

## Exercise 1.1

Page No: 5

1. Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation R in the set  $A = \{1, 2, 3, \dots, 13, 14\}$  defined as  $R = \{(x, y) : 3x - y = 0\}$

(ii) Relation R in the set N of natural numbers defined as  $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

(iii) Relation R in the set  $A = \{1, 2, 3, 4, 5, 6\}$  as  $R = \{(x, y) : y \text{ is divisible by } x\}$

(iv) Relation R in the set Z of all integers defined as  $R = \{(x, y) : x - y \text{ is an integer}\}$

(v) Relation R in the set A of human beings in a town at a particular time given by

- (a)  $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$
- (b)  $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$
- (c)  $R = \{(x, y) : x \text{ is exactly 7 cm taller than } y\}$
- (d)  $R = \{(x, y) : x \text{ is wife of } y\}$
- (e)  $R = \{(x, y) : x \text{ is father of } y\}$

**Solution:**

(i)  $R = \{(x, y) : 3x - y = 0\}$

$A = \{1, 2, 3, 4, 5, 6, \dots, 13, 14\}$

Therefore,  $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\} \dots(1)$

As per reflexive property:  $(x, x) \in R$ , then R is reflexive)  
Since there is no such pair, so R is not reflexive.

As per symmetric property:  $(x, y) \in R$  and  $(y, x) \in R$ , then R is symmetric.  
Since there is no such pair, R is not symmetric

As per transitive property: If  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ . Thus R is transitive.

From (1),  $(1, 3) \in R$  and  $(3, 9) \in R$  but  $(1, 9) \notin R$ , R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

(ii)  $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$  in set  $N$  of natural numbers.

Values of  $x$  are 1, 2, and 3

So,  $R = \{(1, 6), (2, 7), (3, 8)\}$

As per reflexive property:  $(x, x) \in R$ , then  $R$  is reflexive

Since there is no such pair,  $R$  is not reflexive.

As per symmetric property:  $(x, y) \in R$  and  $(y, x) \in R$ , then  $R$  is symmetric.

Since there is no such pair, so  $R$  is not symmetric

As per transitive property: If  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ . Thus  $R$  is transitive.

Since there is no such pair, so  $R$  is not transitive.

Therefore,  $R$  is neither reflexive, nor symmetric and nor transitive.

(iii)  $R = \{(x, y) : y \text{ is divisible by } x\}$  in  $A = \{1, 2, 3, 4, 5, 6\}$

From above we have,

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

As per reflexive property:  $(x, x) \in R$ , then  $R$  is reflexive.

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$  and  $(6, 6) \in R$ . Therefore,  $R$  is reflexive.

As per symmetric property:  $(x, y) \in R$  and  $(y, x) \in R$ , then  $R$  is symmetric.

$(1, 2) \in R$  but  $(2, 1) \notin R$ . So  $R$  is not symmetric.

As per transitive property: If  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ . Thus  $R$  is transitive.

Also  $(1, 4) \in R$  and  $(4, 4) \in R$  and  $(1, 4) \in R$ , So  $R$  is transitive.

Therefore,  $R$  is reflexive and transitive but nor symmetric.

(iv)  $R = \{(x, y) : x - y \text{ is an integer}\}$  in set  $Z$  of all integers.

Now,  $(x, x)$ , say  $(1, 1) = x - y = 1 - 1 = 0 \in Z \Rightarrow R$  is reflexive.

$(x, y) \in R$  and  $(y, x) \in R$ , i.e.,  
 $x - y$  and  $y - x$  are integers  $\Rightarrow R$  is symmetric.

$(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$  i.e.,

$x - y$  and  $y - z$  and  $x - z$  are integers.

$(x, z) \in R \Rightarrow R$  is transitive

Therefore,  $R$  is reflexive, symmetric and transitive.

(v)

(a)  $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

For reflexive:  $x$  and  $x$  can work at same place

$(x, x) \in R$

$R$  is reflexive.

For symmetric:  $x$  and  $y$  work at same place so  $y$  and  $x$  also work at same place.

$(x, y) \in R$  and  $(y, x) \in R$

$R$  is symmetric.

For transitive:  $x$  and  $y$  work at same place and  $y$  and  $z$  work at same place, then  $x$  and  $z$  also work at same place.

$(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z) \in R$

$R$  is transitive

Therefore,  $R$  is reflexive, symmetric and transitive.

(b)  $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$

$(x, x) \in R \Rightarrow R$  is reflexive.

$(x, y) \in R$  and  $(y, x) \in R \Rightarrow R$  is symmetric.

Again,

$(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z) \in R \Rightarrow R$  is transitive.

Therefore,  $R$  is reflexive, symmetric and transitive.

(c)  $R = \{(x, y) : x \text{ is exactly } 7 \text{ cm taller than } y\}$

$x$  can not be taller than  $x$ , so  $R$  is not reflexive.

$x$  is taller than  $y$  then  $y$  can not be taller than  $x$ , so  $R$  is not symmetric.

Again,  $x$  is 7 cm taller than  $y$  and  $y$  is 7 cm taller than  $z$ , then  $x$  can not be 7 cm taller than  $z$ , so  $R$  is not transitive.

Therefore,  $R$  is neither reflexive, nor symmetric and nor transitive.

(d)  $R = \{(x, y) : x \text{ is wife of } y\}$

$x$  is not wife of  $x$ , so  $R$  is not reflexive.

$x$  is wife of  $y$  but  $y$  is not wife of  $x$ , so  $R$  is not symmetric.

Again,  $x$  is wife of  $y$  and  $y$  is wife of  $z$  then  $x$  can not be wife of  $z$ , so  $R$  is not transitive.

Therefore,  $R$  is neither reflexive, nor symmetric and nor transitive.

(e)  $R = \{(x, y) : x \text{ is father of } y\}$

$x$  is not father of  $x$ , so  $R$  is not reflexive.

$x$  is father of  $y$  but  $y$  is not father of  $x$ , so  $R$  is not symmetric.

Again,  $x$  is father of  $y$  and  $y$  is father of  $z$  then  $x$  cannot be father of  $z$ , so  $R$  is not transitive.

Therefore,  $R$  is neither reflexive, nor symmetric and nor transitive.

**2. Show that the relation  $R$  in the set  $R$  of real numbers, defined as  $R = \{(a, b) : a \leq b^2\}$  is neither reflexive nor symmetric nor transitive.**

**Solution:**

$R = \{(a, b) : a \leq b^2\}$ , Relation  $R$  is defined as the set of real numbers.

$(a, a) \in R$  then  $a \leq a^2$ , which is false.  $R$  is not reflexive.

$(a, b) = (b, a) \in R$  then  $a \leq b^2$  and  $b \leq a^2$ , it is false statement.  $R$  is not symmetric.

Now,  $a \leq b^2$  and  $b \leq c^2$ , then  $a \leq c^2$ , which is false.  $R$  is not transitive

Therefore,  $R$  is neither reflexive, nor symmetric and nor transitive.

**3. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as  $R = \{(a, b) : b = a + 1\}$  is reflexive, symmetric or transitive.**

**Solution:**  $R = \{(a, b) : b = a + 1\}$

$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

When  $b = a$ ,  $a = a + 1$ : which is false, So R is not reflexive.

If  $(a, b) = (b, a)$ , then  $b = a + 1$  and  $a = b + 1$ : Which is false, so R is not symmetric.

Now, if  $(a, b)$ ,  $(b, c)$  and  $(a, c)$  belongs to R then  $b = a + 1$  and  $c = b + 1$  which implies  $c = a + 2$ : Which is false, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

**4. Show that the relation R in R defined as  $R = \{(a, b) : a \leq b\}$ , is reflexive and transitive but not symmetric.**

**Solution:**

$a \leq a$ : which is true,  $(a, a) \in R$ , So R is reflexive.

$a \leq b$  but  $b \leq a$  (false):  $(a, b) \in R$  but  $(b, a) \notin R$ , So R is not symmetric.

Again,  $a \leq b$  and  $b \leq c$  then  $a \leq c$ :  $(a, b) \in R$  and  $(b, c) \in R$  and  $(a, c) \in R$ , So R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

**5. Check whether the relation R in R defined by  $R = \{(a, b) : a \leq b^3\}$  is reflexive, symmetric or transitive.**

**Solution:**  $R = \{(a, b) : a \leq b^3\}$

$a \leq a^3$ : which is true,  $(a, a) \in R$ , So R is not reflexive.

$a \leq b^3$  but  $b \leq a^3$  (false):  $(a, b) \in R$  but  $(b, a) \notin R$ , So R is not symmetric.

Again,  $a \leq b^3$  and  $b \leq c^3$  then  $a \leq c^3$  (false) :  $(a, b) \in R$  and  $(b, c) \in R$  and  $(a, c) \notin R$ , So R is not transitive.

Therefore, R is neither reflexive, nor transitive and nor symmetric.

**6. Show that the relation  $R$  in the set  $\{1, 2, 3\}$  given by  $R = \{(1, 2), (2, 1)\}$  is symmetric but neither reflexive nor transitive.**

**Solution:**

$$R = \{(1, 2), (2, 1)\}$$

$(x, x) \notin R$ .  $R$  is not reflexive.

$(1, 2) \in R$  and  $(2, 1) \in R$ .  $R$  is symmetric.

Again,  $(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z)$  does not imply to  $R$ .  $R$  is not transitive.

Therefore,  $R$  is symmetric but neither reflexive nor transitive.

**7. Show that the relation  $R$  in the set  $A$  of all the books in a library of a college, given by  $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$  is an equivalence relation.**

**Solution:**

Books  $x$  and  $x$  have same number of pages.  $(x, x) \in R$ .  $R$  is reflexive.

If  $(x, y) \in R$  and  $(y, x) \in R$ , so  $R$  is symmetric.

Because, Books  $x$  and  $y$  have same number of pages and Books  $y$  and  $x$  have same number of pages.

Again,  $(x, y) \in R$  and  $(y, z) \in R$  and  $(x, z) \in R$ .  $R$  is transitive.

Therefore,  $R$  is an equivalence relation.

**8. Show that the relation  $R$  in the set  $A = \{1, 2, 3, 4, 5\}$  given by  $R = \{(a, b) : |a - b| \text{ is even}\}$ , is an equivalence relation. Show that all the elements of  $\{1, 3, 5\}$  are related to each other and all the elements of  $\{2, 4\}$  are related to each other. But no element of  $\{1, 3, 5\}$  is related to any element of  $\{2, 4\}$ .**

**Solution:**

$$A = \{1, 2, 3, 4, 5\} \text{ and } R = \{(a, b) : |a - b| \text{ is even}\}$$

$$\text{We get, } R = \{(1, 3), (1, 5), (3, 5), (2, 4)\}$$

For  $(a, a)$ ,  $|a - b| = |a - a| = 0$  is even. Therefore,  $R$  is reflexive.

If  $|a - b|$  is even, then  $|b - a|$  is also even.  $R$  is symmetric.

Again, if  $|a - b|$  and  $|b - c|$  is even then  $|a - c|$  is also even.  $R$  is transitive.

Therefore, R is an equivalence relation.

(b) We have to show that, Elements of  $\{1, 3, 5\}$  are related to each other.

$$|1 - 3| = 2$$

$$|3 - 5| = 2$$

$$|1 - 5| = 4$$

All are even numbers.

Elements of  $\{1, 3, 5\}$  are related to each other.

Similarly,  $|2 - 4| = 2$  (even number), elements of  $\{2, 4\}$  are related to each other.

Hence no element of  $\{1, 3, 5\}$  is related to any element of  $\{2, 4\}$ .

**9. Show that each of the relation R in the set  $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$ , given by**

**(i)  $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$**

**(ii)  $R = \{(a, b) : a = b\}$**

**is an equivalence relation. Find the set of all elements related to 1 in each case.**

**Solution:**

(i)  $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$

So,  $A = \{0, 1, 2, 3, \dots, 12\}$

Now  $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

$$R = \{(4, 0), (0, 4), (5, 1), (1, 5), (6, 2), (2, 6), \dots, (12, 9), (9, 12), \dots, (8, 0), (0, 8), \dots, (8, 4), (4, 8), \dots, (12, 12)\}$$

Here,  $(x, x) = |4-4| = |8-8| = |12-12| = 0$  : multiple of 4.

R is reflexive.

$|a - b|$  and  $|b - a|$  are multiple of 4.  $(a, b) \in R$  and  $(b, a) \in R$ .

R is symmetric.

And  $|a - b|$  and  $|b - c|$  then  $|a - c|$  are multiple of 4.  $(a, b) \in R$  and  $(b, c) \in R$  and  $(a, c) \in R$   
R is transitive.

Hence R is an equivalence relation.

(ii) Here,  $(a, a) = a = a$ .

$(a, a) \in R$ . So  $R$  is reflexive.

$a = b$  and  $b = a$ .  $(a, b) \in R$  and  $(b, a) \in R$ .

$R$  is symmetric.

And  $a = b$  and  $b = c$  then  $a = c$ .  $(a, b) \in R$  and  $(b, c) \in R$  and  $(a, c) \in R$   
 $R$  is transitive.

Hence  $R$  is an equivalence relation.

Now set of all elements related to 1 in each case is

(i) Required set =  $\{1, 5, 9\}$

(ii) Required set =  $\{1\}$

**10. Give an example of a relation. Which is**

**(i) Symmetric but neither reflexive nor transitive.**

**(ii) Transitive but neither reflexive nor symmetric.**

**(iii) Reflexive and symmetric but not transitive.**

**(iv) Reflexive and transitive but not symmetric.**

**(v) Symmetric and transitive but not reflexive.**

**Solution:**

(i) Consider a relation  $R = \{(1, 2), (2, 1)\}$  in the set  $\{1, 2, 3\}$

$(x, x) \notin R$ .  $R$  is not reflexive.

$(1, 2) \in R$  and  $(2, 1) \in R$ .  $R$  is symmetric.

Again,  $(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z)$  does not imply to  $R$ .  $R$  is not transitive.

Therefore,  $R$  is symmetric but neither reflexive nor transitive.

(ii) Relation  $R = \{(a, b) : a > b\}$

$a > a$  (false statement).

Also  $a > b$  but  $b > a$  (false statement) and

If  $a > b$  but  $b > c$ , this implies  $a > c$

Therefore,  $R$  is transitive, but neither reflexive nor symmetric.



(iii)  $R = \{a, b\}$ : a is friend of b}

a is friend of a. R is reflexive.

Also a is friend of b and b is friend of a. R is symmetric.

Also if a is friend of b and b is friend of c then a cannot be friend of c. R is not transitive.

Therefore, R is reflexive and symmetric but not transitive.

(iv) Say R is defined in R as  $R = \{(a, b) : a \leq b\}$

$a \leq a$ : which is true,  $(a, a) \in R$ , So R is reflexive.

$a \leq b$  but  $b \leq a$  (false):  $(a, b) \in R$  but  $(b, a) \notin R$ , So R is not symmetric.

Again,  $a \leq b$  and  $b \leq c$  then  $a \leq c$ :  $(a, b) \in R$  and  $(b, c) \in R$  and  $(a, c) \in R$ , So R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

(v)  $R = \{(a, b) : a \text{ is sister of } b\}$  (suppose a and b are female)

a is not sister of a. R is not reflexive.

a is sister of b and b is sister of a. R is symmetric.

Again, a is sister of b and b is sister of c then a is sister of c.

Therefore, R is symmetric and transitive but not reflexive.

**11. Show that the relation R in the set A of points in a plane given by  $R = \{(P, Q) : \text{distance of the point P from the origin is same as the distance of the point Q from the origin}\}$ , is an equivalence relation. Further, show that the set of all points related to a point  $P \neq (0, 0)$  is the circle passing through P with origin as centre.**

**Solution:**  $R = \{(P, Q) : \text{distance of the point P from the origin is the same as the distance of the point Q from the origin}\}$

Say "O" is origin Point.

Since the distance of the point P from the origin is always the same as the distance of the same point P from the origin.

$OP = OP$

So  $(P, P) \in R$ .  $R$  is reflexive.

Distance of the point  $P$  from the origin is the same as the distance of the point  $Q$  from the origin

$OP = OQ$  then  $OQ = OP$   
 $R$  is symmetric.

Also  $OP = OQ$  and  $OQ = OR$  then  $OP = OR$ .  $R$  is transitive.

Therefore,  $R$  is an equivalent relation.

**12. Show that the relation  $R$  defined in the set  $A$  of all triangles as  $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$ , is equivalence relation. Consider three right angle triangles  $T_1$  with sides 3, 4, 5,  $T_2$  with sides 5, 12, 13 and  $T_3$  with sides 6, 8, 10. Which triangles among  $T_1$ ,  $T_2$  and  $T_3$  are related?**

**Solution:**

**Case 1:**

$T_1, T_2$  are triangle.

$R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$

**Check for reflexive:**

As We know that each triangle is similar to itself, so  $(T_1, T_1) \in R$   
 $R$  is reflexive.

**Check for symmetric:**

Also two triangles are similar, then  $T_1$  is similar to  $T_2$  and  $T_2$  is similar to  $T_1$ , so  $(T_1, T_2) \in R$  and  $(T_2, T_1) \in R$   
 $R$  is symmetric.

**Check for transitive:**

Again, if then  $T_1$  is similar to  $T_2$  and  $T_2$  is similar to  $T_3$ , then  $T_1$  is similar to  $T_3$ , so  $(T_1, T_2) \in R$  and  $(T_2, T_3) \in R$  and  $(T_1, T_3) \in R$   
 $R$  is transitive

Therefore,  $R$  is an equivalent relation.

**Case 2:** It is given that  $T_1, T_2$  and  $T_3$  are right angled triangles.

$T_1$  with sides 3, 4, 5

$T_2$  with sides 5, 12, 13 and

$T_3$  with sides 6, 8, 10

Since, two triangles are similar if corresponding sides are proportional.

Therefore,  $3/6 = 4/8 = 5/10 = 1/2$

Therefore,  $T_1$  and  $T_3$  are related.

**13. Show that the relation  $R$  defined in the set  $A$  of all polygons as  $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$ , is an equivalence relation. What is the set of all elements in  $A$  related to the right angle triangle  $T$  with sides 3, 4 and 5?**

**Solution:**

**Case I:**

$R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$

**Check for reflexive:**

$P_1$  and  $P_1$  have same number of sides, So  $R$  is reflexive.

**Check for symmetric:**

$P_1$  and  $P_2$  have same number of sides then  $P_2$  and  $P_1$  have same number of sides, so  $(P_1, P_2) \in R$  and  $(P_2, P_1) \in R$   
 $R$  is symmetric.

**Check for transitive:**

Again,  $P_1$  and  $P_2$  have same number of sides, and  $P_2$  and  $P_3$  have same number of sides, then also  $P_1$  and  $P_3$  have same number of sides .

So  $(P_1, P_2) \in R$  and  $(P_2, P_3) \in R$  and  $(P_1, P_3) \in R$   
 $R$  is transitive

Therefore,  $R$  is an equivalent relation.

Since 3, 4, 5 are the sides of a triangle, the triangle is right angled triangle. Therefore, the set  $A$  is the set of right angled triangle.

**14. Let  $L$  be the set of all lines in  $XY$  plane and  $R$  be the relation in  $L$  defined as  $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$ . Show that  $R$  is an equivalence relation. Find the set of all lines related to the line  $y = 2x + 4$ .**

**Solution:**

$L_1$  is parallel to itself i.e.,  $(L_1, L_1) \in R$

$R$  is reflexive

Now, let  $(L_1, L_2) \in R$

$L_1$  is parallel to  $L_2$  and  $L_2$  is parallel to  $L_1$

$(L_2, L_1) \in R$ , Therefore,  $R$  is symmetric

Now, let  $(L_1, L_2), (L_2, L_3) \in R$

$L_1$  is parallel to  $L_2$ . Also,  $L_2$  is parallel to  $L_3$

$L_1$  is parallel to  $L_3$

Therefore,  $R$  is transitive

Hence,  $R$  is an equivalence relation.

Again, The set of all lines related to the line  $y = 2x + 4$ , is the set of all its parallel lines.

Slope of given line is  $m = 2$ .

As we know slope of all parallel lines are same.

Hence, the set of all related to  $y = 2x + 4$  is  $y = 2x + k$ , where  $k \in R$ .

**15. Let  $R$  be the relation in the set  $\{1, 2, 3, 4\}$  given by  $R = \{(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)\}$ . Choose the correct answer.**

**(A)  $R$  is reflexive and symmetric but not transitive.**

**(B)  $R$  is reflexive and transitive but not symmetric.**

**(C)  $R$  is symmetric and transitive but not reflexive.**

**(D)  $R$  is an equivalence relation.**

**Solution:**

Let  $R$  be the relation in the set  $\{1, 2, 3, 4\}$  given by  $R = \{(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)\}$ .

Step 1:  $(1, 1), (2, 2), (3, 3), (4, 4) \in R$ .  $R$  is reflexive.

Step 2:  $(1, 2) \in R$  but  $(2, 1) \notin R$ .  $R$  is not symmetric.

Step 3: Consider any set of points,  $(1, 3) \in R$  and  $(3, 2) \in R$  then  $(1, 2) \in R$ . So  $R$  is transitive.

Option (B) is correct.

**16. Let  $R$  be the relation in the set  $N$  given by  $R = \{(a, b) : a = b - 2, b > 6\}$ . Choose the correct answer.**

**(A)  $(2, 4) \in R$  (B)  $(3, 8) \in R$  (C)  $(6, 8) \in R$  (D)  $(8, 7) \in R$**

**Solution:**  $R = \{(a, b) : a = b - 2, b > 6\}$

(A) Incorrect : Value of  $b = 4$ , not true.

(B) Incorrect :  $a = 3$  and  $b = 8 > 6$   
 $a = b - 2 \Rightarrow 3 = 8 - 2$  and  $3 = 6$ , which is false.

(C) Correct:  $a = 6$  and  $b = 8 > 6$   
 $a = b - 2 \Rightarrow 6 = 8 - 2$  and  $6 = 6$ , which is true.

(D) Incorrect :  $a = 8$  and  $b = 7 > 6$   
 $a = b - 2 \Rightarrow 8 = 7 - 2$  and  $8 = 5$ , which is false.

Therefore, option (C) is correct.

$\in R$  but  $(2, 1) \notin R$

## Exercise 1.2

Page No: 10

1. Show that the function  $f : \mathbb{R}^* \rightarrow \mathbb{R}^*$  defined by  $f(x) = 1/x$  is one-one and onto, where  $\mathbb{R}^*$  is the set of all non-zero real numbers. Is the result true, if the domain  $\mathbb{R}^*$  is replaced by  $\mathbb{N}$  with co-domain being same as  $\mathbb{R}^*$ ?

**Solution:**

Given:  $f : \mathbb{R}^* \rightarrow \mathbb{R}^*$  defined by  $f(x) = 1/x$

**Check for One-One**

$$f(x_1) = \frac{1}{x_1} \text{ and } f(x_2) = \frac{1}{x_2}$$

$$\text{If } f(x_1) = f(x_2) \text{ then } \frac{1}{x_1} = \frac{1}{x_2}$$

This implies  $x_1 = x_2$

Therefore,  $f$  is one-one function.

**Check for onto**

$$f(x) = 1/x$$

$$\text{or } y = 1/x$$

$$\text{or } x = 1/y$$

$$f(1/y) = y$$

Therefore,  $f$  is onto function.

Again, If  $f(x_1) = f(x_2)$

Say,  $n_1, n_2 \in \mathbb{R}$

$$\frac{1}{n_1} = \frac{1}{n_2}$$

So  $n_1 = n_2$

Therefore,  $f$  is one-one

Every real number belonging to co-domain may not have a pre-image in  $\mathbb{N}$ . for example,  $1/3$  and  $3/2$  are not belongs  $\mathbb{N}$ . So  $\mathbb{N}$  is not onto.

2. Check the injectivity and surjectivity of the following functions:

(i)  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^2$

(ii)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^2$

(iii)  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$

(iv)  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^3$

(v)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^3$

**Solution:**

(i)  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^2$

For  $x, y \in \mathbb{N} \Rightarrow f(x) = f(y)$  which implies  $x^2 = y^2$   
 $\Rightarrow x = y$

Therefore  $f$  is injective.

There are such numbers of co-domain which have no image in domain  $\mathbb{N}$ .

Say,  $3 \in \mathbb{N}$ , but there is no pre-image in domain of  $f$ . such that  $f(x) = x^2 = 3$ .

$f$  is not surjective.

Therefore,  $f$  is injective but not surjective.

(ii) Given,  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^2$

Here,  $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$

$$f(-1) = f(1) = 1$$

But  $-1$  not equal to  $1$ .

$f$  is not injective.

There are many numbers of co-domain which have no image in domain  $\mathbb{Z}$ .

For example,  $-3 \in$  co-domain  $\mathbb{Z}$ , but  $-3 \notin$  domain  $\mathbb{Z}$

$f$  is not surjective.

Therefore,  $f$  is neither injective nor surjective.

(iii)  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$

$$f(-1) = f(1) = 1$$

But -1 not equal to 1.

f is not injective.

There are many numbers of co-domain which have no image in domain  $\mathbb{R}$ .

For example,  $-3 \in$  co-domain  $\mathbb{R}$ , but there does not exist any  $x$  in domain  $\mathbb{R}$  where  $x^2 = -3$   
f is not surjective.

Therefore, f is neither injective nor surjective.

(iv)  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^3$

$$\begin{aligned} \text{For } x, y \in \mathbb{N} \Rightarrow f(x) = f(y) \text{ which implies } x^3 &= y^3 \\ \Leftrightarrow x &= y \end{aligned}$$

Therefore f is injective.

There are many numbers of co-domain which have no image in domain  $\mathbb{N}$ .

For example,  $4 \in$  co-domain  $\mathbb{N}$ , but there does not exist any  $x$  in domain  $\mathbb{N}$  where  $x^3 = 4$ .  
f is not surjective.

Therefore, f is injective but not surjective.

(v)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^3$

$$\begin{aligned} \text{For } x, y \in \mathbb{Z} \Rightarrow f(x) = f(y) \text{ which implies } x^3 &= y^3 \\ \Leftrightarrow x &= y \end{aligned}$$

Therefore f is injective.

There are many numbers of co-domain which have no image in domain  $\mathbb{Z}$ .

For example,  $4 \in$  co-domain  $\mathbb{N}$ , but there does not exist any  $x$  in domain  $\mathbb{Z}$  where  $x^3 = 4$ .  
f is not surjective.

Therefore, f is injective but not surjective.



**3. Prove that the Greatest Integer Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = [x]$ , is neither one-one nor onto, where  $[x]$  denotes the greatest integer less than or equal to  $x$ .**

**Solution:**

Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = [x]$   
 $f(x) = 1$ , because  $1 \leq x \leq 2$

$$f(1.2) = [1.2] = 1$$

$$f(1.9) = [1.9] = 1$$

But  $1.2 \neq 1.9$

$f$  is not one-one.

There is no fraction proper or improper belonging to co-domain of  $f$  has any pre-image in its domain.

For example,  $f(x) = [x]$  is always an integer

for 0.7 belongs to  $\mathbb{R}$  there does not exist any  $x$  in domain  $\mathbb{R}$  where  $f(x) = 0.7$   
 $f$  is not onto.

Hence proved, the Greatest Integer Function is neither one-one nor onto.

**4. Show that the Modulus Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = |x|$ , is neither one-one nor onto, where  $|x|$  is  $x$ , if  $x$  is positive or 0 and  $|x|$  is  $-x$ , if  $x$  is negative.**

**Solution:**

$f : \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = |x|$ , defined as

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$f$  contains values like  $(-1, 1), (1, 1), (-2, 2), (2, 2)$

$$f(-1) = f(1), \text{ but } -1 \neq 1$$

$f$  is not one-one.

$\mathbb{R}$  contains some negative numbers which are not images of any real number since  $f(x) = |x|$  is always non-negative. So  $f$  is not onto.

Hence, Modulus Function is neither one-one nor onto.

5. Show that the Signum Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

**Solution:** Signum Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

$$f(1) = f(2) = 1$$

This implies, for  $n > 0$ ,  $f(x_1) = f(x_2) = 1$

$$x_1 \neq x_2$$

$f$  is not one-one.

$f(x)$  has only 3 values,  $(-1, 0, 1)$ . Other than these 3 values of co-domain  $\mathbb{R}$  has no any pre-image its domain.

$f$  is not onto.

Hence, Signum Function is neither one-one nor onto.

6. Let  $A = \{1, 2, 3\}$ ,  $B = \{4, 5, 6, 7\}$  and let  $f = \{(1, 4), (2, 5), (3, 6)\}$  be a function from  $A$  to  $B$ . Show that  $f$  is one-one.

**Solution:**

$$A = \{1, 2, 3\}$$

$$B = \{4, 5, 6, 7\} \text{ and}$$

$$f = \{(1, 4), (2, 5), (3, 6)\}$$

$$f(1) = 4, f(2) = 5 \text{ and } f(3) = 6$$

Here, also distinct elements of  $A$  have distinct images in  $B$ .

Therefore,  $f$  is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i)  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 3 - 4x$

(ii)  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 1 + x^2$

**Solution:**

(i)  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 3 - 4x$

If  $x_1, x_2 \in \mathbb{R}$  then

$$f(x_1) = 3 - 4x_1 \text{ and}$$

$$f(x_2) = 3 - 4x_2$$

If  $f(x_1) = f(x_2)$  then  $x_1 = x_2$

Therefore,  $f$  is one-one.

Again,

$$f(x) = 3 - 4x$$

$$\text{or } y = 3 - 4x$$

$$\text{or } x = (3-y)/4 \text{ in } \mathbb{R}$$

$$f((3-y)/4) = 3 - 4((3-y)/4) = y$$

$f$  is onto.

Hence  $f$  is onto or bijective.

(ii)  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 1 + x^2$

If  $x_1, x_2 \in \mathbb{R}$  then

$$f(x_1) = 1 + x_1^2 \text{ and}$$

$$f(x_2) = 1 + x_2^2$$

If  $f(x_1) = f(x_2)$  then  $x_1^2 = x_2^2$

This implies  $x_1 \neq x_2$

Therefore,  $f$  is not one-one

Again, if every element of co-domain is image of some element of Domain under  $f$ , such that

$$f(x) = y$$

$$f(x) = 1 + x^2$$

$$y = f(x) = 1 + x^2$$

$$\text{or } x = \pm\sqrt{1-y}$$

$$\text{Therefore, } f(\sqrt{1-y}) = 2 - y \neq y$$

Therefore,  $f$  is not onto or bijective.

**8. Let  $A$  and  $B$  be sets. Show that  $f : A \times B \rightarrow B \times A$  such that  $f(a, b) = (b, a)$  is bijective function.**

**Solution:**

**Step 1:** Check for Injectivity:

Let  $(a_1, b_1)$  and  $(a_2, b_2) \in A \times B$  such that

$$f(a_1, b_1) = (a_2, b_2)$$

This implies,  $(b_1, a_1)$  and  $(b_2, a_2)$

$$b_1 = b_2 \text{ and } a_1 = a_2$$

$$(a_1, b_1) = (a_2, b_2) \text{ for all } (a_1, b_1) \text{ and } (a_2, b_2) \in A \times B$$

Therefore,  $f$  is injective.

**Step 2:** Check for Surjectivity:

Let  $(b, a)$  be any element of  $B \times A$ . Then  $a \in A$  and  $b \in B$

This implies  $(a, b) \in A \times B$

For all  $(b, a) \in B \times A$ , there exists  $(a, b) \in A \times B$

Therefore,  $f : A \times B \rightarrow B \times A$  is bijective function.

9. Let  $f : \mathbf{N} \rightarrow \mathbf{N}$  be defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbf{N}$$

State whether the function  $f$  is bijective. Justify your answer

**Solution:**

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbf{N}$$

For  $n = 1, 2$

$$f(1) = (n+1)/2 = (1+1)/2 = 1 \text{ and}$$

$$f(2) = (n)/2 = (2)/2 = 1$$

$$f(1) = f(2), \text{ but } 1 \neq 2$$

$f$  is not one-one.

For a natural number, "a" in co-domain  $\mathbf{N}$

**If  $n$  is odd**

$n = 2k + 1$  for  $k \in \mathbf{N}$ , then  $4k + 1 \in \mathbf{N}$  such that

$$f(4k+1) = (4k+1+1)/2 = 2k + 1$$

**If  $n$  is even**

$n = 2k$  for some  $k \in \mathbf{N}$  such that

$$f(4k) = 4k/2 = 2k$$

$f$  is onto

Therefore,  $f$  is onto but not bijective function.

10. Let  $A = \mathbb{R} - \{3\}$  and  $B = \mathbb{R} - \{1\}$ . Consider the function  $f : A \rightarrow B$  defined by  $f(x) = \frac{(x-2)}{(x-3)}$   
Is  $f$  one-one and onto? Justify your answer.

**Solution:**  $A = \mathbb{R} - \{3\}$  and  $B = \mathbb{R} - \{1\}$

$f : A \rightarrow B$  defined by  $f(x) = \frac{(x-2)}{(x-3)}$

Let  $(x, y) \in A$  then

$$f(x) = \frac{x-2}{x-3} \text{ and } f(y) = \frac{y-2}{y-3}$$

For  $f(x) = f(y)$

$$\begin{aligned} \frac{x-2}{x-3} &= \frac{y-2}{y-3} \\ (x-2)(y-3) &= (y-2)(x-3) \\ xy-3x-2y+6 &= xy-3y-2x+6 \\ -3x-2y &= -3y-2x \\ -3x+2x &= -3y+2y \\ -x &= -y \\ x &= y \end{aligned}$$

Again,  $f(x) = \frac{(x-2)}{(x-3)}$

or  $y = f(x) = \frac{(x-2)}{(x-3)}$

$y = \frac{(x-2)}{(x-3)}$

$y(x-3) = x-2$

$xy-3y = x-2$

$x(y-1) = 3y-2$

or  $x = \frac{(3y-2)}{(y-1)}$

$$\text{Now, } f\left(\frac{(3y-2)}{(y-1)}\right) = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3} = y$$

$f(x) = y$

Therefore,  $f$  is onto function.

11. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = x^4$ . Choose the correct answer.  
(A)  $f$  is one-one onto (B)  $f$  is many-one onto  
(C)  $f$  is one-one but not onto (D)  $f$  is neither one-one nor onto.

**Solution:**

$f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = x^4$

let  $x$  and  $y$  belongs to  $\mathbb{R}$  such that,  $f(x) = f(y)$

$$x^4 = y^4 \text{ or } x = \pm y$$

$f$  is not one-one function.

$$\text{Now, } y = f(x) = x^4 \text{ Or } x = \pm y^{1/4}$$

$$f(y^{1/4}) = y \text{ and } f(-y^{1/4}) = -y$$

Therefore,  $f$  is not onto function.

**Option D is correct.**

12. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = 3x$ . Choose the correct answer.  
(A)  $f$  is one-one onto (B)  $f$  is many-one onto  
(C)  $f$  is one-one but not onto (D)  $f$  is neither one-one nor onto.

**Solution:**  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = 3x$

let  $x$  and  $y$  belongs to  $\mathbb{R}$  such that  $f(x) = f(y)$

$$3x = 3y \text{ or } x = y$$

$f$  is one-one function.

$$\text{Now, } y = f(x) = 3x$$

$$\text{Or } x = y/3$$

$$f(x) = f(y/3) = y$$

Therefore,  $f$  is onto function.

**Option (A) is correct.**

### Exercise 1.3

Page No: 18

1. Let  $f : \{1, 3, 4\} \rightarrow \{1, 2, 5\}$  and  $g : \{1, 2, 5\} \rightarrow \{1, 3\}$  be given by  $f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(1, 3), (2, 3), (5, 1)\}$ . Write down  $g \circ f$ .

**Solution:**

Given function,  $f : \{1, 3, 4\} \rightarrow \{1, 2, 5\}$  and  $g : \{1, 2, 5\} \rightarrow \{1, 3\}$  be given by

$f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(1, 3), (2, 3), (5, 1)\}$

**Find  $g \circ f$ .**

At  $f(1) = 2$  and  $g(2) = 3$ ,  $g \circ f$  is

$$g \circ f(1) = g(f(1)) = g(2) = 3$$

At  $f(3) = 5$  and  $g(5) = 1$ ,  $g \circ f$  is

$$g \circ f(3) = g(f(3)) = g(5) = 1$$

At  $f(4) = 1$  and  $g(1) = 3$ ,  $g \circ f$  is

$$g \circ f(4) = g(f(4)) = g(1) = 3$$

Therefore,  $g \circ f = \{(1, 3), (3, 1), (4, 3)\}$

2. Let  $f$ ,  $g$  and  $h$  be functions from  $R$  to  $R$ . Show that

$$(f + g) \circ h = f \circ h + g \circ h$$

$$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

**Solution:**

$$\text{LHS} = (f + g) \circ h$$

$$= (f+g)(h(x))$$

$$= f(h(x)) + g(h(x))$$

$$= f \circ h + g \circ h$$

$$= \text{RHS}$$



Again,

$$\text{LHS} = (f \circ g) \circ h$$

$$= f \circ g(h(x))$$

$$= f(h(x)) \circ g(h(x))$$

$$= (f \circ h) \circ (g \circ h)$$

$$= \text{RHS}$$

**3. Find  $g \circ f$  and  $f \circ g$ , if**

(i)  $f(x) = |x|$  and  $g(x) = |5x - 2|$

(ii)  $f(x) = 8x^3$  and  $g(x) = x^{1/3}$ .

**Solution:**

(i)  $f(x) = |x|$  and  $g(x) = |5x - 2|$

$$g \circ f = (g \circ f)(x) = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$f \circ g = (f \circ g)(x) = f(g(x)) = f(|5x - 2|) = ||5x - 2|| = |5x - 2|$$

(ii)  $f(x) = 8x^3$  and  $g(x) = x^{1/3}$ .

$$g \circ f = (g \circ f)(x) = g(f(x)) = g(8x^3) = (8x^3)^{1/3} = 2x$$

$$f \circ g = (f \circ g)(x) = f(g(x)) = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$$

**4. If  $f(x) = \frac{4x+3}{6x-4}$ ,  $x \neq 2/3$ , Show that  $f \circ f(x) = x$ , for all  $x \neq 2/3$ . What is the inverse of  $f$ .**

**Solution:**

$$f(x) = \frac{4x+3}{6x-4}, x \neq 2/3,$$

$$\begin{aligned}
 &= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \\
 &= \frac{16x+12+18x-12}{24x+18-24x+16} \\
 &= \frac{34x}{34} \\
 &= x
 \end{aligned}$$

Therefore,  $f \circ f(x) = x$  for all  $x \neq 2/3$ .

Again,  $f \circ f = I$

The inverse of the given function,  $f$  is  $f$ .

**5. State with reason whether following functions have inverse**

(i)  $f : \{1, 2, 3, 4\} \rightarrow \{10\}$  with  
 $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii)  $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$  with  
 $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii)  $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$  with  
 $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

**Solution:**

(i)  $f : \{1, 2, 3, 4\} \rightarrow \{10\}$  with  $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

$f$  has many-one function like  $f(1) = f(2) = f(3) = f(4) = 10$ , therefore  $f$  has no inverse.

(ii)  $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$  with  $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

$g$  has many-one function like  $g(5) = g(7) = 4$ , therefore  $g$  has no inverse.

(iii)  $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$  with  $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

All elements have different images under  $h$ . So  $h$  is one-one onto function, therefore,  $h$  has an inverse.

6. Show that  $f : [-1, 1] \rightarrow \mathbb{R}$ , given by  $f(x) = x/(x+2)$  is one-one. Find the inverse of the function  $f : [-1, 1] \rightarrow \text{Range } f$ .

(Hint: For  $y \in \text{Range } f$ ,  $y = f(x) = x/(x+2)$ , for some  $x$  in  $[-1, 1]$ , i.e.,  $x = 2y/(1-y)$ .)

**Solution:**

Given function:  $f(x) = x/(x+2)$

Let  $x, y \in [-1, 1]$

Let  $f(x) = f(y)$

$$x/(x+2) = y/(y+2)$$

$$xy + 2x = xy + 2y$$

$$x = y$$

$f$  is one-one.

Again,

Since  $f : [-1, 1] \rightarrow \text{Range } f$  is onto

$$\text{say, } y = x/(x+2)$$

$$yx + 2y = x$$

$$x(1 - y) = 2y$$

$$\text{or } x = 2y/(1-y)$$

$$x = f^{-1}(y) = 2y/(1-y); y \text{ not equal to } 1$$

$f$  is onto function, and  $f^{-1}(x) = 2x/(1-x)$ .

7. Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 4x + 3$ . Show that  $f$  is invertible. Find the inverse of  $f$ .

**Solution:**

Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 4x + 3$

Say,  $x, y \in \mathbb{R}$

Let  $f(x) = f(y)$  then

$$4x + 3 = 4y + 3$$

$$x = y$$

$f$  is one-one function.

Let  $y \in \text{Range of } f$

$$y = 4x + 3$$

$$\text{or } x = (y-3)/4$$

$$\text{Here, } f((y-3)/4) = 4((y-3)/4) + 3 = y$$

This implies  $f(x) = y$

So  $f$  is onto

Therefore,  $f$  is invertible.

$$\text{Inverse of } f \text{ is } x = f^{-1}(y) = (y-3)/4.$$

**8. Consider  $f : \mathbb{R}_+ \rightarrow [4, \infty)$  given by  $f(x) = x^2 + 4$ . Show that  $f$  is invertible with the inverse  $f^{-1}$  of  $f$  given by  $f^{-1}(y) = \sqrt{y-4}$ , where  $\mathbb{R}_+$  is the set of all non-negative real numbers.**

**Solution:**

Consider  $f : \mathbb{R}_+ \rightarrow [4, \infty)$  given by  $f(x) = x^2 + 4$

Let  $x, y \in \mathbb{R} \rightarrow [4, \infty)$  then

$$f(x) = x^2 + 4 \text{ and}$$

$$f(y) = y^2 + 4$$

$$\text{if } f(x) = f(y) \text{ then } x^2 + 4 = y^2 + 4$$

$$\text{or } x = y$$

$f$  is one-one.

$$\text{Now } y = f(x) = x^2 + 4 \text{ or } x = \sqrt{y-4} \text{ as } x > 0$$

$$f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y$$

$$f(x) = y$$

$f$  is onto function.

Therefore,  $f$  is invertible and Inverse of  $f$  is  $f^{-1}(y) = \sqrt{y-4}$ .

9. Consider  $f : \mathbb{R}_+ \rightarrow [-5, \infty)$  given by  $f(x) = 9x^2 + 6x - 5$ . Show that  $f$  is invertible with

$$f^{-1}(y) = \left( \frac{(\sqrt{y+6})-1}{3} \right)$$

**Solution:**

Consider  $f : \mathbb{R}_+ \rightarrow [-5, \infty)$  given by  $f(x) = 9x^2 + 6x - 5$

Consider  $f : \mathbb{R}_+ \rightarrow [4, \infty)$  given by  $f(x) = x^2 + 4$

Let  $x, y \in \mathbb{R} \rightarrow [-5, \infty)$  then

$$f(x) = 9x^2 + 6x - 5 \text{ and}$$

$$f(y) = 9y^2 + 6y - 5$$

$$\text{if } f(x) = f(y) \text{ then } 9x^2 + 6x - 5 = 9y^2 + 6y - 5$$

$$9(x^2 - y^2) + 6(x - y) = 0$$

$$9\{(x-y)(x+y)\} + 6(x - y) = 0$$

$$(x - y)(9(x+y) + 6) = 0$$

$$\text{either } x - y = 0 \text{ or } 9(x+y) + 6 = 0$$

Say  $x - y = 0$ , then  $x = y$ . So  $f$  is one-one.

$$\text{Now, } y = f(x) = 9x^2 + 6x - 5$$

Solving this quadratic equation, we have

$$x = \frac{-6 \pm 6\sqrt{y+6}}{18} \text{ or } x = \frac{\sqrt{y+6}-1}{3}$$

$$\text{So, } f(x) = f\left(\frac{\sqrt{y+6}-1}{3}\right) = 9\left(\frac{\sqrt{y+6}-1}{3}\right)^2 + 6\left(\frac{\sqrt{y+6}-1}{3}\right) - 5$$

$$= y + 7 - 2\sqrt{y+6} + 2\sqrt{y+6} - 2 - 5 = y$$

$f(x) = y$ , therefore,  $f$  is onto.

$$f(x) \text{ is invertible and } f^{-1}(x) = \frac{\sqrt{y+6}-1}{3} .$$

10. Let  $f : X \rightarrow Y$  be an invertible function. Show that  $f$  has unique inverse.

(Hint: suppose  $g_1$  and  $g_2$  are two inverses of  $f$ . Then for all  $y \in Y, fog_1(y) = 1_Y(y) = fog_2(y)$ . Use one-one ness of  $f$ )

**Solution:**

Given,  $f : X \rightarrow Y$  be an invertible function. And  $g_1$  and  $g_2$  are two inverses of  $f$ .

For all  $y \in Y$ , we get

$$fog_1(y) = 1_Y(y) = fog_2(y)$$

$$f(g_1(y)) = f(g_2(y))$$

$$g_1(y) = g_2(y)$$

$$g_1 = g_2$$

Hence  $f$  has unique inverse.

11. Consider  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  given by  $f(1) = a, f(2) = b$  and  $f(3) = c$ . Find  $f^{-1}$  and show that  $(f^{-1})^{-1} = f$ .

**Solution:**

Consider  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  given by  $f(1) = a, f(2) = b$  and  $f(3) = c$

$$\text{So } f = \{(a, 1), (b, 2), (c, 3)\}$$

$$\text{Hence } f^{-1}(a) = 1, f^{-1}(b) = 2 \text{ and } f^{-1}(c) = 3$$

$$\text{Now, } f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$$

$$\text{Therefore, inverse of } f^{-1} = (f^{-1})^{-1} = \{(1, a), (2, b), (3, c)\} = f$$

$$\text{Hence } (f^{-1})^{-1} = f.$$

13. If  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = (3 - x^3)^{\frac{1}{3}}$ , then  $fof(x)$  is

- (A)  $x^{1/3}$       (B)  $x^3$       (C)  $x$       (D)  $(3 - x^3)$

**Solution:**

$f: \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = (3 - x^3)^{\frac{1}{3}}$ , then

$$\begin{aligned} f \circ f(x) &= f(f(x)) \\ &= f\left((3 - x^3)^{\frac{1}{3}}\right) \\ &= \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}} \\ &= \left[3 - (3 - x^3)\right]^{\frac{1}{3}} \\ &= (x^3)^{\frac{1}{3}} = x \end{aligned}$$

Option (C) is correct.

14. Let  $f: \mathbb{R} - \{-4/3\} \rightarrow \mathbb{R}$  be a function defined as  $f(x) = \frac{4x}{3x+4}$ . The inverse of  $f$  is the map  $g: \text{Range } f \rightarrow \mathbb{R} - \{-4/3\}$  given by

- (A)  $g(y) = 3y/(3-4y)$                       (B)  $g(y) = 4y/(4-3y)$   
 (C)  $g(y) = 4y/(3-4y)$                       (D)  $g(y) = 3y/(4-3y)$

**Solution:**

Let  $f: \mathbb{R} - \{-4/3\} \rightarrow \mathbb{R}$  be a function defined as  $f(x) = \frac{4x}{3x+4}$ . And  $\text{Range } f \rightarrow \mathbb{R} - \{-4/3\}$

$$y = f(x) = \frac{4x}{3x+4}$$

$$y(3x + 4) = 4x$$

$$3xy + 4y = 4x$$

$$x(3y - 4) = -4y$$

$$x = 4y/(4-3y)$$

Therefore,  $f^{-1}(y) = g(y) = 4y/(4-3y)$ . Option (B) is the correct answer.

### Exercise 1.4

Page No: 24

1. Determine whether or not each of the definition of  $*$  given below gives a binary operation. In the event that  $*$  is not a binary operation, give justification for this.

(i) On  $Z^+$ , define  $*$  by  $a * b = a - b$

(ii) On  $Z^+$ , define  $*$  by  $a * b = ab$

(iii) On  $R$ , define  $*$  by  $a * b = ab^2$

(iv) On  $Z^+$ , define  $*$  by  $a * b = |a - b|$

(v) On  $Z^+$ , define  $*$  by  $a * b = a$

**Solution:**

(i) On  $Z^+$ , define  $*$  by  $a * b = a - b$

On  $Z^+ = \{1, 2, 3, 4, 5, \dots\}$

Let  $a = 1$  and  $b = 2$

Therefore,  $a * b = a - b = 1 - 2 = -1 \notin Z^+$

operation  $*$  is not a binary operation on  $Z^+$ .

(ii) On  $Z^+$ , define  $*$  by  $a * b = ab$

On  $Z^+ = \{1, 2, 3, 4, 5, \dots\}$

Let  $a = 2$  and  $b = 3$

Therefore,  $a * b = a b = 2 * 3 = 6 \in Z^+$

operation  $*$  is a binary operation on  $Z^+$

(iii) On  $R$ , define  $*$  by  $a * b = ab^2$

$R = \{-\infty, \dots, -1, 0, 1, 2, \dots, \infty\}$

Let  $a = 1.2$  and  $b = 2$



Therefore,  $a * b = ab^2 = (1.2) \times 2^2 = 4.8 \in \mathbb{R}$

Operation  $*$  is a binary operation on  $\mathbb{R}$ .

**(iv) On  $\mathbb{Z}^+$ , define  $*$  by  $a * b = |a - b|$**

On  $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \dots\}$

Let  $a = 2$  and  $b = 3$

Therefore,  $a * b = a b = 2 * 3 = 6 \in \mathbb{Z}^+$

operation  $*$  is a binary operation on  $\mathbb{Z}^+$

**(v) On  $\mathbb{Z}^+$ , define  $*$  by  $a * b = a$**

On  $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \dots\}$

Let  $a = 2$  and  $b = 1$

Therefore,  $a * b = 2 \in \mathbb{Z}^+$

Operation  $*$  is a binary operation on  $\mathbb{Z}^+$ .

**2. For each operation  $*$  defined below, determine whether  $*$  is binary, commutative or associative.**

**(i) On  $\mathbb{Z}$ , define  $a * b = a - b$**

**(ii) On  $\mathbb{Q}$ , define  $a * b = ab + 1$**

**(iii) On  $\mathbb{Q}$ , define  $a * b = ab/2$**

**(iv) On  $\mathbb{Z}^+$ , define  $a * b = 2^{ab}$**

**(v) On  $\mathbb{Z}^+$ , define  $a * b = a^b$**

**(vi) On  $\mathbb{R} - \{-1\}$ , define  $a * b = a/(b+1)$**

**Solution:**

**(i) On  $\mathbb{Z}$ , define  $a * b = a - b$**

Step 1: Check for commutative

Consider  $*$  is commutative, then

$$a * b = b * a$$

Which means,  $a - b = b - a$  (not true)

Therefore,  $*$  is not commutative.

Step 2: Check for Associative.

Consider  $*$  is associative, then

$$(a * b) * c = a * (b * c)$$

$$\text{LHS} = (a * b) * c = (a - b) * c$$

$$= a - b - c$$

$$\text{RHS} = a * (b * c) = a - (b - c)$$

$$= a - (b - c)$$

$$= a - b + c$$

This implies  $\text{LHS} \neq \text{RHS}$

Therefore,  $*$  is not associative.

**(ii) On  $\mathbb{Q}$ , define  $a * b = ab + 1$**

Step 1: Check for commutative

Consider  $*$  is commutative, then

$$a * b = b * a$$

Which means,  $ab + 1 = ba + 1$

or  $ab + 1 = ab + 1$  (which is true)

$$a * b = b * a \text{ for all } a, b \in \mathbb{Q}$$

Therefore,  $*$  is commutative.

Step 2: Check for Associative.

Consider  $*$  is associative, then

$$(a * b) * c = a * (b * c)$$

$$\text{LHS} = (a * b) * c = (ab + 1) * c$$

$$= (ab + 1)c + 1$$

$$= abc + c + 1$$

$$\text{RHS} = a * (b * c) = a * (bc + 1)$$

$$= a(bc + 1) + 1$$

$$= abc + a + 1$$

This implies  $\text{LHS} \neq \text{RHS}$

Therefore,  $*$  is not associative.

**(iii) On  $\mathbb{Q}$ , define  $a * b = ab/2$**

Step 1: Check for commutative

Consider  $*$  is commutative, then

$$a * b = b * a$$

Which means,  $ab/2 = ba/2$

or  $ab/2 = ab/2$  (which is true)

$$a * b = b * a \text{ for all } a, b \in \mathbb{Q}$$

Therefore,  $*$  is commutative.

Step 2: Check for Associative.

Consider \* is associative, then

$$(a * b) * c = a * (b * c)$$

$$\text{LHS} = (a * b) * c = (ab/2) * c$$

$$= \frac{\frac{ab}{2} \times c}{2}$$

$$= abc/4$$

$$\text{RHS} = a * (b * c) = a * (bc/2)$$

$$= \frac{a \times \frac{bc}{2}}{2}$$

$$= abc/4$$

This implies LHS = RHS

Therefore, \* is associative binary operation.

**(iv) On  $\mathbb{Z}^+$ , define  $a * b = 2^{ab}$**

Step 1: Check for commutative

Consider \* is commutative, then

$$a * b = b * a$$

Which means,  $2^{ab} = 2^{ba}$

or  $2^{ab} = 2^{ab}$  (which is true)

$a * b = b * a$  for all  $a, b \in \mathbb{Z}^+$

Therefore, \* is commutative.

Step 2: Check for Associative.

Consider \* is associative, then

$$(a * b) * c = a * (b * c)$$

$$\text{LHS} = (a * b) * c = (2^{ab}) * c$$

$$= 2^{2^{ab} c}$$

$$\text{RHS} = a * (b * c) = a * 2^{bc}$$

$$= 2^{2^{bc} a}$$

This implies  $\text{LHS} \neq \text{RHS}$

Therefore,  $*$  is not associative binary operation.

**(v) On  $\mathbb{Z}^+$ , define  $a * b = a^b$**

Step 1: Check for commutative

Consider  $*$  is commutative, then

$$a * b = b * a$$

Which means,  $a^b = b^a$

Which is not true

$$a * b = b * a \text{ for all } a, b \in \mathbb{Z}^+$$

Therefore,  $*$  is not commutative.

Step 2: Check for Associative.

Consider  $*$  is associative, then

$$(a * b) * c = a * (b * c)$$

$$\text{LHS} = (a^b) * c$$

$$= (a^b)^c$$

$$\text{RHS} = a * (b * c) = a * (b^c)$$

$$= a^{b^c}$$

This implies  $LHS \neq RHS$

Therefore,  $*$  is not associative.

**(vi) On  $R - \{-1\}$ , define  $a * b = a/(b+1)$**

Step 1: Check for commutative

Consider  $*$  is commutative, then

$$a * b = b * a$$

Which means,  $a/(b+1) = b/(a+1)$

Which is not true

Therefore,  $*$  is commutative.

Step 2: Check for Associative.

Consider  $*$  is associative, then

$$(a * b) * c = a * (b * c)$$

$$LHS = (a * b) * c = (a/(b+1)) * c$$

$$= \frac{\frac{a}{b+1}}{c}$$

$$= a/(c(b+1))$$

$$RHS = a * (b * c) = a * (b/(c+1))$$

$$= \frac{\frac{a}{b}}{c+1}$$

$$= a(c+1)/b$$

This implies  $LHS \neq RHS$

Therefore,  $*$  is not associative binary operation.

3. Consider the binary operation  $\wedge$  on the set  $\{1, 2, 3, 4, 5\}$  defined by  $a \wedge b = \min \{a, b\}$ . Write the operation table of the operation  $\wedge$ .

**Solution:**

The binary operation  $\wedge$  on the set, say  $A = \{1, 2, 3, 4, 5\}$  defined by  $a \wedge b = \min \{a, b\}$ . the operation table of the operation  $\wedge$  as follow:

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4. Consider a binary operation  $*$  on the set  $\{1, 2, 3, 4, 5\}$  given by the following multiplication table (Table 1.2).

- (i) Compute  $(2 * 3) * 4$  and  $2 * (3 * 4)$
  - (ii) Is  $*$  commutative?
  - (iii) Compute  $(2 * 3) * (4 * 5)$ .
- (Hint: use the following table)

Table 1.2

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

**Solution:**

- (i) Compute  $(2 * 3) * 4$  and  $2 * (3 * 4)$

From table:  $(2 * 3) = 1$  and  $(3 * 4) = 1$

$$(2 * 3) * 4 = 1 * 4 = 1 \text{ and}$$

$$2 * (3 * 4) = 2 * 1 = 1$$

**(ii) Is  $*$  commutative?**

Consider  $2 * 3$ , we have  $2 * 3 = 1$  and  $3 * 2 = 1$

Therefore,  $*$  is commutative.

**(iii) Compute  $(2 * 3) * (4 * 5)$ .**

From table:  $(2 * 3) = 1$  and  $(4 * 5) = 1$

$$\text{So } (2 * 3) * (4 * 5) = 1 * 1 = 1$$

**5. Let  $*$ ' be the binary operation on the set  $\{1, 2, 3, 4, 5\}$  defined by  $a *' b = \text{H.C.F. of } a \text{ and } b$ . Is the operation  $*$ ' same as the operation  $*$  defined in Exercise 4 above? Justify your answer.**

**Solution:** Let  $A = \{1, 2, 3, 4, 5\}$  and  $a *' b = \text{H.C.F. of } a \text{ and } b$ . Plot a table values, we have

$*'$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Operation  $*$ ' same as the operation  $*$ .

**6. Let  $*$  be the binary operation on  $\mathbb{N}$  given by  $a * b = \text{L.C.M. of } a \text{ and } b$ . Find**

**(i)  $5 * 7, 20 * 16$**

**(ii) Is  $*$  commutative?**

**(iii) Is  $*$  associative?**

**(iv) Find the identity of  $*$  in  $\mathbb{N}$**



(v) Which elements of  $\mathbb{N}$  are invertible for the operation  $*$ ?

**Solution:**

(i)  $5 * 7 = \text{LCM of } 5 \text{ and } 7 = 35$

$$20 * 16 = \text{LCM of } 20 \text{ and } 16 = 80$$

(ii) Is  $*$  commutative?

$$a * b = \text{L.C.M. of } a \text{ and } b$$

$$b * a = \text{L.C.M. of } b \text{ and } a$$

$$a * b = b * a$$

Therefore  $*$  is commutative.

(iii) Is  $*$  associative?

For  $a, b, c \in \mathbb{N}$

$$(a * b) * c = (\text{L.C.M. of } a \text{ and } b) * c = \text{L.C.M. of } a, b \text{ and } c$$

$$a * (b * c) = a * (\text{L.C.M. of } b \text{ and } c) = \text{L.C.M. of } a, b \text{ and } c$$

$$(a * b) * c = a * (b * c)$$

Therefore, operation  $*$  associative.

(iv) Find the identity of  $*$  in  $\mathbb{N}$

Identity of  $*$  in  $\mathbb{N} = 1$

$$\text{because } a * 1 = \text{L.C.M. of } a \text{ and } 1 = a$$

(v) Which elements of  $\mathbb{N}$  are invertible for the operation  $*$ ?

Only the element 1 in  $\mathbb{N}$  is invertible for the operation  $*$  because  $1 * 1/1 = 1$

**7. Is  $*$  defined on the set  $\{1, 2, 3, 4, 5\}$  by  $a * b = \text{L.C.M. of } a \text{ and } b$  a binary operation? Justify your answer.**

**Solution:**

The operation  $*$  defined on the set  $\{1, 2, 3, 4, 5\}$  by  $a * b = \text{L.C.M. of } a \text{ and } b$

Suppose,  $a = 2$  and  $b = 3$

$$2 * 3 = \text{L.C.M. of } 2 \text{ and } 3 = 6$$

But 6 does not belong to the set A.

Therefore, given operation  $*$  is not a binary operation.

**8. Let  $*$  be the binary operation on  $N$  defined by  $a * b = \text{H.C.F. of } a \text{ and } b$ . Is  $*$  commutative? Is  $*$  associative? Does there exist identity for this binary operation on  $N$ ?**

**Solution:**

The operation  $*$  be the binary operation on  $N$  defined by  $a * b = \text{H.C.F. of } a \text{ and } b$

$$a * b = \text{H.C.F. of } a \text{ and } b = \text{H.C.F. of } b \text{ and } a = b * a$$

Therefore, operation  $*$  is commutative.

$$\text{Again, } (a * b) * c = (\text{HCF of } a \text{ and } b) * c = \text{HCF of } (\text{HCF of } a \text{ and } b) \text{ and } c = a * (b * c)$$

$$(a * b) * c = a * (b * c)$$

Therefore, the operation is associative.

$$\text{Now, } 1 * a = a * 1 \neq a$$

Therefore, there does not exist any identity element.

**9. Let  $*$  be a binary operation on the set  $Q$  of rational numbers as follows:**

**(i)  $a * b = a - b$**

**(ii)  $a * b = a^2 + b^2$**

**(iii)  $a * b = a + ab$**

**(iv)  $a * b = (a - b)^2$**

**(v)  $a * b = ab/4$**

**(vi)  $a * b = ab^2$**

**Find which of the binary operations are commutative and which are associative.**

**Solution:**

**(i)  $a * b = a - b$**

$$a * b = a - b = -(b - a) = -b * c \neq b * a \text{ (Not commutative)}$$

$$(a * b) * c = (a - b) * c = (a - (b - c)) = a - b + c \neq a * (b * c) \text{ (Not associative)}$$

**(ii)  $a * b = a^2 + b^2$**

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a \text{ (operation is commutative)}$$

Check for associative:

$$(a * b) * c = (a^2 + b^2) * c^2 = (a^2 + b^2) + c^2$$

$$a * (b * c) = a * (b^2 + c^2) = a^2 * (b^2 + c^2)^2$$

$$(a * b) * c \neq a * (b * c) \text{ (Not associative)}$$

**(iii)  $a * b = a + ab$**

$$a * b = a + ab = a(1 + b)$$

$$b * a = b + ba = b(1+a)$$

$$a * b \neq b * a$$

The operation  $*$  is not commutative

Check for associative:

$$(a * b) * c = (a + ab) * c = (a + ab) + (a + ab)c$$

$$a * (b * c) = a * (b + bc) = a + a(b + bc)$$

$$(a * b) * c \neq a * (b * c)$$

The operation  $*$  is not associative

**(iv)  $a * b = (a - b)^2$**

$$a * b = (a - b)^2$$

$$b * a = (b - a)^2$$

$$a * b = b * a$$

The operation  $*$  is commutative.

Check for associative:

$$(a * b) * c = (a - b)^2 * c = ((a - b)^2 - c)^2$$

$$a * (b * c) = a * (b - c)^2 = (a - (b - c)^2)^2$$

$$(a * b) * c \neq a * (b * c)$$

The operation  $*$  is not associative

**(v)  $a * b = ab/4$**

$$b * a = ba/2 = ab/2$$

$$a * b = b * a$$

The operation  $*$  is commutative.

Check for associative:

$$(a * b) * c = ab/4 * c = abc/16$$

$$a * (b * c) = a * (bc/4) = abc/16$$

$$(a * b) * c = a * (b * c)$$

The operation  $*$  is associative.

**(vi)  $a * b = ab^2$**

$$b * a = ba^2$$

$$a * b \neq b * a$$

The operation  $*$  is not commutative.

Check for associative:

$$(a * b) * c = (ab^2) * c = ab^2 c^2$$

$$a * (b * c) = a * (b c^2) = ab^2 c^4$$

$$(a * b) * c \neq a * (b * c)$$

The operation  $*$  is not associative.

10. Find which of the operations given above has identity.

**Solution:** Let  $I$  be the identity.

(i)  $a * I = a - I \neq a$

(ii)  $a * I = a^2 - I^2 \neq a$

(iii)  $a * I = a + aI \neq a$

(iv)  $a * I = (a - I)^2 \neq a$

(v)  $a * I = aI/4 \neq a$

Which is only possible at  $I = 4$  i.e.  $a * I = aI/4 = a(4)/4 = a$

(vi)  $a * I = aI^2 \neq a$

Above identities does not have identity element except (V) at  $b = 4$ .

11. Let  $A = N \times N$  and  $*$  be the binary operation on  $A$  defined by  
 $(a, b) * (c, d) = (a + c, b + d)$

Show that  $*$  is commutative and associative. Find the identity element for  $*$  on  $A$ , if any.

**Solution:**  $A = N \times N$  and  $*$  is a binary operation defined on  $A$ .

$$(a, b) * (c, d) = (a + c, b + d)$$

$$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$$

The operation  $*$  is commutative

$$\begin{aligned} \text{Again, } ((a, b) * (c, d)) * (e, f) &= (a + c, b + d) * (e, f) \\ &= (a + c + e, b + d + f) \end{aligned}$$

$$(a, b) * ((c, d) * (e, f)) = (a, b) * (c + e, d + f) = (a + c + e, b + d + f)$$

$$\Rightarrow ((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d) * (e, f))$$

The operation  $*$  is associative.

Let  $(e, f)$  be the identity function, then

$$(a, b) * (e, f) = (a + e, b + f)$$

For identity function,  $a = a + e \Rightarrow e = 0$  and  $b = b + f \Rightarrow f = 0$

As zero is not a part of set of natural numbers. So identity function does not exist.

As  $0 \notin \mathbb{N}$ , therefore, identity-element does not exist.

**12. State whether the following statements are true or false. Justify.**

**(i) For an arbitrary binary operation  $*$  on a set  $N$ ,  $a * a = a \forall a \in N$ .**

**(ii) If  $*$  is a commutative binary operation on  $N$ , then  $a * (b * c) = (c * b) * a$**

**Solution:**

(i) Given:  $*$  being a binary operation on  $N$ , is defined as  $a * a = a \forall a \in N$

Here operation  $*$  is not defined, therefore, the given statement is not true.

(ii) Operation  $*$  being a binary operation on  $N$ .

$$c * b = b * c$$

$$(c * b) * a = (b * c) * a = a * (b * c)$$

Thus,  $a * (b * c) = (c * b) * a$ , therefore the given statement is true.

**13. Consider a binary operation  $*$  on  $N$  defined as  $a * b = a^3 + b^3$ . Choose the correct answer.**

**(A) Is  $*$  both associative and commutative?**

**(B) Is  $*$  commutative but not associative?**

**(C) Is  $*$  associative but not commutative?**

**(D) Is  $*$  neither commutative nor associative?**

**Solution:**

A binary operation  $*$  on  $N$  defined as  $a * b = a^3 + b^3$ ,

Also,  $a * b = a^3 + b^3 = b^3 + a^3 = b * a$

The operation  $*$  is commutative.

Again,  $(a * b) * c = (a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$

$a * (b * c) = a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$

$$\Rightarrow (a * b) * c \neq a * (b * c)$$

The operation  $*$  is not associative.

Therefore, option (B) is correct.

## Miscellaneous Exercise

Page No: 29

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = 10x + 7$ . Find the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g \circ f = f \circ g = I_{\mathbb{R}}$ .

**Solution:**

Firstly, Find the inverse of  $f$ .

Let say,  $g$  is inverse of  $f$  and

$$y = f(x) = 10x + 7$$

$$y = 10x + 7$$

$$\text{or } x = (y-7)/10$$

$$\text{or } g(y) = (y-7)/10; \text{ where } g : \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{Now, } g \circ f = g(f(x)) = g(10x + 7)$$

$$= \frac{(10x+7)-7}{10}$$

$$= x$$

$$= I_{\mathbb{R}}$$

$$\text{Again, } f \circ g = f(g(x)) = f((y-7)/10)$$

$$= 10((y-7)/10) + 7$$

$$= y - 7 + 7 = y$$

$$= I_{\mathbb{R}}$$

Since  $g \circ f = f \circ g = I_{\mathbb{R}}$ .  $f$  is invertible, and

Inverse of  $f$  is  $x = g(y) = (y-7)/10$

2. Let  $f : \mathbb{W} \rightarrow \mathbb{W}$  be defined as  $f(n) = n - 1$ , if  $n$  is odd and  $f(n) = n + 1$ , if  $n$  is even. Show that  $f$  is invertible. Find the inverse of  $f$ . Here,  $\mathbb{W}$  is the set of all whole numbers.

**Solution:**

$f : W \rightarrow W$  be defined as  $f(n) = n - 1$ , if  $n$  is odd and  $f(n) = n + 1$ , if  $n$  is even.

Function can be defined as:

$$f(n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

**$f$  is invertible, if  $f$  is one-one and onto.**

**For one-one:**

**There are 3 cases:**

for any  $n$  and  $m$  two real numbers:

**Case 1:**  $n$  and  $m$  : both are odd

$$\begin{aligned} f(n) &= n + 1 \\ f(m) &= m + 1 \\ \text{If } f(n) &= f(m) \\ \Rightarrow n + 1 &= m + 1 \\ \Rightarrow n &= m \end{aligned}$$

**Case 2:**  $n$  and  $m$  : both are even

$$\begin{aligned} f(n) &= n - 1 \\ f(m) &= m - 1 \\ \text{If } f(n) &= f(m) \\ \Rightarrow n - 1 &= m - 1 \\ \Rightarrow n &= m \end{aligned}$$

**Case 3:**  $n$  is odd and  $m$  is even

$$\begin{aligned} f(n) &= n + 1 \\ f(m) &= m - 1 \\ \text{If } f(n) &= f(m) \\ \Rightarrow n + 1 &= m - 1 \\ \Rightarrow m - n &= 2 \text{ (not true, because Even - Odd } \neq \text{ Even )} \end{aligned}$$

Therefore,  $f$  is one-one



**Check for onto:**

$$f(n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Say  $f(n) = y$ , and  $y \in W$

**Case 1: if  $n = \text{odd}$**

$$f(n) = n - 1$$

$$n = y + 1$$

Which show, if  $n$  is odd,  $y$  is even number.

**Case 2: If  $n$  is even**

$$f(n) = n + 1$$

$$y = n + 1$$

$$\text{or } n = y - 1$$

If  $n$  is even, then  $y$  is odd.

In any of the cases  $y$  and  $n$  are whole numbers.

This shows,  $f$  is onto.

Again, For inverse of  $f$

$$f^{-1} : y = n - 1$$

$$\text{or } n = y + 1 \text{ and } y = n + 1$$

$$\Rightarrow n = y - 1$$

$$f^{-1}(n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Therefore,  $f^{-1}(y) = y$ . This show inverse of  $f$  is  $f$  itself.

3. If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 - 3x + 2$ , find  $f(f(x))$ .

**Solution:**

Given:  $f(x) = x^2 - 3x + 2$

$$f(f(x)) = f(x^2 - 3x + 2)$$

$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= x^4 - 6x^3 + 10x^2 - 3x$$

4. Show that the function  $f : \mathbb{R} \rightarrow \{x \in \mathbb{R} : -1 < x < 1\}$  defined by  $f(x) = \frac{x}{1+|x|}$ ,  $x \in \mathbb{R}$  is one one and onto function.

**Solution:**

The function  $f : \mathbb{R} \rightarrow \{x \in \mathbb{R} : -1 < x < 1\}$  defined by  $f(x) = \frac{x}{1+|x|}$ ,  $x \in \mathbb{R}$

**For one-one:**

Say  $x, y \in \mathbb{R}$

As per definition of  $|x|$ ;

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\text{So } f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

For  $x \geq 0$

$$f(x) = x/(1+x)$$

$$f(y) = y/(1+y)$$

If  $f(x) = f(y)$ , then

$$x/(1+x) = y/(1+y)$$

$$x(1+y) = y(1+x) \\ \Rightarrow x = y$$

For  $x < 0$

$$f(x) = x/(1-x)$$

$$f(y) = y/(1-y)$$

If  $f(x) = f(y)$ , then

$$x/(1-x) = y/(1-y)$$

$$x(1-y) = y(1-x)$$
$$\Rightarrow x = y$$

In both the conditions,  $x = y$ .

Therefore,  $f$  is one-one.

**Again for onto:**

$$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

For  $x < 0$

$$y = f(x) = x/(1-x)$$

$$y(1-x) = x$$

$$\text{or } x(1+y) = y$$

$$\text{or } x = y/(1+y) \dots(1)$$

**For  $x \geq 0$**

$$y = f(x) = x/(1+x)$$

$$y(1+x) = x$$

$$\text{or } x = y/(1-y) \dots(2)$$

Now we have two different values of  $x$  from both the case.

Since  $y \in \{x \in \mathbb{R} : -1 < x < 1\}$   
The value of  $y$  lies between  $-1$  to  $1$ .

If  $y = 1$

$x = y/(1-y)$  (not defined)

If  $y = -1$

$x = y/(1+y)$  (not defined)

So  $x$  is defined for all the values of  $y$ , and  $x \in \mathbb{R}$

This shows that,  $f$  is onto.

**Answer:  $f$  is one-one and onto.**

**5. Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3$  is injective.**

**Solution:**

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3$   
Let  $x, y \in \mathbb{R}$  such that  $f(x) = f(y)$

This implies,  $x^3 = y^3$

$x = y$

$f$  is one-one. So  $f$  is injective.

**6. Give examples of two functions  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $g \circ f$  is injective but  $g$  is not injective.**

(Hint : Consider  $f(x) = x$  and  $g(x) = |x|$ )

**Solution:**

Given: two functions are  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{Z}$

Let us say,  $f(x) = x$  and  $g(x) = |x|$

$g \circ f = (g \circ f)(x) = f(f(x)) = g(x)$

Here  $g \circ f$  is injective but  $g$  is not.

Let us take an example to show that  $g$  is not injective: Since  $g(x) = |x|$

$g(-1) = |-1| = 1$  and  $g(1) = |1| = 1$

But  $-1 \neq 1$

7. Give examples of two functions  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $g \circ f$  is injective but  $g$  is not injective.

(Hint : Consider  $f(x) = x + 1$  and  $g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$  )

**Solution:**

Given: Two functions  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{Z}$

Say  $f(x) = x + 1$

And  $g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$

**Check if  $f$  is onto:**

$f : \mathbb{N} \rightarrow \mathbb{N}$  be  $f(x) = x + 1$

say  $y = x + 1$

or  $x = y - 1$

for  $y = 1$ ,  $x = 0$ , does not belong to  $\mathbb{N}$

Therefore,  $f$  is not onto.

**Find  $g \circ f$**

For  $x = 1$ ;  $g \circ f = g(x + 1) = 1$  (since  $g(x) = 1$ )

For  $x > 1$ ;  $g \circ f = g(x + 1) = (x + 1) - 1 = x$  (since  $g(x) = x - 1$ )

So we have two values for  $g \circ f$ .

As  $g \circ f$  is a natural number, as  $y = x$ ,  $x$  is also a natural number. Hence  $g \circ f$  is onto.

8. Given a non empty set  $X$ , consider  $P(X)$  which is the set of all subsets of  $X$ .

Define the relation  $R$  in  $P(X)$  as follows:

For subsets  $A, B$  in  $P(X)$ ,  $A R B$  if and only if  $A \subset B$ . Is  $R$  an equivalence relation on  $P(X)$ ? Justify your answer.

**Solution:**

$A \subset A \therefore R$  is reflexive.

$A \subset B \neq B \subset A \therefore R$  is not commutative.

If  $A \subset B, B \subset C$ , then  $A \subset C \therefore R$  is transitive

Therefore,  $R$  is not equivalent relation

**9. Given a non-empty set  $X$ , consider the binary operation  $*$  :  $P(X) \times P(X) \rightarrow P(X)$  given by  $A * B = A \cap B \forall A, B$  in  $P(X)$ , where  $P(X)$  is the power set of  $X$ . Show that  $X$  is the identity element for this operation and  $X$  is the only invertible element in  $P(X)$  with respect to the operation  $*$ .**

**Solution:**

Let  $T$  be a non-empty set and  $P(T)$  be its power set. Let any two subsets  $A$  and  $B$  of  $T$ .

$$A \cup B \subset T$$

So,  $A \cup B \in P(T)$

Therefore,  $\cup$  is an binary operation on  $P(T)$ .

Similarly, if  $A, B \in P(T)$  and  $A - B \in P(T)$ , then the intersection of sets and difference of sets are also binary operation on  $P(T)$  and  $A \cap T = A = T \cap A$  for every subset  $A$  of sets

$$A \cap T = A = T \cap A \text{ for all } A \in P(T)$$

$T$  is the identity element for intersection on  $P(T)$ .

**10. Find the number of all onto functions from the set  $\{1, 2, 3, \dots, n\}$  to itself.**

**Solution:** The number of onto functions that can be defined from a finite set  $A$  containing  $n$  elements onto a finite set  $B$  containing elements =  $2^n - n$ .

**11. Let  $S = \{a, b, c\}$  and  $T = \{1, 2, 3\}$ . Find  $F^{-1}$  of the following functions  $F$  from  $S$  to  $T$ , if it exists.**

(i)  $F = \{(a, 3), (b, 2), (c, 1)\}$

(ii)  $F = \{(a, 2), (b, 1), (c, 1)\}$

**Solution:** (i)  $F = \{(a, 3), (b, 2), (c, 1)\}$

$$F(a) = 3, F(b) = 2 \text{ and } F(c) = 1$$

$$F^{-1}(3) = a, F^{-1}(2) = b \text{ and } F^{-1}(1) = c$$

$$F^{-1} = \{(3, a), (2, b), (1, c)\}$$

$$(ii) F = \{(a, 2), (b, 1), (c, 1)\}$$

Since element  $b$  and  $c$  have the same image  $1$  i.e.  $(b, 1), (c, 1)$ .

Therefore,  $F$  is not one-one function.

**12. Consider the binary operations  $*$  :  $R \times R \rightarrow R$  and  $\circ$  :  $R \times R \rightarrow R$  defined as  $a * b = |a - b|$  and  $a \circ b = a, \forall a, b \in R$ . Show that  $*$  is commutative but not associative,  $\circ$  is associative but not commutative. Further, show that  $\forall a, b, c \in R, a * (b \circ c) = (a * b) \circ (a * c)$ . [If it is so, we say that the operation  $*$  distributes over the operation  $\circ$ ]. Does  $\circ$  distribute over  $*$ ? Justify your answer.**

**Solution:**

**Step 1: Check for commutative and associative for operation  $*$ .**

$$a * b = |a - b| \text{ and } b * a = |b - a| = |a - b|$$

Operation  $*$  is commutative.

$$a*(b*c) = a*|b-c| = |a-(b-c)| = |a-b+c| \text{ and}$$

$$(a*b)*c = |a-b|*c = |a-b-c|$$

$$\text{Therefore, } a*(b*c) \neq (a*b)*c$$

Operation  $*$  is not associative.

**Step 2: Check for commutative and associative for operation  $\circ$ .**

$$a \circ b = a \quad \forall a, b \in R \text{ and } b \circ a = b$$

This implies  $a \circ b \neq b \circ a$

Operation  $\circ$  is not commutative.

$$\text{Again, } a \circ (b \circ c) = a \circ b = a \text{ and } (a \circ b) \circ c = a \circ c = a$$

$$\text{Here } a \circ (b \circ c) = (a \circ b) \circ c$$

Operation  $\circ$  is associative.

### Step 3: Check for the distributive properties

If  $*$  is distributive over  $\circ$  then,  $a*(b \circ c) = a*b = |a-b|$

RHS:

$$(a*b) \circ (a*b) = (a-b) \circ (a-b) = |a-b|$$

= LHS

And,  $a \circ (b * c) = (a \circ b) * (a \circ b)$

LHS

$$a \circ (b * c) = a \circ (|b-c|) = a$$

RHS

$$(a \circ b) * (a \circ b) = a * a = |a-a| = 0$$

LHS  $\neq$  RHS

Hence, operation  $\circ$  does not distribute over.

**13. Given a non-empty set  $X$ , let  $*$  :  $P(X) \times P(X) \rightarrow P(X)$  be defined as  $A * B = (A - B) \cup (B - A)$ ,  $\forall A, B \in P(X)$ . Show that the empty set  $\phi$  is the identity for the operation  $*$  and all the elements  $A$  of  $P(X)$  are invertible with  $A^{-1} = A$ . (Hint :  $(A - \phi) \cup (\phi - A) = A$  and  $(A - A) \cup (A - A) = A * A = \phi$ ).**

**Solution:**  $x \in P(x)$

$$\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$$

And

$$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$$

$\phi$  is the identity element for the operation  $*$  on  $P(x)$ .

Also  $A * A = (A - A) \cup (A - A)$

$$= \phi \cup \phi = \phi$$

Every element  $A$  of  $P(X)$  is invertible with  $A^{-1} = A$ .



14. Define a binary operation  $*$  on the set  $\{0, 1, 2, 3, 4, 5\}$  as

$$a * b = \begin{cases} a + b & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element  $a \neq 0$  of the set is invertible with  $6 - a$  being the inverse of  $a$ .

**Solution:**

Let  $x = \{0, 1, 2, 3, 4, 5\}$  and operation  $*$  is defined as

$$a * b = \begin{cases} a + b & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Let us say,  $e \in X$  is the identity for the operation  $*$ , if  $a * e = a = e * a \forall a \in X$

$$\begin{cases} a + b = 0 = b + a, & \text{if } a + b < 6 \\ a + b - 6 = 0 = b + a - 6, & \text{if } a + b \geq 6 \end{cases}$$

That is  $a = -b$  or  $b = 6 - a$ , which shows  $a \neq -b$

Since  $x = \{0, 1, 2, 3, 4, 5\}$  and  $a, b \in X$

Inverse of an element  $a \in x$ ,  $a \neq 0$ , and  $a^{-1} = 6 - a$ .

15. Let  $A = \{-1, 0, 1, 2\}$ ,  $B = \{-4, -2, 0, 2\}$  and  $f, g : A \rightarrow B$  be functions defined by  $f(x) = x^2 - x$ ,  $x \in A$  and  $g(x) = 2|x - \frac{1}{2}| - 1$ ,  $x \in A$ . Are  $f$  and  $g$  equal?

Justify your answer. (Hint: One may note that two functions  $f : A \rightarrow B$  and  $g : A \rightarrow B$  such that  $f(a) = g(a) \forall a \in A$ , are called equal functions).

**Solution:**

Given functions are:  $f(x) = x^2 - x$  and  $g(x) = 2|x - \frac{1}{2}| - 1$

At  $x = -1$

$$f(-1) = 1^2 + 1 = 2 \text{ and } g(-1) = 2|-1 - \frac{1}{2}| - 1 = 2$$

At  $x = 0$

$$F(0) = 0 \text{ and } g(0) = 0$$

At  $x = 1$

$$F(1) = 0 \text{ and } g(1) = 0$$

At  $x = 2$   
 $F(2) = 2$  and  $g(2) = 2$

So we can see that, for each  $a \in A$ ,  $f(a) = g(a)$

This implies  $f$  and  $g$  are equal functions.

**16. Let  $A = \{1, 2, 3\}$ . Then number of relations containing  $(1, 2)$  and  $(1, 3)$  which are reflexive and symmetric but not transitive is**

- (A) 1            (B) 2            (C) 3            (D) 4

**Solution:**

Option (A) is correct.

As 1 is reflexive and symmetric but not transitive.

**17. Let  $A = \{1, 2, 3\}$ . Then number of equivalence relations containing  $(1, 2)$  is**

- (A) 1            (B) 2            (C) 3            (D) 4

**Solution:**

Option (B) is correct.

**18. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the Signum Function defined as**

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be the Greatest Integer Function given by  $g(x) = [x]$ , where  $[x]$  is greatest integer less than or equal to  $x$ . Then, does  $f \circ g$  and  $g \circ f$  coincide in  $(0, 1]$ ?

**Solution:**

Given:

$f : \mathbb{R} \rightarrow \mathbb{R}$  be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be the Greatest Integer Function given by  $g(x) = [x]$ , where  $[x]$  is

greatest integer less than or equal to  $x$ .

Now, let say  $x \in (0, 1]$ , then

$$\begin{aligned} [x] &= 1 \text{ if } x = 1 \text{ and} \\ [x] &= 0 \text{ if } 0 < x < 1 \end{aligned}$$

Therefore:

$$\begin{aligned} f \circ g(x) &= f(g(x)) = f([x]) \\ &= \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0, 1) \end{cases} \\ &= \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0, 1) \end{cases} \end{aligned}$$

$$\begin{aligned} \text{Gof}(x) &= g(f(x)) = g(1) = [1] = 1 \\ \text{For } x > 0 \end{aligned}$$

When  $x \in (0, 1)$ , then  $f \circ g = 0$  and  $\text{gof} = 1$   
But  $f \circ g(1) \neq \text{gof}(1)$

This shows that,  $f \circ g$  and  $\text{gof}$  do not coincide in  $(0, 1]$ .

**19. Number of binary operations on the set  $\{a, b\}$  are**

- (A) 10      (B) 16      (C) 20      (D) 8

**Solution:**

Option (B) is correct.

$$A = \{a, b\} \text{ and}$$

$$A \times A = \{(a,a), (a,b), (b,b), (b,a)\}$$

Number of elements = 4

So, number of subsets =  $2^4 = 16$ .