

### Miscellaneous Examples

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$$\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$$

1. Prove that the determinant is independent of  $\theta$ .

**Solution:**

$$\text{Let } \Delta = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$$

$$\begin{aligned} \Delta &= x \begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin \theta \begin{vmatrix} -\sin \theta & 1 \\ \cos \theta & x \end{vmatrix} + \cos \theta \begin{vmatrix} -\sin \theta & -x \\ \cos \theta & 1 \end{vmatrix} \\ &= x(-x^2 - 1) - \sin \theta(-x \sin \theta - \cos \theta) + \cos \theta(-\sin \theta + x \cos \theta) \\ &= -x^3 - x + x(\sin^2 \theta + \cos^2 \theta) \\ &= -x^3 \end{aligned}$$

Which is independent of  $\theta$  (Proved)

2. Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

**Solution:**

Start with LHS:

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

Multiplying R1 by a R2 by b and R3 by c, we have

$$\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking out common elements

$$\frac{abc}{abc} \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Interchanging  $C_1$  and  $C_3$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$$

Interchanging  $C_2$  and  $C_3$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

=RHS (proved)

### 3. Evaluate

$$\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$$

**Solution:**

$$\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$$

$$= \cos \alpha \cos \beta (\cos \alpha \cos \beta - 0) - \cos \alpha \sin \beta (-\cos \alpha \sin \beta - 0) - \sin \alpha (-\sin \alpha \sin^2 \beta - \sin \alpha \cos^2 \beta)$$

$$= \cos^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta)$$

$$= \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta)$$

$$= \cos^2 \alpha + \sin^2 \alpha = 1$$

**4. If a, b and c are real numbers, and**

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

**Show that either  $a + b + c = 0$  or  $a = b = c$**

**Solution:**

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying:  $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Since  $\Delta = 0$

This implies,

Either  $2(a + b + c) = 0$  or

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Case 1: If  $2(a + b + c) = 0$

Then  $(a + b + c) = 0$

Case 2:

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Applying:  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$$

$$\Rightarrow (b-c)(c-b) - (b-a)(c-a) = 0$$

$$\Rightarrow bc - b^2 - c^2 + bc - bc + ab + ac - a^2 = 0$$

$$\Rightarrow -a^2 - b^2 - c^2 + ab + bc + ca = 0$$

$$\Rightarrow a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ca = 0$$

$$\Rightarrow (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ca) = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

Above expression only possible, if  $(a - b) = 0$  and  $(b - c) = 0$  and  $(c - a) = 0$

That is  $a = b$  and  $b = c$  and  $c = a$

Therefore, we have result, either  $a+b+c = 0$  or  $a=b=c$ .

### 5. Solve the equation

$$\Delta = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

**Solution:**

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying:  $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$(3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Case 1: Either  $3x + a = 0$

then  $x = -a/3$

Case 2: or

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying:

$C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

$$a^2 = 0$$

or  $a = 0$

Not possible, as we are given  $a \neq 0$ .

So,  $x = -a/3$  is only the solution.

**6. Prove that**

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

**Solution:**

LHS:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b, and c from all the row1, row 2 and row 3 respectively.

$$= abc \begin{vmatrix} a & c & (a+c) \\ (a+b) & b & a \\ b & (b+c) & c \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2 - R_3$$

$$= abc \begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1$$

$$= abc \begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix} = abc \begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

$$= abc(-2b)(-ac-ac) = 4a^2b^2c^2$$

RHS (Proved)

7. If  $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$  find  $(AB)^{-1}$ .

**Solution:**

As we know,  $(AB)^{-1} = B^{-1}A^{-1} \dots(1)$

First find inverse of matrix B.

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) + (-2)(2-0) = 1 \neq 0 \text{ (Inverse of B is possible)}$$

Find cofactors of B:

$$B_{11} = 3, B_{12} = 1, B_{13} = 2$$

$$B_{21} = 2, B_{22} = 1, B_{23} = 2 \text{ and}$$

$$B_{31} = 6, B_{32} = 2, B_{33} = 5$$

So adj. of B is

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now,

$$B^{-1} = \frac{1}{|B|} (\text{adj. } B) = \frac{1}{1} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

From equation (1),

$$(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

8. Let  $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ , verify that

(i)  $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(ii)  $(A^{-1})^{-1} = A$

**Solution:**

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= -13 \neq 0 \text{ (Inverse of A exists)}$$

Cofactors of A are:

$$A_{11} = 14, A_{12} = 11, A_{13} = -5$$

$$A_{21} = 11, A_{22} = 4, A_{23} = -3$$

$$A_{31} = -5, A_{32} = -3, A_{33} = -1$$

So, adjoint of A is

$$\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Now,  $A^{-1}$

$$\frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Again,

$$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

$$= 169 \neq 0 \text{ (Inverse of A exists)}$$

Cofactors of B are:

$$B_{11} = -13, B_{12} = 26, B_{13} = -13$$

$$B_{21} = 26, B_{22} = -39, B_{23} = -13$$

$$B_{31} = -13, B_{32} = -13, B_{33} = -65$$

Therefore, Inverse of B is

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$



Find:  $\text{adj } A^{-1}$

$$A^{-1} = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$|A^{-1}| = -1/13 \neq 0$  (After solving the determinant we get the value. Try at your own)

Inverse of  $A^{-1}$  exists.

Let say cofactors of  $A^{-1}$  are represented as  $C_{ij}$ , we have

$$C_{11} = \frac{-1}{13}, C_{12} = \frac{2}{13}, C_{13} = \frac{-1}{13}$$

$$C_{21} = \frac{2}{13}, C_{22} = \frac{-3}{13}, C_{23} = \frac{-1}{13} \text{ and}$$

$$C_{31} = \frac{-1}{13}, C_{32} = \frac{-1}{13}, C_{33} = \frac{-5}{13}$$

Therefore:

$$(A^{-1})^{-1} = C^{-1} = \frac{1}{|C|} (\text{adj. } C)$$

Which implies,

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= A$$

Which is again given matrix A.

$$(i) \quad (\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$$

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$(ii) \quad (A^{-1})^{-1} = A$$

$$\frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

### 9. Evaluate

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

#### Solution:

Consider,

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation:  $[R_1 \rightarrow R_1 + R_2 + R_3]$

$$\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking  $2(x+y)$  common from first row

$$2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation:  $[C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1]$

$$2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$

$$2(x+y) \cdot 1 \begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$$

$$2(x+y) \{-x^2 + y(x-y)\}$$

$$-2(x+y)(x^2 - xy + y^2)$$

$$=-2(x^3 + y^3)$$

$$\Delta = -2(x^3 + y^3)$$

**10. Evaluate**

$$\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

**Solution:**

Consider

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$\begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix}$$

$$\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

$$=xy$$

$$\Rightarrow \Delta = xy$$

Using properties of determinants in Exercises 11 to 15, prove that:

11.

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

**Solution:**

LHS

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

Operation:  $[C_3 \rightarrow C_3 + C_1]$

$$\begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$

$$(\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$(\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$$

=

$$(\alpha+\beta+\gamma) \begin{vmatrix} \beta-\alpha & (\beta-\alpha)(\beta+\alpha) \\ \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix}$$

=

$$(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} 1 & (\beta+\alpha) \\ 1 & (\gamma+\alpha) \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha)$$

$$= (\alpha + \beta + \gamma)[-(\alpha - \beta)](\gamma - \alpha)[-(\beta - \gamma)]$$

$$= (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$$

= RHS

12.

$$\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x)$$

**Solution:**

LHS=

$$\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix}$$

=

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

We have two determinants, say  $\Delta_1$  and  $\Delta_2$

$$= \Delta_1 + \Delta_2$$

$$\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$= \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$$

$$= (y-x)(z-x) \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z+x-y-x)$$

$$= (x-y)(y-z)(z-x)$$

Again:

$$\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$pxyz\Delta_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= -pxyz \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

$$= pxyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= pxyz \Delta_1$$

Therefore:  $\Delta_1 + \Delta_2$

LHS

$$= (y-x)(z-x)(z-y) + pxyz(y-x)(z-x)(z-y)$$

$$= (1+pxyz)(y-x)(z-x)(z-y)$$

= RHS  
(Proved)

**13. Prove that**

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

**Solution:**

LHS

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Operation:  $[C_1 \rightarrow C_1 + C_2 + C_3]$

$$\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$(a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$(a+b+c) \cdot 1 \begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$$

$$= (a+b+c) [(2b+a)(2c+a) - (a-b)(a-c)]$$

$$= (a+b+c) [4bc + 2ab + a^2 - a^2 + ac + ab - bc]$$

$$= 3(a+b+c)(ab+bc+ac)$$

= RHS

Hence Proved.

**14. Prove that**

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

**Solution:**

LHS

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1]$

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

=

$$1 \begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$$

$$= 7+3p-6-3p$$

= 1

=RHS

Hence Proved.



15. Prove that

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$$

**Solution:**

LHS

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

=

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Operation:  $[C_3 \rightarrow C_3 + (\sin \delta)C_1]$

=

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta + \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta + \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta + \sin \gamma \sin \delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta \end{vmatrix}$$

$$= \cos \delta \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \\ \sin \beta & \cos \beta & \cos \beta \\ \sin \gamma & \cos \gamma & \cos \gamma \end{vmatrix}$$

Column 2 and column 3 are identical, as per determinant property, value is zero.

= 0

= RHS

**16. Solve the system of equations**

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

**Solution:**

Let

$$\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$$

We have

$$\begin{aligned} 2u + 3v + 10w &= 4; \\ 4u - 6v + 5w &= 1; \text{ and} \\ 6u + 9v - 20w &= 2 \end{aligned}$$

Below is the matrix from the given equations:  $AX = B$

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Let say

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Then,

$$|A| = 1200 \neq 0$$

$A^{-1}$  exists.

Cofactors of A are:

$$\begin{aligned} A_{11} &= 75, A_{12} = 110, A_{13} = 72 \\ A_{21} &= 150, A_{22} = -100, A_{23} = 0 \text{ and} \\ A_{31} &= 75, A_{32} = 30, A_{33} = -24 \end{aligned}$$

$$\text{adj. } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Inverse of A is

$$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Resubstitute the values, to get answer in the form of x, y and z.

Since  $AX = B$

$$X = A^{-1} B$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

=

$$\frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

Which means:

$$u = \frac{1}{2}, v = \frac{1}{3} \text{ and } w = \frac{1}{5}$$

This implies:

$$x = \frac{1}{u} = 2$$

$$y = \frac{1}{v} = 3$$

$$z = \frac{1}{w} = 5$$

Answer!

Choose the correct answer in Exercise 17 to 19.

17. If  $a, b, c$  are in A.P., then the determinant is 
$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$
.

- (A) 0      (B) 1      (C)  $x$       (D)  $2x$

**Solution:**

Option (A) is correct.

Explanation:

Since  $a, b, c$  are in A.P.

So,  $b - a = c - b$

Let

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Operation:  $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$$

Row 2 and row 3 are identical, so value is zero.

= 0

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

18. If  $x, y, z$  are non-zero real numbers, then the inverse of matrix  $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$  is:

(A)

$$\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

(B)

$$xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

(C)

$$\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

(D)

$$\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution:** Option (A) is correct.

Explanation:

Let  $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$$|A| = xyz \neq 0 ; (A^{-1} \text{ exists})$$

Now: Cofactors of  $A$  are:

$$A_{11} = yz, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = xz, A_{23} = 0 \text{ and}$$

$$A_{31} = 0, A_{32} = 0, A_{33} = xy$$

Therefore:

$$A^{-1} = \frac{\text{adj.}A}{|A|} = \frac{1}{xyz} \begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$$

$$= \begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$$

19. Let  $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$ , where  $0 \leq \theta \leq 2\pi$ . then:

(A)  $\text{Det (A)} = 0$

(B)  $\text{Det (A)} \in (2, \infty)$

(C)  $\text{Det (A)} \in (2, 4)$

(D)  $\text{Det (A)} \in [2, 4]$

**Solution:**

Option (D) is correct.

Explanation:

Let  $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$

$$|A| = 2 + 2 \sin^2 \theta \neq 0 ; (A^{-1} \text{ exists})$$

Since :

$$-1 \leq \sin \theta \leq 1$$

$$0 \leq \sin^2 \theta \leq 1$$

(The value of  $\theta$  cannot be negative)

$$\text{So, } 0 \leq 2 \sin^2 \theta \leq 2$$

Add 2 in all the expressions:

$$2 \leq 2 + 2 \sin^2 \theta \leq 4$$

Which is equal to

$$2 \leq \text{Det. } A \leq 4$$

