kg
Radius of the hollow cylinder, r = 40 cm = 0.4 m
Force applied, F = 30 N
Moment of inertia of the hollow cylinder about its axis:
$I = mr^2$
$= 3 \times (0.4)^2 = 0.48$ kg m^2
Torque, $T = F \times r = 30 \times 0.4 = 12$ Nm
Also, we know that:

Torque = moment of inertia x acceleration
$T = I\alpha$
(a) Therefore, $\alpha = T / I = 12 / 0.48 = 25$ rad s^{-2}
(b) Linear acceleration = $R\alpha = 0.4 \times 25 = 10$ m s^{-2}

Q15. To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

Solution:

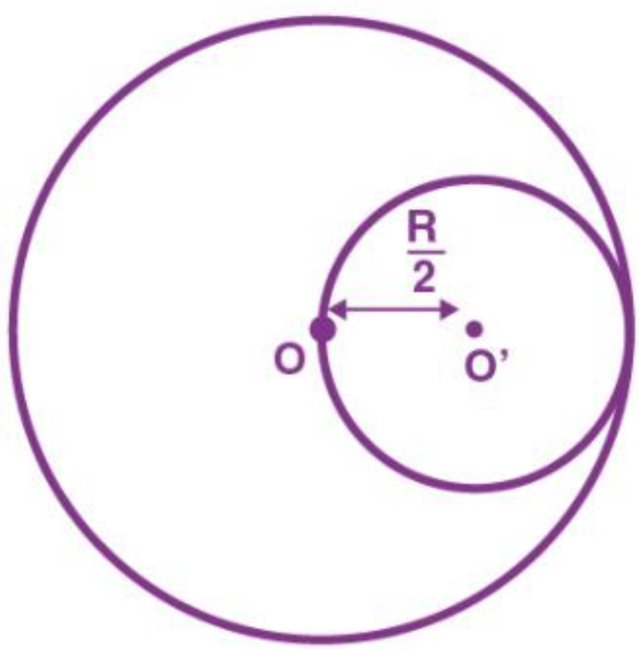
Given,

The angular speed of the rotor, $\omega = 200$ rad/s
Torque, T = 180 Nm
Therefore, power of the rotor (P):
$P = T\,\omega$
$= 200 \times 180$
$= 36.0$ kW
Therefore, the engine requires 36.0 kW of power.

Q.16. From a uniform disc of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body

Solution:

Let the mass/unit area of the original disc = σ
Radius of the original disc $=2r$
Mass of the original disc, $m = \pi(2r^2)\,\sigma = 4\,\pi r^2\sigma$ . . . . (i)
The disc with the cut portion is shown in the following figure:



Radius of the smaller disc = r

Mass of the smaller disc, $m' = \pi\, r^2\sigma$

=> $m' = m/4$ [From equation (i)]
Let O' and O be the respective centers of the disc cut off from the original and the original disc.
According to the definition of center of mass, the center of mass of the original disc is concentrated at O, while that of the smaller disc is supposed at O′.

We know that :
OO′= $R/2 = r_2$
After the smaller circle has been cut out, we are left with two systems whose masses are:
–*m'* (= m/4) concentrated at O′, *and m* (concentrated at O).
(The negative sign means that this portion has been removed from the original disc.)
Let *X* be the distance of the center of mass from O.
We know :
$X = (m_1 r_1 + m_2 r_2) / (m_1 + m_2)$
$X = [m \times 0 - m' \times (r / 2)] / (M + (-M')) = -R / 6$
(The negative sign indicates that the center of mass is R/6 towards the left of O.)

Q17. A metre stick is balanced on a knife-edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

Solution:

The centre of mass of the meter rule shifts to 45 cm mark from 50 cm mark when 2 coins are added at 12 cm mark.

Mass of two coins, *m* = 10 g

Distance at which the coins are placed from the new support d_1= 45 – 12 = 33 cm

Distance of the centre of mass from the new support= 50- 45 = 5 cm

To find the mass of the scale, we should use the balancing moments

Moment due to the coins, $r_1 = m \times d_1$

Moment due to the mass of the scale $r_2 = M \times d_2$

$r_1 = r_2$

$m \times d_1 = M \times d_2$

Substituting we get

$10 \times 33 = M \times d_2$

$M = (10 \times 33) /5 = 330/5 = 66$ g

Q18. A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?

Solution:

(a)Let the mass of the ball = m
let the height of the ball = h
let the final velocity of the ball at the bottom of the plane = v

At the top of the plane, the ball possesses Potential energy = mgh

At the bottom of the plane, the ball possesses rotational and translational kinetic energies.
Thus, total kinetic energy = $(1/2)mv^2 + (1/2) I \omega^2$

Using the law of conservation of energy, we have:
$(1/2)mv^2 + (1/2)I\omega^2 = mgh$

For a solid sphere, the moment of inertia about its centre, $I = (2/5)mr^2$
Thus, equation (i) becomes:

$(1/2)mv^2 + (1/2)[(2/5)mr^2]\omega^2 = mgh$
$(1/2)v^2 + (1/5)r^2\omega^2 = gh$
Also , we know $v = r\omega$
$\therefore$ We have : $(1/2)v^2 + (1/5)v^2 = gh$
$v^2(7/10) = gh$

$$v = \sqrt{\frac{10}{7}gh}$$

Since the height of both the planes is the same, the velocity of the ball will also be the same irrespective of which plane it is rolled down.

(b) Let the inclinations of the two planes be θ_1 and θ_2, , where:
$\theta_1 < \theta_2$
The acceleration of the ball as it rolls down the plane with an inclination of θ_1 is:
$g \sin\theta_1$
let R_1 be the normal reaction to the sphere.

Similarly, the acceleration in the ball as it rolls down the plane with an inclination of θ_2 is:
$g \sin\theta_2$
Let R_2 be the normal reaction to the ball.
Here, $\theta_2 > \theta_1$; $\sin\theta_2 > \sin\theta_1$ . . . . (1)
$\therefore a_2 > a_1$ . . .. (2)

Initial velocity, u = 0
Final velocity, v = Constant
Using the first equation of motion:
$v = u + at$
$\therefore t \propto (1/a)$
For inclination θ_1 : $t_1 \propto (1/a_1)$
For inclination θ_2 : $t_2 \propto (1/a_2)$

As $a_2 > a_1$ we have:
$t_2 < t_1$
(c)Therefore, the ball will take a greater amount of time to reach the bottom of the inclined plane having the smaller inclination.

Q19. A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

Solution:

Given

The radius of the ring, r = 2 m
Mass of the ring, m = 100 kg
Velocity of the hoop, v = 20 cm/s = 0.2 m/s
Total energy of the loop = Rotational K.E + Translational K.E..
$E_T = (1/2)mv^2 + (1/2)\, I\, \omega^2$
We know, the moment of inertia of a ring about its centre, $I = mr^2$
$E_T = (1/2)mv^2 + (1/2)\,(mr^2)\omega^2$
Also, we know $v = r\omega$
$\therefore E_T = (1/2)mv^2 + (1/2)mr^2\omega^2$
$\Rightarrow (1/2)mv^2 + (1/2)mv^2 = mv^2$
Thus the amount of energy required to stop the ring = total energy of the loop.
$\therefore$ The amount of work required, $W = mv^2 = 100 \times (0.2)^2 = 4$ J.

Q.20. The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two-thirds of its kinetic energy of translation. Find the average angular velocity of the molecule

Solution:

Given,

Mass of one oxygen molecule, $m = 5.30 \times 10^{-26}$ kg

Thus, the mass of each oxygen atom = m/2

Moment of inertia of it, $I = 1.94 \times 10^{-46}$ kg m^2
Velocity of the molecule, v = 500 m/s
The distance between the two atoms in the molecule = 2r

Thus, moment of inertia I, is calculated as:
$I = (m/2)r^2 + (m/2)r^2 = mr^2$
$r = (I / m)^{1/2}$
$\Rightarrow (1.94 \times 10^{-46} / 5.36 \times 10^{-26})^{1/2} = 0.60 \times 10^{-10}$ m
Given,
$K.E_{rot} = (2/3)K.E_{trans}$
$(1/2)\, I\, \omega^2 = (2/3) \times (1/2) \times mv^2$
$mr^2\omega^2 = (2/3)mv^2$
Therefore, $\omega = (2/3)^{1/2}\,(v/r)$
$= (2/3)^{1/2}\,(500 / 0.6 \times 10^{-10}) = 6.88 \times 10^{12}$ rad/s.

Q21. A solid cylinder rolls up an inclined plane of the angle of inclination 30°. At the bottom of the inclined plane, the centre of mass of the cylinder has a speed of 5 m/s.
(a) How far will the cylinder go up the plane?
(b) How long will it take to return to the bottom?

Solution:

Given,

initial velocity of the solid cylinder, v = 5 m/s
Angle of inclination, $\theta = 30°$
Assuming that the cylinder goes up to a height of h. We get :
$(\frac{1}{2}) mv^2 + (\frac{1}{2}) I \omega^2 = mgh$

$(\frac{1}{2}) mv^2 + (\frac{1}{2}) (\frac{1}{2} mr^2) \omega^2 = mgh$
$3/4\ mv^2 = mgh$ (since $v = r\omega$)
$h = 3v^2 / 4g = (3 \times 5^2) / 4 \times 9.8 = 75/39.2 = 1.913$ m

let d be the distance the cylinder covers up the plane, this means :
$\sin\theta = h/d$
$d = h/\sin\theta = 1.913/\sin 30° = 3.826$m
Now, the time required to return back:

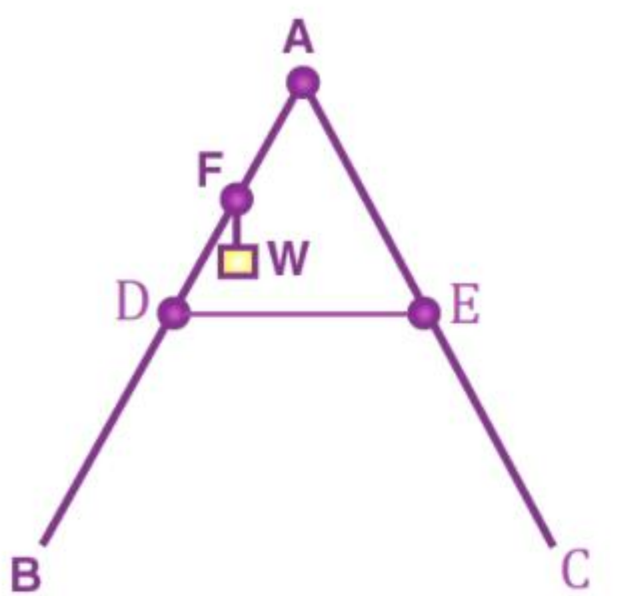
$$t = \sqrt{\frac{2d(1+\frac{K^2}{r^2})}{g\sin\Theta}}$$

$$= \sqrt{\frac{2*3.826(1+\frac{1}{2})}{9.8\sin 30^\circ}}$$

= 1.53 s

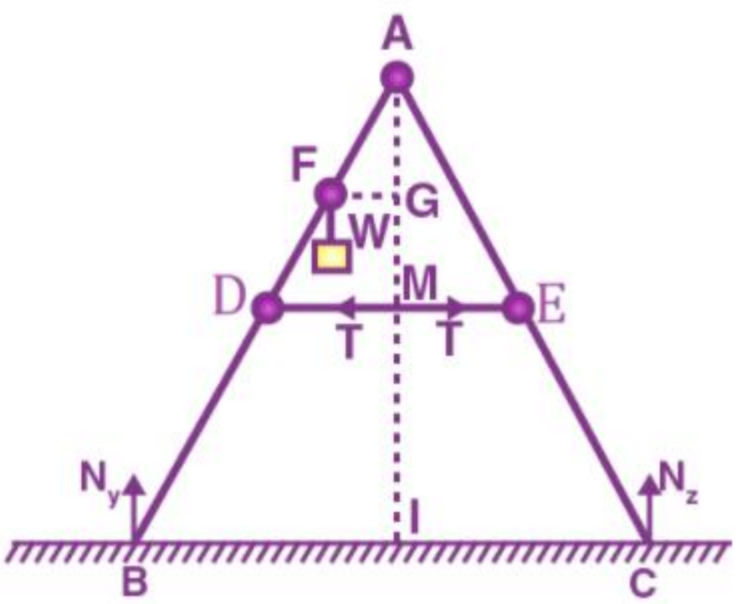
Thus, the cylinder takes 1.53 s to return to the bottom.

Q22. As shown in Figure the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied halfway up. A weight 40 kg is suspended from a point F, 1.2 m from B along with the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take g = 9.8 m/s^2) (Hint: Consider the equilibrium of each side of the ladder separately.)



Solution:

The above situation can drawn as :



Solution:

Here,

N_B = Force being applied by floor point B on the ladder
N_C = Force being applied by floor point C on the ladder
T = Tension in string.
BA= CA = 1.6 m
DE = 0.5 m
BF = 1.2 m
Mass of the weight, m = 40 kg

Now,

Make a perpendicular from A on the floor BC. This will intersect DE at mid-point H.
ΔABI and ΔAIC are similar
∴BI = IC
This makes I the mid-point of BC.
DE || BC
BC = 2 × DE = 1 m AF
BA – BF= 1.6 – 1.2 = 0.4 m . . . . (1)
D is the mid-point of AB.
Thus, we can write:
AD = (1/2) × BA = 0.8 m . . . . (2)

Using equations (1) and (2), we get:
DF = 0.4 m
Thus, F is the mid-point of AD.

FG || DH and F is the mid-point of AD. This will make G the mid-point of AH.
ΔAFG and ΔADH are similar
∴ FG / DH = AF / AD
FG / DH = 0.4 / 0.8 = 1 / 2
FG = (1/2) DH

H is the midpoint of the rope. Therefore, DH = 0.5/2 = 0.25 m
= (1/2) × 0.25 = 0.125 m

In ΔADH:
$AH = (AD^2 – DH^2)^{1/2}$
$= (0.8^2 – 0.25^2)^{1/2} = 0.76$ m

For translational equilibrium of the ladder, the downward force should be equal to the upward force.
$N_C + N_B = mg = 392$ N . . .. (3) [mg = 9.8 x 40]
Rotational equilibrium of the ladder about A is:
$-N_B \times BI + FG \times mg + N_C \times CI – T \times AG + AG \times T = 0$
$-N_B \times 0.5 + 392 \times 0.125 + N_C \times 0.5 = 0$
$(N_C – N_B) \times 0.5 = 49$
$N_C – N_B = 98$ . . . . (4)
Adding equation (3) and equation (4), we get:
$N_C = 245$ N
$N_Y = 147$ N
Rotational equilibrium about AB

Considering the moment about A
$-N_B \times BI + FG \times mg + T \times AG = 0$
-245× 0.5 + 392 × 0.125 + 0.76 x T = 0
∴ T = 96.7 N.

Q23. A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.
(a) What is his new angular speed? (Neglect friction.)
(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Solution:

(a) Given,

Mass of each weight = 5 kg

Moment of inertia of the man-platform system = 7.6 kg m^2

Moment of inertia when his arms are fully stretched to 90 cm:
$2 \times m r^2$
$= 2 \times 5 \times (0.9)^2$
= 8.1 kg m^2

Initial moment of inertia of the system, $I_i = 7.6 + 8.1 = 15.7$ kg m^2

Angular speed, ω_i = 30 rev / min
=> Angular momentum, $L_i = I_i\omega_i = 15.7 \times 30$

= 471 . . . . (i)

Moment of inertia when he folds his hands inward to 15 cm:
$2 \times mr^2$
$= 2 \times 5 \times (0.2)^2 = 0.4$ kg m^2

Final moment of inertia, $I_f = 7.6 + 0.4 = 8.0$ kg m^2
let final angular speed = ω_f
=> Final angular momentum, $L_f = I_f\omega_f = 8.0\omega_f$ . . . . . . (ii)

According to the principle of conservation of angular momentum:
$I_i\omega_i = I_f\omega_f$
$\therefore \omega_f = 471/8 = 58.88$ rev/min

(b) There is a change in kinetic energy, with the decrease in the moment of inertia kinetic energy increases. The extra kinetic energy is supplied to the system by the work done by the man in folding his arms inside.

Q24. A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.
(Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.)

Solution:

Given, Velocity, v = 500 m/s

Mass of bullet, $m = 10$ g $= 10 \times 10^{-3}$ kg

Width of the door, $L = 1$ m
Radius of the door, $r = 1/2$
Mass of the door, $M = 10$ kg

Angular momentum imparted by the bullet on the door:
$\alpha = mvr$
$= (10\times 10^{-3}) \times (500) \times (1/2) = 2.5$ kg m^2 s^{-1} …(i)

Now, Moment of inertia of the door :
$I = ML^2 / 3$
$= (1/3) \times 12 \times 1^2 = 4$ kgm^2
We know, $\alpha = I\omega$
$\therefore \omega = \alpha / I$
$= 2.5/4 = 0.625$ rad /s

Q25. Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident.

(a) What is the angular speed of the two-disc system?

(b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$

Solution:

(a)Given,

Let the moment of inertia of the two turntables be I_1 and I_2 respectively.

Let the angular speed of the two turntables be ω_1 and ω_2 respectively.

Thus we have;

Angular momentum of turntable 1, $L_1 = I_1\omega_1$
Angular momentum of turntable 2, $L_2 = I_2\omega_2$
=> total initial angular momentum $L_i = I_1\omega_1 + I_2\omega_2$

When the two turntables are combined together:
Moment of inertia of the two turntable system, $I = I_1 + I_2$
Let ω be the angular speed of the system.
=> final angular momentum, $L_T = (I_1 + I_2)\ \omega$
According to the principle of conservation of angular momentum, we have:
$L_i = L_T$
$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$
Therefore , $\omega = (I_1\omega_1 + I_2\omega_2) / (I_1 + I_2)$ . . . . (1)

(b) Kinetic energy of turntable 1, $K.E_1 = (1/2)\ I_1\omega_1^2$
Kinetic energy of turntable 2, $K.E_1 = (1/2)\ I_2\omega_2^2$
Total initial kinetic energy, $KE_I = (1/2)\ (I_1\omega_1^2 + I_2\omega_2^2)$
When the turntables are combined together, their moments of inertia add up.
Moment of inertia of the system, $I = I_1 + I_2$
Angular speed of the system = ω
Final kinetic energy KE_F : $= (1/2)\ (I_1 + I_2)\ \omega^2$Using the value of ω from (1)
$= (1/2)\ (I_1 + I_2)\ [\ (I_1\omega_1 + I_2\omega_2) / (I_1 + I_2)\ ]^2$
$KE_F = (1/2)\ (I_1\omega_1 + I_2\omega_2)^2 / (I_1 + I_2)$

Now, $KE_I - KE_F$
$= (I_1\omega_1^2 + I_2\omega_2^2)\ (1/2) - [\ (1/2)\ (I_1\omega_1 + I_2\omega_2)^2 / (I_1 + I_2)\ ]$

Solving the above equation, we get :
$= I_1\ I_2\ (\omega_1 - \omega_2)^2 / 2(I_1 + I_2)$
As $(\omega_1 - \omega_2)^2$ will only yield a positive quantity and I_1 and I_2 are both positive, the RHS will be positive.
Which means $KE_I - KE_F > 0$
Or, $KE_I > KE_F$

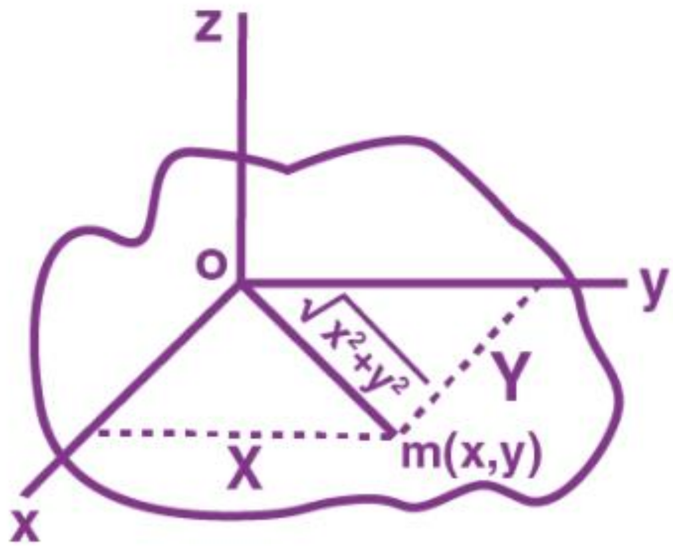
Some of the kinetic energy was lost overcoming the forces of friction when the two turntables were brought in contact.

Q26. (a) Prove the theorem of perpendicular axes. (Hint: Square of the distance of a point (x, y) in the x–y plane from an axis through the origin and perpendicular to the plane is x^2+y^2).
(b) Prove the theorem of parallel axes. (Hint: If the centre of mass of a system of n particles is chosen to be the origin $\sum m_i\ r_i = 0$).

Solution:

(a) According to the theorem of perpendicular axes, the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point where the perpendicular axis passes through it.

Let us consider a planar body. An axis perpendicular to the body through a point O is taken as the z-axis. Two mutually perpendicular axes lying in the plane of the body and concurrent with z-axis, i.e., passing through O, are taken as the x and y-axes.



Suppose at the point R, m particles are then the moment of inertia about Z axis of the lamina

$I_z = \sum m(x^2 + y^2)$

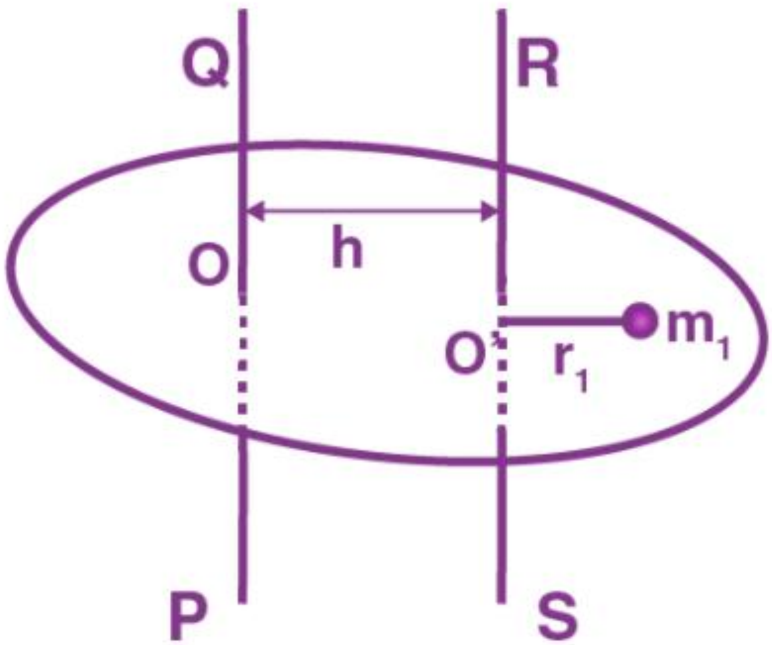
$= \sum mx^2 + \sum m\, y^2$

Moment of inertia about x-axis, $I_x = \sum mx^2$

Moment of inertia about y-axis, $I_y = \sum my^2$

$I_z = I_x + I_y$
Thus, the theorem is verified.

(b) According to the theorem of parallel axes, the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its center of mass and the product of its mass and the square of the distance between the two parallel axes.



Suppose a rigid body is made up of n number of particles, having masses m_1, m_2, m_3, … , m_n, at perpendicular distances r_1, r_2, r_3, … , r_n respectively from the center of mass C of the rigid body.

Let r_i be the perpendicular distance of the particle of mass m_i from KL, then

$$I_{KL} = \sum_i m_i r_i^2$$

The perpendicular distance of mass mi from the axis AB= h+ ri

$$I_{AB} = \sum_i m_i (h + r_i)^2$$

$$I_{AB} = \sum_i m_i (h^2 + r_i^2 + 2hr_i)$$

$$I_{AB} = \sum_i m_i h^2 + \sum_i m_i r_i^2 + \sum_i m_i 2hr_i$$

As the body is balanced about the centre of mass, the algebraic sum of the moments of the weights of all particles about an axis passing through C must be zero.

$\sum_i m_i 2hr_i$ = 0

$$I_{AB} = I_{KL} + \sum_i m_i h^2$$

Therefore, $\sum m_i$ = M

M = Total mass of the rigid body

Therefore, $I_{AB} = I_{KL} + Mh^2$

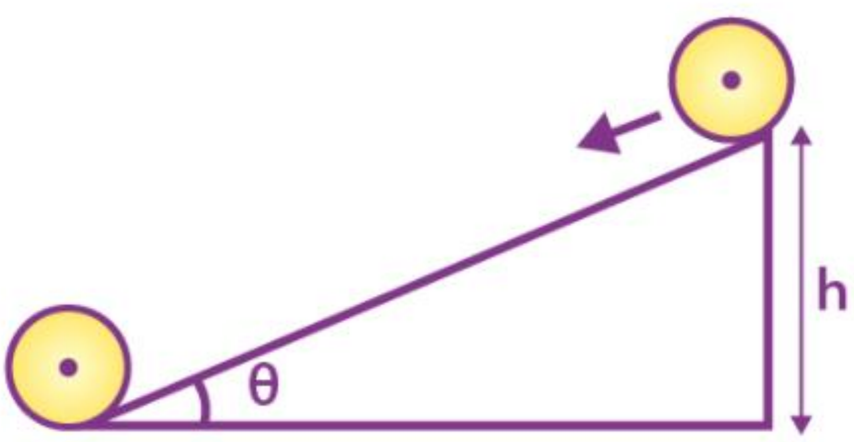
Therefore, the theorem is verified.

Q27. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by

$v^2 = 2gh/(1+k^2/R^2)$ using dynamical consideration (i.e. by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Solution:

The above situation can be represented as :



Here,

R = the body's radius

g = Acceleration due to gravity

K = the body's radius of gyration

v = the body's translational velocity

m = Mass of the body
h = Height of the inclined plane

Total energy at the top of the plane, E_T (potential energy) = mgh

Total energy at the bottom of the plane, $E_b = KE_{rot} + KE_{trans}$

$= (1/2)\, I\, \omega^2 + (1/2)\, mv^2$

We know, $I = mk^2$ and $\omega = v / R$

Thus, we have $E_b = \frac{1}{2}mv^2 + \frac{1}{2}(mk^2)(\frac{v^2}{R^2})$

$= \frac{1}{2}mv^2(1 + \frac{k^2}{R^2})$

According to the law of conservation of energy:

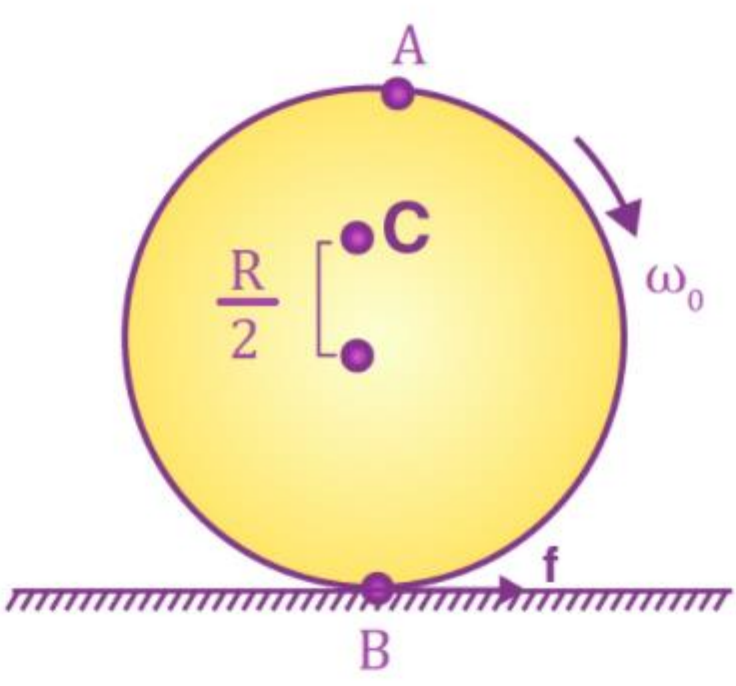
$E_T = E_b$

mgh = $\frac{1}{2}mv^2(1 + \frac{k^2}{R^2})$

$\therefore v^2 = 2gh / [\,1 + (k^2 / R^2)\,]$

Thus, the given relation is proved.

Q28. A disc rotating about its axis with angular speed ω_o is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Figure. Will the disc roll in the direction indicated?



Solution:

The respective linear velocities are :

For point A, $v_A = r\omega_o$ in the direction of the arrow

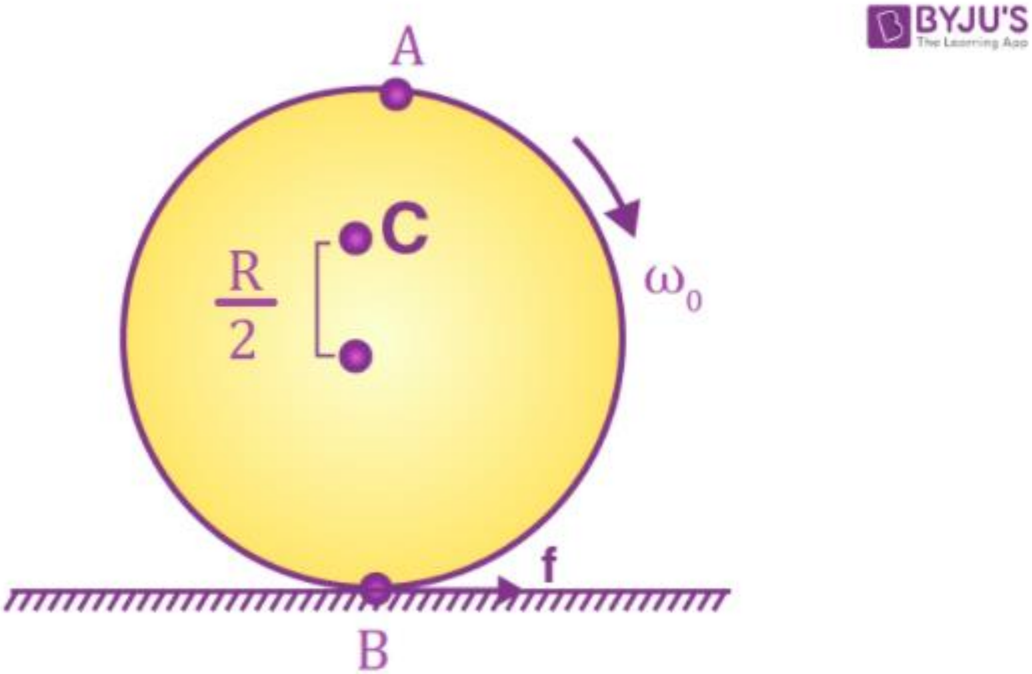
For point B, $v_B = r\omega_0$ in the direction opposite to the arrow

For point C, $v_c = (R/2)\omega_o$ in the same direction as that of v_A

Firstly there is no tangential push given to the disc in the initial state. Secondly, the force of friction was the only means of tangential force, but that too is absent as the surface is frictionless. Therefore, the disc cannot roll ahead.

Q29. Explain why friction is necessary to make the disc in Figure roll in the direction indicated.
(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.
(b) What is the force of friction after perfect rolling begins?

Solution:



(a) The frictional force acting on the disc must be opposite to the direction of velocity. Frictional force acts towards left as it opposes the direction of velocity at point B which is towards right.

(b) As force of friction acts in the direction opposite to the velocity at point B, perfect rolling starts only when the force of friction at that point equals zero. This makes the force of friction acting on the disc equal to zero.

Q30. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with an initial angular speed equal to 10 π rad s^{-1}. Which of the two will start to roll earlier? The coefficient of kinetic friction is μ_k = 0.2.

Solution:

Given,

Radii of the ring and the disc, r = 10 cm = 0.10 m
Initial angular speed, $\omega_0 = 10\pi$ rad s^{-1}
Coefficient of kinetic friction, $\mu_k = 0.2$

The motion of the two objects is caused by the force of friction. According to Newton's second, the force of friction, f = ma
$\mu_k mg = ma$
Where,
a = Acceleration produced in the disc and the ring
m = Mass
$\therefore a = \mu_k g$ . . . . (1)

Using the first equation of motion :
v = u + at
$= 0 + \mu_k gt$
$= \mu_k gt$ . . . . (2)

The frictional force applies a torque in perpendicularly outward direction and reduces the initial angular speed.
Torque, $T = -I\alpha$
Where, α = Angular acceleration
$\mu_k mgr = -I\alpha$
$\therefore \alpha = -\mu_k mgr / I$ . . . . (3)

According to the first equation of rotational motion, we have :
$\omega = \omega_0 + \alpha t$
$= \omega_0 + (-\mu_k mgr / I)t$ . . . .(4)
Rolling starts when linear velocity, $v = r\omega$
$\therefore v = r(\omega_0 - \mu_k mgrt / I)$ …(5)

Using equation (2) and equation (5), we have:
$\mu_k gt = r(\omega_0 - \mu_k mgrt / I)$
$= r\omega_0 - \mu_k mgr^2 t / I$ . . . . (6)

For the ring:
$I = mr^2$
$\therefore \mu_k gt = r\omega_0 - \mu_k mgr^2 t / mr^2$
$= r\omega_0 - \mu_k gt$
$2\mu_k gt = r\omega_0$
$\therefore t = r\omega_0 / 2\mu_k g$
$= (0.1 \times 10 \times 3.14) / (2 \times 0.2 \times 10) = 0.80$ s . . . . (7)

For the disc: $I = (1/2)mr^2$
$\therefore \mu_k gt = r\omega_0 - \mu_k mgr^2 t / (1/2)mr^2$
$= r\omega_0 - 2\mu_k gt$
$3\mu_k gt = r\omega_0$

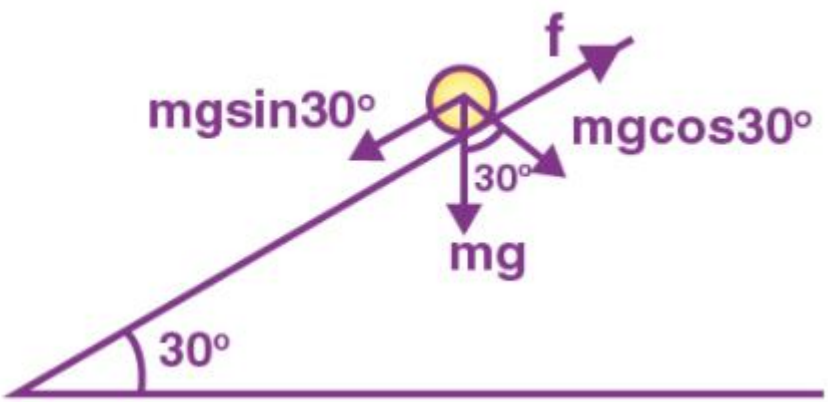
$\therefore$ $t = r\omega_0 / 3\mu_k g$
$= (0.1 \times 10 \times 3.14) / (3 \times 0.2 \times 9.8) = 0.53$ s .....(8)

Since $t_D > t_R$, the disc will start rolling before the ring.

Q31. A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30º. The coefficient of static friction μ_s = 0.25.
(a) How much is the force of friction acting on the cylinder?
(b) What is the work done against friction during rolling?
(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?

Solution:

The above situation can be depicted as:



Given,

mass, m = 10 kg
Radius, r = 15 cm = 0.15 m
Co-efficient of kinetic friction, μ_s = 0.25
Angle of inclination, θ = 30°

We know, moment of inertia of a solid cylinder about its geometric axis, $I = (1/2)mr^2$

The acceleration of the cylinder is given as:

$a = mg \sin\theta / [m + (I/r^2)]$
$= mg \sin\theta / [m + \{ (½)mr^2 / r^2 \}]$
$= (2/3)\, g \sin 30°$
$a = (2/3) \times 9.8 \times (1/2) = 3.26\ ms^{-2}$

(a) Using Newton's second law of motion, we can write net force as:
$f_{NET} = ma$
$mg \sin 30° - f = ma$
$f = mg \sin 30° - ma$

$= 10 \times 9.8 \times (1/2) - 10 \times 3.26$
$= 49 – 32.6 = 16.3N$

(b) There is no work done against friction during rolling.

(c) We know for rolling without skidding :
$\mu = (1/3) \tan \theta$
$\tan \theta = 3\mu = 3 \times 0.25$
$\therefore \theta = \tan^{-1} (0.75) = 36.87°$

Q32. Read each statement below carefully, and state, with reasons, if it is true or false;
(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.
(b) The instantaneous speed of the point of contact during rolling is zero.
(c) The instantaneous acceleration of the point of contact during rolling is zero.
(d) For perfect rolling motion, work done against friction is zero.
(e) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Solution:

(a) False. The direction of frictional force is opposite to the direction of motion of the centre of mass. In case of rolling object, the centre of mass moves backwards so the frictional force acts in the forward direction

(b) True. During rolling the point of the body in contact with the ground does not move ahead (this would be slipping) instead it only touches the ground for an instant and lifts off following a curve. Thus, only if the point of contact remains in touch with the ground and moves forward will the instantaneous speed not be equal to zero.

(c) False. For a rolling object instantaneous acceleration will have a value it is not zero

(d) True. This is because during perfect rolling frictional force is zero so work done against it is zero.

(d) True. This is because during perfect rolling frictional force is zero so work done against it is zero.

(e) True. Rolling occurs only when there is a frictional force to provide the torque so in the absence of friction the wheel simply slips down the plane under the influence of its weight.

Q33. Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:

(i) Show $p = p'_i + m_i V$

Where p_i is the momentum of the i^{th} particle (of mass m_i) and $p'_i = m_i v_i'$. Note v_i' is the velocity of the i^{th} particle with respect to the centre of mass.

Also, verify using the definition of the centre of mass that $\Sigma p'_i = 0$

(ii) Prove that $K = K' + ½MV^2$

Where K is the total kinetic energy of the system of particles, K′ is the total kinetic energy of the system when the particle velocities are taken relative to the center of mass and $MV^2/2$ is the kinetic energy of the translation of the system as a whole.

(iii) Show L = L’+ R × MV where L’ = Σ$r'_i \times p'_i$ is the angular momentum of the system about the centre of mass with velocities considered with respect to the centre of mass. Note $r'_i = r_i - R$, rest of the notation is the standard notation used in the lesson. Note L’ and MR × V can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.

(iv) Prove that :

$dL'/dt = \sum r'_i \times dp'/dt$

Further prove that :

$dL'/dt = \tau'_{ext}$

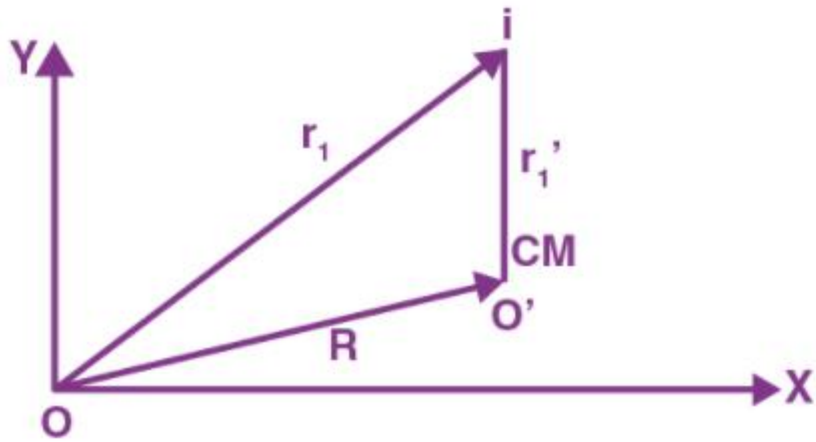
Where τ'_{ext} is the sum of all external torques acting on the system about the centre of mass. (Clue : Apply Newton’s Third Law and the definition of centre of mass . Consider that internal forces between any two particles act along the line connecting the particles.)

Solution:

Here $\vec{r_i} = \vec{r'}_i + \vec{R} + R$ ......(1)

and, $\vec{V_i} = \vec{V'}_i + \vec{V}$ ....(2)

Where, $\vec{r'}_i$ and $\vec{v'}_i$ represent the radius vector and velocity of the i^{th} particle referred to centre of mass O’ as the new origin and $\vec{V}$ is the velocity of centre of mass with respect to O.



(i) Momentum of i[th] particle

$\vec{p'} = m_i \vec{V'}_i$

= $m_i(\vec{V'}_i + \vec{V})$ [From equation (1)]

Or, $\vec{P} = m_i\vec{V} + \vec{P_i}$

(ii) Kinetic energy of system of particles

K = $\frac{1}{2}\sum m_i V_i^2$

= $\frac{1}{2}\sum m_i \vec{V_i}.\vec{V_i}$

= $\frac{1}{2}\sum m_i(\vec{V'}_i + \vec{V})(\vec{V'}_i + \vec{V})$

= $\frac{1}{2}\sum m_i(\vec{V'}_i^2 + \vec{V}^2 + 2\vec{V'}_i\vec{V})$

= $\frac{1}{2}\sum m_i V'^2_i + \frac{1}{2}\sum m_i V^2 + \sum m_i \vec{V'}_i \vec{V}$

= ½ MV^2 + K'

Where M = $\sum m_i$ = total mass of the system.

K' = $\frac{1}{2}\sum m_i V'^2_i$

= kinetic energy of motion about the centre of mass.

Or, ½ Mv^2 = kinetic energy of motion of centre of mass.(Proved)

Since, $\sum_i m_i \vec{V'}_i \vec{V}$ = $\sum \; m_i \frac{d\vec{r_i}}{dt}\vec{V}$

=0

(iii) Total angular momentum of the system of particles.

$\vec{L} = \vec{r_i} \times \vec{p}$

$= (\vec{r_i} + \vec{R}) \times \sum_i m_i(\vec{V'}_i + \vec{V})$

$= \sum_i(\vec{R} \times m_i\vec{V}) + \sum_i(\vec{r'}_i \times m_i\vec{V'}_i) + (\sum_i m_i\vec{r'}_i) \times \vec{V} + \vec{R}\times \sum_i m_i\vec{V_i}$

$= \sum_i(\vec{R} \times m_i\vec{V}) + \sum_i(\vec{r'}_i \times m_i\vec{V'}_i) + (\sum_i m_i\vec{r'}_i) \times \vec{V} + \vec{R} \times \frac{d}{dt}(\sum_i m_i\vec{r'}_i)$

However, we know $\sum_i m_i\vec{r'}_i = 0$

Since, $\sum_i m_i\vec{r'}_i = \sum_i m_i(\vec{r'}_i - \vec{R}) = M\vec{R} - M\vec{R} = 0$

According to the definition of centre of mass,

$\sum_i(\vec{R} \times m_i\vec{V}) = \vec{R} \times M\vec{V}$

Such that, $\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r'}_i \times \vec{P_i}$

Given, $\vec{L} = \sum \vec{r'}_i \times \vec{p'}_i$

Thus, we have ; $\vec{L} = \vec{R} \times M\vec{V} + \vec{L'}$

(iv) From previous solution

$\vec{L'} = \sum \vec{r_i} \times \vec{P_i}$ $\frac{d\vec{L'}}{dt} = \sum \vec{r_i} \times \frac{d\vec{P_i}}{dt} + \sum \frac{d\vec{r'}_i}{dt} \times \vec{P_i}$

$= \sum \vec{r'}_i \times \frac{d\vec{P'}_i}{dt}$

$= \sum \vec{r'}_i \times \vec{F_i^{ext}} = \vec{T'}_{ext}$

Since, $\sum \frac{d\vec{r'}_i}{dt} \times \vec{P_i} = \sum \frac{d\vec{r'}_i}{dt} \times m\vec{v_i} = 0$

Total torque = $\vec{\tau'}_{ext} = \sum \vec{r'}_i \times \vec{F}_i^{ext}$

= $\sum(\vec{r'}_i + \vec{R}) \times \vec{F}_i^{ext}$

= $\vec{\tau'}_{ext} + \vec{T}_0^{(ext)}$

Where, $\vec{\tau'}_{ext}$ is the net torque about the centre of mass as origin and $\vec{\tau}_0^{ext}$ is about the origin O.

$\vec{\tau'}_{ext}$ = $\sum \vec{r'}_i \times \vec{F}_i^{ext}$

= $\sum \vec{r'}_i \times \frac{d\vec{P'}_i}{dt}$

= $\frac{d}{dt} \sum(\vec{r'}_i \times \vec{P'}_i)$ = $\frac{d\vec{L'}}{dt}$

Thus we have , $\frac{d\vec{L'}}{dt}$ = $\vec{\tau'}_{ext}$