JEE Main 2021(July) Paper

(Memory Based)

Date of Exam: 20^{th} July 2021
Time: 03:00 p.m.-06:00 p.m.
Subject: Mathematics

Q1. $\int_{-\pi/2}^{\pi/2}([x]-[\sin x])dx =$ (Where [.] represents G.I.F)

(1) -2 (2) 1
(3) 0 (4) -1

Solution:

$I=\int_{-\pi/2}^{\pi/2}([x]-[\sin x])dx$

Using the property $\int_{-a}^{a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(-x)dx$

$I=\int_0^{\pi/2}([x]+[-x])dx-\int_0^{\pi/2}([\sin x]+[-\sin x])dx=0$

Q2. If $\lim\limits_{x\to 0}\frac{\alpha x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-\beta\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)+\gamma x^2\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)}{x^3}=10$

Solution:

$\lim\limits_{x\to 0}\frac{\alpha x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-\beta\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)+\gamma x^2\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)}{x^3}=10$

$\Rightarrow \alpha-\beta=0, \Rightarrow \alpha=\beta$

$\Rightarrow \alpha+\frac{\beta}{2}+\gamma=0 \Rightarrow \gamma=-\frac{3\beta}{2}$

$\Rightarrow \frac{\alpha}{2}-\frac{\beta}{3}-\gamma=10$

$\Rightarrow \beta=6, \alpha=6, \gamma=-9$

So, the value of $\alpha+\beta+\gamma=3$

Q3. The value of x satisfying the equation $\log_{(x+1)}(2x^2+7x+5)+\log_{(2x+5)}(x+1)^2=4$ is:

(1) -2 (2) 2
(3) -4 (4) 4

Solution:

$\log_{(x+1)}\big((2x+5)(x+1)\big)+\log_{(2x+5)}(x+1)^2=4$

$1+\log_{(x+1)}(2x+5)+2\log_{(2x+5)}(x+1)=4$

Put $\log_{(x+1)}(2x+5)=t$

$\therefore 1+t+\frac{2}{t}=4 \Rightarrow t^2+t+2=4t \Rightarrow t^2-3t+2=0$

$t=1, t=2$

For $t=1$

$2x+5=x+1$

$\Rightarrow x=-4$(Rejected)

For $t=2$

$2x+5=(x+1)^2$

$x=2, x=-2$(Rejected)

Q4. Mean of 6 observations is 10 and their variance is $\frac{20}{3}$. If observations are 15, 11, 10, 7, a, b then |a-b| is equal to:

(1) 2
(2) 1
(3) 3
(4) 4

Solution:

Mean=10

$\frac{7+10+11+15+a+b}{6} = 10$

$a+b = 17$

Variance $=\frac{20}{3}$

$\frac{49+100+121+225+a^2+b^2}{6} - 100 = \frac{20}{3}$

$a^2+b^2 = 145$

$(a+b)^2 = 289$

$ab = 72$

$(a-b)^2 = (a+b)^2 - 4ab$

$(a-b)^2 = 289 - 288 = 1$

$|a-b| = 1$

Q5. If $f(x) = x+1$, then find $\lim_{n\to\infty} \frac{1}{n}\left[f\ (0) + f\ \left(\frac{5}{n}\right)\ + f\ \left(\frac{10}{n}\right) \dots\dots\dots\dots + f\ \left(\frac{5(n-1)}{n}\right)\right]$

(1) $\frac{7}{2}$
(2) $\frac{3}{2}$
(3) $\frac{5}{2}$
(4) $\frac{1}{2}$

Solution:

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right) = \int_0^1 f(5x) = \int_0^1 (5x+1).dx$

$= \left(\frac{5x^2}{2} + x\right)\Big|_0^1 = \frac{5}{2} + 1 = \frac{7}{2}$

Q6. Sum of 21 terms of series $\log_{9^{\frac{1}{2}}} x + \log_{9^{\frac{1}{3}}} x + \log_{9^{\frac{1}{4}}} x \cdots\cdots$ is 252, then the value of x is:

(1) 7
(2) 243
(3) 9
(4) 81

Solution:

$2\log_9 x + 3\log_9 x + 4\log_9 x$ ....21 terms

$= (2+3+4+5 \dots. +22)\log_9 x = \frac{21}{2}(2+22)\log_9 x$

$= 21 \times 12 \log_9 x$

$= 252 \log_9 x$

Given sum $=252 \Rightarrow \log_9 x = 1$

$\Rightarrow x = 9$

Q7. If $Re[(1+\cos\theta+2i\sin\theta)]=4$, then the value of θ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$

(3) $-\frac{\pi}{2}$ (4) π

Solution:

$\frac{1}{1+\cos^2\theta+2isin\,\theta}\times\frac{1+\cos\theta-2i\sin\theta}{1+\cos\theta-2i\sin\theta}$

$=\frac{1+\cos\theta-2i\sin\theta}{(1+\cos\theta)^2+4\sin^2\theta}$

$\Rightarrow\frac{1+\cos\theta}{1+\cos^2\theta+2\cos\theta+4\sin^2\theta}=4$

$\Rightarrow\frac{1+\cos\theta}{1+\cos^2\theta+2\cos\theta+4-4\cos^2\theta}=4$

$\Rightarrow\frac{1+\cos\theta}{5+2\cos\theta-3\cos^2\theta}=4$

$\Rightarrow 1+\cos\theta=2\theta+8\cos\theta-12\cos^2\theta$

$\Rightarrow 12\cos^2\theta-7\cos\theta-19=0$

$\Rightarrow 12\cos^2\theta-19\cos\theta+12\cos\theta-19=0$

$\Rightarrow \cos\theta(12\cos\theta-19)+1(12\cos\theta-19)=0$

$\Rightarrow \cos\theta=-1$ or $\cos\theta=\frac{19}{12}$ (rejected)

$\Rightarrow \theta=\pi$

Q8. If $x=ay-1=z-2$, and $x=3y-2=bz-2$ lie in same plane, then the value of a,b is

(1) $a=2,b=3$ (2) $a=1,b=1$

(3) $b=1,a\in\mathbb{R}-\{0\}$ (4) $a=3,b=2$

Solution:

$\frac{x}{1}=\frac{y-\frac{1}{a}}{\frac{1}{a}}=\frac{z-2}{1},x=\frac{y-\frac{2}{3}}{\frac{1}{3}}=\frac{z-\frac{2}{b}}{\frac{1}{b}}$

$(\overrightarrow{a_1}-\overrightarrow{a_2})\cdot(\overrightarrow{b_1}-\overrightarrow{b_2})=0$

$$\begin{vmatrix}0 & \frac{1}{a}-\frac{2}{3} & 2-\frac{2}{b}\\ 1 & \frac{1}{a} & 1\\ 1 & \frac{1}{3} & \frac{1}{b}\end{vmatrix}=0$$

$\Rightarrow\frac{1}{ab}-\frac{1}{a}=0$

$\Rightarrow b=1,a\in\mathbb{R}-\{0\}$

Q9. If $P(\bar{A} \cap B) + P(A \cap \bar{B}) = 1 - k$

$P(\bar{A} \cap C) + P(A \cap \bar{C}) = 1 - 2k$

$P(\bar{B} \cap C) + P(\bar{C} \cap B) = 1 - k$

$P(A \cap B \cap C) = k^2, k \in (0,1)$.

Then the value of P (at least one of A, B, C) is

(1) $> \frac{1}{2}$ (2) $\left[\frac{1}{8}, \frac{1}{4}\right]$

(3) $< \frac{1}{4}$ (4) $\frac{1}{4}$

Solution:

$P(A) + P(B) - 2P(A \cap B) = 1 - k$

$P(A) + P(C) - 2P(A \cap C) = 1 - 2k$

$P(B) + P(C) - 2P(B \cap C) = 1 - k$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= \frac{3-4k}{2} + k^2 = \frac{2k^2-4k+3}{2}$

$\because$ the value of $2k^2 - 4k + 3$ is greater than 1

$\therefore P(A \cup B \cup C) = \frac{1}{2}$

Q10. If $f(x) = \frac{5x+3}{6x+a}$ and $f(f(x)) = x$ then the value of a is:

(1) -5 (2) 5

(3) 6 (4) -6

Solution:

$f(f(x)) = \frac{5f(x)+3}{6f(x)+a} = x \Rightarrow 5f(x) + 3 = 6xf(x) + ax$

$\frac{25x+15}{6x+a} + 3 = 6x\left(\frac{5x+3}{6x+a}\right) + ax$

$\Rightarrow 25x + 15 + 18x + 3a = 30x^2 + 18x + 6ax^2 + a^2x$

$\Rightarrow (30a + 6a)x^2 + (a^2 - 25)x - (3a + 15) = 0$

$\Rightarrow 6(a+5)x^2 + (a^2 - 25)x - (3a + 15) = 0, \Rightarrow a + 5 = 0 \Rightarrow a = -5$

Q11. If $g(t) = \begin{cases} \max(t^3 - 6t^2 + 9t - 3, 0), & t \in [0,3] \\ 4 - t, & t \in (3,4] \end{cases}$ then the number of points at which $g(t)$ is non differentiable is:

(1) 1 (2) 3

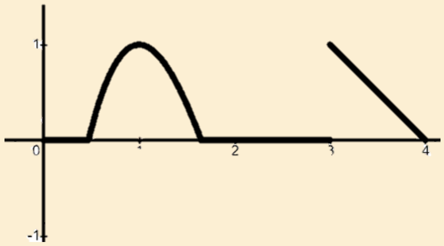
(3) 2 (4) 4

Solution:

$y = t^3 - 6t^2 + 9t - 3$

$y' = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3)$



So total 3 points

Q12. A: if 2+4 = 7, then 3+4 = 8
B: if 3+5 = 8, then earth is flat
C: if A and B are true, then 5+4 = 11
(1) A is true, B and C are false
(2) B is true, A and C are false
(3) C is true, A and B are false
(4) B is false, A and C are true

Solution:
Truth table $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A is true, B is false, C is true

Q13. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B \sum_{r=1}^{2021} A^r$ then value of $|B|$ is
(1) 2021
(2) 2021^2
(3) -2021
(4) 0

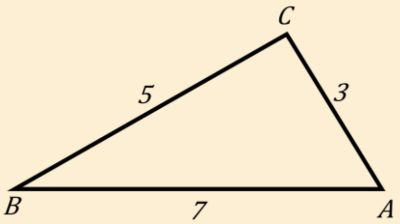
Solution:
$A = I, \quad B = I + I + I + \cdots$, 2021 times
$B = 2021 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$|B| = (2021)^2$

Q14. If element of matrix A=$[a_{ij}]_{3\times3}$ where A=$\begin{cases} (-1)^{j-i} & i < j \\ 2 & i = j \\ (-1)^{i+j} & i > j \end{cases}$, then the value of $|3Adj(2A^{-1})|$ is:
(1) 72
(2) 36
(3) 108
(4) 48

Solution:
$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$
So, $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 2 \end{vmatrix}$
$= 2(4-1) + 1(-2+1) + 1(1-2)$
$= 2(3) + 1(-1) + 1(-1) = 4$
$|3Adj(2A^{-1})| = 3^3|Adj(2A^{-1})| = 3^3 \times |2A^{-1}|^2$
$= 3^3 \times 2^6 \times |A^{-1}|^2 = 3^3 \times 3^6 \times \frac{1}{|A|^2}$=108

Q15. In a $\triangle ABC$, if $|\overrightarrow{AB}| = 7$, $|\overrightarrow{BC}| = 5$, and $|\overrightarrow{CA}| = 3$. If the projection of $\overrightarrow{BC}$ on $\overrightarrow{CA}$ is $\left(\frac{n}{2}\right)$, then the value of n is:

Solution:
$|\overrightarrow{AB}| = 7, \quad |\overrightarrow{BC}| = 5, \quad |\overrightarrow{CA}| = 3$



Projection of $\overrightarrow{BC}$ on $\overrightarrow{CA}$ is $= |\overrightarrow{BC}| \cos \angle BCA$

$5\left|\left(\frac{3^2+5^2-7^2}{2\cdot 3\cdot 5}\right)\right| = 4\left|-\frac{15}{30}\right| = \frac{5}{2}$

Q16. The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ is:

(1) $\frac{220}{21}$
(2) $\frac{110}{21}$
(3) $\frac{55}{21}$
(4) $\frac{20}{11}$

Solution:

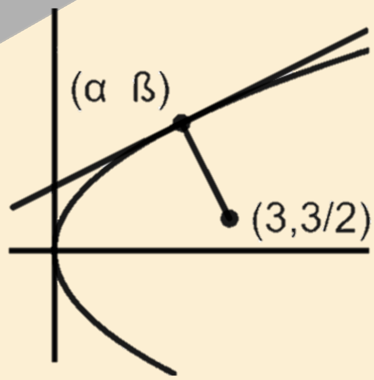
$\tan\left(\tan^{-1}\frac{\frac{6}{5}}{1-\frac{9}{25}} + \tan^{-1}\frac{5}{12}\right)$

$= \tan\left(\tan^{-1}\frac{15}{8} + \tan^{-1}\frac{5}{12}\right) = \frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8}\cdot\frac{5}{12}} = \frac{220}{21}$

Q17. If (α, β) is the point on $y^2 = 6x$, that is closest to $\left(3, \frac{3}{2}\right)$ then find $2(\alpha + \beta)$

(1) 6
(2) 9
(3) 7
(4) 5

Solution:
Equation of normal at (α, β) is



$y - \beta = -\frac{\beta}{2a}(x - \alpha)$

$\Rightarrow 3y - 3\beta = -\beta(x - \alpha)$

For shortest distance normal will pass through $\left(3, \frac{3}{2}\right)$

$\therefore \frac{9}{2} - 3\beta = -3\beta - \beta\alpha \Rightarrow \alpha\beta = \frac{9}{2} \cdots (1)$

Now from $y^2 = 6x \Rightarrow \beta^2 = 6\alpha \cdots (2)$

From (1) and (2) we get $\alpha = \frac{3}{2}, \beta = 3$

Q18. Two circles pass through $(-1,4)$ and their centres lie on $x^2+y^2+2x+4y=4$. If r_1 & r_2 are maximum and minimum radii and $\frac{r_1}{r_2}=a+b\sqrt{2}$, then the value of $a+b$ is

Solution:

Given circle

$(x+1)^2+(y+2)^2=3^2$

Any point on the circle is $(3\cos\theta-1, 3\sin\theta-2)$

Radius of the circles which passes through $(-1,4)$

$r=\sqrt{(3\cos\theta-1+1)^2+(3\sin\theta-2-4)^2}$

$=\sqrt{9\cos^2\theta+9\sin^2\theta+36-36\sin\theta}$

$=\sqrt{45-36\sin\theta}$

$r_{max}=9=r_1$ & $r_{min}=3=r_2$

$\Rightarrow \frac{r_1}{r_2}=\frac{9}{3}=3$

$\Rightarrow a+b=3$

Q19. If ΔABC is right angled triangle with sides a, b & c and smallest angle θ. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are also the sides of right-angle triangle then find $\sin\theta$

(1) $\sqrt{\frac{3-\sqrt{5}}{2}}$

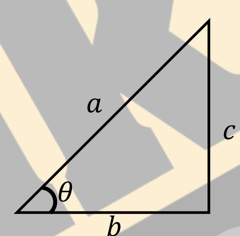
(2) $\frac{3-\sqrt{5}}{2}$

(3) $\sqrt{\frac{3+\sqrt{5}}{2}}$

(4) $\frac{3+\sqrt{5}}{2}$

Solution:

Let $a>b>c$



$\sin\theta=\frac{c}{a}$

$\frac{1}{a}<\frac{1}{b}<\frac{1}{c}$

$\frac{1}{c^2}=\frac{1}{a^2}+\frac{1}{b^2}$

$1=\frac{c^2}{a^2}+\frac{c^2}{b^2}$

$1=\frac{c^2}{a^2}+\frac{c^2}{a^2-c^2}$ (As $b^2+c^2=a^2$)

$1=\sin^2\theta+\frac{1}{\frac{a^2}{c^2}-1}=\sin^2\theta+\frac{1}{\operatorname{cosec}^2\theta-1}$

$\frac{1-\sin^2\theta+1}{\operatorname{cosec}^2\theta-1}=1\Rightarrow \sin^2\theta+\operatorname{cosec}^2\theta=3$