Exercise 5.1 Page No: 159

1. Prove that the function f(x) = 5x - 3 is continuous at x = 0 at x = -3 and at x = 5.

Solution:
Given function is f(x) = 5x – 3

Continuity at x = 0,

$\lim_{x\to 0} f(x) = \lim_{x\to 0}(5x-3)$

= 5 (0) – 3
= 0 – 3
= – 3

Again, f(0) = 5 (0) – 3 = 0 – 3 = – 3

As $\lim_{x\to 0} f(x) = f(x)$, therefore, f(x) is continuous at x = 0.

Continuity at x = -3,

$\lim_{x\to -3} f(x) = \lim_{x\to -3}(5x-3)$ = 5 (– 3) – 3 = – 18

And f(-3)= 5 (– 3) – 3 = – 18

As $\lim_{x\to -3} f(x) = f(x)$, therefore, is continuous at x = -3

Continuity at x = 5,

$\lim_{x\to 5} f(x) = \lim_{x\to 5}(5x-3)$

= 5 (5) – 3 = 22

And f(5) = 5 (5) – 3 = 22

Therefore, $\lim_{x\to 5} f(x) = f(x)$, so, f(x) is continuous at x = -5.

2. Examine the continuity of the function f(x) = 2x^2 – 1 at x = 3.

Solution:
Given function f(x) = 2x^2 – 1

Check Continuity at x = 3,

$\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x^2 - 1)$

= 2(3)^2 – 1 = 17

And f(3) = 2(3)^2 – 1 = 17

Therefore, $\lim_{x\to 3} f(x) = f(x)$ so f(x) is continuous at x = 3.

3. Examine the following functions for continuity:

(a) $f(x) = x - 5$

(b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

(d) $f(x) = |x - 5|$

Solution:

(a) Given function is $f(x) = x - 5$

We know that, f is defined at every real number k and its value at k is $k - 5.$

Also observed that $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k - y = f(k)$

As, $\lim_{x\to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every real number and it is a continuous function.

(b) Given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we have

$\lim_{x\to k} f(x) = \lim_{x\to k} \frac{1}{x-5} = \frac{1}{k-5}$

and $f(k) = \frac{1}{k-5}$

As, $\lim_{x\to k} f(x) = f(k)$

Therefore,

f(x) is continuous at every point of domain of f and it is a continuous function.

(c) Given function is $f(x)=\frac{x^2-25}{x+5}, x\neq -5$

For any real number, $k\neq -5$, we get

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{x^2-25}{x+5}=\lim_{x\to k}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to k}(x-5)=k-5$$

And $f(k)=\frac{(k+5)(k-5)}{k+5}=k-5$

As, $\lim_{x\to k} f(x)=f(k)$, therefore, $f(x)$ is continuous at every point of domain of f and it is a continuous function.

(d) Given function is $f(x)=|x-5|$

Domain of $f(x)$ is real and infinite for all real x

Here $f(x)=|x-5|$ is a modulus function.

As, every modulus function is continuous.

Therefore, f is continuous in its domain R.

4. Prove that the function $f(x)=x^n$ is continuous at $x=n$ where n is a positive integer.

Solution: Given function is $f(x)=x^n$ where n is a positive integer.

Continuity at $x=n$, $\lim_{x\to n} f(x)=\lim_{x\to n}(x^n)=n^n$

And $f(n)=n^n$

As, $\lim_{x\to n} f(x)=f(x)$, therefore, $f(x)$ is continuous at $x=n$.

5. Is the function f defined by $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$ **continuous at** $x=0,$ **at** $x=1,$ **at** $x=2$?

Solution: Given function is $f(x)=\begin{cases} x, & \text{if} & x\le 1 \\ 5, & \text{if} & x>1 \end{cases}$

Step 1: At $x=0$, We know that, f is defined at 0 and its value 0.

Then $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x=0$ and $f(0)=0$

Therefore, $f(x)$ is continuous at $x=0$.

Step 2: At $x=1$, Left Hand limit (LHL) of f $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x)=1$

Right Hand limit (RHL) of f $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x)=5$

Here $\lim\limits_{x\to 1^-} f(x)\ne\lim\limits_{x\to 1^+} f(x)$

Therefore, $f(x)$ is not continuous at $x=1$.

Step 3: At $x=2$, f is defined at 2 and its value at 2 is 5.

$\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}(5)=5$, therefore, $\lim\limits_{x\to 2} f(x)=f(2)$

Therefore, $f(x)$ is continuous at $x=2$.

Find all points of discontinuity of f, where f is defined by:

6. $f(x)=\begin{cases} 2x+3, & x\le 2 \\ 2x-3, & x>2 \end{cases}$

Solution: Given function is $f(x)=\begin{cases} 2x+3, & \text{if} & x\le 2 \\ 2x-3, & \text{if} & x>2 \end{cases}$

Here $f(x)$ is defined for $x\le 2$ or $(-\infty,2)$ and also for $x>2$ or $(2,\infty)$.

Therefore, Domain of f is $(-\infty,2)\cup(2,\infty)=(-\infty,\infty)$ = R

Therefore, For all $x < 2, f(x) = 2x + 3$ is a polynomial and hence continuous and for all $x > 2, f(x) = 2x - 3$ is a continuous and hence it is also continuous on R – {2}.

Now Left Hand limit = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3)$ = 2 x 2 + 3 = 7

Right Hand limit = $\lim_{x \to 2+} f(x) = \lim_{x \to 2^-} (2x - 3)$ = 2 x 2 – 3 = 1

As, $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^-} f(x)$

Therefore, $\lim_{x \to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at only x = 2.

Find all points of discontinuity of f, where f is defined by:

7. $$f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$$

$$f(x) = \begin{cases} |x| + 3, & \text{if } \quad x \le 2 \\ -2x, & \text{if } \quad x > 2 \\ 6x + 2, & \text{if } \quad x \ge 3 \end{cases}$$

Solution: Given function is

Here $f(x)$ is defined for $x \le -3$ or $(-\infty, -3)$ and for $-3 < x < 3$ and also for $x \ge 3$ or $(3, \infty)$.

Therfore, Domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty) = (-\infty, \infty)$ = R

Therfore, For all $x < -3, f(x) = |x| + 3 = -x + 3$ is a polynomial and hence continuous and

for all $x(-3 < x < 3), f(x) = -2x$ is a continuous and a continuous function and also

for all $x > 3, f(x) = 6x + 2$.

Therefore, f(x) is continuous on R – {-3, 3}.

And, x = -3 and x = 3 are partitioning points of domain R.

Now, Left Hand limit = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (|x|+3) = \lim_{x\to 3^-} (-x+3) = 3+3 = 6$

Right Hand limit = $\lim_{x\to 3+} f(x) = \lim_{x\to 3^-} (-2x) = (-2)(-3) = 6$

And $f(-3) = |-3|+3 = 3+3 = 6$

Therefore, $f(x)$ is continuous at $x=-3$.

Again,Left Hand limit = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x) = -2(3) = -6$

Right Hand limit = $\lim_{x\to 3+} f(x) = \lim_{x\to 3^-} (6x+2) = 6(3)+2 = 20$

As, $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^-} f(x)$

Therefore, $\lim_{x\to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at only x = 3.

Find all points of discontinuity of f, where f is defined by:

8.

$$f(x)=\begin{cases}\dfrac{|x|}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Solution: Given function is $f(x)=\begin{cases}\dfrac{|x|}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

f(x) = |x|/x can also be defined as,

$\frac{x}{x}=1$ if $x>0$ and $\frac{-x}{x}=-1$ if $x<0$

$\Rightarrow f(x)=1$ if $x>0$, $f(x)=-1$ if $x<0$ and $f(x)=0$ if $x=0$

We get that, domain of $f(x)$ is R as $f(x)$ is defined for $x>0$, $x<0$ and $x=0$.

For all $x>0$, $f(x)=1$ is a constant function and continuous.

For all $x<0$, $f(x)=-1$ is a constant function and continuous.

Therefore $f(x)$ is continuous on R – {0}.

Now,

Left Hand limit = $\lim_{x\to0^-} f(x)=\lim_{x\to0^-}(-1)=-1$

Right Hand limit = $\lim_{x\to0+} f(x)=\lim_{x\to0^+}(1)=1$

As, $\lim_{x\to0^-} f(x)\neq\lim_{x\to0^+} f(x)$

Therefore, $\lim_{x\to0} f(x)$ does not exist and f(x) is discontinuous at only x = 0.

Find all points of discontinuity of f, where f is defined by:

9.

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\geq0\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\geq0\end{cases}$$

At $x=0$, L.H.L. = $\lim_{x\to0^-}\frac{x}{|x|}=-1$ And $f(0)=-1$

R.H.L. = $\lim_{x\to0^+} f(x)=-1$

As, L.H.L. = R.H.L. = $f(0)$

Therefore, $f(x)$ is a continuous function.

Now,

for $x = c < 0$ $\lim_{x \to c^-} \frac{x}{|x|} = -1 = f(c)$

Therefore, $\lim_{x \to c^-} = f(x)$

Therefore, $f(x)$ is a continuous at $x = c < 0$

Now, for $x = c > 0$ $\lim_{x \to c^-} f(x) = 1 = f(c)$

Therefore, $f(x)$ is a continuous at $x = c > 0$

Answer: The function is continuous at all points of its domain.

Find all points of discontinuity of f, where f is defined by:

10.

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Solution: Given function is

$$f(x) = \begin{cases} x+1, & \text{if} \quad x \geq 1 \\ x^2+1, & \text{if} \quad x < 1 \end{cases}$$

We know that, f(x) being polynomial is continuous for $x \geq 1$ and $x < 1$ for all $x \in$ R.

Check Continuity at x = 1

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = \lim_{h \to 0} (1+h+1) = 2$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = \lim_{h \to 0} \left((1-h)^2+1\right) = 2$

And $f(1) = 2$

As, L.H.L. = R.H.L. = $f(1)$

Therefore, $f(x)$ is a continuous at $x = 1$ for all $x \in$ R.

Hence, $f(x)$ has no point of discontinuity.

Find all points of discontinuity of $f,$ where f is defined by:
11.

$$f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$$

At $x=2,$ L.H.L. = $\lim_{x\to 2^-}\left(x^3-3\right)=8-3=5$

R.H.L. = $\lim_{x\to 2^+}\left(x^2+1\right)=4+1=5$

$f(2)=2^3-3=8-3=5$

As, L.H.L. = R.H.L. = $f(2)$

Therefore, $f(x)$ is a continuous at $x=2$

Now, for $x=c<0$ $\lim_{x\to c}\left(x^3-3\right)=c^3-3=f(c)$ and

$\lim_{x\to c}\left(x^2+1\right)=c^2+1=f(c)$

Therefore, $\lim_{x\to c^-}=f(x)$

This implies, $f(x)$ is a continuous for all $x\in$ R.

Hence the function has no point of discontinuity.

Find all points of discontinuity of f, where f is defined by:

12. $f(x)=\begin{cases} x^{10}-1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

Solution: Given function is

$f(x)=\begin{cases} x^{10}-1, & \text{if } \quad x \le 1 \\ x^2, & \text{if } \quad x > 1 \end{cases}$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(x^{10}-1\right)=0$

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(x^2\right)=1$

$f(1)=1^{10}-1=0$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$ $\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1=f(c)$ and for $x=c>1$ $\lim_{x\to c}\left(x^2\right)=c^2=f(1)$

Therefore, $f(x)$ is a continuous for all $x\in$ R – {1}

Hence for all given function $x=1$ is a point of discontinuity.

13. Is the function defined by $f(x)=\begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 11 \end{cases}$ **a continuous function?**

Solution: Given function is $f(x)=\begin{cases} x+5, & \text{if } \quad x \le 1 \\ x-5, & \text{if } \quad x > 1 \end{cases}$

At $x=1$, L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+5)=6$

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-5)=-4$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x=1$

Now, for $x=c<1$

$\lim_{x\to c}(x+5)=c+5=f(c)$ and

for $x=c>1$ $\lim_{x\to c}(x-5)=c-5=f(c)$

Therefore, f(x) is a continuous for all x ∈ R – {1}

Hence f(x) is not a continuous function.

14. **Discuss the continuity of the function** f, **where** f **is defined by:**

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}3, & \text{if} \quad 0\le x\le 1\\ 4, & \text{if} \quad 1<x<3\\ 5, & \text{if} \quad 3\le x\le 10\end{cases}$$

In interval, $0\le x\le 1$, $f(x)=3$

Therefore, f is continuous in this interval.

At x =1,

L.H.L. = $\lim_{x\to 1^-} f(x)=3$ and R.H.L. = $\lim_{x\to 1^+} f(x)=4$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at x = 1.

At $x=3$, L.H.L. = $\lim_{x\to 3^-} f(x)=4$ and R.H.L. = $\lim_{x\to 3^+} f(x)=5$

As, L.H.L. $\neq$ R.H.L.

Therefore, f(x) is discontinuous at x = 3

Hence, f is discontinuous at x = 1 and x = 3.

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$$

15.

Solution: Given function is

$$f(x)=\begin{cases}2x, & \text{if} \quad x<0\\ 0, & \text{if} \quad 0\le x\le 1\\ 4x, & \text{if} \quad x>1\end{cases}$$

At x =0, L.H.L. = $\lim\limits_{x\to 0^-} 2x=0$ and R.H.L. = $\lim\limits_{x\to 0^+}(0)=0$

As, L.H.L. = R.H.L.

Therefore, f(x) is continuous at x = 0

At x = 1, L.H.L. = $\lim\limits_{x\to 1^-}(0)=0$ and R.H.L. = $\lim\limits_{x\to 1^+}(4x)=4$

As, L.H.L. $\neq$ R.H.L.

Therefore, f(x) is discontinuous at x = 1.

When x<0,

f(x) is a polynomial function and is continuous for all x < 0.

When $x>1, f(x)=4x$

It is being a polynomial function is continuous for all $x>1$.

Hence, x = 1 is a point of discontinuity.

Discuss the continuity of the function f, where f is defined by

16.
$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

Solution: Given function is

$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

At x = -1,

L.H.L. = $\lim_{x\to -1^-} f(x) = -2$ and R.H.L. = $\lim_{x\to -1^+} f(x) = -2$

As, L.H.L. = R.H.L.

Therefore, $f(x)$ is continuous at $x=-1$

At x = 1,

L.H.L. = $\lim_{x\to 1^-} f(x) = 2$ and R.H.L. = $\lim_{x\to 1^+} f(x) = 2$

As, L.H.L. = R.H.L.

Therefore, f(x) is continuous at x = 1.

17. Find the relationship between a and b so that the function f defined by
$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+1, & \text{if } x>3\end{cases}$$
is continuous at x = 3

Solution: Given function is

$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$$

Check Continuity at $x=3$.

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(ax+1)=\lim_{h\to 0}\{a(3-h)+1\}=\lim_{h\to 0}(3a-ah+1)=3a+1$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(bx+3)=\lim_{h\to 0}\{b(3+h)+3\}=\lim_{h\to 0}(3b+bh+3)=3b+3$$

Also $f(3)=3a+1$

Therefore, $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$

$\Rightarrow 3b+3=3a+1$

$\Rightarrow a-b=\frac{2}{3}$

18. For what value of λ is the function defined by

$$f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Solution: Since $f(x)$ is continuous at $x=0$.
Therefore,

L.H.L.

$$\lim_{x\to 0^-} f(x)=f(0)=\lambda(x^2-2x)=\lambda(0-0)=0$$

R.H.L

And $\lim_{x\to 0^-} f(x)=f(0)=4x+1=4\times 0+1=1$

Here, L.H.L. ≠ R.H.L.

This implies 0 = 1, which is not possible.

Again, f(x) is continuous at x = 1.

Therefore,

$$\lim_{x\to 1^-} f(x) = f(-1) = \lambda(x^2 - 2x) = \lambda(1+2) = 3\lambda$$

And $\lim_{x\to 1^+} f(x) = f(1) = 4x+1 = 4\times 1+1 = 5$

Let us say, L.H.L. = R.H.L.

$\Rightarrow 3\lambda = 5$

$\Rightarrow \lambda = \frac{5}{3}$

The value of is 3/5.

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution: For any real number, x,

$[x]$ denotes the fractional part or decimal part of x.

For example,

[2.35] = 0.35

[–5.45] = 0.45

[2] = 0

[–5] = 0

The function g : R -> R defined by $g(x) = x - [x] \forall x \in \infty$ is called the fractional part function.

The domain of the fractional part function is the set R of all real numbers , and

[0, 1) is the range of the set.

So, given function is discontinuous function.

20. Is the function $f(x)=x^2-\sin x+5$ continuous at $x=\pi$?

Solution: Given function is $f(x)=x^2-\sin x+5$

L.H.L. = $\lim_{x\to\pi^-}\left(x^2-\sin x+5\right)=\lim_{x\to\pi^-}\left[(\pi-h)^2-\sin(\pi-h)+5\right]=\pi^2+5$

R.H.L. = $\lim_{x\to\pi^+}\left(x^2-\sin x+5\right)=\lim_{x\to\pi^-}\left[(\pi+h)^2-\sin(\pi+h)+5\right]=\pi^2+5$

And $f(\pi)=\pi^2-\sin\pi+5=\pi^2+5$

Since L.H.L. = R.H.L. = $f(\pi)$

Therefore, f is continuous at $x=\pi$

21. Discuss the continuity of the following functions:

(a) $f(x)=\sin x+\cos x$

(b) $f(x)=\sin x-\cos x$

(c) $f(x)=\sin x.\cos x$

Solution: **(a)** Let "a" be an arbitrary real number then

$\lim_{x\to a^+} f(x)\Rightarrow\lim_{h\to0} f(a+h)$

Now,

$\lim_{h\to0} f(a+h)=\lim_{h\to0}\sin(a+h)+\cos(a+h)$

$=\lim_{h\to0}(\sin a\cos h+\cos a\sin h+\cos a\cos h-\sin a\sin h)$

$=\sin a\cos 0+\cos a\sin 0+\cos a\cos 0-\sin a\sin 0$

{As cos 0 = 1 and sin 0 = 0}

$=\sin a+\cos a=f(a)$

Similarly,

$$\lim_{x\to a^-} f(x) = f(a)$$

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^-} f(x)$$

Therefore, f(x) is continuous at x = a.

As, "a" is an arbitrary real number, therefore, $f(x) = \sin x + \cos x$ is continuous.

(b) Let "a" be an arbitrary real number then $\lim_{x\to a^+} f(x) \Rightarrow \lim_{h\to 0} f(a+h)$
Now,

$\lim_{h\to 0} f(a+h) =$ $\lim_{h\to 0} \sin(a+h) - \cos(a-h)$

$\Rightarrow$ $\lim_{h\to 0}(\sin a\cos\ h + \cos a\sin\ h - \cos a\cos\ h - \sin a\sin\ h)$

$= \sin a\cos 0 + \cos a\sin 0 - \cos a\cos 0 - \sin a\sin 0$

$= \sin a + 0 - \cos a - 0$

$= \sin a - \cos a = f(a)$

Similarly, $\lim_{x\to a^-} f(x) = f(a)$

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^-} f(x)$$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $f(x) = \sin x - \cos x$ is continuous.

(c) Let "a" be an arbitrary real number then $\lim_{x\to a^+} f(x) \Rightarrow \lim_{h\to 0} f(a+h)$

$Now, \lim_{h\to 0} f(a+h) =$ $\lim_{h\to 0} \sin(a+h).\cos(a+h)$

$= \lim_{h\to 0}(\sin a\cos\ h + \cos a\sin\ h)(\cos a\cos\ h - \sin a\sin\ h)$

= $\left(\sin a\cos 0+\cos a\sin 0\right)\left(\cos a\cos 0-\sin a\sin 0\right)$

= $\left(\sin a+0\right)\left(\cos a-0\right)$

= $\sin a.\cos a=f\left(a\right)$

Similarly, $\lim_{x\to a^-} f\left(x\right)=f\left(a\right)$

$\lim_{x\to a^-} f\left(x\right)=f\left(a\right)=\lim_{x\to a^-} f\left(x\right)$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $f\left(x\right)=\sin x.\cos x$ is continuous.

22. Discuss the continuity of cosine, cosecant, secant and cotangent functions.

Solution:

Continuity of cosine:

Let say "a" be an arbitrary real number then

$\lim_{x\to a^-} f\left(x\right)\Rightarrow\lim_{x\to a^+}\cos x\Rightarrow\lim_{h\to 0}\cos\left(a+h\right)$

Which implies, $\lim_{h\to 0}\left(\cos a\cos\ h-\sin a\sin\ h\right)$

= $\cos a\lim_{h\to 0}\cos\ h-\sin a\lim_{h\to 0}\sin h$

= $\cos a\times 1-\sin a\times 0$ = $\cos a=f\left(a\right)$

$\lim_{x\to a} f\left(x\right)=f\left(a\right)$ for all $a\in$ R

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, $\cos x$ is continuous.

Continuity of cosecant:

Let say "a" be an arbitrary real number then

$f(x) = \cos ec\ x = \frac{1}{\sin x}$ and

domain $x = \mathrm{R} - (x\pi), x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\sin x} = \frac{1}{\lim_{h \to 0} \sin(a+h)}$

$= \frac{1}{\lim_{h \to 0} (\sin a \cos\ h + \cos a \sin\ h)}$

$= \frac{1}{\sin a \cos 0 + \cos a \sin 0}$

$= \frac{1}{\sin a(1) + \cos a(0)}$

$= \frac{1}{\sin a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, “a” is an arbitrary real number, therefore, $f(x) = \cos ec\ x$ is continuous.

Continuity of secant:

Let say “a” be an arbitrary real number then

$f(x) = \sec\ x = \frac{1}{\cos x}$ and domain $x = \mathrm{R} - (2x+1)\frac{\pi}{2}, x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\cos x} = \frac{1}{\lim_{h \to 0} \cos(a+h)}$

$= \frac{1}{\lim_{h \to 0} (\cos a \cos\ h - \sin a \sin\ h)}$

$= \frac{1}{\cos a \cos 0 - \sin a \sin 0}$

$= \frac{1}{\cos a(1) - \sin a(0)} = \frac{1}{\cos a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, “a” is an arbitrary real number, therefore, $f(x) = \sec x$ is continuous.

Continuity of cotangent:

Let say “a” be an arbitrary real number then

$f(x) = \cot x = \frac{1}{\tan x}$ and domain $x = \mathrm{R} - (x\pi), x \in$ I

$\Rightarrow \lim_{x \to a} \frac{1}{\tan x} = \frac{1}{\lim_{h \to 0} \tan(a+h)}$

$= \frac{1}{\lim_{h \to 0} \left(\frac{\tan a + \tan h}{1 - \tan a \tan h} \right)} = \frac{1}{\frac{\tan a + 0}{1 - \tan a \tan 0}}$

$= \frac{1-0}{\tan a} = \frac{1}{\tan a} = f(a)$

Therefore, f(x) is continuous at x = a.

Since, “a” is an arbitrary real number, therefore, $f(x) = \cot x$ is continuous.

23. Find all points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$**.**

Solution: Given function is

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

At x = 0,

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(-h)}{-h} = 1$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1 = 1$

$f(0) = 1$

Therefore, f is continuous at $x = 0$.

When $x < 0, \sin x$ and x are continuous, then $\frac{\sin x}{x}$ is also continuous.

When $x > 0, f(x) = x+1$ is a polynomial, then f is continuous.

Therefore, f is continuous at any point.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function.

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x}$$

As we know, sin(1/x) lies between -1 and 1, so the value of sin 1/x be any integer, say m, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x}$$

= 0 x m

= 0

And, f(0) = 0

Since, $\lim_{x\to 0} f(x) = f(0)$, therefore, the function f is continuous at $x = 0$.

25. Examine the continuity of f, where f is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}.$$

Solution:

Given function is

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$$

Let's find the left hand and right hand limits at x =0.

At x = 0, L.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0-h)=\lim_{h\to 0} f(-h)$

$\Rightarrow \lim_{h\to 0}\sin(-h)-\cos(-h) = \lim_{h\to 0}(-\sin h-\cos h)=-0-1=-1$

R.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0+h)=\lim_{h\to 0} f(h)$

$\Rightarrow \lim_{h\to 0}\sin(h)-\cos(h) = \lim_{h\to 0}(\sin h-\cos h)=0-1=-1$

And, given f(0) = -1

Thus, $\lim_{h\to 0^-} f(x)=\lim_{h\to 0^+} f(x)=f(0)$

Therefore, f(x) is continuous at x = 0.

Find the values of k so that the function f is continuous at the indicated point in Exercise 26 to 29.

26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ **at** $x=\dfrac{\pi}{2}$.

Solution:
Given function is

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

So, $x\to\dfrac{\pi}{2}$

This implies, $x\neq\dfrac{\pi}{2}$

Putting $x=\dfrac{\pi}{2}+h$ where $h\to 0$

$$=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=\lim_{h\to 0}\frac{-k\sin h}{\pi-\pi-2h}$$

$$=\lim_{h\to 0}\frac{-k\sin h}{-2h}$$

$$=\frac{k}{2}\times\lim_{h\to 0}\frac{\sin h}{h}$$

$=\dfrac{k}{2}$ ……….(1)

And $f\left(\dfrac{\pi}{2}\right)=3$ ……….(2)

$f(x)=3$ when $x=\dfrac{\pi}{2}$ [Given]

As we know, f(x) is continuous at x= π/2.

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

From equation (1) and equation (2), we have

$$\frac{k}{2} = 3$$

k = 6

Therefore, the value of k is 6.

27. $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ **at** $x = 2$.

Solution:
Given function is

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2+h) = 3$$

$\lim_{x\to 2^-} f(x) = 3$ and $f(2) = 3$

$$k \times 2^2 = 3$$

This implies, $k = \frac{3}{4}$

when k=3/4, then $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{3}{4}(2-h)^2 = 3$

Therefore, $f(x)$ is continuous at $x = 2$ when $k = \frac{3}{4}$.

28. $f(x)=\begin{cases} kx+1, & \text{if } x\le\pi \\ \cos x, & \text{if } x>\pi \end{cases}$ **at** $x=\pi.$

Solution:
Given function is:

$$f(x)=\begin{cases} kx+1, & \text{if } x\le\pi \\ \cos x, & \text{if } x>\pi \end{cases}$$

$$\lim_{x\to\pi^+} f(x)=\lim_{h\to0} f(\pi+h)=\lim_{h\to0}\cos(\pi+h)=-\cos h=-\cos 0=-1$$

and $\lim_{x\to\pi^-} f(x)=\lim_{h\to0} f(\pi-x)=\lim_{h\to0}\cos(\pi-h)=-\cos h=-\cos 0=-1$

Again,

$$\lim_{x\to\pi} f(x)=\lim_{h\to0}(k\pi+1)$$

As given function is continuous at $x=\pi$, we have

$$\lim_{x\to\pi^+} f(x)=\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi} f(x)$$

$$\Rightarrow k\pi+1=-1$$

$$\Rightarrow k\pi=-2$$

$$\Rightarrow k=\frac{-2}{\pi}$$

The value of k is -2/π.

29. $f(x)=\begin{cases} kx+1, & \text{if } x\le5 \\ 3x-5, & \text{if } x>5 \end{cases}$ **at** $x=5.$

Solution:
Given function is

$$f(x)=\begin{cases} kx+1, & \text{if } x\le5 \\ 3x-5, & \text{if } x>5 \end{cases}$$

When x< 5, $f(x)=kx+1$: A polynomial is continuous at each point x < 5.

When x > 5, $f(x)=3x-5$: A polynomial is continuous at each point x > 5.

Now $f(5)=5k+1=3(5+h)-5$

$\lim_{x\to 5^-} f(x)=\lim_{h\to 0} f(5+h)=15+3h-5$ ……….(1)

= $10+3h=10+3\times 0=10$

$\lim_{x\to 5^-} f(x)=\lim_{h\to 0} f(5-h)=k(5-h)+1=5k-nk+1=5k+1$ …….(2)

Since function is continuous, therefore, both the equations are equal,

Equate both the equations and find the value of k,

$10=5k+1$

$5k=9$

$k=\frac{9}{5}$

30. Find the values of a and b such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

Solution:

Given function is:

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

For x < 2 ; function is f(x) =5 ; which is a constant.

Function is continuous.

For 2 < x < 10; function f(x) = ax + b; a polynomial.

Function is continuous.

For x > 10; function is f(x) =21; which is a constant.

Function is continuous.

Now, for continuity at $x = 2,$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{h\to 0}(5) = \lim_{h\to 0}\{a(2+h)+b\} = 5$$

$\Rightarrow 2a + b = 5$ ……….(1)

For continuity at $x = 10,$ $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$$\Rightarrow \lim_{h\to 0}(21) = \lim_{h\to 0}\{a(10-h)+b\} = 21$$

$\Rightarrow 10a + b = 21$ ……….(2)

Solving equation (1) and equation (2), we get

a = 2 and b = 1.

31. Show that the function defined by f(x) = cos (x^2) is a continuous function.

Solution:
Given function is :
f(x) = cos (x^2)

Let g(x) = cos x and h(x) = x^2, then

goh(x) = g(h(x))

= g(x^2)

= cos (x^2)

=f(x)

This implies, goh(x) = f(x)
Now,
g(x) = cos x is continuous and
h(x) = x^2 (a polynomial)

[We know that, if two functions are continuous then their composition is also continuous]

So, goh(x) is also continuous.

Thus f(x) is continuous.

32. Show that the function defined by f(x) = |cos x| is a continuous function.

Solution: Given function is
f(x) = |cos x|

f(x) is a real and finite for all x ∈ R and Domain of f(x) is R.

Let $g(x) = \cos x$ and $h(x) = |x|$

Here, g(x) and h(x) are cosine function and modulus function are continuous for all real x.

Now, $(goh)x = g\{h(x)\} = g(|x|) = \cos|x|$ is also is continuous being a composite function of two continuous functions, but not equal to f(x).

Again, $(hog)x = h\{g(x)\} = h(\cos x) = |\cos x| = f(x)$ [Using given]

Therefore, $f(x) = |\cos x| = (hog)x$ is composite function of two continuous functions is continuous.

33. Examine that $\sin|x|$ is a continuous function.

Solution:

Let $f(x) = |x|$ and $g(x) = \sin x$, then
$(gof)x = g\{f(x)\} = g(|x|) = \sin|x|$

Now, f and g are continuous, so their composite, (gof) is also continuous.

Therefore, $\sin|x|$ is continuous.

34. Find all points of discontinuity of f defined by f(x) = |x| - |x +1|
Solution:

Given function is $f(x)=|x|-|x+1|$

When x < -1: $f(x)=-x-\{-(x+1)\}$ = $-x+x+1=1$

When $-1\le x<0$; $f(x)=-x-(x+1)=-2x-1$

When $x\ge 0,$; $f(x)=x-(x+1)=-1$

So, we have a function as:

$$f(x)=\begin{cases} 1, & \text{if} \quad x<-1 \\ -2x-1, & \text{if} \quad -1\le x<0 \\ -1, & \text{if} \quad x\ge 0 \end{cases}$$

Checking the continuity at x = -1 and x = 0

At $x=-1,$ L.H.L. = $\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-} 1=1$

R.H.L. = $\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(-2x-1)=1$

And $f(-1)=-2\times 1-1=1$

Therefore, at $x=-1,$ $f(x)$ is continuous.

At $x=0,$ L.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-2x-1)=-1$ and R.H.L. = $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(-1)=-1$

And $f(0)=-1$

Therefore, at x = 0, f(x) is continuous.

Hence, there are no points of discontinuity for f(x).