ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર–ક્રમાંક
મશબ/1211/414/છ, તા. 19-1-2012 થી–મંજૂર

MATHEMATICS

Standard 12

(Semester III)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

Price : ₹ 94.00

Gujarat State Board of School Textbooks
'Vidyayan', Sector 10-A, Gandhinagar-382 010

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place this textbook of **Mathematics** before the students for **Standard 12 (Semester III)** prepared according to the new syllabus.

The manuscript has been fully reviewed by experts and teachers teaching at this level. Following the suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to make this textbook interesting, useful and free from errors. However, we welcome suggestions, to enhance the quality of the textbook.

H. K. Patel GAS
Director

Dr Nitin Pethani
Executive President

Date : 22-08-2013

Gandhinagar

First Edition : 2012, Reprint : 2012, 2013

Published by : H. K. Patel, Director, on behalf of Gujarat State Board of School Textbooks, 'Vidyayan', Sector 10-A, Gandhinagar

Printed by :

FUNDAMENTAL DUTIES

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(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement.

INDEX

About This Textbook...

We are happy to publish the textbook for semester III for standard XII, in continuation of textbooks of semester I and semester II for standard XI prepared on the base of NCERT syllabus and NCF 2005.

This textbook has been prepared originally in English as in the case of textbooks of semester I and II for standard XI. The manuscript has been thoroughly examined by learned teachers from schools and colleges in a workshop in the month of October. The suggestions and proper amendments had been accepted and the revised manuscript has been translated in Gujarati. The Gujarati version was also examined by teachers from schools and colleges and the necessary amendments were made. The English manuscript and the translated version in Gujarati were read by language experts and the corrections were made. This way the final draft of the manuscript was prepared. It was reviewed in the office of Gujarat Higher Secondary and Secondary Education Board by invited expert teachers in the presence of the writers. The suggestions made by them were incorporated and the manuscript was finalized.

In chapter 1, there are explanation of relations, types of relations, equivalence classes, functions, one-one and onto functions, inverse functions and binary operations as functions. These points are explained in the textbook as in the NCERT textbooks. In chapter 2, we explain trigonometric inverse functions and their graphs and related theorems. The result of these chapters are very useful in the study of calculus. In chapter 3, we deal with the information about determinants and their theorems and their applications to solve a system of linear equations and in coordinate geometry. In chapter 4 we apply this information for the system of linear equations. The Echelon method to find the inverse of a matrix is an important point of this chapter. To find the inverse of a matrix without the use of determinants, Echelon method is useful. In semester III we began with the study of the limits. In chapter 5, we proceed further with the idea of limit and study continuity and differentiation in details. This chapter is very useful in the study of application in semester IV. Chapter 6 is the beginning of indefinite integration. Study of integration and differentiation is a base for calculus. Lebnitz and Newton both studied them in different directions and connected them. We have started with differentiation and taken as a basis of integration. Chapter 7 and 8 are the sections of statistics. We have continuously the study of probability which began in standard IX and carried it further upto binomial distribution. Chapters 8 is about linear programming.

We have studied graphs of linear inequalities in semester I of standard XI. We use them to solve practical problems in this chapter.

Attractive four-colour title, four-colour printing and figures make this textbook very useful and valuable. Plenty of illustrations and exercises are useful to explain concepts and variety of problems. They will help the students in achieving good marks in semester examination and competitive examinations.

We thank all who helped to prepare this textbook. We hope that all students, teachers and parents all would like this textbook. Positive suggestions to enhance the quality of this textbook are welcomed.

– Authors

Born	22 December, 1887 Erode, Madras Presidency
Died	26 April, 1920 (aged 32) Chetput, Madras, Madras Presidency
Nationality	India
Fields	Mathematics
Alma mater	Government Arts College Pachaiyappa's College The University of Cambridge
Academic advisors	G. H. Hardy J. E. Littlewood
Known for	Landau–Ramanujan constant Mock theta functions Ramanujan conjecture Ramanujan prime Ramanujan–Soldner constant Ramanujan theta function Ramanujan's sum Rogers–Ramanujan identities
Influences	G. H. Hardy

Shreenivas Ramanujan

Government of India has declared to celebrate the year 2012 as 'a year of Mathematics' in memory of Shree Ramanujan on his 125th birthday.

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RELATIONS AND FUNCTIONS 1

The roots of education are bitter but the fruit is sweet.

– Gauss

Mathematicians do not study objects but relations between them. Thus they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevent. They are interested in form only.

– Henri Poincare

1.1 Relations :

Last year we have studied the concept of a relation and a function. We also studied algebraic operations on functions and graphs of relations and functions. We will develop these concepts further in this chapter.

The word 'relation' is used in the context of social obligations also. We will relate the concept of the word 'relation' as used in social and family terms with the word relation as used in mathematics.

We define a relation of the set of human beings H as

$S = \{(x, y) \mid x \in H, y \in H, x \text{ is a brother of } y.\}$

Dev is a brother of Rucha. So ordered pair (Dev, Rucha) $\in$ S.

Let C be the set of all captains of Indian cricket team till 2011.

Let $S = \{(x, y) \mid x \text{ precedes } y, x, y \in C\}$

Then (Kapildev, M. S. Dhoni) $\in$ S.

But (M. S. Dhoni, Kapildev) $\notin$ S.

In the set of natural numbers N, x precedes y, if $y = x + k$ for some $k \in N$. Let $S = \{(x, y) \mid x \text{ precedes } y, x \in N, y \in N\}$. Then $(3, 5) \in S$ as $5 = 3 + 2$. But $(5, 3) \notin S$.

If S is a relation in A i.e. $S \subset (A \times A)$ and $(x, y) \in S$, we say x is related to y by S or xSy.

Let S be a relation in N defined as follows :

$S = \{(x, y) \mid |x - y| \text{ is an even positive integer } x, y \in N\}$, then whenever $(x, y) \in S$, $(y, x) \in S$. **(Why ?)**

Also note that $(x, x) \notin S$.

Now we will define various types of relations.

Void or Empty relation : A relation in the set A with no elements is called an empty relation. $\emptyset \subset (A \times A)$. $\emptyset$ is a relation called empty relation.

The relation S in N defined by

$S = \{(x, y) \mid x + y = 0, x \in N, y \in N\}$ is an empty relation as sum of two positive integers can never be zero.

Universal Relation : A relation in the set A which is $A \times A$ itself is called a universal relation.

The relation S in R defined by

$S = \{(x, y) \mid x \leq y \text{ or } y \leq x\}$ is universal relation because of the law of trichotomy.

A relation is defined on the set of all living human beings by

$S = \{(x, y) \mid$ Difference between ages of x and y is less than 200 years$\}$. Obviously S is the universal relation.

Reflexive Relation : If S is a relation in the set A and aSa, $\forall a \in A$ i.e. $(a, a) \in S$, $\forall a \in A$, we say S is a reflexive relation.

For example similarity of triangles, congruence of triangles, equality of numbers, subsets in a power set ($A \subset A$ for all $A \in P(U)$) are examples of reflexive relations.

$<$ is not a reflexive relation in R. Infact $a < a$ is false for all $a \in R$.

But $\leq$ is reflexive relation on R. $a \leq a$, $\forall a \in R$.

Symmetric Relation : If S is a relation in a set A and if $aSb \Rightarrow bSa$

i.e. $(a, b) \in S \Rightarrow (b, a) \in S \quad \forall a, b \in A$, we say S is a symmetric relation in A.

If $ABC \leftrightarrow PQR$ is a similarity relation in the set of triangles in a plane, then $PQR \leftrightarrow ABC$ is a similarity.

In the set of all non-zero integers, we define relation S by $(a, b) \in S \Leftrightarrow d$ divides $a - b$ where d is a fixed non-zero integer.

If m divides $a - b$, then m divides $b - a$. $(a, b) \in S \Rightarrow (b, a) \in S$. If $\Delta PQR \cong \Delta ABC$ then $\Delta ABC \cong \Delta PQR$. These are examples of symmetric relations.

For unequal sets A and B, $A \subset B$ does not imply $B \subset A$.

So $\subset$ is not a symmetric relation in P(U).

Transitive relation : If S is a relation in the set A and if aSb and $bSc \Rightarrow aSc$, $\forall a, b, c \in A$

i.e. $(a, b) \in S$ and $(b, c) \in S \Rightarrow (a, c) \in S$, $\forall a, b, c \in A$, then we say that S is a transitive relation in A.

$\subset$ is a transitive relation in P(U) as $A \subset B$ and $B \subset C \Rightarrow A \subset C$. $\forall A, B, C \in P(U)$.

Similarly $<$ is a transitive relation in R, as $a < b$ and $b < c \Rightarrow a < c \quad \forall a, b, c \in R$.

Equivalence Relation : If a relation S in a set A is reflexive, symmetric and transitive, it is called an equivalence relation in A.

If S is an equivalence relation and $(x, y) \in S$ then we will write, $x \sim y$.

For example equality is an equivalence relation in R, congruence of triangle is an equivalence relation on a set of coplaner triangles.

Example 1 : Prove that congruence $\equiv$ is an equivalence relation in Z.

$x \equiv y \pmod{m}$ (Read : x is congruent to y modulo m) $\Leftrightarrow m$ divides $x - y$, $m \in Z - \{0\}$.

Solution : Reflexivity : $a \equiv a \pmod{m}$ as $a - a = 0$ is divisible by any non-zero integer m.

(**Note :** 0 is divisible by any non-zero real number. But no real number is divisible by 0.)

Symmetry : If $a \equiv b \pmod{m}$, then m dividies $a - b$.

Let $a - b = mn \qquad n \in Z$

$\therefore \quad b - a = -mn = m(-n) \qquad -n \in Z$

$\therefore \quad b \equiv a \pmod{m}$

$\therefore$ If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$

$\therefore \equiv$ is a symmetric relation in Z.

Transitivity : If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $m \mid (a - b)$ and $m \mid (b - c)$.

($m \mid (a - b)$ means m divides $(a - b)$)

$\therefore$ for some $k \in Z$, $t \in Z$ $a - b = mk$ and $b - c = mt$

$\therefore \quad a - b + b - c = mk + mt$

$\therefore \quad a - c = m(k + t) \qquad k + t \in Z$

$\therefore$ $a \equiv c \pmod{m}$

If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$

$\therefore$ Congruence relation is an equivalence relation in Z.

Example 2 : Prove that similarity is an equivalence relation in the set of all triangles in a plane.

Solution : For any ΔABC, $\Delta ABC \sim \Delta ABC$ for the correspondence $ABC \leftrightarrow ABC$.

If $\Delta ABC \sim \Delta PQR$, then $\Delta PQR \sim \Delta ABC$.

Also, if $\Delta ABC \sim \Delta PQR$ and $\Delta PQR \sim \Delta XYZ$, then $\Delta ABC \sim \Delta XYZ$.

$\therefore$ $\sim$ is an equivalence relation.

(**Note :** Similarly congruence is an equivalence relation in the set of all triangles in plane.)

Example 3 : A = {the set of all lines in plane}

S = $\{(x, y) \mid x = y$ or x is a line parallel to line y.$\}$

Is S an equivalence relation in A ?

Solution : $(l, l) \in$ S as $l = l$. So, S is reflexive. **(given)**

Let $(l, m) \in$ S. So $l \parallel m$ or $l = m$.

If $l \parallel m$, then $m \parallel l$ or if $l = m$, then $m = l$.

$\therefore$ If $(l, m) \in$ S then $(m, l) \in$ S.

$\therefore$ S is symmetric.

Let $(l, m) \in$ S and $(m, n) \in$ S.

If l, m, n are distinct lines, then $l \parallel m$ and $m \parallel n$ and hence $l \parallel n$.

If $l \parallel m$ and $m = n$ or if $l = m$ and $m \parallel n$, then $l \parallel n$.

If $l = m$ and $m = n$, then $l = n$

$\therefore$ If $(l, m) \in$ S and $(m, n) \in$ S, then $(l, n) \in$ S.

$\therefore$ S is transitive.

So, S is reflexive, symmetric and transitive.

$\therefore$ S is an equivalence relation.

Example 4 : Prove that the relation S = $\{(a, b) \mid |a - b|$ is even.$\}$ is an equivalence relation in the set A = {1, 2, 3, 4, 5, 6, 7}.

Solution : | odd integer − odd integer | = | even integer − even integer | = an even integer

$\therefore$ S = {(1, 3), (3, 1), (1, 5) (5, 1), (3, 5), (5, 3), (1, 7), (7, 1), (3, 7), (7, 3), (5, 7), (7, 5), (2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)}

Since $(x, x) \in$ S, $\forall x \in$ A. S is reflexive.

Let $(x, y) \in$ S.

Hence $|x - y|$ is even.

$|x - y| = |y - x|$. So $|y - x|$ is even. Hence $(x, y) \in$ S $\Rightarrow (y, x) \in$ S. So S is symmetric.

Let $(x, y) \in$ S and $(y, z) \in$ S.

If $|x - y|$ and $|y - z|$ are even, then x and y have same parity (both even or both odd) and y and z have same parity. Thus x and z have same parity.

$\therefore$ $|x - z|$ is even.

$\therefore$ $(x, z) \in$ S, if $(x, y) \in$ S and $(y, z) \in$ S

$\therefore$ S is transitive.

So, S is reflexive, symmetric and transitive.

$\therefore$ S is an equivalence relation.

Antisymmetric Relation : If S is a relation in A and if $(a, b) \in$ S and $(b, a) \in$ S $\Rightarrow a = b$, $\forall a, b \in$ A then S is said be an antisymmetric relation.

$\subset$ is an antisymmetric relation in the set P(U) as $A \subset B$ and $B \subset A \Rightarrow A = B$, $\forall A, B \in$ P(U)

$\leq$ is an antisymmetric relation in R because $a \leq b$ and $b \leq a \Rightarrow a = b$ $\forall a, b \in$ R

Example 5 : Give an example of a relation which is **(1)** reflexive and symmetric but not transitive **(2)** reflexive and transitive but not symmetric **(3)** symmetric and transitive but not reflexive.

Solution :

(1) A = the set of all lines in plane.

S = $\{(x, y) \mid x = y$ or x is perpendicular to y, $x, y \in$ A$\}$ is a relation in A.

Since $l = l$, $(l, l) \in$ S. So S is reflexive.

If $(l, m) \in$ S, then $l = m$ or l is perpendicular to m.

$\therefore$ $m = l$ or m is perpendicular to l.

$\therefore$ $(m, l) \in$ S.

$\therefore$ $(l, m) \in$ S $\Rightarrow (m, l) \in$ S.

So S is symmetric.

Let $(l, m) \in$ S and $(m, n) \in$ S and $l \neq m$, $m \neq n$, $l \neq n$

Hence $l \perp m$ and $m \perp n$. So $l \parallel n$, as $l \neq n$.

$\therefore$ $(l, n) \notin$ S

$\therefore$ S is reflexive and symmetric but not transitive.

(2) $\leq$ in R is reflexive and transitive but not symmetric.

$a \leq a \quad \forall a \in$ R. So, S is reflexive.

$a \leq b$ and $b \leq c \Rightarrow a \leq c \quad \forall a, b\ c \in$ R. So S is transitive.

but if $a \leq b$, then $b \nleq a$, unless $a = b$.

$\therefore$ S is not symmetric.

Thus $(3, 5) \in$ S, but $(5, 3) \notin$ S where S is the relation $\leq$.

$\therefore$ S is reflexive and transitive but not symmetric.

(3) Let A = $\{1, 2, 3\}$.

S = $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$

S is symmetric and transitive but not reflexive as $(3, 3) \notin$ S

Example 6 : Give an example of a relation which is **(1)** reflexive but not symmetric or transitive **(2)** symmetric but not reflexive or transitive **(3)** transitive but not reflexive or symmetric.

Solution : (1) Let A = $\{1, 2, 3\}$.

S = $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

$(1, 1), (2, 2), (3, 3)$ are in S. Hence S is reflexive.

$(1, 2) \in$ S but $(2, 1) \notin$ S. Hence S is not symmetric.

$(1, 2) \in$ S, $(2, 3) \in$ S but $(1, 3) \notin$ S.

$\therefore$ S is not transitive.

$\therefore$ S is reflexive but neither symmetric nor transitive.

(2) Let A = $\{1, 2, 3\}$, S = $\{(1, 2), (2, 1)\}$

S is symmetric but neither reflexive nor transitive.

(3) Consider $<$ in the set R.

$a < b$ and $b < c \Rightarrow a < c \quad \forall a, b\ c \in$ R. So, S is transitive.

but $a \nless a$ and if $a < b$ then $b \nless a$. So, S is neither reflexive nor symmetric.

$\therefore$ $<$ is transitive but neither reflexive nor symmetric.

Example 7 : Give an example of a relation which is not reflexive, not symmetric, not transitive.

Solution : Let A = $\{1, 2, 3\}$, S = $\{(1, 1), (2, 2), (1, 2), (2, 3)\}$.

$(3, 3) \notin$ S. So S is not reflexive.

$(1, 2) \in$ S but $(2, 1) \notin$ S. So S is not symmetric.

$(1, 2) \in$ S and $(2, 3) \in$ S but $(1, 3) \notin$ S. So S is not transitive.

$\therefore$ S is not reflexive, not symmetric, not transitive.

Example 8 : Following is a proof that a relation which is symmetric and transitive is also reflexive. Find what is wrong with it.

Let $x\mathrm{S}y$

$\therefore$ $y\mathrm{S}x$ **(Symmetry)**

Since $x\mathrm{S}y$ and $y\mathrm{S}x$, so $x\mathrm{S}x$ **(Transitivity)**

$\therefore$ S is reflexive.

Solution : This is not correct argument.

There may be some x such that $x\mathrm{S}y$ is not true for any y in set A.

Then the argument fails.

For example let A = {1, 2, 3, 4}

S = {(1, 1), (2, 2), (1, 2), (2, 1), (1, 3), (3, 1), (3, 3), (2, 3), (3, 2)}

$(4, 4) \notin$ S. This is because for no x, $(x, 4) \in$ S.

$\therefore$ S is not reflexive even though it is symmetric and transitive..

Example 9 : A relation S is said to be circular if $x\mathrm{S}y$ and $y\mathrm{S}z$ implies $z\mathrm{S}x$. Prove that if a relation is reflexive and circular, it is an equivalence relation.

Solution : S is reflexive. **(given)**

Let $x\mathrm{S}y$. We already have $y\mathrm{S}y$.

$\therefore$ $x\mathrm{S}y$ and $y\mathrm{S}y \Rightarrow y\mathrm{S}x$

$\therefore$ $x\mathrm{S}y \Rightarrow y\mathrm{S}x$

$\therefore$ S is symmetric.

Let $x\mathrm{S}y$ and $y\mathrm{S}z$.

$\therefore$ $z\mathrm{S}x$ **(S is circular)**

$\therefore$ $x\mathrm{S}z$ **(S is symmetric)**

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

Arbitrary Union : Let I be a non-empty set of real numbers. Let A_i be a set corresponding to $i \in$ I

Then we define $\bigcup\limits_{i \in \mathrm{I}} \mathrm{A}_i = \{x \mid x \in \mathrm{A}_i$ for at least one $i \in \mathrm{I}\}$

$\bigcap\limits_{i \in \mathrm{I}} \mathrm{A}_i = \{x \mid x \in \mathrm{A}_i$ for all $i \in \mathrm{I}\}$

For example, let I = [0, 1]. Let $\mathrm{A}_i = [0, i]$

Then $\bigcup\limits_{i \in \mathrm{I}} \mathrm{A}_i = [0, 1]$

$\bigcap\limits_{i \in \mathrm{I}} \mathrm{A}_i = \{0\}$

Equivalence Classes : Let S be an equivalence relation in a set A. If $x\mathrm{S}y$, we say $x \sim y$ (x is equivalent to y) (Read $\sim$ as wiggle)

Let $\mathrm{A}_p = \{x \mid x \sim p, x \in \mathrm{A}\}$

Let us prove the following :

if $p \sim q$, $A_p = A_q$ and if p is not equivalent to q, $A_p \cap A_q = \emptyset$

If $A_p \cap A_q \neq \emptyset$, let $x \in (A_p \cap A_q)$

$\therefore \quad x \in A_p$ and $x \in A_q$

$\therefore \quad x \sim p$ and $x \sim q$

$\therefore \quad p \sim x$ and $x \sim q$

$\therefore \quad p \sim q$

$\therefore \quad p \in A_q$ and $q \in A_p$

$\therefore \quad A_p \subset A_q$ and $A_q \subset A_p$.

$\therefore \quad A_p = A_q$

Now, if $A_p \cap A_q \neq \emptyset$, then $A_p = A_q$

Also, $p \sim p$.

$\therefore \quad p \in A_p \quad \forall p \in A.$

$$\bigcup_{p \in A} A_p = A$$

Thus an equivalence relation 'partitions' A into disjoint sets A_p such that

(i) $A_p \cap A_q = \emptyset$, if p is not equivalent to q.

(ii) $\bigcup_{p \in A} A_p = A$

These sets A_p are called equivalence classes corresponding to the equivalence relation $\sim$.

Conversely any partition of A gives rise to an equivalence relation in A.

We define $x \sim y$ if x and y are in the same class A_p.

$x \sim x$ as x and x belong to the same classes A_p.

If $x \sim y$, then $y \sim x$ because if x and y belong to the same class, then y and x also belong to the same class.

If $x \sim y$ and $y \sim z$, then x and y, y and z belong to the same class. Hence x and z belong to same class.

Hence $x \sim z$

$\therefore \quad \sim$ is an equivalence relation.

Example 10 : We define $a \equiv b \pmod 2$, if $a - b$ is even. Prove $\equiv$ is an equivalence relation in Z. Find equivalence classes.

Solution : $a \equiv a$ as 2 divides 0, or 0 is even.

If $a \equiv b$, then $b \equiv a$ as $a - b$ is even $\Leftrightarrow b - a$ is even.

If $a \equiv b$ and $b \equiv c$, then $a \equiv c$ since $a - b$ is even and $b - c$ is even implies

$a - c = a - b + b - c$ is even.

$\therefore \quad \equiv$ is an equivalence relation.

$1, 3, 5, \ldots \in A_1$ say. $(1 \equiv 3,\ 3 \equiv 5$ etc.)

$2, 4, 6, \ldots \in A_2$ say. $(2 \equiv 4,\ 4 \equiv 6$ etc.)

All integers are divided into two equivalence classes,

A_1 = the set of odd integers and A_2 = the set of all even integers.

Example 11 : Let $Z = A_1 \cup A_2 \cup A_3$ where $A_1 = \{....1, 4, 7,...\}$
$A_2 = \{....2, 5, 8,...\}$
$A_3 = \{....3, 6, 9,...\}$

Define an equivalence relation whose equivalence classes are A_1, A_2 and A_3.

Solution : Let us define aSb if $3 \mid (a - b)$ or $a \equiv b \pmod 3$.

Then $\equiv$ is an equivalence relation as

$a \equiv a$ as 3 divides $a - a = 0$, so aSa

$a \equiv b \pmod 3 \Rightarrow 3 \mid (a - b)$

$\Rightarrow 3 \mid (b - a)$

$\Rightarrow b \equiv a \pmod 3$

$\therefore \quad aSb \Rightarrow bSa$

$3 \mid (a - b)$ and $3 \mid (b - c)$ implies $3 \mid [(a - b) + (b - c)] = a - c$. Hence aSb and $bSc \Rightarrow aSc$. S is an equivalence relation. So we can write $a \sim b$, if aSb. For this equivalence relation,

$A_1 = \{....1, 4, 7, 10,...\}$, $A_2 = \{....2, 5, 8...\}$, $A_3 = \{....3, 6, 9...\}$ are equivalence classes. For this relation, difference $x - y$ is divisible by 3, if x and y belong to the same class.

Example 12 : Let L be the set of all lines in the XY-plane and S be the relation defined in L as $S = \{(L_1, L_2) \mid L_1 = L_2$ or L_1 is parallel to $L_2\}$. Prove S is an equivalence relation and obtain equivalence classes containing (i) X-axis (ii) Y-axis.

Solution : We have seen that S is an equivalence relation.

The equivalence class of lines containing X-axis is the set of lines $y = b$, $b \in R$.

The equivalence class of lines containing Y-axis is the set of lines $x = a$, $a \in R$.

Example 13 : Show that the set $S = \{(P, Q) \mid$ distance of $P(x, y)$ and $Q(x_1, y_1)$ from origin is same. $P, Q \in R^2\}$ is an equivalence relation. What is the equivalence class containing (1, 0) ?

Solution : $d(P, O) = d(P, O)$. So $(P, P) \in S$. So S is reflexive.

If $d(P, O) = d(Q, O) = r$, then $d(Q, O) = d(P, O) = r$. So S is symmetric.

If $d(P, O) = d(Q, O) = r$ and $d(Q, O) = d(R, O) = r$, then $d(P, O) = d(R, O) = r$

$\therefore \quad (P, Q) \in S, (Q, R) \in S \Rightarrow (P, R) \in S$. Hence S is transitive.

$\therefore \quad$ S is an equivalence relation.

$d(A(1, 0), O) = 1$

The equivalence class containing (1, 0) consists of all points at distance 1 from origin i.e. unit circle.

Exercise 1.1

1. **Determine which of the following relations is reflexive, symmetric or transitive ?**
   (1) $A = \{1, 2, 3,...., 10\}$. $S = \{(x, y) \mid y = 2x\}$
   (2) $A = N$, $S = \{(x, y) \mid y$ divides $x\}$
   (3) $A = \{1, 2, 3, 4, 5, 6\}$, $S = \{(x, y) \mid y$ divides $x\}$
   (4) $A = Z$, $S = \{(x, y) \mid x - y \in Z\}$
   (5) $A = R$, $S = \{(x, y) \mid y = x + 1\}$
2. aSb if $6 \mid (a - b)$, $a, b \in Z$. Prove that S is an equivalence relation and write down equivalence classes.
3. Prove $\subset$ is reflexive, antisymmetric and transitive in P(U).
4. (1) $f : N \rightarrow N, f(x) = x^2$ is a function. We define xSy if $f(x) = f(y)$. Is S an equivalence relation ? What are equivalence classes ?
   (2) If $f : Z \rightarrow Z, f(x) = x^2$, what are equivalence classes for this equivalence relation ?

5. $f : N \times N \to N \times N$, $f((m, n)) = ((n, m))$. We say $(a, b)S(c, d)$ if $f((a, b)) = f((c, d))$. Is S an equivalence relation ? What is the equivalence class containing (1, 2) ?

6. Let L be the set of lines in XY plane. Define a relation S in L by $xSy \Leftrightarrow x = y$ or $x \perp y$ or $x \parallel y$.

Is S an equivalence relation ? If so, what are equivalence classes ? What is the equivalence class containing X-axis ? What happens if L is the set of all lines in space ?

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1.2 One-one and onto Functions

We have studied the concept of a special type of relation called a function.

Remember, if $A \neq \emptyset$ and $B \neq \emptyset$ and if $f \subset (A \times B)$ and $f \neq \emptyset$ such that for every $x \in A$, there is one and only one $y \in B$ such that $(x, y) \in f$, then f is a function.

Thus f is a relation whose domain is A. We also studied graphs of functions and algebraic operations of addition, subtraction, multiplication and division of functions.

Consider following two functions :

$f : N \to N$, $f(x) = x^2$

$\therefore$ $f = \{(1, 1), (2, 4), (3, 9), (4, 16),....\}$

Here $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

$g : Z \to Z$ $g(x) = x^2$

Then $g = \{(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4).....\}$

But $-1 \neq 1$ and $g(-1) = g(1) = 1$.

Functions like f are called one-one functions and functions like g are called many-one functions.

Let us give a formal definition.

One-one function : If $f : A \to B$ is a function and if $\forall x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, we say $f : A \to B$ is a one-one function, also called an injective function.

Generally we deal with equality with ease rather than working with an inequation. Using contrapositive of defining statement, we can say that if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in A$, then $f : A \to B$ is a one-one function.

For a function $f : A \to A$, $S = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is an equivalence relation in A.

Obviously $f(x_1) = f(x_1)$ **(Reflexive)**

$f(x_1) = f(x_2) \Rightarrow f(x_2) = f(x_1)$ **(Symmetry)**

$f(x_1) = f(x_2)$ and $f(x_2) = f(x_3) \Rightarrow f(x_1) = f(x_3)$ **(Transitivity)**

$\therefore$ S is an equivalence relation.

For a one-one function $f : A \to A$, the equivalence class containing x_1 is $\{x_1\}$ only.

So $A = \bigcup_{x \in A} \{x\}$. Also $A_i = \{x_i\}$ is the partition of A corresponding to this equivalence relation.

Consider $f : \{1, 2, 3, 4, 5\} \to \{2, 3, 6, 7, 8\}$

$f = \{(1, 2), (2, 2), (3, 3), (4, 6), (5, 6)\}$. f is not a one-one function as $1 \neq 2$ and $f(1) = f(2) = 2$.

Many-one function : If $f : A \to B$ is a function and if $\exists x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, then $f : A \to B$ is said to be a many-one function.

See that this defining statement is the negation of the statement used to define a one-one function.

We define $f(C) = \{y \mid y = f(x), x \in C, C \subset A, C \neq \emptyset\}$ and

$$f^{-1}(D) = \{x \mid y = f(x), x \in A, y \in D, D \subset B\}$$

See that $f(C)$ and $f^{-1}(D)$ are merely symbols.

We note that $f(C)$ is never empty. Set $f^{-1}(D)$ could be $\emptyset$.

In this example if $C = \{2, 3, 4\}$, $f(C) = \{2, 3, 6\}$

If $C = \{1, 2\}$, $\quad f(C) = \{2\}$

If $D = \{8\}$, $\quad f^{-1}(D) = \emptyset$

If $D = \{2\}$, $\quad f^{-1}(D) = \{1, 2\}$

If $D = \{2, 6\}$, $\quad f^{-1}(D) = \{1, 2, 4, 5\}$

In fact $f(A)$ is the range of $f : A \to B$.

$f^{-1}(D)$ is the set of pre-images of the elements of D.

$f^{-1}(B) = A$

Let us see some examples.

Example 14 : Determine whether $f : N \to N$, $f(x) = 2x$ is one-one or not.

Solution : Let $x_1, x_2 \in N$.

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$\therefore$ $f : N \to N$, $f(x) = 2x$ is one-one.

Example 15 : If $f : R \to Z$, $f(x) = [x]$ = integer part of x (or floor function $\lfloor x \rfloor$), is $f : R \to Z$ one-one ?

Solution : No. $f(2.1) = [2.1] = 2$

$f(2.23) = [2.23] = 2$

$\therefore$ $f : R \to Z$, $f(x) = [x]$ is not one-one.

Example 16 : Is $f : R \to R^+ \cup \{0\}$, $f(x) = |x|$ one-one ?

Solution : No. $f(-1) = f(1) = 1$

$\therefore$ $f : R \to R^+ \cup \{0\}$, $f(x) = |x|$ is not one-one.

Example 17 : If $f : N \cup \{0\} \to N \cup \{0\}$, $f(x) = x - 3\left[\frac{x}{3}\right]$, is f one-one ? Find equivalence classes for the relation $S = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$.

Solution : $f(1) = 1 - 3\left[\frac{1}{3}\right] = 1$, $f(2) = 2$, $f(3) = 3 - 3 = 0$, $f(4) = 4 - 3\left[\frac{4}{3}\right] = 1$,

$f(5) = 5 - 3\left[\frac{5}{3}\right] = 2$, $f(6) = 6 - 3\left[\frac{6}{3}\right] = 0$.

In fact $f(n)$ = the remainder when n is divided by 3.

$\therefore$ $f(1) = f(4) = f(7) = f(10) = ... = 1$

$f(2) = f(5) = f(8) = f(11) = ... = 2$

$f(3) = f(6) = f(9) = f(12) = ... = 0$

$\therefore$ $f : N \cup \{0\} \to N \cup \{0\}$, $f(x) = x - 3\left[\frac{x}{3}\right]$ is not one-one.

The equivalence classes are $\{1, 4, 7, 10,...\}$, $\{2, 5, 8, 11,...\}$, $\{0, 3, 6, 9, 12,...\}$

Onto Function : If the range of the function $f : A \to B$ is B, we say that f is an onto function or surjective function or more precisely f is a function from A onto B.

If $R_f = f(A) = B$, f is onto.

Thus, if there exists at least one $x \in A$ corresponding to every $y \in B$, such that $y = f(x)$, $f : A \to B$ is an onto function. If $\exists y \in B$, for which there is no $x \in A$ such that $y = f(x)$, $f : A \to B$ is not an onto function.

Example 18 : Give one example each of a function which is (1) one-one and onto, (2) one-one and not onto, (3) many-one and onto, (4) many-one and not onto.

Solution : (1) $f : N \to E$, E being the set of even natural numbers, $f(x) = 2x$.

$f = \{(1, 2), (2, 4), (3, 6),.....\}$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

$R_f = \{2, 4, 6,...\} = E$

Infact every $y \in E$ is of the form $2n$ for some $n \in N$ and $f(n) = 2n = y$

$\therefore$ $R_f = E$

$\therefore$ f is an onto function.

(2) $f : N \to N$, $f(x) = 2x$

$f = \{(1, 2), (2, 4), (3, 6),...\}$

f is one-one as in (1).

$\therefore$ $R_f = \{2n \mid n \in N\} = E$, the set of even natural numbers.

$\therefore$ $R_f = E \neq N$

$\therefore$ f is not an onto function.

(3) $f : R \to Z$, $f(x) = [x]$

$f(1.1) = 1$, $f(1.3) = 1$

$\therefore$ f is many-one.

But $R_f = Z$, since for every $n \in Z$, $f(n) = n$. Thus every integer is in the range of f.

$\therefore$ f is onto.

(4) $f : Z \to Z$, $f(x) = x^2$

$f(-1) = f(1) = 1$. So f is not one-one, but it is many-one.

$R_f = \{0, 1, 4, 9,...\} \neq Z$

$\therefore$ f is not onto.

One-one and onto function :

If $f : A \to B$ is a one-one and onto function, it is called a bijective function.

Example 19 : Prove that $f : R \to R$, $f(x) = ax + b$ $a \neq 0$ is a bijective function.

Solution : Let $f(x_1) = f(x_2)$

$\therefore$ $ax_1 + b = ax_2 + b$

$\therefore$ $ax_1 = ax_2$

$\therefore$ $x_1 = x_2$ $\qquad (a \neq 0)$

$\therefore$ f is one-one.

$y = ax + b \Leftrightarrow x = \dfrac{y-b}{a}$ $\qquad (a \neq 0)$

$\therefore$ For every $y \in R$, $\exists x \in R$ such that,

$$f(x) = f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y - b + b = y$$

$\therefore$ Range of f is R.

$\therefore$ $f : R \rightarrow R$ is onto.

$\therefore$ $f : R \rightarrow R$ is a bijective function.

Example 20 : In how many points does a horizontal line intersect the graph of $y = f(x)$, if f is one-one ?

Solution :

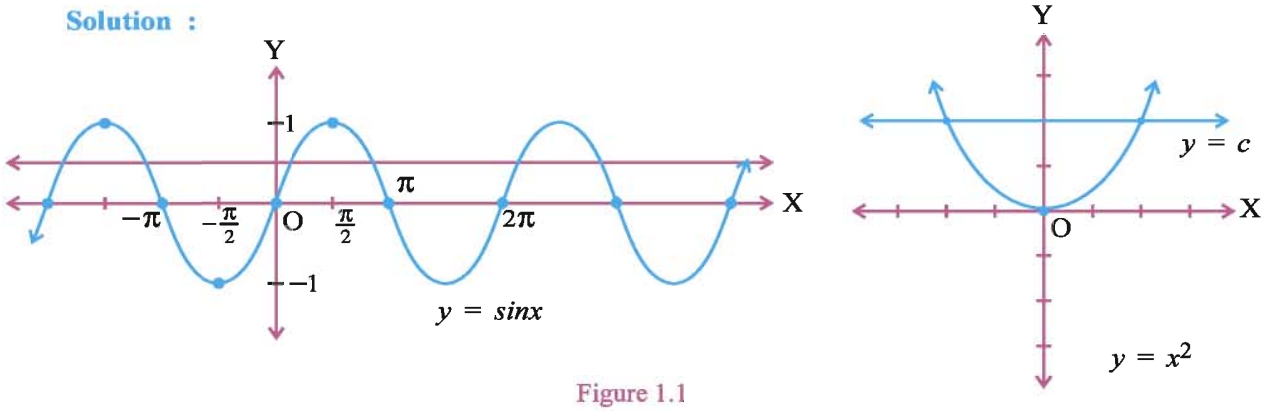

Figure 1.1

The graph of a one-one function $f : A \rightarrow B$ is intersected by a horizontal line $y = c$ in at most one point.

For $f : R \rightarrow R$, the graph of $f(x) = x^2$ is intersected by a horizontal line $y = c$ in two points in general ($c > 0$). For $x_1 \neq x_2$, we should have $f(x_1) \neq f(x_2)$. So if restrict the function to $f : R^+ \rightarrow R^+$, $f(x) = x^2$, it is one-one. The same thing happens in the case of graph of $y = sinx$. If $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[0, \frac{\pi}{2}\right]$ etc, the graph of $y = sinx$ is intersected by line $y = c$ $(-1 \leq c \leq 1)$ in at most one point. Otherwise the line $y = c$ intersects the graph of $y = sinx$ in infinitely many points. $(-1 \leq c \leq 1)$

Example 21 : If $A = \{x_1, x_2, x_3,..., x_n\}$, prove any function $f : A \rightarrow A$ is injective if and only if it surjective.

Solution : Let $f : A \rightarrow A$ be one-one.

$\therefore$ $f(x_1), f(x_2),..., f(x_n)$ are all distinct elements of A.

But A has n elements $x_1, x_2,..., x_n$ only.

$\therefore$ $f(x_1), f(x_2),..., f(x_n)$ must be $x_1, x_2, x_3,..., x_n$ in some order.

$\therefore$ $R_f = A$

$\therefore$ $f : A \rightarrow A$ is onto.

Conversely, suppose $f : A \rightarrow A$ is onto.

$\therefore$ $R_f = \{x_1, x_2, x_3,..., x_n\}$

Now, $\{f(x_1), f(x_2),..., f(x_n)\} = \{x_1, x_2, x_3,..., x_n\}$.

$\therefore$ No $f(x_i)$ can be equal to $f(x_j)$. $(i \neq j)$

(If some $f(x_i) = f(x_j)$, R_f will not contain all $x_1, x_2, x_3,..., x_n$.)

$\therefore$ f is one-one.

Example 22 : If $f : \{x_1, x_2,..., x_m\} \rightarrow \{y_1, y_2,..., y_n\}$ is one-one, prove that $m \leq n$.

Solution : f is one-one.

$\therefore$ $f(x_1), f(x_2),..., f(x_m)$ are m distinct elements from amongst $\{y_1, y_2,..., y_n\}$

$\therefore$ $m \leq n$

Example 23 : If $f : \{x_1, x_2,..., x_m\} \rightarrow \{y_1, y_2,..., y_n\}$ is onto, prove that $m \geq n$.

Solution : Some of $f(x_1), f(x_2),..., f(x_m)$ may be equal but they must form the set $\{y_1, y_2,..., y_n\}$.

$\therefore$ If $m < n$, at most m elements out of $\{y_1, y_2,..., y_n\}$ will be in the range, not all $y_1, y_2,..., y_n$.

$\therefore$ $m \geq n$

(**Note :** If A, B are finite sets and $f : A \rightarrow B$ is bijective then $n(A) = n(B)$.

Example 24 : $f : N \rightarrow Z$, $f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$

Prove that f is bijective.

Solution : $f = \{(1, 0), (2, 1), (3, -1), (4, 2),...\}$

as $f(1) = -\frac{1-1}{2} = 0$ (1 odd)

$f(2) = \frac{2}{2} = 1$ (2 even) etc.

$\therefore$ If n is a positive integer, $f(2n) = \frac{2n}{2} = n$. Since $2n \in N$, $2n \in D_f$, $2n$ is even.

If n is a negative integer or zero, $f(-2n + 1) = -\left(\frac{-2n+1-1}{2}\right) = n$.

If n is a negative integer or zero, $-2n + 1 \in N$. $-2n + 1$ is odd.

$\therefore$ All integers are in the range of given $f : N \rightarrow Z$.

$\therefore$ $R_f = Z$. So f is surjective.

$f(n) = \frac{n}{2}$ or $-\left(\frac{n-1}{2}\right)$

$\frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$, $-\frac{n_1-1}{2} = -\frac{n_2-1}{2} \Rightarrow n_1 = n_2$

and $\frac{n_1}{2} = -\frac{n_2-1}{2} \Rightarrow n_1 + n_2 = 1$, impossible.

$\therefore$ $f(n_1) \neq f(n_2)$ for any $n_1, n_2 \in N$.

$\therefore$ f is one-one.

$\therefore$ f is bijective.

Example 25 : Prove that $f : R - \{2\} \rightarrow R - \{2\}$, $f(x) = \frac{2x-1}{x-2}$ is one-one and onto.

Solution : $f(x_1) = f(x_2) \Rightarrow \frac{2x_1-1}{x_1-2} = \frac{2x_2-1}{x_2-2}$

$\Rightarrow 2x_1x_2 - x_2 - 4x_1 + 2 = 2x_1x_2 - x_1 - 4x_2 + 2$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let $x \in R - \{2\}$.

Let $y = f(x) = \frac{2x-1}{x-2}$ where $x = R - \{2\}$

$\therefore$ $xy - 2y = 2x - 1$

$\therefore$ $(y - 2)x = 2y - 1$ $(y \neq 2)$

$\therefore$ $x = \frac{2y-1}{y-2}$

$\therefore$ For every $y \in R - \{2\}$, there is $x \in R - \{2\}$ such that,

$$y = f(x), \text{ since } f(x) = f\left(\frac{2y-1}{y-2}\right) = \frac{2\left(\frac{2y-1}{y-2}\right)-1}{\frac{2y-1}{y-2}-2}$$

$$= \frac{4y-2-y+2}{2y-1-2y+4}$$

$$= y$$

$\therefore$ $R_f = R - \{2\}$

$\therefore$ f is onto.

Example 26 : $f : N \times N \to N$, $f((m, n)) = m + n$. Is f one-one ? Is f onto ?

Solution : $f((1, 2)) = 1 + 2 = 3$, $f((2, 1)) = 2 + 1 = 3$

but $(1, 2) \neq (2, 1)$.

$\therefore$ f is not one-one.

$m \geq 1, n \geq 1 \Rightarrow m + n \geq 2$

$\therefore$ $f((m, n)) \geq 2, \forall (m, n) \in N \times N$

$\therefore$ $1 \notin R_f$

$\therefore$ f is not onto.

Example 27 : $f : N \times N \to N \times N$, $f((m, n)) = (n, m)$. Prove f is bijective.

Solution : $f((m_1, n_1)) = f((m_2, n_2)) \Rightarrow (n_1, m_1) = (n_2, m_2)$

$\Rightarrow n_1 = n_2, m_1 = m_2$

$\Rightarrow (m_1, n_1) = (m_2, n_2)$

$\therefore$ f is one-one.

$\forall (m, n) \in N \times N, f((n, m)) = (m, n)$

$\therefore$ $R_f = N \times N$

$\therefore$ f is onto.

Exercise 1.2

Are following functions one-one ? Are they onto ? (1 to 11)

1. $f : R \to R$, $f(x) = 5x + 7$
2. $f : R \to R$, $f(x) = 2 - 3x$
3. $f : R \to R$, $f(x) = x^2 + 4x + 5$
4. $f : R \to R$, $f(x) = x^2 - x - 2$
5. $f : N \to N$, $f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{n+1}{2} & n \text{ is odd} \end{cases}$
6. $f : R \to (-1, 1)$, $f(x) = \frac{x}{1+|x|}$
7. $f : A \times B \to A$, $f((a, b)) = a$, A and B are not singleton, $A \neq \emptyset$, $B \neq \emptyset$.
8. $f : R \to R$, $f(x) = x^3$
9. $f : Z \to Z$, $f(n) = \begin{cases} n + 2 & n \text{ is even} \\ 2n + 1 & n \text{ is odd.} \end{cases}$

10. $f : Z \to Z,\ f(n) = \begin{cases} n+1 & n \text{ even} \\ n-3 & n \text{ odd.} \end{cases}$

11. $f : Z \to Z,\ f(n) = \begin{cases} n-2 & n \text{ even} \\ 2n+2 & n \text{ odd.} \end{cases}$

12. How many one-one functions are there from $\{1, 2, 3,..., n\}$ to itself ?

13. $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}$

How many onto functions $f : A_i \to A_i$ $(i = 1, 2, 3)$ are there ? Can you generalize the result ?

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1.3 Composite Functions

We have studied the concept of composite functions. Let us revise it.

If $f : A \to B$ and $g : B \to C$ are two functions, their composite function $gof : A \to C$ is defined by

$$(gof)(x) = g(f(x))$$

If $f : A \to B$ and $g : C \to D$ are functions and $R_f \subset D_g$, $gof : A \to D$ is defined by

$$(gof)(x) = g(f(x))$$

Example 28 : If $f : N \to N$, $f(x) = 2x + 3$ and $g : N \to N$, $g(x) = 5x + 7$, find gof and fog.

Solution : $gof : N \to N$

$$(gof)(x) = g(f(x)) = g(2x + 3) = 5(2x + 3) + 7 = 10x + 22$$

$fog : N \to N$

$$(fog)(x) = f(g(x)) = f(5x + 7) = 2(5x + 7) + 3 = 10x + 17$$

In general, $gof \neq fog$.

Example 29 : If $f : R \to R$, $f(x) = x^3$ and $g : R \to R$ $g(x) = x^5$, prove that $gof = fog$.

Solution : $gof : R \to R$, $(gof)(x) = g(f(x)) = g(x^3) = (x^3)^5 = x^{15}$

$fog : R \to R$, $(fog)(x) = f(g(x)) = f(x^5) = (x^5)^3 = x^{15}$

Here $fog = gof$

(**Note :** Obviously $(a^m)^n = (a^n)^m = a^{mn}$)

Example 30 : $f : \{1, 2, 4, 5\} \to \{2, 3, 6, 7\}$

$f = \{(1, 2), (2, 3), (4, 6), (5, 7)\}$ and

$g : \{2, 3, 6, 7, 8\} \to \{1, 3, 5, 6\}$

$g = \{(2, 1), (3, 1), (6, 1), (7, 5), (8, 6)\}$. Find gof and fog whichever is possible.

Solution : $R_f = \{2, 3, 6, 7\} \subset D_g = \{2, 3, 6, 7, 8\}$

$\therefore$ gof exists.

$\therefore$ $gof = \{(1, 1), (2, 1), (4, 1), (5, 5)\}$

as $(gof)(1) = g(f(1)) = g(2) = 1$, $(gof)(2) = g(f(2)) = g(3) = 1$ etc.

$R_g = \{1, 5, 6\} \not\subset D_f = \{1, 2, 4, 5\}$

$\therefore$ fog does not exist.

Example 31 : If $f : A \to B$ and $g : B \to C$ are one-one functions, prove that $gof : A \to C$ is one-one.

Solution : $(gof)(x_1) = (gof)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$ $\quad (x_1, x_2 \in A)$

$\Rightarrow f(x_1) = f(x_2)$ $\quad$ (g is one-one)

$\Rightarrow x_1 = x_2$ $\quad$ (f is one-one)

$\therefore$ $gof : A \to C$ is one-one.

Example 32 : If $f : A \to B$ is onto B and $g : B \to C$ is onto C, prove that, $gof : A \to C$ is onto C.

Solution : Let $y \in C$.

Since $g : B \to C$ is onto C, there exists $z \in B$ such that $g(z) = y$.

Now, $f : A \to B$ is onto B and $z \in B$.

$\therefore$ $\exists x \in A$ such that $f(x) = z$

$\therefore$ $g(z) = y \Rightarrow g(f(x)) = y$

$\therefore$ $(gof)(x) = y$

$\therefore$ For every $y \in C$, $\exists x \in A$ such that $(gof)(x) = y$

$\therefore$ $gof : A \to C$ is onto C.

Example 33 : If $gof : A \to C$ is one-one, can you say $f : A \to B$ and $g : B \to C$ are one-one ?

Solution : No.

Let $f : A \to B$, $A = \{1, 2, 3, 4, 5\}$, $B = \{5, 6, 7, 8, 9, 10, 11\}$

$f = \{(1, 5), (2, 6), (3, 7), (4, 8), (5, 9)\}$

Let $g : B \to B$, $g(x) = x + 1$, if $x \neq 10$ or 11

$g(10) = g(11) = 5$

Then $gof : A \to B$, $gof = \{(1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$ is one-one.

But $g : B \to B$ is not one-one.

[**Note :** Here we have taken B = C.]

Example 34 : If $f : A \to B$ and $g : B \to C$ are two functions and $gof : A \to C$ is one-one, then prove that $f : A \to B$ is one-one.

Solution : Let $f(x_1) = f(x_2)$ $\quad x_1, x_2 \in A$

$\therefore$ $g(f(x_1)) = g(f(x_2))$ $\quad (f(x_1) \in B, f(x_2) \in B)$

$\therefore$ $(gof)(x_1) = (gof)(x_2)$

$\therefore$ $x_1 = x_2$ $\quad$ (gof is one-one)

$\therefore$ $f : A \to B$ is one-one.

Example 35 : If $gof : A \to C$ is onto C, are $f : A \to B$ and $g : B \to C$ onto C ?

Solution : No. Let $f : \{1, 2, 3, 4\} \to \{2, 3, 4, 5, 6, 7\}$, $f(x) = x + 1$

$g : \{2, 3, 4, 5, 6, 7\} \to \{4, 6, 8, 10\}$, $g(x) = 2x$ $\quad$ if $x \neq 6$ or 7

$g(6) = g(7) = 10$

Then $gof : \{1, 2, 3, 4\} \rightarrow \{4, 6, 8, 10\}$,

$gof = \{(1, 4), (2, 6), (3, 8), (4, 10)\}$

$\therefore$ gof is onto C. But $f : A \rightarrow B$ is not onto as $6, 7 \notin R_f$

Example 36 : If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two functions and if $gof : A \rightarrow C$ is onto C, prove that g is onto C.

Solution : $gof : A \rightarrow C$ is onto C.

Let $z \in C$

$\therefore$ $\exists x \in A$ such that $(gof)(x) = z$

$\therefore$ $g(f(x)) = z$

$x \in A$ and $f : A \rightarrow B$ is a function.

$\therefore$ $f(x) \in B$. Let $y = f(x)$.

$\therefore$ $g(y) = z$, where $y \in B$.

$\therefore$ For every $z \in C$. $\exists y \in B$ such that $g(y) = z$

$\therefore$ $g : B \rightarrow C$ is onto C.

Exercise 1.3

1. $f : R \rightarrow R$, $g : R \rightarrow R$, $h : R \rightarrow R$ are functions.

   Prove : (i) $(fog)oh = fo(goh)$ (2) $(f + g)oh = foh + goh$

2. Find gof and fog for

   (1) $f : R \rightarrow R$, $f(x) = |x|$, $g : R \rightarrow R$, $g(x) = x^2$

   (2) $f : R^+ \rightarrow R^+$, $f(x) = x^3$, $g : R^+ \rightarrow R^+$, $g(x) = x^{\frac{1}{3}}$

3. $f : R^+ \rightarrow R$, $f(x) =$ cube root of $(3 - x^3)$. Find fof.

4. $f : R \rightarrow R$, $f(x) = x^2 - x - 2$. Find fof.

5. $f : R - \{-1\} \rightarrow R - \{-1\}$, $f(x) = \frac{1-x}{1+x}$. Find fof.

6. $f : R \rightarrow R$ is signum function.

   $$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

   $g : R \rightarrow Z$, $g(x) = [x]$. Find fog and gof.

7. $f : Z \rightarrow Z$ and $g : Z \rightarrow Z$ are defined as follows :

   $$f(n) = \begin{cases} n + 2 & n \text{ even} \\ 2n - 1 & n \text{ odd} \end{cases} \qquad g(n) = \begin{cases} 2n & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

   Find fog and gof.

8. (1) If $A \neq \emptyset$, $B \neq \emptyset$ and $f : A \rightarrow B$ is a one-one function, prove that there exists a function $g : B \rightarrow A$ such that $gof = I_A$. (I is identity function) (g is called left inverse of f.)

(2) If $A \neq \emptyset$, $B \neq \emptyset$ and $f : A \rightarrow B$ is a function onto B, prove that $\exists$ a function $g : B \rightarrow A$ such that $fog = I_B$. (g is called right inverse of f.)

(3) Combine results (1) and (2) if $f : A \rightarrow B$ is a bijective function.

*

1.4 Inverse of a Function

We have $3 \cdot 1 = 3$ as 1 is multiplicative identity. $3 \cdot \frac{1}{3} = 1$ and so $\frac{1}{3}$ is multiplicative inverse of 3. Similarly we have seen in XIth standard that for a function $f : A \rightarrow B$, $foI_A = f$ and $I_Bof = f$ where I_A and I_B are identity functions on A and B respectively. So does there exist a function $g : B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$? The answer is yes under some conditions. We define inverse of a function.

Definition : If $f : A \rightarrow B$ is a function and if there exists a function $g : B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$ we say $g : B \rightarrow A$ is the inverse function of $f : A \rightarrow B$ and denote g by f^{-1}.

The question arises why 'the' inverse ? **We must prove that $g : B \rightarrow A$ is unique before we call it the inverse of $f : A \rightarrow B$ and assign a symbol f^{-1}.**

Uniqueness : Suppose $g : B \rightarrow A$ and $h : B \rightarrow A$ are two inverses of $f : A \rightarrow B$.

$\therefore \quad gof = I_A,\ fog = I_B,\ hof = I_A,\ foh = I_B.$

$\quad g = goI_B = go(foh) = (gof)oh = I_Aoh = h$

Also $g : B \rightarrow A$, $h : B \rightarrow A$ are functions.

$\therefore$ Inverse of a function $f : B \rightarrow A$, if it exists, is unique.

When does the inverse of a function exist ? This is reflected in the following theorems.

Theorem 1.1 : If $f : A \rightarrow B$ has inverse $g : B \rightarrow A$, then $f : A \rightarrow B$ is one-one and onto.

Proof : For $x_1, x_2 \in A$. let $f(x_1) = f(x_2)$

$\therefore \quad g(f(x_1)) = g(f(x_2))$ $\qquad$ $(f(x_1), f(x_2) \in B)$

$\therefore \quad (gof)(x_1) = (gof)(x_2)$

$\therefore \quad I_A(x_1) = I_A(x_2)$ $\qquad$ ($g : B \rightarrow A$ is the inverse of $f : A \rightarrow B$)

$\therefore \quad x_1 = x_2$

$\therefore \quad f : A \rightarrow B$ is one-one.

Let $y \in B$

$\therefore \quad I_B(y) = y$

$\therefore \quad (fog)(y) = y$ $\qquad$ $(fog = I_B)$

$\therefore \quad f(g(y)) = y$

$g : B \rightarrow A$ is a function. $y \in B$. Hence $g(y) \in A$.

Let $g(y) = x$. So $f(g(y)) = f(x) = y$

$\therefore \quad x \in A$ and $f(x) = y$

$\therefore$ For every $y \in B$, there exists $x \in A$ such that $y = f(x)$.

$\therefore \quad f : A \rightarrow B$ is onto B.

Theorem 1.2 : If $f : A \rightarrow B$ is one-one and onto, it has an inverse $g : B \rightarrow A$.

Proof : Let $f(x) = y \qquad x \in A$

Define $g(y) = x$

Since $f : A \to B$ is onto, for every $y \in B$ there exists $x \in A$ such that $f(x) = y$ and this x is unique as $f : A \to B$ is one-one.

$\therefore$ $g : B \to A$ is a function.

$(gof)(x) = g(f(x)) = g(y) = x$

$(fog)(y) = f(g(y)) = f(x) = y$

$\therefore$ $gof = I_A$ and $fog = I_B$.

$\therefore$ g is the inverse of f.

A result :

If $f : A \to B$ and $g : B \to C$ are one-one and onto, $gof : A \to C$ is one-one and onto and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof : We know $gof : A \to C$ is one-one and onto. (Ex. 31, 32)

$(gof)^{-1} : C \to A$ exists and $(gof)^{-1} : C \to A$ is a function.

$f^{-1} : B \to A$ and $g^{-1} : C \to B$ are functions.

$\therefore$ $f^{-1}og^{-1} : C \to A$ is a function.

$$\begin{aligned}
(gof)\, o\, (f^{-1}og^{-1}) &= go((fof^{-1})og^{-1}) \\
&= go(I_B og^{-1}) \\
&= gog^{-1} \\
&= I_C \\
(f^{-1}og^{-1})\, o\, (gof) &= f^{-1}o((g^{-1}og)\, of) \\
&= f^{-1}o(I_B of) \\
&= f^{-1}of \\
&= I_A
\end{aligned}$$

$\therefore$ $(gof)^{-1} = f^{-1}og^{-1}$

Example 37 : For $f : N \to E$, $f(x) = 2x$, find f^{-1} and verify $fof^{-1} = I_E$, $f^{-1}of = I_N$ where E is the set of even natural numbers.

Solution : $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ $f : N \to E$ is one-one.

if $y \in E$, $y = 2n$ For some n, $n \in N$

$f(n) = 2n = y$

$\therefore$ For every $y \in E$, $\exists n \in N$ such that $f(n) = y$

$\therefore$ $f : N \to E$ is onto.

$y = f(x) = 2x \Rightarrow x = \frac{y}{2} \Rightarrow f^{-1}(y) = \frac{y}{2}$ $\qquad (x = f^{-1}(y))$

$\therefore$ $f^{-1} : E \to N$, $f^{-1}(y) = \frac{y}{2}$ or $f^{-1}(x) = \frac{x}{2}$

Verification is left to the reader.

Example 38 : $f : R \to R$, $f(x) = ax + b$ $\quad a \neq 0$. Find the inverse of $f : R \to R$.

Solution : $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b$

$\Rightarrow ax_1 = ax_2$

$\Rightarrow x_1 = x_2$ $\qquad (a \neq 0)$

$\therefore$ f is one-one.

Let $y \in$ R.

$y = ax + b \Rightarrow x = \dfrac{y-b}{a} \in$ R $\qquad (a \neq 0)$

$\therefore$ For every $y \in$ R, $\exists x \in$ R such that $f(x) = f\left(\left(\dfrac{y-b}{a}\right)\right) = a\left(\dfrac{y-b}{a}\right) + b = y$

$\therefore$ f is onto R.

$\therefore$ f^{-1} : R $\rightarrow$ R, $\quad x = f^{-1}(y) = \dfrac{y-b}{a}$

or we may write f^{-1} : R $\rightarrow$ R, $f^{-1}(x) = \dfrac{x-b}{a}$

Example 39 : If f : $\mathrm{R}^+ \rightarrow \mathrm{R}^+$, $f(x) = x^2$, find f^{-1}.

Solution : $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = x_2 \qquad (x_1, x_2 \in \mathrm{R}^+)$

$\therefore$ f is one-one.

Let $y \in \mathrm{R}^+$

$\therefore$ $\exists x \in \mathrm{R}^+$ such that $x = \sqrt{y}$ so that $f(x) = x^2 = y$.

$\therefore$ For every $y \in \mathrm{R}^+$, $\exists x \in \mathrm{R}^+$ such that $f(x) = y$.

$\therefore$ f is onto R^+.

$\therefore$ f^{-1} : $\mathrm{R}^+ \rightarrow \mathrm{R}^+$, $f^{-1}(y) = \sqrt{y}$

or we may write f^{-1} : $\mathrm{R}^+ \rightarrow \mathrm{R}^+$, $f^{-1}(x) = \sqrt{x}$

Example 40 : f : $\mathrm{R} - \left\{-\frac{3}{2}\right\} \rightarrow \mathrm{R} - \left\{\frac{3}{2}\right\}$, $\quad f(x) = \dfrac{3x+2}{2x+3}$. Find f^{-1}.

Solution : Let $f(x_1) = f(x_2) \qquad x_1, x_2 \in \mathrm{R} - \left\{-\frac{3}{2}\right\}$

$\therefore$ $\dfrac{3x_1+2}{2x_1+3} = \dfrac{3x_2+2}{2x_2+3}$

$\therefore$ $6x_1x_2 + 9x_1 + 4x_2 + 6 = 6x_1x_2 + 9x_2 + 4x_1 + 6$

$\therefore$ $5x_1 = 5x_2$

$\therefore$ $x_1 = x_2$

$\therefore$ f is one-one.

Let $x \in \mathrm{R} - \left\{-\frac{3}{2}\right\}$ and $y = \dfrac{3x+2}{2x+3}$

$\therefore$ $2xy + 3y = 3x + 2$

$\therefore$ $(2y - 3)x = 2 - 3y$

$\therefore$ $x = \dfrac{2-3y}{2y-3} \qquad y \neq \frac{3}{2}$

$\therefore$ For every $y \in \mathrm{R} - \left\{\frac{3}{2}\right\}$, there exists $x \in \mathrm{R} - \left\{-\frac{3}{2}\right\}$ such that $f(x) = y$.

$\therefore$ f is onto.

$\therefore$ f^{-1} : $\mathrm{R} - \left\{\frac{3}{2}\right\} \rightarrow \mathrm{R} - \left\{-\frac{3}{2}\right\}$, $f^{-1}(y) = -\dfrac{3y-2}{2y-3}$ or

f^{-1} : $\mathrm{R} - \left\{\frac{3}{2}\right\} \rightarrow \mathrm{R} - \left\{-\frac{3}{2}\right\}$, $f^{-1}(x) = -\dfrac{3x-2}{2x-3}$.

Example 41 : If $f : A \to B$ is one-one and onto. Prove $(f^{-1})^{-1}$ exists and $(f^{-1})^{-1} = f$.

Solution : By definition of inverse if $f^{-1} : B \to A$ has inverse $h : A \to B$, it must satisfy $hof^{-1} = I_B$ and $foh^{-1} = I_A$. But $f : A \to B$ does satisfy these conditions and inverse is unique, if it exists.

$\therefore$ $(f^{-1})^{-1}$ exist and $(f^{-1})^{-1} = f$.

Example 42 : $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$, $f : A \to B$, $f(x) = x^2$. Find f^{-1} and verify $f^{-1}of = I_A$, $fof^{-1} = I_B$.

Solution : $f = \{(1, 1), (2, 4), (3, 9)\}$

$\therefore$ f is one-one.

$R_f = \{1, 4, 9\} = B$

$\therefore$ f is onto B.

$\therefore$ $f^{-1} : B \to A$, $f^{-1}(x) = \sqrt{x}$. $f^{-1} = \{(1, 1), (4, 2), (9, 3)\}$.

$\therefore$ $fof^{-1} = \{(1, 1), (4, 4), (9, 9)\} = I_B$.

$\therefore$ $f^{-1}of = \{(1, 1), (2, 2), (3, 3)\} = I_A$.

Example 43 : For $f : R \to \{x \mid x \geq 5, x \in R\}$, $f(x) = x^2 + 4x + 9$, find f^{-1} if possible.

Solution :
$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^2 + 4x_1 + 9 = x_2^2 + 4x_2 + 9 \\ &\Rightarrow x_1^2 - x_2^2 + 4(x_1 - x_2) = 0 \\ &\Rightarrow (x_1 - x_2)(x_1 + x_2 + 4) = 0 \\ &\Rightarrow x_1 = x_2 \text{ or } x_1 + x_2 + 4 = 0 \end{aligned}$$

Let $x_1 = 0$, $x_2 = -4$ **(To make $x_1 + x_2 + 4 = 0$)**

Then $f(0) = 9$, $f(-4) = 16 - 16 + 9 = 9$

$\therefore$ f is not one-one.

$\therefore$ f^{-1} does not exist.

Example 44 : If $f : R - \{-1\} \to R - \{-1\}$, $f(x) = \frac{1-x}{1+x}$. Prove that f^{-1} exists and show that $f = f^{-1}$.

Solution :
$$\begin{aligned} (fof)(x) &= f(f(x)) \\ &= f\left(\frac{1-x}{1+x}\right) \\ &= \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} \\ &= \frac{1+x-1+x}{1+x+1-x} \\ &= x \end{aligned}$$

$\therefore$ $fof = I_A$, where $A = R - \{-1\}$

$\therefore$ By uniqueness of inverse and the definition of f^{-1}, f^{-1} exists and $f = f^{-1}$.

Note : Examples mark with * are only for information, not for examination.

***Example 45 :** If f, g, h are functions from A to A and if fog *and* goh are bijective, prove that f, g, h are bijective.

Solution : (1) First of all we prove that f, g, h are one-one.

Let $g(x_1) = g(x_2)$ $\quad x_1, x_2 \in A$

$\therefore$ $f(g(x_1)) = f(g(x_2))$ $\quad g(x_1) \in A,\ g(x_2) \in A$

$\therefore$ $(fog)(x_1) = (fog)(x_2)$

$\therefore$ $x_1 = x_2$ **(fog is one-one)**

$\therefore$ $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

$\therefore$ $g : A \rightarrow A$ is one-one.

Let $h(x_1) = h(x_2)$ $\quad x_1, x_2 \in A$

$\therefore$ $g(h(x_1)) = g(h(x_2))$ $\quad h(x_1) \in A,\ h(x_2) \in A$

$\therefore$ $(goh)(x_1) = (goh)(x_2)$

$\therefore$ $x_1 = x_2$ **(goh is one-one)**

$\therefore$ $h(x_1) = h(x_2) \Rightarrow x_1 = x_2$

$\therefore$ $h : A \rightarrow A$ is one-one.

Let $f(x_1) = f(x_2)$ $\quad x_1, x_2 \in A$

Since goh is onto A, $\exists y_1, y_2 \in A$ such that,

$(goh)(y_1) = x_1$, $(goh)(y_2) = x_2$

$\therefore$ $f((goh)(y_1)) = f((goh)(y_2))$ **($f(x_1) = f(x_2)$)**

$\therefore$ $(fog)(h(y_1)) = (fog)(h(y_2))$

$\therefore$ $h(y_1) = h(y_2)$ **(fog is one-one)**

$\therefore$ $g(h(y_1)) = g(h(y_2))$ $\quad h(y_1), h(y_2) \in A$

$\therefore$ $(goh)(y_1) = (goh)(y_2)$

$\therefore$ $x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ $f : A \rightarrow A$ is one-one.

(2) Now we prove f, g, h are onto A.

Let $y \in A$

Since fog is onto A, $\exists z \in A$ such that

$(fog)(z) = y$

$\therefore$ $f(g(z)) = y$

Let $g(z) = x$. Then $x \in A$. Also $f(x) = y$ and $x \in A$

For every $y \in A$, $\exists x \in A$ such that $f(x) = y$.

$\therefore$ f is onto A.

Similarly, since goh is onto A, $\exists z \in A$ such that

$(goh)(z) = y$

$\therefore$ $g(h(z)) = y$

Let $h(z) = x$. Then $g(x) = y$ where $x \in A$

$\therefore$ g is onto A.

Let $y \in$ A. Now $g(y) \in$ A.

Since goh is onto A, $\exists x \in$ A such that

$(goh)(x) = g(y)$

$\therefore$ $g(h(x)) = g(y)$

But g is one-one.

$\therefore$ $h(x) = y$

$\therefore$ For every $y \in$ A, $\exists x \in$ A such that $h(x) = y$.

$\therefore$ h is onto A.

***Example 46 :** f : A $\rightarrow$ B and g : B $\rightarrow$ C and h : B $\rightarrow$ C are functions.

(1) Prove if f is surjective and $gof = hof$, then $g = h$.

(2) Give an example in which $gof = hof$ but $g \neq h$.

Solution : (1) Let $y \in$ B. f is onto B.

$\therefore$ $\exists x \in$ A such that $f(x) = y$

$\therefore$ $g(f(x)) = g(y)$ **($f(x) \in$ B)**

$\therefore$ $h(f(x)) = g(y)$ **($gof = hof$)**

$\therefore$ $h(y) = g(y)$

Since $y \in$ B is arbitrary and g : B $\rightarrow$ C and h : B $\rightarrow$ C are functions, $g = h$.

(2) f : {1, 2, 3, 4} $\rightarrow$ {5, 6, 7}

f = {(1, 5), (2, 6), (3, 6), (4, 5)}

Let g : {5, 6, 7} $\rightarrow$ {6, 8}, g = {(5, 6), (6, 8), (7, 8)}

Let h : {5, 6, 7} $\rightarrow$ {6, 8}, h = {(5, 6), (6, 8), (7, 6)}

gof = {(1, 6), (2, 8), (3, 8), (4, 6)}

hof = {(1, 6), (2, 8), (3, 8), (4, 6)}

$\therefore$ $gof = hof$. But $g \neq h$

***Example 47 :** If f : A $\rightarrow$ B, g : A $\rightarrow$ B are functions and h : B $\rightarrow$ C is a function.

(1) Prove if $hof = hog$ and h is one-one, then $f = g$.

(2) Give an example where $hof = hog$ but $f \neq g$.

Solution : (1) hof : A $\rightarrow$ C and hog : A $\rightarrow$ C are functions.

$(hof)(x) = (hog)(x)$ for $\forall x \in$ A

$\therefore$ $h(f(x)) = h(g(x))$

$\therefore$ $f(x) = g(x)$ $\forall x \in$ A **(h is one-one)**

$\therefore$ $f = g$

(2) f : {1, 2, 3} $\rightarrow$ {4, 5}, f = {(1, 4), (2, 4), (3, 4)}

g : {1, 2, 3} $\rightarrow$ {4, 5}, g = {(1, 5), (2, 5), (3, 5)}

h : {4, 5} $\rightarrow$ {6, 7}, h = {(4, 6), (5, 6)}

hof = {(1, 6), (2, 6), (3, 6)} = hog, but $f \neq g$.

Exercise 1.4

Find f^{-1} if it exists : (1 to 6)

1. $f : R \to R, \quad f(x) = 2x + 3.$
2. $f : Z \to Z, \quad f(x) = x - 7.$
3. $f : R^+ \to R^+, \quad f(x) = x^3.$
4. $f : \{1, 2, 3, 4,..., n\} \to \{2, 4, 6,..., 2n\}, f(n) = 2n.$
5. $f : Z \to Z \times \{0, 1\}, f(n) = \begin{cases} \left(\frac{n}{2}, 0\right) & n \text{ even.} \\ \left(\frac{n-1}{2}, 1\right) & n \text{ odd.} \end{cases}$
6. $f : Z \to N, f(n) = \begin{cases} 4n & n > 0, \quad n \text{ even} \\ 4|n| + 1 & n \le 0, \quad n \text{ even} \\ 4n + 2 & n > 0, \quad n \text{ odd} \\ 4|n| + 3 & n < 0, \quad n \text{ odd} \end{cases}$

   (**Hint :** f is not onto. $3 \notin R_f$)
7. For $f : A \to B$, $\exists$ a function $g : B \to A$ such that $gof = I_A$. Prove f is one-one.
8. For $f : A \to B$, $\exists$ a function $h : B \to A$ such that $foh = I_B$. Prove f is onto B.
9. **Examine if following functions have an inverse. Find inverse, if it exists :**

   (1) $f : R \to R, \quad f(x) = \lfloor x \rfloor$ **(Floor function)**

   (2) $f : R \to R^+ \cup \{0\} \quad f(x) = |x|$

   (3) $f : R \to [0, 1), \quad f(x) = x - [x]$

   (4) $f : R \to Z, \quad f(x) = \lceil x \rceil$ **(Ceiling function)**

   (5) $f : C \to C, \quad f(z) = \bar{z}$ **(C = set of complex numbers)**

   (6) $f : N \times N \to N, \quad f((m, n)) = m + n$

   (7) $f : N \times N \to N \times N, \quad f((m, n)) = (n, m)$

*

1.5 Binary Operations

We know that addition of two natural numbers is a natural number.

i.e. $a \in N, b \in N \Rightarrow a + b \in N$.

Similarly $a - b \in Z$ if $a, b \in Z$

$a \times b \in Z$ if $a, b \in Z$

Thus there is a non-empty set X and an ordered pair of elements (a, b) of $X \times X$ giving a unique element of X obtained by so called 'addition', 'multiplication' etc. These are called binary operations on X.

Binary Operation : Let $A \neq \emptyset$. A function $* : A \times A \to A$ is called a binary operation. Instead of notation like $f((a, b))$ or $*(a, b)$, we use the notation $a * b$ for the image of this function

for (a, b) and call $*$ a binary opration on A. Thus, corresponding to $(a, b) \in A \times A$, a unique element $a * b$ of A can be obtained by $*$.

Thus $+$ is a binary operation on N, Z, Q, R, C.

$\times$ is a binary operation on N, Z, Q, R, C.

$-$ is a binary operation on Z, Q, R, C as $a - b$ does not necessarily belong to N if $a \in$ N, $b \in$ N.

For example $3 \in$ N, $7 \in$ N, but $3 - 7 = -4 \notin$ N.

Similarly $\div$ is a binary operation on Q $-$ {0}, R $-$ {0}, C $-$ {0}. If $b = 0$, $\frac{a}{b}$ is not defined in Q or in R or in C.

If $a \in$ N, $b \in$ N, then $\frac{a}{b} \notin$ N unless $b \mid a$.

Hence division is not a binary opertion on N.

Commutative law : If $*$ is a binary operation on set A and if $a * b = b * a$, $\forall a, b \in$ A, we say $*$ is a commutative operation.

$+$ is commutative on N.

$-$ is not commutative on Z as $a - b \neq b - a$, $a, b \in$ Z.

Associative law : If $*$ is a binary operation on A and if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in$ A, we say $*$ is an associative binary operation on A.

What is the need of this law ?

See that $(a + b) + c = a + (b + c)$ i.e. $+$ is associative on R. Hence we can write $a + b + c$ without ambiguity for this expression.

$(a - b) - c \neq a - (b - c)$ $\forall a, b, c \in$ R

Hence '$-$' is not associative on R. So we have to specify brackets while using '$-$' for three real numbers.

Identity Element : If $*$ is a binary operation on A and if there exists an element e in A such that $a * e = e * a = a$, $\forall a \in$ A, we say e is an identity element for $*$.

$0 + a = a + 0 = a$, $\forall a \in$ R

$1 \cdot a = a \cdot 1 = a$, $\forall a \in$ R

$\therefore$ 0 is the additive identity and 1 is the multiplicative identity in R.

$a - 0 \neq 0 - a$ for $a \in$ R unless $a = 0$.

$\therefore$ '$-$' has no additive identity.

Inverse of an element : If $*$ is a binary operation on A with an identity element e and if corresponding to $a \in$ A, there exists an element $a' \in$ A such that $a * a' = a' * a = e$ where e is the identity element for $*$, we say a' is an inverse of a and we denote the inverse a' of a by a^{-1}.

$\therefore$ $a * a^{-1} = a^{-1} * a = e$

In R, every non-zero real number a has an inverse $\frac{1}{a}$ for multiplication.

Every element a has an inverse $-a$ for addition in R.

0 has no inverse for multiplication in R.

Operation Table : If A is a finite set and $n(A)$ is 'small', we can prepare a table as follows :

$*$	a_1	a_2	a_3	a_n
a_1				
a_2				
a_3				
.				
.				
.				
a_n				

$a_i * a_j$ is written at the intersection of the ith row and jth column.

If $*$ is commutative, the table is symmetric about the main diagonal.

Example 48 : $*$ is defined on $N \cup \{0\}$ by $a * b = |a - b|$. Is it a binary operation ?

Solution : Yes. If $a \in N \cup \{0\}$, $b \in N \cup \{0\}$, then $a - b \in Z$ and $|a - b| \in N \cup \{0\}$

$\therefore$ $*$ is a binary opeation.

Example 49 : Determine whether following operations $*$ are commutative or not ? associative or not ?

(1) On $N \cup \{0\}$, $a * b = 2^{ab}$

(2) On R^+, $a * b = \frac{a}{b+1}$

Solution : (1) $a * b = 2^{ab} = 2^{ba} = b * a \quad \forall a, b \in N \cup \{0\}$

$\therefore$ $*$ is commutative.

$(2 * 3) * 4 = 2^6 * 4 = 2^{2^6 \cdot 4} = 2^{256}$

$2 * (3 * 4) = 2 * 2^{12} = 2^{2 \cdot 2^{12}} = 2^{2^{13}}$

$\therefore$ $*$ is not associative.

(2) $a * b = \frac{a}{b+1}$, $b * a = \frac{b}{a+1}$

$$\frac{a}{b+1} = \frac{b}{a+1} \Rightarrow a^2 + a = b^2 + b$$
$$\Rightarrow (a - b)(a + b) + (a - b) = 0$$
$$\Rightarrow (a - b)(a + b + 1) = 0$$

$\therefore$ If $a = b$ or $a + b + 1 = 0$, then $a * b = b * a$.

$2 * 3 = \frac{2}{4} = \frac{1}{2} \qquad 3 * 2 = \frac{3}{3} = 1$

$\therefore$ $*$ is not commutative.

$$(2 * 3) * 4 = \frac{2}{4} * 4 = \frac{1}{2} * 4 = \frac{\frac{1}{2}}{4+1} = \frac{1}{10}$$

$$2 * (3 * 4) = 2 * \frac{3}{5} = \frac{2}{\frac{3}{5}+1} = \frac{10}{8} = \frac{5}{4}$$

$\therefore$ $*$ is not associative.

Example 50 : $\wedge : R \times R \rightarrow R$ is defined by $\wedge(a, b) = a \wedge b = min(a, b)$.

Prepare the operation table for $\wedge$ for the subset $\{2, 3, 4, 7, 8\}$.

Solution :

$\wedge$	2	3	4	7	8
2	2	2	2	2	2
3	2	3	3	3	3
4	2	3	4	4	4
7	2	3	4	7	7
8	2	3	4	7	8

Example 51 : Define * on {2, 4, 6, 8} by $a * b = g.c.d.\ (a, b)$.

Prepare the operation table for *. Is * commutative ?

Solution :

g.c.d.	2	4	6	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

Obviously $g.c.d.\ (a, b) = g.c.d.(b, a)$

$\therefore$ * is commutative.

See that the table is symmetric about dotted diagonal.

Example 52 : * is the binary operation on N defined by $a * b = l.c.m.\ (a, b)$

(1) Find 8 * 10, 5 * 3, 12 * 24.

(2) Is * commutative ?

(3) Is * associative ?

(4) Find the identity for *, if it exists.

(5) Find inverse of those elements for which it exists.

Solution : (1) $8 * 10 = l.c.m.\ (8, 10) = 40$

$5 * 3 = l.c.m.\ (5, 3) = 15$

$12 * 24 = l.c.m.\ (12, 24) = 24$

(2) $l.c.m.\ (a, b) = l.c.m.\ (b, a)$

$\therefore$ * is commutative.

(3) * is associative.

(4) $a * e = a, \quad \forall a \in N$ means $l.c.m.\ (a, e) = a, \quad \forall a \in N$

$\therefore \quad e \mid a \quad \forall a \in N$. In special case $e \mid 1$. So, $e = 1$

Also, $l.c.m.\ (a, 1) = a$.

$\therefore$ 1 is the identity for *l.c.m.* operation.

(5) $l.c.m.\ (a, b) \geq a$ and $l.c.m.\ (a, b) \geq b$.

$\therefore$ $l.c.m.\ (a, b) \neq 1$ unless $a = b = 1$. Inverse of 1 only exists and it is 1.

Example 53 : Let $X \neq \emptyset$. Prove that union and intersection are binary operations on P(X). Are they commutative ? Are they associative ? Find the identity and inverse if any for $\cup$ and $\cap$.

Solution : $A \cup B \in P(X)$ and $A \cap B \in P(X)$ if $A, B \in P(X)$.

$\therefore$ $\cup$ and $\cap$ and are binary operations on P(X).

Let $A, B, C \in P(X)$.

$A \cup B = B \cup A \quad A \cap B = B \cap A$

and $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

$\therefore$ $\cup$ and $\cap$ and are associative.

Also $A \cup \emptyset = \emptyset \cup A = A$ for all $A \in P(X)$

$\therefore$ $\emptyset$ is the identity for union.

$A \cap X = X \cap A = A$ for all $A \in P(X)$

$\therefore$ X is the identity for intersection.

$A \cup B = \emptyset \Leftrightarrow A = B = \emptyset$.

$\therefore$ $\emptyset$ is the only element of P(X) having $\emptyset$ as the inverse for union.

$(A \cap B) \subset A$. Hence $A \cap B \neq X$ unless $A = B = X$.

$\therefore$ X is the only element of P(X) having inverse X for intersection.

Example 54 : Define $a * b = a + 2b$ on N. Is * commutative ? Is * associative ? Is there any identity or inverse for any element in N ?

Solution : $2 * 3 = 2 + 6 = 8$

$3 * 2 = 3 + 4 = 7$

$\therefore$ * is not commutative.

$(2 * 3) * 4 = 8 * 4 = 8 + 8 = 16$

$2 * (3 * 4) = 2 * 11 = 2 + 22 = 24$

$\therefore$ * is not associative.

If $a * e = e * a = a$, then $a + 2e = e + 2a = a \quad \forall a \in N$

$\therefore$ $a + 2e = a$

$\therefore$ $e = 0$

But $0 \notin N$.

$\therefore$ * has no identity and therefore there is no question of inverse.

Example 55 : * is defined on Z by $a * b = a + b + 1$. Is * associative ? Find the identity and inverse of any element, if it exists.

Solution : $(a * b) * c = (a + b + 1) * c$

$= a + b + 1 + c + 1 = a + b + c + 2$

$a * (b * c) = a * (b + c + 1) = a + (b + c + 1) + 1 = a + b + c + 2$

$\therefore$ * is associative.

Let $a * e = e * a = a$ for $\forall a \in Z$

$\therefore$ $a + e + 1 = a$

$\therefore$ $e = -1$

Also, $a * (-1) = a + (-1) + 1 = a$. Also $(-1) * a = (-1) + a + 1 = a$.

$\therefore$ -1 is the identity for *.

$a * b = a + b + 1 = -1 \Rightarrow b = -2 - a$

Also $a * (-a - 2) = a + (-a - 2) + 1 = -1$

$\therefore$ $-a - 2$ is the inverse of a.

Example 56 : Prove if $*$ is an associative binary operation having identity e and if a has an inverse, the inverse is unique.

Solution : Suppose a has two inverses a' and a''.

$\therefore$ $a * a' = a' * a = e$

$a * a'' = a'' * a = e$

Now $a' = a' * e = a' * (a * a'')$

$= (a' * a) * a''$

$= e * a''$

$= a''$

$\therefore$ The inverse is unique.

Example 57 : Define $*$ on R by $a * b = a + b - (ab)^2$.

(1) Prove $*$ is commutative but not associative.

(2) Find the identity element for $*$.

(3) Prove that 1 has two inverses for $*$.

(4) Prove if $a \in$ R, a has at most two inverses.

(5) Which elements have no inverse ? Which have only one inverse ? Which have two inverses ? Find the unique inverse if there is any.

Solution : (1) $a * b = a + b - (ab)^2 = b + a - (ba)^2 = b * a$

$\therefore$ $*$ is commutative.

$(2 * 3) * (-2) = (2 + 3 - 36) * (-2) = (-31) * (-2)$

$= -31 - 2 - (62)^2$

$= -33 - 3844$

$= -3877$

$2 * (3 * (-2)) = 2 * (3 - 2 - (-6)^2) = 2 * (-35)$

$= 2 + (-35) - 4900$

$= -4933$

$\therefore$ $*$ is not associative.

(2) $a * e = a + e - (ae)^2 = e + a - (ae)^2 = a \Rightarrow e - a^2e^2 = 0 \quad \forall a \in \text{R} \Rightarrow e = 0$

(Take in particular $a = 0$)

$a * 0 = a + 0 - 0 = a = 0 * a$

$\therefore$ 0 is the identity for $*$.

(3) Let $1^{-1} = a$.

$1 * a = 1 + a - a^2 = 0$

$\therefore$ $a^2 - a - 1 = 0$

$\therefore$ $a = \frac{1 \pm \sqrt{5}}{2}$

$\therefore$ $1^{-1} = \frac{\sqrt{5}+1}{2}$ or $\frac{1-\sqrt{5}}{2}$

$\therefore$ 1 has two inverses.

(4) Let b be inverse of a, $a \in$ R.

$\therefore$ $a * b = 0$

$\therefore$ $a + b - a^2b^2 = 0$ **(0 is identity)**

$\therefore$ $b^2a^2 - b - a = 0$

This is a quadratic equation in b.

$\therefore$ It has at most two real roots as $\Delta = 1 + 4a^3$ and Δ may be positive or negative or zero.

$\therefore$ Every element a can have at most two inverses.

If $4a^3 < -1$ or $a < \left(-\frac{1}{4}\right)^{\frac{1}{3}}$, $\Delta < 0$

$\therefore$ a has no inverse.

If $4a^3 > -1$, a has two inverses.

If $a^3 = \frac{-1}{4}$, a has only one inverse.

$\therefore$ If $a = \sqrt[3]{\frac{-1}{4}}$, a has only one inverse, namely $b = \frac{1 \pm \sqrt{1+4a^3}}{2a^2} = \frac{1}{2a^2}$

$\therefore$ $a * \frac{1}{2a^2} = a + \frac{1}{2a^2} - \left(\frac{1}{2a}\right)^2 = a + \frac{1}{2a^2} - \frac{1}{4a^2} = a + \frac{1}{4a^2} = \frac{4a^3+1}{4a^2} = 0$

$\therefore$ $a = \sqrt[3]{\frac{-1}{4}}$ has only one inverse namely $\frac{1}{2a^2}$.

(**Note :** Here $*$ is not associative. Hence uniqueness of inverse cannot be asserted.)

Miscellaneous Examples :

Example 58 : A relation S is said to be triangular, if xSy and xS$z \Rightarrow y$Sz.

Prove S is an equivalence relation $\Leftrightarrow$ S is reflexive and triangular.

Solution : Suppose S is an equivalence relation.

$\therefore$ S is reflexive.

Let xSy and xSz

$\therefore$ ySx and xSz **(S is symmetric)**

$\therefore$ ySz **(S is transitive)**

$\therefore$ xSy and xS$z \Rightarrow y$Sz

$\therefore$ S is triangular.

Conversely let S be reflexive and triangular.

Let xSy. Also xSx.

$\therefore$ ySx

$\therefore$ xS$y \Rightarrow y$Sx

$\therefore$ S is symmetric.

Let xSy and ySz

$\therefore$ ySx and ySz **(S is symmetric)**

$\therefore$ xSz

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

Example 59 : In R, let xSy if $x - y \in$ Z. Prove that S is an equivalence relation. What are equivalence classes ?

Solution : $x - x \in Z$ as $0 \in Z$

$\therefore$ xSx

$\therefore$ S is reflexive.

If $x - y \in Z$, then $y - x \in Z$

$\therefore$ $xSy \Rightarrow ySx$

$\therefore$ S is symmetric.

If $x - y \in Z$ and $y - z \in Z$, then

$x - y + y - z = x - z \in Z$

$\therefore$ If xSy and ySz, then xSz

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

So now we can denote S by $\sim$.

Now $x \sim y \Leftrightarrow x - y$ is an integer.

Like if $x = 7.82$, $y = 2.82$, then $x - y = 5 \in Z$

$\therefore$ $x \sim y$

$x - [x] = 7.82 - 7 = 0.82$

$y - [y] = 5.82 - 5 = 0.82$ must be same, if $x \sim y$.

$x - [x]$ consists of those real numbers whose decimal expressions after decimal point are identical.

$x - [x] = y - [y]$ or equivalently $x - y = [x] - [y]$.

The equivalence class of x consists of those real numbers y for which $x - y = [x] - [y]$

Example 60 : Prove $f : R - \{-2\} \to R - \{1\}$, $f(x) = \frac{x}{x+2}$ is one-one and onto. Find f^{-1}.

Solution : $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$

$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let $y \in R - \{1\}$, $x \in R - \{-2\}$

Let $y = \frac{x}{x+2}$

$\therefore$ $xy + 2y = x$

$x(y - 1) = -2y$

$x = \frac{-2y}{y-1} = \frac{2y}{1-y}$ $\quad (y \in R - \{1\})$

$\therefore$ For every $y \in R - \{1\}$, $\exists x \in R - \{-2\}$ such that $y = f(x)$

$\therefore$ $R_f = R - \{1\}$

$\therefore$ f is onto $R - \{1\}$.

$\therefore$ $f^{-1} : R - \{1\} \to R - \{-2\}$, $f^{-1}(x) = \frac{2x}{1-x}$

Example 61 : $*$ is defined on R by $a * b = a + b - ab$. Is there an identity for $*$? What is inverse of $a \in R$, if it exists ?

Solution : $a * e = e * a = a, \forall a \in R \Rightarrow a + e - ae = a \quad \forall a \in R$

$\Rightarrow e - ae = 0 \quad \forall a \in R$

$\Rightarrow e = 0$ **(Take $a = 0$ in particular)**

Also $a * 0 = 0 * a = a + 0 - 0 = a$

$\therefore$ 0 is the identity for $*$.

Now $a * b = a + b - ab = 0 \Rightarrow (1 - a)b = -a$

$$\Rightarrow b = \frac{a}{a-1}, \text{ if } a \neq 1$$

If $a \neq 1$, a^{-1} exists and $a^{-1} = \frac{a}{a-1}$

Example 62 : Define relation S on $Z - \{0\} \times Z - \{0\}$ by $(a, b)S(c, d) \Leftrightarrow ad = bc$. Prove that it is an equivalence relation. What about equivalence classes ?

Solution : $(a, b)S(a, b)$ as $ab = ba$

$\therefore$ S is reflexive.

If $(a, b)S(c, d)$, then $ad = bc$

$\therefore$ $cb = da$

$\therefore$ $(c, d)S(a, b)$

$\therefore$ S is symmetric.

Let $(a, b)S(c, d)$ and $(c, d)S(e, f)$

$\therefore$ $ad = bc$ and $cf = de$

$\therefore$ $ade = bce$ and $acf = ade$

$\therefore$ $acf = bce$

$\therefore$ $af = be$, since $c \neq 0$

$\therefore$ $(a, b)S(e, f)$

$\therefore$ S is transitive

$\therefore$ S is an equivalence relation.

In fact $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

$\therefore$ $\frac{2}{4} \sim \frac{3}{6} \sim \frac{1}{2} \sim \frac{5}{10} \ldots$

$\therefore$ The equivalance class of fractions (a, b) consists of non-zero rational number $\frac{a}{b}$.

Example 63 : Let $*$ be defined by $a * b = \frac{ab}{10}$ for $a, b \in Q^+$

Find the identity element. Find 4^{-1} and $(4 * 5)^{-1}$.

Solution : $a * b = a \Rightarrow \frac{ab}{10} = a \Rightarrow b = 10$ **(as $a \neq 0$)**

Also $a * 10 = 10 * a = \frac{a \cdot 10}{10} = a$

$\therefore$ 10 is the identity for $*$.

Let $4 * a = 10$

$\therefore$ $\frac{4a}{10} = 10$

$\therefore \quad a = 25$

$\therefore \quad 4^{-1} = 25$ $\qquad \left(4 * 25 = \frac{4 \cdot 25}{10} = 10\right)$

$\therefore \quad 4 * 5 = \frac{4 \cdot 5}{10} = 2$

Now $2 * a = 10 \Rightarrow \frac{2a}{10} = 10$

$\Rightarrow a = 50$

$\therefore \quad (4 * 5)^{-1} = 2^{-1} = 50$

Exercise 1

1. Prove that there is only one relation in $\{1, 2, 3\}$ which is reflexive and symmetric but not transitive and which contains $(1, 2)$ and $(1, 3)$.
2. Prove that the number of equivalence relations in $\{1, 2, 3\}$ containing $(1, 2)$ is two.
3. S is defined on R by, $(a, b) \in S \Leftrightarrow 1 + ab > 0 \quad \forall a, b \in R$
   Prove S is reflexive and symmetric but not transitive.
   (**Hint :** Take $a = \frac{1}{3}$, $b = \frac{-1}{2}$, $c = -8$. $(a, b) \in S$, $(b, c) \in S$ and $(a, c) \notin S$)
4. $A = \{1, 2, 3,..., 14, 15\}$, $S = \{(x, y) \mid y = 5x, x, y \in A\}$
   Determine whether S is reflexive, symmetric or transitive.
5. The relation S is defined on R as follows :
   $S = \{(a, b) \mid a \leq b^2, a, b \in R\}$
   Prove S is not reflexive, not symmetric and not transitive.
6. Let $S \subset (R \times R)$. $S = \{(A, B) \mid d(A, B) < 2\}$. Prove S is not transitive.
7. S is defined on $N \times N$ by
   $(a, b)\ S\ (c, d) \Leftrightarrow ad(b + c) = bc(a + d)$. Prove that S is an equivalence relation.
8. **Determine whether following functions are injective or not ? surjective or not ?**
   (1) $f : R \to R, \quad f(x) = \begin{cases} 2x + 1 & x \geq 0 \\ x^2 & x < 0 \end{cases}$
   (2) $f : R \to R, \quad f(x) = \begin{cases} -x + 1 & x \geq 0 \\ x^2 & x < 0 \end{cases}$
   (3) $f : Z \to Z, \quad f(n) = \begin{cases} n - 1 & n \text{ odd} \\ n & n \text{ even} \end{cases}$
   (4) $f : Z \to Z, \quad f(n) = \begin{cases} n & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$
   (5) $f : R \times (R - \{0\}) \to R, f((x, y)) = \frac{x}{y}$
   (6) $f : Z \to Z, f(n) = \begin{cases} n & n \text{ even} \\ 2n + 3 & n \text{ odd} \end{cases}$ (**Hint :** Is $3 \in R_f$?)
   (7) $f : [-1, 1] \to [-1, 1], f(x) = x\,|x|$
   (8) $f : N \to N \cup \{0\}, f(n) = n + (-1)^n$

(9) $f : N - \{1\} \to N, f(n)$ = largest prime divisor of n.

(10) $f : R - \{3\} \to R - \{1\}, f(x) = \dfrac{x-2}{x-3}$

(11) $f : R \to R, f(x) = x - [x]$

9. $f : [0, 1] \to [0, 1], f(x) = \begin{cases} x & x \in Q \\ 1 - x & x \notin Q \end{cases}$

Prove $(fof)(x) = x$.

10. $f : Z \to Z, f(n) = 5n$ and

$g : Z \to Z, g(n) = \begin{cases} \frac{n}{5} & \text{if } 5 \mid n \\ 0 & \text{otherwise.} \end{cases}$ Find gof and fog.

11. $f : R \to R, f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

and $g : R \to R, g(x) = [x]$. Prove $(fog)(x) = (gof)(x) \quad \forall x \in [-1, 0)$

12. If $f : A \to B$ and $g : B \to A$ are two functions such that $gof = I_A$, then prove that f is one-one and g is onto A.

13. **Prove for functions $f : A \to B$ and $g : B \to C$**

(1) If $gof : A \to C$ is onto C, $g : B \to C$ is onto C.

(2) If $gof : A \to C$ is one-one, $f : A \to B$ is one-one.

(3) If $gof : A \to C$ is onto and $g : B \to C$ is one-one, $f : A \to B$ is onto.

(4) If $gof : A \to C$ is one-one and $f : A \to B$ is onto B, $g : B \to C$ is one-one.

14. $f : R^+ \cup \{0\} \to R^+ \cup \{0\}, f(x) = \sqrt{x}$, $g : R \to R$, $g(x) = x^2 - 1$. Find fog or gof whichever exists.

15. If $f : N \cup \{0\} \to N \cup \{0\}, f(n) = \begin{cases} n + 1 & n \text{ even} \\ n - 1 & n \text{ odd.} \end{cases}$ Prove $f = f^{-1}$.

16. $f : R \to (-1, 1), f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$. Find f^{-1}, if it exists.

17. $f : R - \left\{\frac{2}{3}\right\} \to R, f(x) = \dfrac{4x+3}{6x-4}$. Prove $(fof)(x) = x$. What can you say about f^{-1} ?

18. $*$ is defined on R by $a * b = a + b + ab$. Is $*$ commutative ? Is it associative ?

Answer the same question if $a * b = a - b + ab$.

19. **Examine whether following binary operations are commutative or not and associative or not :**

(1) $a * b = a^b$ on N

(2) $a * b = g.c.d.\ (a, b)$ on N

(3) $a * b = a - b$ on Q

(4) $a * b = a^2 b$ on Q

(5) $a * b = a + b - 5$ on R

(6) $a * b = \frac{a}{b+1}$ on R − {−1}

(7) $a * b = \frac{a+b}{2}$ on Q

(8) $a * b = \frac{a-b}{2}$ on Q

(9) $a * b = a + b - 2$ on Z

(10) $a * b = a + 2b - 3$ on Z.

20. Find the identity element for following binary operations and inverse of any element in case it exists (provided identity exists) :

(1) $a * b = a + b + ab$ on Q − {−1}

(2) $a * b = \frac{ab}{2}$ on Q − {0}

(3) $a * b = a + b - 2$ on Z

(4) $a * b = a + b - ab$ on R − {1}

(5) $a * b = \sqrt{|a^2 - b^2|}$ on R

(6) $a * b = 3a + 4b - 2$ on R

(7) $a * b = a + 3b^2$ on Z

(8) $a * b = g.c.d.\ (a, b)$ on N.

(9) A * B = A ∩ B on P(X) for a non-empty set X.

(10) A * B = A ∪ B on P(X) for a non-empty set X.

Section A (1 mark)

1. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The relation S = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)} on {1, 2, 3, 4, 5} is ..... ☐

(a) symmetric only (b) reflexive only

(c) transitive only (d) an equivalence relation

(2) If A = {1, 2, 3}, then the number of equivalence relation containing (1, 3) is... ☐

(a) 1 (b) 2 (c) 3 (d) 8

(3) S is defined in Z by $(x, y) \in S \Leftrightarrow |x - y| \leq 1$. S is... ☐

(a) reflexive and transitive but not symmetric.

(b) reflexive and symmetric but not transitive.

(c) symmetric and transitive but not reflexive.

(d) an equivalence relation

(4) If S is defined on R − {0} by $(x, y) \in S \Leftrightarrow xy \geq 0$. Then S is... ☐

(a) an equivalence relation (b) reflexive only

(c) symmetric only (d) transitive only

(5) Which of the following defined on Z is not an equivalence relation... ☐

(a) $(x, y) \in S \Leftrightarrow x \geq y$ (b) $(x, y) \in S \Leftrightarrow x = y$

(c) $(x, y) \in S \Leftrightarrow x - y$ is a multiple of 3 (d) $(x, y) \in S$ if $|x - y|$ is even

(6) If $a * b = a^2 + b^2$ on Z, then $(2 * 3) * 4 =$ .....

(a) 13 (b) 16 (c) 185 (d) 13

(7) If $a * b = a^2 + b^2 + ab + 2$ on Z, then $3 * 4 =$ .....

(a) 40 (b) 39 (c) 25 (d) 41

(8) If $a * b = \frac{ab}{2}$ on Q^+, then the identity for $*$ is .....

(a) 2 (b) 3 (c) 0 (d) 1

(9) If $a * b = \frac{ab}{3}$ on Q^+, then the inverse of a $(a \neq 0)$ for $*$ is .....

(a) $\frac{3}{a}$ (b) $\frac{9}{a}$ (c) $\frac{1}{a}$ (d) $\frac{2}{a}$

(10) The number of binary operations on $\{1, 2\}$ is .....

(a) 16 (b) 8 (c) 2 (d) 4

(11) The number of binary operations on $\{1, 2, 3,..., n\}$ is .....

(a) 2^n (b) n^{n^2} (c) n^3 (d) n^{2n}

(12) If $a * b = a + b + ab$ on $R - \{-1\}$, then a^{-1} is .....

(a) a^3 (b) $\frac{1}{a}$ (c) $\frac{-a}{a+1}$ (d) $\frac{1}{a^2}$

(13) For $a * b = a + b + 10$ on Z, the identity is .....

(a) 0 (b) -5 (c) -10 (d) 1

(14) The number of commutative binary operations on $\{1, 2\}$ is .....

(a) 8 (b) 4 (c) 16 (d) 27

(15) If $a * b = \frac{ab}{100}$ on Q^+, inverse of 0.1 is .....

(a) 100000 (b) 10000 (c) 1000 (d) 10

Section B (2 marks)

(16) $A = [-1, 1]$, $B = [0, 1]$, $C = [-1, 0]$

$S_1 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in A\}$

$S_2 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in B\}$

$S_3 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in C\}$

$S_4 = \{(x, y) \mid x^2 + y^2 = 1, x \in B, y \in C\}$, then

(a) S_1 is not a graph of a function. (b) S_2 is not a graph of a function.

(c) S_3 is not a graph of a function. (d) S_4 is not a graph of a function.

(17) $f : R \to R$, $f(x) = 3^x + 3^{|x|} =$ .....

(a) one-one and onto (b) one-one but not onto

(c) many-one and onto (d) many-one and not onto

(18) $f : R - \{q\} \to R - \{1\}$, $f(x) = \frac{x-p}{x-q}$, $p \neq q$, then f is .....

(a) one-one and onto (b) many-one and not onto

(c) one-one and not onto (d) many-one and onto

(19) $f : [-1, 1] \to [-1, 1], f(x) = -x\,|x|$ is .....

(a) one-one and onto (b) many-one and onto

(c) many-one and not onto (d) one-one and not onto

(20) If $f : R \to R, f(x) = 2x - 3$, then...

(a) $f^{-1}(x) = \frac{1}{2x-3}$ (b) $f^{-1}(x) = \frac{x+3}{2}$

(c) f^{-1} does not exist (d) $f^{-1}(x) = 3x - 2$

(21) $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is a bijection if...

(a) $f(x) = |x|$ (b) $f(x) = sinx$ (c) $f(x) = x^2$ (d) $f(x) = cosx$

(22) $f : R \to R, f(x) = x^2 + 2x + 3$ is...

(a) a bijection (b) one-one but not onto

(c) onto but not one-one (d) many-one and not onto

(23) If $a * b = ab + 1$ on R, is...

(a) commutative, but not associative (b) associative, but not commutative

(c) neither commutative nor associative (d) both commutative and associative

(24) If $a * b = a^2 + b^2$ on Z, then $*$ is...

(a) commutative and associative (b) commutative and not associative

(c) not commutative and associative (d) neither commutative nor associative

(25) If $a * b = a + b - ab$ on $Q - \{1\}$, then the identity and the inverse of a for $*$ are respectively...

(a) 0 and $\frac{a}{a-1}$ (b) 1 and $\frac{a-1}{a}$ (c) -1 and a (d) 0, $\frac{1}{a}$

(26) If $a * b = \frac{ab}{3}$ on Q^+, then $3 * \left(\frac{1}{5} * \frac{1}{2}\right)$ is...

(a) $\frac{5}{160}$ (b) $\frac{1}{30}$ (c) $\frac{3}{160}$ (d) $\frac{3}{60}$

(27) If Δ is defined on P(X) $(X \neq \emptyset)$ by, $A \Delta B = (A \cup B) - (A \cap B)$, then...

(a) identity for Δ is $\emptyset$ and inverse of A is A

(b) identity for Δ is A and inverse of A is $\emptyset$

(c) identity for Δ is A' and inverse of A is A

(d) identity for Δ is X and inverse of A is $\emptyset$

Section C (3 marks)

(28) S is defined on $N \times N$ by $((a, b), (c, d)) \in S \Leftrightarrow a + d = b + c$...

(a) S is reflexive, but not symmetric (b) S is reflexive and transitive only

(c) S is an equivalence relation (d) S is transitive only

(29) Let S be the relation on the set A = {5, 6, 7, 8},
S = {(5, 6), (6, 6), (5, 5), (8, 8), (5, 7), (7, 7), (7, 6)}, then...

(a) S is reflexive and symmetric but not transtive
(b) S is reflexive and transitive but not symmetric
(c) S is symmetric and transitive but not reflexive
(d) S is an equivalence relation.

(30) If $f : R^+ \to R$, $f(x) = \frac{x}{x+1}$ is .....

(a) one-one and onto
(b) one-one and not onto
(c) not one-one and not onto
(d) onto but not one-one

(31) If $f : R \to R$, $f(x) = [x]$, $g : R \to R$, $g(x) = sinx$, $h : R \to R$, $g(x) = 2x$, then $ho(gof)$ = ...

(a) $sin[x]$
(b) $[sin2x]$
(c) $2(sin[x])$
(d) $sin2[x]$

(32) If $f : R \to (-1, 1)$, $f(x) = \frac{-x|x|}{1+x^2}$, then f^{-1} = ...

(a) $\frac{1}{x^2+1}$
(b) $-\text{signum } x \sqrt{\frac{|x|}{1-|x|}}$
(c) $-\frac{\sqrt{x}}{1-x}$
(d) $\frac{x^2}{x^2+1}$

(33) $f : R \to R$, $f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$

$g : R \to R$, $g(x) = 1 + x - [x]$, then for all x, $f(g(x))$ = ...

(a) 1
(b) 2
(c) 0
(d) −1

Section D (4 marks)

(34) If $f : \{x \mid x \geq 1, x \in R\} \to \{x \mid x \geq 2, x \in R\}$, $f(x) = x + \frac{1}{x}$, $f^{-1}(x)$ = ...

(a) $\frac{x+\sqrt{x^2-4}}{2}$
(b) $\frac{x-\sqrt{x^2-4}}{2}$
(c) $\frac{x^2+1}{x}$
(d) $\sqrt{x^2-4}$

(35) If $f : R \to R$, $f(x) = x - [x]$, then $f^{-1}(x)$ = ...

(a) does not exist
(b) is x
(c) is $[x]$
(d) $x - [x]$

(36) If $f(x) = \frac{x}{\sqrt{1+x^2}}$, then $(fo(fof))(x)$ = ...

(a) $\frac{x}{1+x^2}$
(b) $\frac{1+x^2}{x}$
(c) $\frac{x}{\sqrt{1+2x^2}}$
(d) $\frac{x}{\sqrt{1+3x^2}}$

(37) $f : R \to R$, $f(x) = x^2$, $g : R \to R$, $g(x) = 2^x$, then $\{x \mid (fog)(x) = (gof)(x)\}$ = ...

(a) {0}
(b) {0, 1}
(c) R
(d) {0, 2}

(38) $f : R \rightarrow Z, f(x) = [x]$ is .....

(a) one-one and onto and has an inverse (b) many-one and not onto, no inverse

(c) many-one and onto, no inverse (d) one-one and not onto, no inverse

(39) A = {0, 1, 2, 3, 4, 5, 6}. If $a, b \in$ A, $a * b$ = remainder when ab is divided by 7. From binary operation table of *, inverse of 2 is .....

(a) 1 (b) 5 (c) 6 (d) 4

*

Summary

We have studied the following points in this chapter :

1. Relation and equivalence relation.
2. One-one and onto functions
3. Composition of functions
4. Inverse of a function
5. Binary Operations on a set

Srinivasa Ramanujan

Born in Erode, Madras Presidency, to a poor Brahmin family, Ramanujan first encountered formal mathematics at age 10. He demonstrated a natural ability, and was given books on advanced trigonometry written by S. L. Loney. He mastered them by age 12, and even discovered theorems of his own, including independently re-discovering Euler's identity. He demonstrated unusual mathematical skills at school, winning accolades and awards. By 17, Ramanujan conducted his own mathematical research on Bernoulli numbers and the Euler–Mascheroni constant. He received a scholarship to study at Government College in Kumbakonam, but lost it when he failed his non-mathematical coursework. He joined another college to pursue independent mathematical research, working as a clerk in the Accountant-General's office at the Madras Port Trust Office to support himself. In 1912–1913, he sent samples of his theorems to three academics at the University of Cambridge. Only Hardy recognised the brilliance of his work, subsequently inviting Ramanujan to visit and work with him at Cambridge. He became a Fellow of the Royal Society and a Fellow of Trinity College, Cambridge, dying of illness, malnutrition and possibly liver infection in 1920 at the age of 32.

During his short lifetime, Ramanujan independently compiled nearly 3900 results (mostly identities and equations). Although a small number of these results were actually false and some were already known, most of his claims have now been proven correct. He stated results that were both original and highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research. However, the mathematical mainstream has been rather slow in absorbing some of his major discoveries. The Ramanujan Journal, an international publication, was launched to publish work in all areas of mathematics influenced by his work.

INVERSE TRIGONOMETRIC FUNCTIONS 2

No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it untill it also gives the impression of being beautiful.

– George Boole

Mathematics consists of proving the most obvious things in the least obvious way.

– George Polya

2.1 Introduction

We have studied that a function has an inverse if and only if it is one-one and onto. There are many functions which are not one-one or not onto or both and hence they cannot have an inverse function. In class XI, we have studied that all trigonometric functions are periodic and hence they are all many-one functions. Therefore, they cannot have an inverse. In order to have inverse of these functions, we must restrict their domain and codomain in such a way that they become one-one and onto. With this modified domain and codomain, it can have an inverse.

We know that if $f = \{(x, y) \mid y = f(x), x \in A, y \in B\}$ is one-one and onto, then f^{-1} exists and $f^{-1} = \{(y, x) \mid y = f(x), x \in A, y \in B\}$

Also $fof^{-1} = I_B$ and $f^{-1}of = I_A$

$\therefore \quad x \in A \Rightarrow (f^{-1}of)(x) = x, \; y \in B \Rightarrow (fof^{-1})(y) = y$

In this chapter, we shall discuss the existence of the inverse of trigonometric functions and discuss their properties.

2.2 Inverse of *sine* Function

We know that $sin : R \to R$ is many-one and range of *sine* is $[-1, 1]$. So, it is not onto R. $sine = \{(x, y) \mid y = sinx, x \in R, y \in [-1, 1]\}$ is a many-one function on R and is onto $[-1, 1]$. It is many-one and periodic with period 2π. We can see from its graph that, if the domain of *sine* is taken as, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ or $\left[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$, $k \in Z$, it becomes one-one and remains onto $[-1, 1]$.

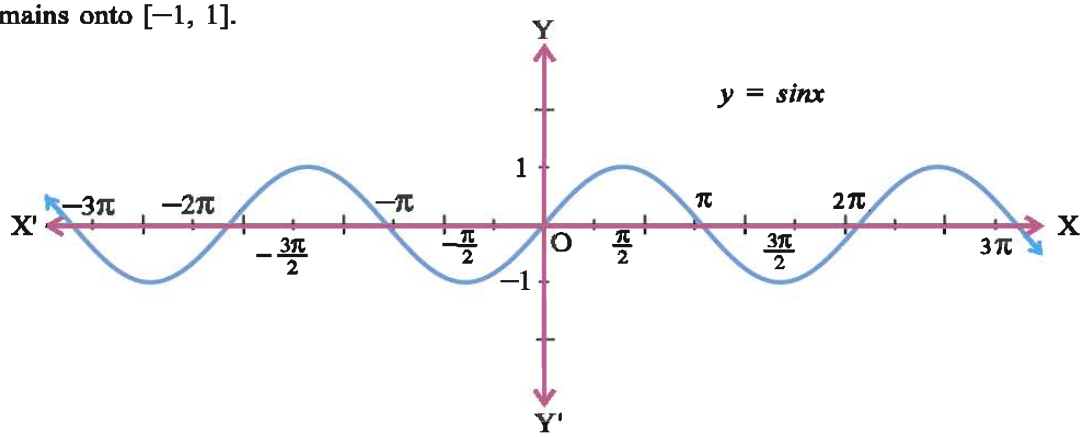

Figure 2.1

So, to define the inverse of *sine* function, we can take any of these intervals as the domain of *sine*. We shall take the domain of *sine* function as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to define the inverse of *sine*. So we consider the function $sin = \left\{(x, y) \mid y = sinx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \in [-1, 1]\right\}$. **This is a one-one and onto function. Therefore, it will have an inverse function. The inverse of *sine* function is denoted by sin^{-1}.**

$\therefore \quad sin^{-1} = \left\{(y, x) \mid y = sinx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \in [-1, 1]\right\}$.

Thus, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$

$\therefore \quad y = sinx \Leftrightarrow sin^{-1}y = x$

The domain of sin^{-1} is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Remember that if $y \in [-1, 1]$, $sin^{-1}y$ is not just any real x for which $sinx = y$ but only that $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $sinx = y$. For instance, given $y = \frac{\sqrt{3}}{2}$, we know that $sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. Although $sin\left(\pi - \frac{\pi}{3}\right) = sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, we can not write $sin^{-1}\frac{\sqrt{3}}{2} = \frac{2\pi}{3}$ because $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Also $\forall\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $sin^{-1}(sin\theta) = \theta$.

$sin(sin^{-1}x) = x$. $\forall x \in [-1, 1]$.

For instance, $sin\left(sin^{-1}\frac{5}{7}\right) = \frac{5}{7}$, because $\frac{5}{7} \in [-1, 1]$. $sin^{-1}\left(sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$ because $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin^{-1}\left(sin\left(\frac{3\pi}{5}\right)\right) \neq \frac{3\pi}{5}$ because $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

If the inverse of $f : A \rightarrow B$ is $f^{-1} : B \rightarrow A$, then we know that,

$fof^{-1} = I_B$ and $f^{-1}of = I_A$

Thus, $sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ has inverse, $sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad sin^{-1}(sinx) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin(sin^{-1}x) = x \quad \forall x \in [-1, 1]$.

We note that,

(1) $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow |x| \leq \frac{\pi}{2}$ and

$y \in [-1, 1] \Leftrightarrow -1 \leq y \leq 1 \Leftrightarrow |y| \leq 1$.

(2) $sin^{-1}x \neq \frac{1}{sinx}$, that is $sin^{-1}x \neq (sinx)^{-1}$

2.3 The Graph of $y = sin^{-1}x$

The domain and the range of sin^{-1} are $[-1, 1]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ respectively. Its graph will be confined between the two vertical lines $x = -1$ and $x = 1$ and the two horizontal lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

We can use our knowledge of the graph of $y = sinx$ to get the graph of $y = sin^{-1}x$. To obtain it, let us first examine how to find the graph of f^{-1} from the graph of f, when inverse of f exists. The graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ are very interestingly related. If point (a, b) is on the graph of $y = f(x)$, then $b = f(a)$ and so $a = f^{-1}(b)$. Therefore, the point (b, a) is on the graph of $y = f^{-1}(x)$. The converse is also true. Hence, A(a, b) is on the graph of $y = f(x)$ if and only if B(b, a) is on the graph of $y = f^{-1}(x)$.

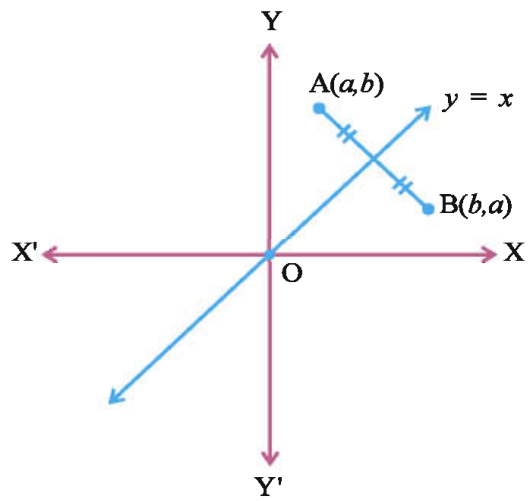

Figure 2.2

We can see that the line $y = x$ is the perpendicular bisector of the line-segment joining A(a, b) and B(b, a). Slope of the segment joining A(a, b) and B(b, a) is $\frac{b-a}{a-b} = -1$. The slope of $y = x$ is 1. Hence $\overleftrightarrow{AB}$ is perpendicular to the line $y = x$. Also the mid-point $\overline{AB}$ is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ and obviously it lies on the line $y = x$.

- The line $y = x$ is perpendicular bisector of $\overline{AB}$. Thus, B(b, a) is the mirror image of A(a, b) in the line $y = x$. Thus, the graph of $y = f^{-1}(x)$ is just the image of the graph of $y = f(x)$ in the line $y = x$.

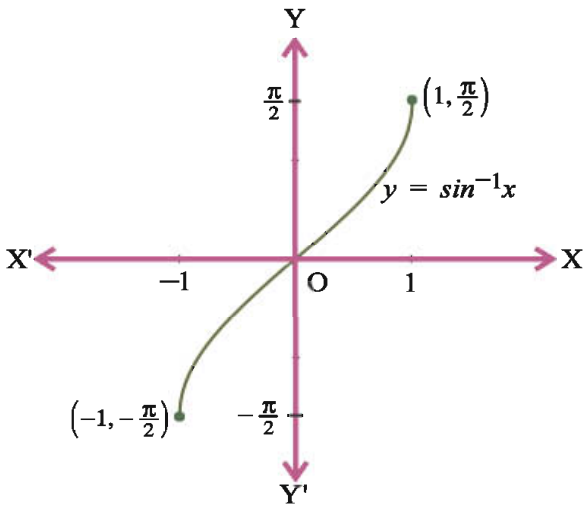

Figure 2.3

Thus, the graph of $y = sin^{-1}x$ is obtained by simply reflecting the graph of sin through the line $y = x$. First draw the graph of $y = sinx$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \in [-1, 1]$ on a piece of paper. Now fold this paper on the line $y = x$. Now turn the paper upside down, interchange the X-axis and Y-axis and look at the graph. What you see is the graph of $y = sin^{-1}x$.

Note : The student herself should perform this activity in the class-room.

For the graph of $y = sinx$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \in [-1, 1]$ and for the graph of $y = sin^{-1}x$, $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 1 : Obtain the value of : (1) $sin^{-1}\left(\frac{1}{2}\right)$, (2) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, (3) $sin^{-1}\left(-\frac{1}{2}\right)$.

Solution : (1) $sin^{-1}\left(\frac{1}{2}\right) = sin^{-1}\left(sin\frac{\pi}{6}\right) = \frac{\pi}{6}$, because $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(2) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = sin^{-1}\left(sin\frac{\pi}{4}\right) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(3) $sin^{-1}\left(-\frac{1}{2}\right). = sin^{-1}\left(-sin\frac{\pi}{6}\right) = sin^{-1}\left(sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$, because $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2.4 Inverse of *cosine* Function

We know that $cos : R \rightarrow R$ is many-one and range of *cosine* is $[-1, 1]$. So, it is not onto. $cos = \{(x, y) \mid y = cosx, x \in R, y \in [-1, 1]\}$ is a many-one function onto $[-1, 1]$ with

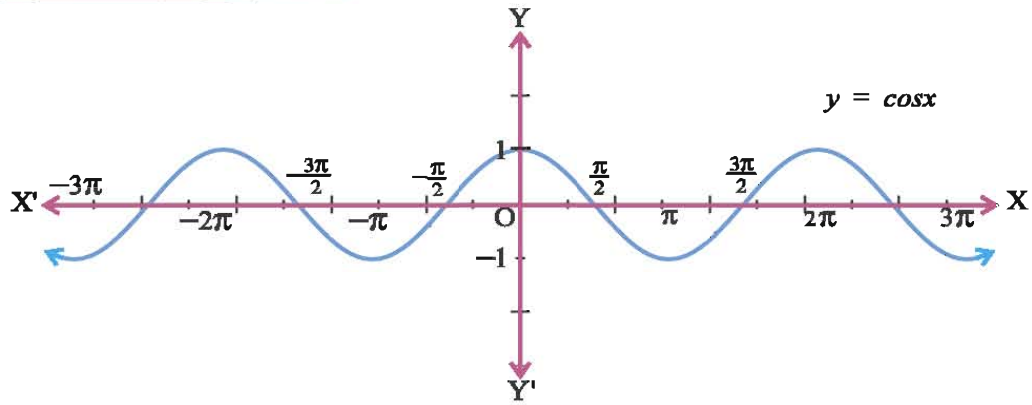

Figure 2.4

period 2π. We see from its graph that it becomes one-one and onto if the domain is restricted to $[0, \pi]$ or $[\pi, 2\pi]$ or $[2\pi, 3\pi]$ or... $[k\pi, (k + 1)\pi]$, $k \in Z$

We shall take the domain of *cosine* function as $[0, \pi]$ to define the inverse of *cosine*. So consider the function **$cos = \{(x, y) \mid y = cosx, x \in [0, \pi], y \in [-1, 1]\}$. This is a one-one and onto function. So, its inverse exists. We denote its inverse by cos^{-1}.**

$\therefore \quad cos^{-1} = \{(y, x) \mid y = cosx, x \in [0, \pi], y \in [-1, 1]\}$. Thus, for $x \in [0, \pi]$ and $y \in [-1, 1]$,

$y = cosx \Leftrightarrow cos^{-1}y = x$.

The domain of cos^{-1} is $[-1, 1]$ and its range is $[0, \pi]$.

Like *sine* function, here also we have to remember that if $y \in [-1, 1]$. $cos^{-1}y$ is not just any real x for which $cosx = y$ but only that $x \in [0, \pi]$, for which $cosx = y$. For instance $cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{6} \in [0, \pi]$. Hence, $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. But, $cos\left(-\frac{\pi}{6}\right) = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. But, $-\frac{\pi}{6} \notin [0, \pi]$.

$\therefore \quad cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \neq -\frac{\pi}{6}$.

$cos : [0, \pi] \rightarrow [-1, 1]$ has the inverse $cos^{-1} : [-1, 1] \rightarrow [0, \pi]$.

So, $cos^{-1}(cosx) = x, \forall x \in [0, \pi]$ and $cos(cos^{-1}x) = x, \forall x \in [-1, 1]$.

Note that $sin^{-1}(sinx)$ and $cos^{-1}(cosx)$ exist, $\forall x \in$ R, but they may not be equal to x. However, each will be equal to x in its appropriate domains. [The above experiment can be done with some appropriate change.]

2.5 The Graph of $y = cos^{-1}x$

We have discussed the method of drawing the graph of the inverse function from the graph of the function. As in the case of the graph of sin^{-1} the graph of cos^{-1} as obtained from the graph of $y = cosx$, $x \in [0, \pi]$ is shown in the figure 2.5.

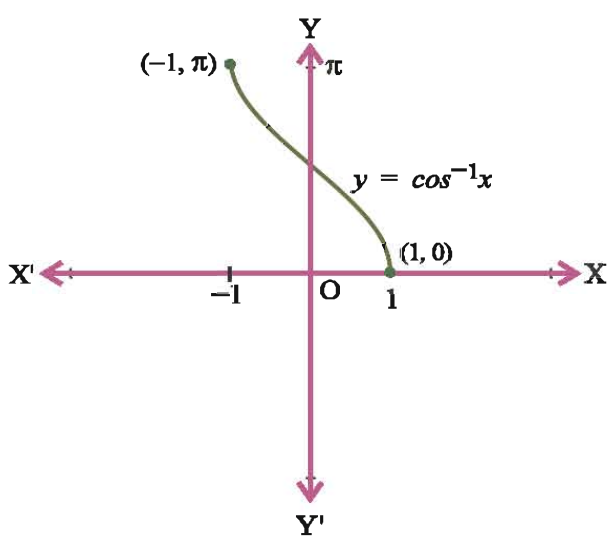

Figure 2.5

Example 2 : Obtain the value of : (1) $cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution : (1) $cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = cos^{-1}\left(cos\frac{\pi}{4}\right) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in [0, \pi]$.

(2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = cos^{-1}\left(cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$, because $\frac{5\pi}{6} \in [0, \pi]$.

2.6 Inverse of *tan* Function

We know that $tan : R - \left\{(2k + 1)\frac{\pi}{2} \mid k \in Z\right\} \to R$ is many-one and range of *tan* is R. So it is onto.

$tan = \left\{(x, y) \mid y = tanx, x \in R - \left\{(2k + 1)\frac{\pi}{2} \mid k \in Z\right\}, y \in R\right\}$ is many-one function with period π and it is onto R. If its domain is restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ or $\left((2k - 1)\frac{\pi}{2}, (2k + 1)\frac{\pi}{2}\right)$, $k \in Z$; it becomes one-one and remains onto R. So we can get its inverse by taking one of these intervals as its domain. We shall take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as the domain and get the inverse which is denoted by tan^{-1}. So, $tan^{-1} = \left\{(y, x) \mid y = tanx, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), y \in R\right\}$.

Thus, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y \in R$,

$$y = tanx \Leftrightarrow tan^{-1}y = x.$$

Domain of tan^{-1} is R and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Note : $tan^{-1}x \neq (tanx)^{-1}$ i.e. $tan^{-1}x \neq \frac{1}{tanx}$. $tan^{-1}x \neq \frac{sin^{-1}x}{cos^{-1}x}$.

$tan^{-1}(tanx) = x, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan(tan^{-1}x) = x, \forall x \in R$.

$tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So, $tan^{-1}(-1) = -\frac{\pi}{4}$

But $tan\left(\frac{3\pi}{4}\right) = -1$ does not imply $tan^{-1}(-1) = \frac{3\pi}{4}$ as $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$tan^{-1}\left(tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$, because $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\left(tan^{-1}\left(\frac{533}{413}\right)\right) = \frac{533}{413}$.

But $tan^{-1}\left(tan\frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$ because $\frac{5\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2.7 The Graph of $y = tan^{-1}x$

The graph of $y = tan^{-1}x$ is obtained by taking the image of $y = tanx$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $y \in R$ in the line $y = x$. We get the graph of $y = tan^{-1}x$ as shown.

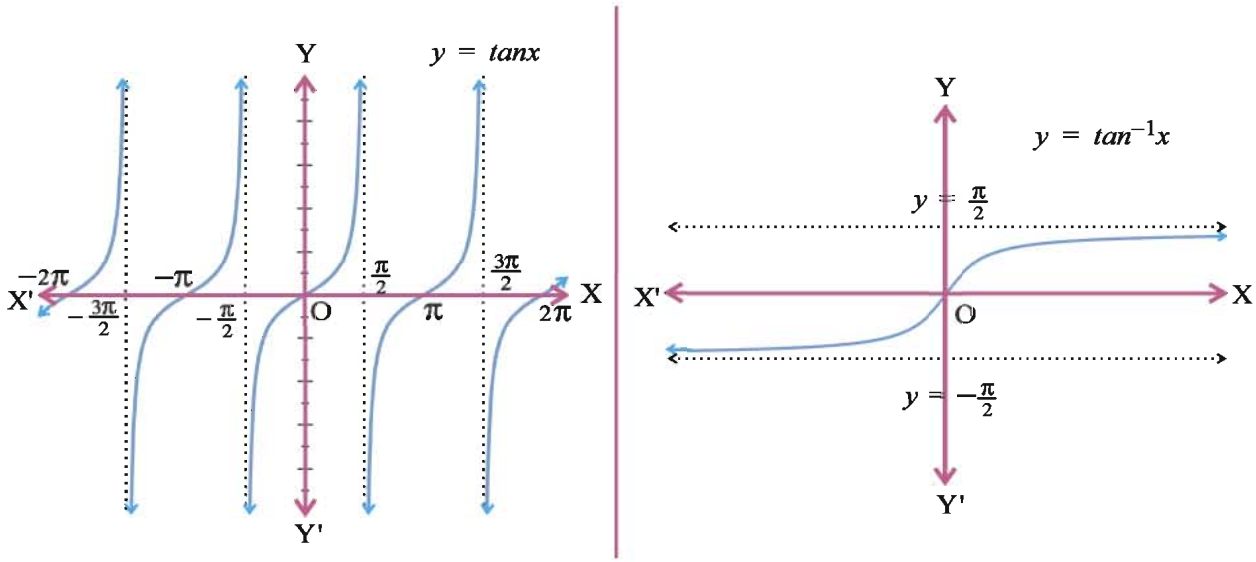

Figure 2.6

2.8 Inverse of *cot* Function

We know that $cot : R - \{k\pi \mid k \in Z\} \rightarrow R$ is many-one and the range of *cot* is R. So *cot* is onto. $cot = \{(x, y) \mid y = cotx, x \in R - \{k\pi \mid k \in Z\}, y \in R\}$ is a many-one, onto and periodic function with period π. The function becomes one-one and onto R, if its domain is restricted to $(0, \pi)$ or $(\pi, 2\pi)$ or $(2\pi, 3\pi)$ or $(k\pi, (k + 1)\pi)$, $k \in Z$. We shall take the domain as $(0, \pi)$ and get the inverse which is denoted by cot^{-1}.

So, $cot^{-1} = \{(y, x) \mid y = cotx, x \in (0, \pi), y \in R\}$.

Thus, for $x \in (0, \pi)$ and $y \in R$,

$$y = cotx \Leftrightarrow cot^{-1}y = x.$$

Domain of cot^{-1} is R and its range is $(0, \pi)$.

$cot^{-1}(cotx) = x$, $x \in (0, \pi)$ and $cot(cot^{-1}x) = x$, $x \in R$.

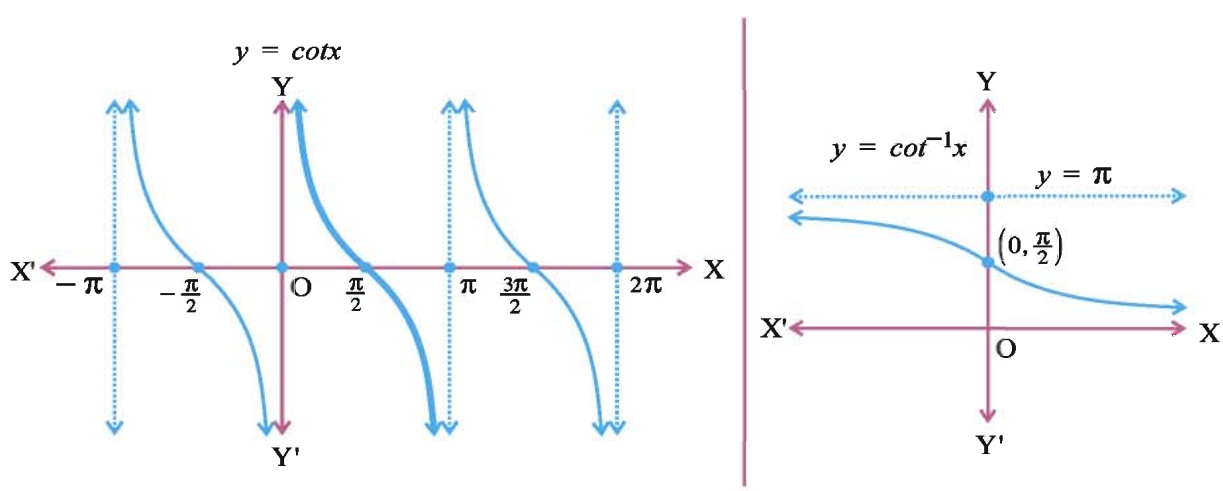

Figure 2.7

Note that $cot^{-1}\left(cot\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$, because $\frac{3\pi}{4} \in (0, \pi)$

Also $cot\left(\frac{3\pi}{4}\right) = -1 \Leftrightarrow cot^{-1}(-1) = \frac{3\pi}{4}$.

$cot\left(-\frac{\pi}{4}\right) = -1$, but $cot^{-1}(-1) \neq -\frac{\pi}{4}$ because $-\frac{\pi}{4} \notin (0, \pi)$.

$cot^{-1}\left(cot\frac{4\pi}{3}\right) \neq \frac{4\pi}{3}$, because $\frac{4\pi}{3} \notin (0, \pi)$.

However, $cot\left(\frac{4\pi}{3}\right) = cot\left(\pi + \frac{\pi}{3}\right) = cot\frac{\pi}{3}$ and $\frac{\pi}{3} \in (0, \pi)$.

So, $cot^{-1}\left(cot\frac{4\pi}{3}\right) = cot^{-1}\left(cot\frac{\pi}{3}\right) = \frac{\pi}{3}$.

The graphs of $y = cotx$ and $y = cot^{-1}x$ are given in figure 2.7.

2.9 The Inverse of *sec* Function

We know that $cos : [0, \pi] \rightarrow [-1, 1]$ is one-one and onto.

$\therefore \quad sec : [0, \pi] - \left\{\frac{\pi}{2}\right\} \rightarrow R - (-1, 1)$ is also one-one and onto.

$\therefore \quad sec = \left\{(x, y) \mid y = secx,\ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\},\ y \in R - (-1, 1)\right\}$ **is one-one and onto.**

Therefore inverse of this function exists and is denoted by sec^{-1}.

So, $sec^{-1} = \left\{(y, x) \mid y = secx,\ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\},\ y \in R - (-1, 1)\right\}$.

Thus, for $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $y \in R - (-1, 1)$, $y = secx \Leftrightarrow sec^{-1}y = x$.

Domain of sec^{-1} **is** $R - (-1, 1)$ **and its range is** $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Also, $sec\left(\frac{\pi}{4}\right) = \sqrt{2}$.

So, $sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

But $sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$ does not imply $sec^{-1}(\sqrt{2}) = -\frac{\pi}{4}$ because $-\frac{\pi}{4} \notin [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

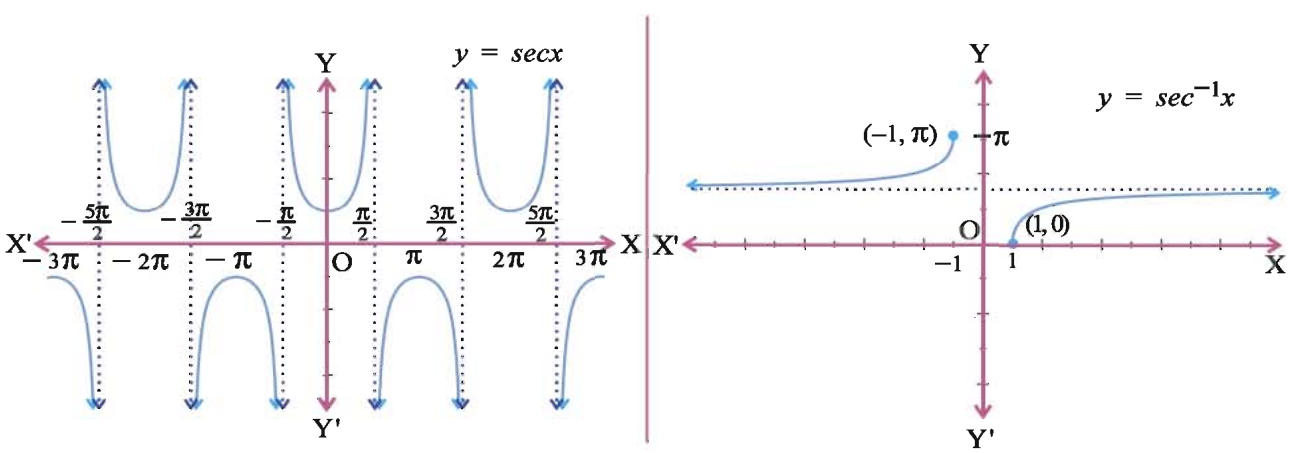

Figure 2.8

For each x, $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $sec^{-1}(secx) = x$ and for each $x \in R - (-1, 1)$, $sec(sec^{-1}x) = x$.

We note that $x \in R - (-1, 1) \Leftrightarrow x \leq -1$ or $x \geq 1 \Leftrightarrow |x| \geq 1$.

The graph of $y = secx$ and $y = sec^{-1}x$ are given in figure 2.8.

2.10 Inverse of *cosec* Function

We know that $sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is one-one and onto.

$\therefore$ $cosec : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to R - (-1, 1)$ is also one-one and onto.

$\therefore$ $cosec = \left\{(x, y) \mid y = cosecx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, y \in R - (-1, 1)\right\}$ **is one-one and onto.**

Therefore, the inverse of this function exists and is denoted by $cosec^{-1}$.

So, $cosec^{-1} = \left\{(y, x) \mid y = cosecx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, y \in R - (-1, 1)\right\}$.

Thus, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $y \in R - (-1, 1)$.

$$y = cosecx \Leftrightarrow cosec^{-1}y = x.$$

Domain of $cosec^{-1}$ is $R - (-1, 1)$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Also, $cosec\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

So, $cosec^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{3}$.

For each $x \in R - (-1, 1)$, $cosec(cosec^{-1}x) = x$ and for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $cosec^{-1}(cosecx) = x$.

The graphs of $y = cosecx$ and $y = cosec^{-1}x$ are given in figure 2.9.

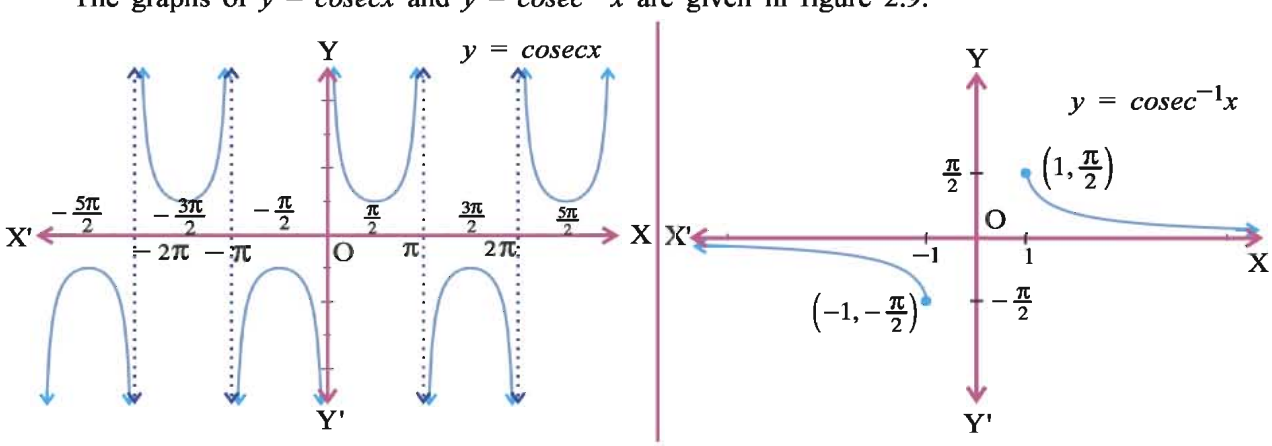

Figure 2.9

Example 3 : Evaluate : (1) $tan^{-1}(\sqrt{3})$ (2) $cot^{-1}(-\sqrt{3})$ (3) $cosec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution : (1) $tan^{-1}(\sqrt{3}) = tan^{-1}\left(tan\frac{\pi}{3}\right) = \frac{\pi}{3}$ $\left(\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(2) $cot^{-1}(-\sqrt{3}) = cot^{-1}\left(-cot\frac{\pi}{6}\right) = cot^{-1}\left(cot\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ $\left(\frac{5\pi}{6} \in (0, \pi)\right)$

(3) $cosec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = cosec^{-1}\left(-cosec\frac{\pi}{3}\right) = cosec^{-1}\left(cosec\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$ $\left(-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}\right)$

Example 4 : Evaluate : (1) $cos^{-1}\left(cos\frac{2\pi}{3}\right)$ (2) $sin^{-1}\left(sin\frac{2\pi}{3}\right)$ (3) $tan^{-1}\left(tan\frac{3\pi}{4}\right)$

(4) $cot^{-1}\left(tan\frac{7\pi}{4}\right)$ (5) $cos^{-1}\left(sin\frac{\pi}{5}\right)$

Solution : (1) $cos^{-1}\left(cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ $\left(\frac{2\pi}{3} \in [0, \pi]\right)$

(2) $sin^{-1}\left(sin\frac{2\pi}{3}\right) = sin^{-1}\left(sin\left(\pi - \frac{\pi}{3}\right)\right)$ $\left(\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$= sin^{-1}\left(sin\frac{\pi}{3}\right)$

$\therefore\ sin^{-1}\left(sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$ $\left(\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(3) $tan^{-1}\left(tan\frac{3\pi}{4}\right) = tan^{-1}\left(tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= tan^{-1}\left(-tan\left(\frac{\pi}{4}\right)\right)$

$= tan^{-1}\left(tan\left(-\frac{\pi}{4}\right)\right)$

$\therefore\ tan^{-1}\left(tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$ $\left(-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(4) $cot^{-1}\left(tan\frac{7\pi}{4}\right) = cot^{-1}\left(tan\left(2\pi - \frac{\pi}{4}\right)\right)$

$= cot^{-1}\left(-tan\frac{\pi}{4}\right)$

$= cot^{-1}\left(cot\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right)$

$= cot^{-1}\left(cot\frac{3\pi}{4}\right)$

$\therefore\ cot^{-1}\left(tan\frac{7\pi}{4}\right) = \frac{3\pi}{4}$ $\left(\frac{3\pi}{4} \in (0, \pi)\right)$

(5) $cos^{-1}\left(sin\frac{\pi}{5}\right) = cos^{-1}\left(cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\right)$

$= cos^{-1}\left(cos\frac{3\pi}{10}\right)$

$\therefore\ cos^{-1}\left(sin\frac{\pi}{5}\right) = \frac{3\pi}{10}$ $\left(\frac{3\pi}{10} \in [0, \pi]\right)$

Example 5 : Find the value of :

(1) $cos\left(2sin^{-1}\frac{3}{4}\right)$ (2) $sin\left(2tan^{-1}\frac{4}{5}\right)$ (3) $tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right)$ (4) $cos\left(3cos^{-1}\frac{2}{3}\right)$

Solution : (1) Consider $cos\left(2sin^{-1}\frac{3}{4}\right)$.

Let $sin^{-1}\frac{3}{4} = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So $sin\theta = \frac{3}{4}$.

So, $cos\left(2sin^{-1}\frac{3}{4}\right) = cos2\theta$

$$= 1 - 2sin^2\theta = 1 - 2\left(\frac{9}{16}\right) = -\frac{1}{8}$$

$\therefore \quad cos\left(2sin^{-1}\frac{3}{4}\right) = -\frac{1}{8}$

(2) Let $tan^{-1}\frac{4}{5} = \theta$. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $tan\theta = \frac{4}{5}$

So, $sin\left(2tan^{-1}\frac{4}{5}\right) = sin2\theta = \frac{2tan\theta}{1+tan^2\theta} = \frac{2\left(\frac{4}{5}\right)}{1+\frac{16}{25}} = \frac{40}{41}$

$\therefore \quad sin\left(2tan^{-1}\frac{4}{5}\right) = \frac{40}{41}$

(3) Let $cos^{-1}\frac{3}{4} = \theta$, $\theta \in [0, \pi]$. Then $\frac{3}{4} = cos\theta$

So, $tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right) = tan^2\left(\frac{\theta}{2}\right) = \frac{1-cos\,\theta}{1+cos\,\theta} = \frac{1-\frac{3}{4}}{1+\frac{3}{4}} = \frac{4-3}{4+3} = \frac{1}{7}$

$\therefore \quad tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right) = \frac{1}{7}$

(4) Let $cos^{-1}\frac{2}{3} = \theta$, $\theta \in [0, \pi]$. Then $cos\theta = \frac{2}{3}$

So, $cos\left(3cos^{-1}\frac{2}{3}\right) = cos3\theta$

$$= 4cos^3\theta - 3cos\theta = 4\left(\frac{8}{27}\right) - 3\left(\frac{2}{3}\right) = \frac{32-54}{27} = -\frac{22}{27}$$

$\therefore \quad cos\left(3cos^{-1}\frac{2}{3}\right) = -\frac{22}{27}$

Example 6 : Express the following in the simplest form :

(1) $tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right), -\pi < x < \pi$ **(2)** $tan^{-1}\left(\frac{cosx}{1+sinx}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution : (1) $tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = tan^{-1}\left(\sqrt{tan^2\frac{x}{2}}\right) = tan^{-1}\left(\left|tan\,\frac{x}{2}\right|\right)$

Case 1 : If $-\pi < x < 0$, then $-\frac{\pi}{2} < \frac{x}{2} < 0$

$\therefore \quad tan\frac{x}{2} < 0$

$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = tan^{-1}\left(-tan\frac{x}{2}\right) = tan^{-1}\left(tan\left(-\frac{x}{2}\right)\right)$

Now, $0 < -\frac{x}{2} < \frac{\pi}{2}$. So, $-\frac{\pi}{2} < -\frac{x}{2} < \frac{\pi}{2}$

$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = -\frac{x}{2}$

Case 2 : If $0 \le x < \pi$, then $0 \le \frac{x}{2} < \frac{\pi}{2}$

$\therefore \quad tan\frac{x}{2} \ge 0$

$$tan^{-1}\left(\left|tan\frac{x}{2}\right|\right) = tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2} \qquad \left(0 \le \frac{x}{2} < \frac{\pi}{2}\right)$$

$$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = \begin{cases} \frac{x}{2} & 0 \le x < \pi \\ -\frac{x}{2} & -\pi < x < 0 \end{cases}$$

(2) $$tan^{-1}\left(\frac{cosx}{1+sinx}\right) = tan^{-1}\left(\frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{cos^2\frac{x}{2} + sin^2\frac{x}{2} + 2sin\frac{x}{2}cos\frac{x}{2}}\right)$$

$$= tan^{-1}\left(\frac{\left(cos\frac{x}{2} + sin\frac{x}{2}\right)\left(cos\frac{x}{2} - sin\frac{x}{2}\right)}{\left(cos\frac{x}{2} + sin\frac{x}{2}\right)^2}\right)$$

$$= tan^{-1}\left(\frac{cos\frac{x}{2} - sin\frac{x}{2}}{cos\frac{x}{2} + sin\frac{x}{2}}\right)$$

$$= tan^{-1}\left(\frac{1 - tan\frac{x}{2}}{1 + tan\frac{x}{2}}\right) \qquad \left(cos\frac{x}{2} \ne 0, \text{ why ?}\right)$$

$$= tan^{-1}\left(tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

Now, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Hence $-\frac{\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$.

$\therefore \quad 0 < \left(\frac{\pi}{4} - \frac{x}{2}\right) < \frac{\pi}{2}$

Thus, $tan^{-1}\left(\frac{cosx}{1+sinx}\right) = tan^{-1}\left(tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2}$

$\therefore \quad tan^{-1}\left(\frac{cosx}{1+sinx}\right) = \frac{\pi}{4} - \frac{x}{2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Exercise 2.1

1. **Evaluate :**

(1) $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $sin^{-1}\left(-\frac{1}{2}\right)$ (3) $sec^{-1}(-2)$

(4) $tan^{-1}(-\sqrt{3})$ (5) $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (6) $cosec^{-1}(-\sqrt{2})$

2. **Evaluate :**

(1) $cos^{-1}\left(sin\frac{\pi}{7}\right)$ (2) $sin^{-1}\left(cos\frac{\pi}{5}\right)$

(3) $tan^{-1}\left(tan\frac{5\pi}{4}\right)$ (4) $sec^{-1}\left(cosec\left(\frac{\pi}{8}\right)\right)$

3. **Evaluate :**

(1) $sin\left(2tan^{-1}\frac{2}{5}\right)$ (2) $tan^2\left(\frac{1}{2}cos^{-1}\frac{2}{3}\right)$

(3) $sin\left(2cos^{-1}\frac{4}{5}\right)$ (4) $tan^2\left(\frac{1}{2}sin^{-1}\frac{2}{3}\right)$ (5) $sin\left(3\,sin^{-1}\frac{1}{2}\right)$

4. Express in the simplest form :

$$tan^{-1}\left(\frac{cosx - sinx}{cosx + sinx}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$$

*

2.11 Values of Inverse Trigonometric Functions for $-x$

We have seen that by restricting the domain and codomain of a trigonometric function, it can be made one-one and onto, which is the necessary and sufficient condition for a function to have its inverse. Also, we have restricted the domain in such a way that the domain of each trigonometric function contains $\left(0, \frac{\pi}{2}\right)$ as its subset. By doing so, we always have the value of each inverse function in $\left(0, \frac{\pi}{2}\right)$, whenever the value of the function is positive. We also make a note that the domain of all the inverse trigonometric functions are such that x belongs to the domain if and only if $-x$ also belongs to it. This is because the domain is $[-1, 1]$ or R or R $-$ $(-1, 1)$, i.e. $|x| \leq 1$ or R or $|x| \geq 1$ respectively. If A is in any of this set, then $x \in A \Leftrightarrow -x \in A$.

The values at x and $-x$ of every trigonometric inverse function are related as shown in the following theorem.

Theorem 2.1 : (1) $sin^{-1}(-x) = -sin^{-1}x, \quad |x| \leq 1$

(2) $cos^{-1}(-x) = \pi - cos^{-1}x, \quad |x| \leq 1$

(3) $tan^{-1}(-x) = -tan^{-1}x, \quad x \in R$

(4) $cot^{-1}(-x) = \pi - cot^{-1}x, \quad x \in R$

(5) $cosec^{-1}(-x) = -cosec^{-1}x, \quad |x| \geq 1$

(6) $sec^{-1}(-x) = \pi - sec^{-1}x, \quad |x| \geq 1$

Proof : (1) $|x| \leq 1$

Suppose $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $x = sin\theta$.

$sin(-\theta) = -sin\theta$

$\therefore \quad sin(-\theta) = -x$ (i)

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} \geq -\theta \geq -\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \leq -\theta \leq \frac{\pi}{2}$

$\therefore \quad -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $|x| = |-x|$. Hence $|x| \leq 1 \Rightarrow |-x| \leq 1$

$\therefore$ By (i), $sin(-\theta) = -x$ $\left(-\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], |-x| \leq 1\right)$

$\therefore \quad sin^{-1}(-x) = -\theta = -sin^{-1}x$

$\therefore \quad sin^{-1}(-x) = -sin^{-1}x$

(2) Suppose $cos^{-1}x = \theta$. $\theta \in [0, \pi]$, $|x| \leq 1$. Then $x = cos\theta$

Also, $cos(\pi - \theta) = -cos\theta$,

$\therefore \quad cos(\pi - \theta) = -x$ (i)

$\theta \in [0, \pi] \Rightarrow 0 \le \theta \le \pi$

$\Rightarrow 0 \ge -\theta \ge -\pi$

$\Rightarrow \pi \ge (\pi - \theta) \ge 0$

$\Rightarrow 0 \le (\pi - \theta) \le \pi$

$\therefore$ $(\pi - \theta) \in [0, \pi]$ and $|x| = |-x|$. Thus $|x| \le 1 \Rightarrow |-x| \le 1$

$\therefore$ By (i), $cos(\pi - \theta) = -x$ $\qquad (\pi - \theta \in [0, \pi],\ |-x| \le 1)$

$\therefore$ $cos^{-1}(-x) = \pi - \theta = \pi - cos^{-1}x$

$\therefore$ $cos^{-1}(-x) = \pi - cos^{-1}x$

(3) Suppose $tan^{-1}x = \theta$. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $x \in$ R. Then $x = tan\theta$

Now, $tan(-\theta) = -tan\theta = -x$ $\qquad$ (i)

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} > -\theta > -\frac{\pi}{2}$

$\therefore$ $-\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x \in$ R. Thus $-x \in$ R

$\therefore$ By (i), $tan(-\theta) = -x$ $\qquad \left(-\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), -x \in R\right)$

$\therefore$ $tan^{-1}(-x) = -\theta = -tan^{-1}x$

$\therefore$ $tan^{-1}(-x) = -tan^{-1}x$

Similarly we can prove **(4)**, **(5)** and **(6)**.

Example 7 : Evaluate :

(1) $sin^{-1}\left(-\frac{1}{2}\right)$ $\qquad$ **(2)** $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\qquad$ **(3)** $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ $\qquad$ **(4)** $cot^{-1}(-1)$

Solution : (1) $sin^{-1}\left(-\frac{1}{2}\right) = -sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$

(2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - cos^{-1}\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(3) $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(4) $cot^{-1}(-1) = \pi - cot^{-1}1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

2.12 Values of Trigonometric Functions for $\frac{1}{x}$

Now we get relations between the values of trigonometric inverse functions at x and at $\frac{1}{x}$, when $x \ne 0$.

Theorem 2.2 : (1) $cosec^{-1}x = sin^{-1}\frac{1}{x}$, $\qquad |x| \ge 1$

(2) $sec^{-1}x = cos^{-1}\frac{1}{x}$, $\qquad |x| \ge 1$

(3) (a) $cot^{-1}x = tan^{-1}\frac{1}{x}$, $\qquad x > 0$

(b) $cot^{-1}x = tan^{-1}\frac{1}{x} + \pi$, $\qquad x < 0$

Proof : (1) Let $cosec^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$. Then $x = cosec\theta$. $|x| \ge 1$

$|x| \geq 1$. So $x \neq 0$ and $\left|\frac{1}{x}\right| \leq 1$.

$cosec\theta = x$

$\therefore \quad sin\theta = \frac{1}{x}$

$\therefore \quad \theta = sin^{-1}\frac{1}{x}$ $\qquad \left(\theta \in \left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}\right) \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left|\frac{1}{x}\right| \leq 1\right)$

$\therefore \quad cosec^{-1}x = sin^{-1}\frac{1}{x}$

(2) Let $sec^{-1}x = \theta$, $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $|x| \geq 1$. Then $x = sec\theta$

$|x| \geq 1$. So $x \neq 0$ and $\left|\frac{1}{x}\right| \leq 1$.

$sec\theta = x$

$\therefore \quad cos\theta = \frac{1}{x}$

$\therefore \quad \theta = cos^{-1}\frac{1}{x}$ $\qquad \left(\theta \in \left([0, \pi] - \left\{\frac{\pi}{2}\right\}\right) \subset [0, \pi], \left|\frac{1}{x}\right| \leq 1\right)$

$\therefore \quad sec^{-1}x = cos^{-1}\frac{1}{x}$

(3) **(a)** Let $cot^{-1}x = \theta$, $\theta \in (0, \pi)$, $x \in R$

$\therefore \quad cot\theta = x$

$x > 0$ and hence $x \neq 0$. So $\frac{1}{x} \in R$.

$\therefore \quad tan\theta = \frac{1}{x}$ and $\theta \in (0, \pi)$

Now, since $x > 0$, $tan\theta = \frac{1}{x} > 0$

Also $0 < \theta < \pi$. So we must have $0 < \theta < \frac{\pi}{2}$. $\qquad (tan\theta > 0)$

Thus, $tan\theta = \frac{1}{x}$, $\theta \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \theta = tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad cot^{-1}x = tan^{-1}\frac{1}{x}$

(b) As we have seen above, if $cot^{-1}x = \theta$, $\theta \in (0, \pi)$, $x \in R$, then $cot\theta = x$.

Since $x < 0$, $cot\theta = x < 0$. Thus, $tan\theta < 0$ and $\theta \in (0, \pi)$.

This means that $\frac{\pi}{2} < \theta < \pi$

$\therefore \quad \frac{\pi}{2} - \pi < (\theta - \pi) < \pi - \pi$

$\therefore \quad -\frac{\pi}{2} < (\theta - \pi) < 0$

i.e. $\theta - \pi \in \left(-\frac{\pi}{2}, 0\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{1}{x} \in R$ as $x \neq 0$

$tan(\theta - \pi) = tan\theta = \frac{1}{x}$ $\qquad$ **(Period of *tan* is π)**

$\therefore \quad tan(\theta - \pi) = \frac{1}{x}$

$\therefore \quad \theta - \pi = tan^{-1}\frac{1}{x}$ $\qquad \left(\theta - \pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \frac{1}{x} \in R\right)$

$\therefore \quad tan^{-1}\frac{1}{x} = cot^{-1}x - \pi$

$\therefore$ For $x < 0$, $cot^{-1}x = tan^{-1}\frac{1}{x} + \pi$.

(**Note :** We can derive from this theorem that

(1) $sin^{-1}x = cosec^{-1}\frac{1}{x}$, $x \in [-1, 1] - \{0\}$

(2) $cos^{-1}x = sec^{-1}\frac{1}{x}$, $x \in [-1, 1] - \{0\}$

(3) (a) $tan^{-1}x = cot^{-1}\frac{1}{x}$, $x > 0$

(b) $tan^{-1}x = cot^{-1}\frac{1}{x} - \pi$, $x < 0$

2.13 Formulae for Value of Trigonometric Inverse Functions for Complementary Numbers :

Theorem 2.3 : (1) $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$, $|x| \le 1$

(2) $cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}$, $|x| \ge 1$

(3) $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbf{R}$

Proof : (1) Let $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $|x| \le 1$. Then $x = sin\theta$

$\therefore \quad cos\left(\frac{\pi}{2} - \theta\right) = x$

Now, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} \ge -\theta \le -\frac{\pi}{2}$

$\Rightarrow \pi \ge \left(\frac{\pi}{2} - \theta\right) \ge 0$

$\Rightarrow 0 \le \left(\frac{\pi}{2} - \theta\right) \le \pi$

$\therefore \quad \left(\frac{\pi}{2} - \theta\right) \in [0, \pi]$ and $|x| \le 1$. Also $cos\left(\frac{\pi}{2} - \theta\right) = x$

$\therefore \quad cos^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - sin^{-1}x$

$\therefore \quad sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

(2) Let $cosec^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $|x| \ge 1$. Then $x = cosec\theta$

$\therefore \quad sec\left(\frac{\pi}{2} - \theta\right) = x$

Now, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$

$\Rightarrow \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}, \theta \ne 0$

$\Rightarrow \pi \ge \left(\frac{\pi}{2} - \theta\right) \ge 0, \theta \ne 0$

$\Rightarrow 0 \le \left(\frac{\pi}{2} - \theta\right) \le \pi, \theta \ne 0$

Also $\frac{\pi}{2} - \theta \ne \frac{\pi}{2}$ as $\theta \ne 0$

$\therefore \quad \frac{\pi}{2} - \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $|x| \ge 1$ and $sec\left(\frac{\pi}{2} - \theta\right) = x$.

$\therefore \quad sec^{-1}x = \frac{\pi}{2} - \theta$

$\therefore \quad \theta + sec^{-1}x = \frac{\pi}{2}$

$\therefore \quad cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}$

or we think in another way as, $cosec^{-1}x + sec^{-1}x = sin^{-1}\frac{1}{x} + cos^{-1}\frac{1}{x}$ $\left(|x| \geq 1 \Rightarrow \frac{1}{|x|} \leq 1\right)$

$= \frac{\pi}{2}$ (By (1))

(3) can be proved similarly as (1).

2.14 Addition and Subtraction Formulae

Theorem 2.4 : If $x > 0$, $y > 0$, then

(1) $tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(2) $tan^{-1}x + tan^{-1}y = \pi + tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy > 1$

(3) $tan^{-1}x + tan^{-1}y = \frac{\pi}{2}$, if $xy = 1$

(4) $tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Proof : Here, $x > 0$, $y > 0$.

Let $tan^{-1}x = \alpha$ and $tan^{-1}y = \beta$, $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad tan\alpha = x > 0$ and $tan\beta = y > 0$

As $tan\alpha$ and $tan\beta$ are positive and $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$

(1) $tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta} = \frac{x+y}{1-xy}$

Let $x > 0$, $y > 0$ and $xy < 1$. Hence, $(1 - xy) > 0$ and $x + y > 0$.

$\therefore \quad \frac{x+y}{1-xy} > 0$. Hence, $tan(\alpha + \beta) > 0$

Also $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < \alpha + \beta < \pi$

But $tan(\alpha + \beta) > 0$. Hence, $\alpha + \beta \in \left(0, \frac{\pi}{2}\right)$

Thus, $tan(\alpha + \beta) = \frac{x+y}{1-xy}$

$\therefore \quad \alpha + \beta = tan^{-1}\frac{x+y}{1-xy}$. $\left((\alpha + \beta) \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\therefore \quad tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

(2) $tan(-\pi + \alpha + \beta) = tan(\alpha + \beta)$ (π is a period of tan)

$= \frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta}$

$\therefore \quad tan(-\pi + \alpha + \beta) = \frac{x+y}{1-xy}$

Now, $x > 0$, $y > 0$. Also, $xy > 1$. So, $1 - xy < 0$

$\therefore \quad \frac{x+y}{1-xy} < 0$

$\therefore \quad tan(-\pi + \alpha + \beta) < 0$

Now $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$.

$\therefore \quad 0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < \alpha + \beta < \pi$

$\therefore \quad -\pi < \alpha + \beta - \pi < 0$

But, as $tan(-\pi + \alpha + \beta) < 0$, $-\frac{\pi}{2} < \alpha + \beta - \pi < 0$.

So, $\alpha + \beta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

Thus, $tan(-\pi + \alpha + \beta) = \frac{x+y}{1-xy}$, $\alpha + \beta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

$\therefore \quad -\pi + \alpha + \beta = tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\therefore \quad \alpha + \beta = tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$

$\therefore \quad tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$

(3) $tan^{-1}x + tan^{-1}y = tan^{-1}x + tan^{-1}\frac{1}{x}$ $\quad (xy = 1)$

$= tan^{-1}x + cot^{-1}x$ $\quad (x > 0)$

$= \frac{\pi}{2}$

(4) As we have noted $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$

Thus, $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$. So $-\frac{\pi}{2} < -\beta < 0$.

$\therefore \quad 0 < \alpha < \frac{\pi}{2}$ and $-\frac{\pi}{2} < -\beta < 0$.

$\therefore \quad -\frac{\pi}{2} < (\alpha - \beta) < \frac{\pi}{2}$

Thus, $(\alpha - \beta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha\, tan\beta}$

$\therefore \quad tan(\alpha - \beta) = \frac{x-y}{1+xy} \qquad \alpha - \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \alpha - \beta = tan^{-1}\left(\frac{x-y}{1+xy}\right)$ $\quad \left(\frac{x-y}{1+xy} \in \mathbf{R} \text{ and } x > 0, y > 0; \text{ so } xy \neq -1\right)$

$\therefore \quad tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Example 8 : Prove :

(1) $tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24} = tan^{-1}\left(\frac{1}{2}\right)$

(2) $cot^{-1}\frac{1}{2} + cot^{-1}\frac{1}{3} = \frac{3\pi}{4}$

(3) $tan^{-1}\frac{1}{7} + tan^{-1}\frac{4}{7} + tan^{-1}\frac{9}{7} = \frac{\pi}{2}$

Solution : (1) L.H.S. $= tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24}$

$$= tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right) \qquad \left(\frac{2}{11}\times\frac{7}{24}<1\right)$$

$$= tan^{-1}\left(\frac{48+77}{264-14}\right) = tan^{-1}\left(\frac{125}{250}\right) = tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

(2) L.H.S. $= cot^{-1}\frac{1}{2} + cot^{-1}\frac{1}{3}$

$$= tan^{-1}2 + tan^{-1}3 \qquad (2 > 0,\ 3 > 0)$$

$$= \pi + tan^{-1}\left(\frac{2+3}{1-2\times 3}\right) \qquad (2\times 3 > 1)$$

$$= \pi + tan^{-1}(-1)$$

$$= \pi - tan^{-1}(1) \qquad (tan^{-1}(-x) = -tan^{-1}x)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4} = \text{R.H.S.}$$

(3) L.H.S. $= tan^{-1}\frac{1}{7} + tan^{-1}\frac{4}{7} + tan^{-1}\frac{9}{7}$

$$= tan^{-1}\left(\frac{\frac{1}{7}+\frac{4}{7}}{1-\frac{1}{7}\times\frac{4}{7}}\right) + tan^{-1}\frac{9}{7} \qquad \left(\frac{1}{7}\times\frac{4}{7} < 1\right)$$

$$= tan^{-1}\left(\frac{7+28}{49-4}\right) + tan^{-1}\frac{9}{7}$$

$$= tan^{-1}\left(\frac{35}{45}\right) + tan^{-1}\left(\frac{9}{7}\right)$$

$$= tan^{-1}\left(\frac{7}{9}\right) + tan^{-1}\left(\frac{9}{7}\right)$$

$$= \frac{\pi}{2} = \text{R.H.S.} \qquad \left(\frac{7}{9}\times\frac{9}{7} = 1\right)$$

Example 9 : Prove that : $3sin^{-1}x = sin^{-1}(3x - 4x^3)$, if $-\frac{1}{2} \le x \le \frac{1}{2}$.

Solution : Let $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $|x| \le 1$. Then $x = sin\theta$

Now, $sin3\theta = 3sin\theta - 4sin^3\theta$

$\therefore \quad sin3\theta = 3x - 4x^3$

Now, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow sin\left(-\frac{\pi}{6}\right) \le sin\theta \le sin\frac{\pi}{6}$

$$\Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad \left(sin \text{ is } \uparrow \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$\Rightarrow -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

Since $sin3\theta = 3x - 4x^3$, $-\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$

$\therefore \quad 3\theta = sin^{-1}(3x - 4x^3)$

$\therefore \quad 3sin^{-1}x = sin^{-1}(3x - 4x^3)$

Example 10 : Prove : **(1)** $tan^{-1}\sqrt{\frac{a-x}{a+x}} = \frac{1}{2}cos^{-1}\frac{x}{a}$, $-a < x < a$, $a \in R^+$

(2) $cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right) = \frac{\pi}{2} - \frac{x}{2}$, $\frac{\pi}{2} < x < \pi$

(3) $tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x$, $0 < x < 1$.

Solution : (1) Consider $tan^{-1}\sqrt{\frac{a-x}{a+x}}$, $-a < x < a$

$-a < x < a \Rightarrow -1 < \frac{x}{a} < 1$ $\qquad (a \in R^+)$

$\therefore \quad \frac{x}{a} \in (-1, 1)$

$\therefore \quad \exists\theta \in (0, \pi)$ such that $cos\theta = \frac{x}{a}$ or $\theta = cos^{-1}\frac{x}{a}$

$0 < \theta < \pi$. So, $0 < \frac{\theta}{2} < \frac{\pi}{2}$

$$\begin{aligned}
\text{Now, } tan^{-1}\sqrt{\frac{a-x}{a+x}} &= tan^{-1}\sqrt{\frac{a-acos\,\theta}{a+acos\,\theta}} \\
&= tan^{-1}\sqrt{\frac{1-cos\,\theta}{1+cos\,\theta}} \\
&= tan^{-1}\sqrt{tan^2\frac{\theta}{2}} \\
&= tan^{-1}\left|\,tan\frac{\theta}{2}\,\right| \\
&= tan^{-1}\left(tan\frac{\theta}{2}\right) && \left(0 < \frac{\theta}{2} < \frac{\pi}{2}\right) \\
&= \frac{\theta}{2} && \left(\frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \\
&= \frac{1}{2}cos^{-1}\frac{x}{a}
\end{aligned}$$

(2) L.H.S. $= cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right)$, $\frac{\pi}{2} < x < \pi$

$$= cot^{-1}\left(\frac{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}+\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}-\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}\right)$$

$$= cot^{-1}\left(\frac{\left|cos\frac{x}{2}+sin\frac{x}{2}\right|+\left|cos\frac{x}{2}-sin\frac{x}{2}\right|}{\left|cos\frac{x}{2}+sin\frac{x}{2}\right|-\left|cos\frac{x}{2}-sin\frac{x}{2}\right|}\right)$$

As, $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\therefore \quad cos\frac{x}{2} < sin\frac{x}{2}$ and $cos\frac{x}{2} > 0$, $sin\frac{x}{2} > 0$

$$= cot^{-1}\left(\frac{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)-\left(cos\frac{x}{2}-sin\frac{x}{2}\right)}{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)+\left(cos\frac{x}{2}-sin\frac{x}{2}\right)}\right) \qquad \left(\left|cos\,\frac{x}{2} - sin\,\frac{x}{2}\right| = -\left(cos\,\frac{x}{2} - sin\,\frac{x}{2}\right)\right)$$

$$= cot^{-1}\left(tan\frac{x}{2}\right)$$

$$= cot^{-1}\left(cot\left(\frac{\pi}{2}-\frac{x}{2}\right)\right) \qquad \left(0 < \frac{\pi}{2} - \frac{x}{2} < \frac{\pi}{4}\right)$$

Now, $-\frac{\pi}{4} > -\frac{x}{2} > -\frac{\pi}{2}$. So, $\frac{\pi}{2} - \frac{\pi}{4} > \frac{\pi}{2} - \frac{x}{2} > 0$.

$\therefore \quad 0 < \frac{\pi}{2} - \frac{x}{2} < \frac{\pi}{4}$.

$\therefore$ L.H.S. $= \frac{\pi}{2} - \frac{x}{2} =$ R.H.S.

(3) Consider $tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$, $0 < x < 1$

Let $\theta = cos^{-1}x$, $\theta \in [0, \pi]$. $x \in (0, 1)$. Then $x = cos\theta$.

$$\text{L.H.S.} = tan^{-1}\left(\frac{\sqrt{1+cos\,\theta}-\sqrt{1-cos\,\theta}}{\sqrt{1+cos\,\theta}+\sqrt{1-cos\,\theta}}\right)$$

$$= tan^{-1}\left(\frac{\sqrt{2cos^2\frac{\theta}{2}}-\sqrt{2sin^2\frac{\theta}{2}}}{\sqrt{2cos^2\frac{\theta}{2}}+\sqrt{2sin^2\frac{\theta}{2}}}\right)$$

$$= tan^{-1}\left(\frac{\left|cos\frac{\theta}{2}\right|-\left|sin\frac{\theta}{2}\right|}{\left|cos\frac{\theta}{2}\right|+\left|sin\frac{\theta}{2}\right|}\right)$$

As, $0 < x < 1 \Rightarrow 0 < cos\theta < 1$

$\Rightarrow cos\frac{\pi}{2} < cos\,\theta < cos0$

$\Rightarrow 0 < \theta < \frac{\pi}{2}$ $\qquad$ (cos is $\downarrow$ in the 1st quadrant)

$\Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{4}$

Also, $-\frac{\pi}{4} < -\frac{\theta}{2} < 0$

So, $0 < \left(\frac{\pi}{4} - \frac{\theta}{2}\right) < \frac{\pi}{4}$

$$\text{L.H.S.} = tan^{-1}\left(\frac{cos\frac{\theta}{2}-sin\frac{\theta}{2}}{cos\frac{\theta}{2}+sin\frac{\theta}{2}}\right) \qquad \left(0 < \frac{\theta}{2} < \frac{\pi}{4}\right)$$

$$= tan^{-1}\left(\frac{1-tan\frac{\theta}{2}}{1+tan\frac{\theta}{2}}\right) \qquad \left(cos\frac{\theta}{2} \neq 0.\ \text{Why ?}\right)$$

$$= tan^{-1}\left(tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$$

$$= \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x = \text{R.H.S.} \qquad \left(\frac{\pi}{4} - \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)\right)$$

2.15 Inter-relations Between the Inverse Functions

(1) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, if $0 < x < 1$.

(2) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$, if $0 < x < 1$.

(3) $\tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$, if $x > 0$

Proof : Suppose, $\sin^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in (0, 1)$. So $\sin\theta = x$

Also, $\sin\theta = x > 0$. Hence, $\theta \in \left(0, \frac{\pi}{2}\right)$.

Also, $\cos^2\theta = 1 - \sin^2\theta = 1 - x^2$

$\therefore \quad \cos\theta = \sqrt{1-x^2}$ $\qquad \left(\cos\theta > 0 \text{ in } \left(0, \frac{\pi}{2}\right)\right)$

$\therefore \quad \theta = \cos^{-1}\sqrt{1-x^2}$ $\qquad \left(\theta \in \left(0, \frac{\pi}{2}\right), 1 < \sqrt{1-x^2} < 1\right)$

$\therefore \quad \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

Also, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\therefore \quad \theta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, as $\theta \in \left(0, \frac{\pi}{2}\right)$.

$\therefore \quad \sin^{-1}x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

Similarly (2) and (3) can be proved.

Example 11 : Prove : $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{15}{17} + \sin^{-1}\frac{36}{85} = \frac{\pi}{2}$

$$\text{L.H.S.} = \sin^{-1}\frac{3}{5} + \cos^{-1}\frac{15}{17} + \sin^{-1}\frac{36}{85}$$

$$= \tan^{-1}\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}} + \tan^{-1}\frac{\sqrt{1-\frac{225}{289}}}{\frac{15}{17}} + \tan^{-1}\frac{\frac{36}{85}}{\sqrt{1-\frac{36^2}{85^2}}}$$

$$= \tan^{-1}\frac{3}{\sqrt{25-9}} + \tan^{-1}\frac{\sqrt{289-225}}{15} + \tan^{-1}\frac{36}{\sqrt{85^2-36^2}}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{36}{77}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{8}{15}}{1-\frac{3}{4}\times\frac{8}{15}}\right) + \tan^{-1}\left(\frac{36}{77}\right) \qquad \left(\frac{3}{4}\times\frac{8}{15} < 1\right)$$

$$= \tan^{-1}\left(\frac{45+32}{60-24}\right) + \tan^{-1}\left(\frac{36}{77}\right)$$

$$= \tan^{-1}\left(\frac{77}{36}\right) + \tan^{-1}\left(\frac{36}{77}\right)$$

$$= \frac{\pi}{2} = \text{R.H.S.} \qquad \left(\frac{77}{36}\times\frac{36}{77} = 1\right)$$

Exercise 2.2

1. Find the value of :

(1) $sin^{-1}\frac{\sqrt{3}}{2} - cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2tan^{-1}(1)$

(2) $3sin^{-1}\frac{1}{2} + 4cos^{-1}\frac{\sqrt{3}}{2} + sec^{-1}1$

(3) $cot^{-1}(1) + 3sin^{-1}\frac{1}{2} - cosec^{-1}(-2) - 3tan^{-1}\frac{1}{\sqrt{3}}$

(4) $5cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - 4tan^{-1}(-\sqrt{3}) + 3sin^{-1}(1)$

(5) $cos\left(sin^{-1}\left(-\frac{4}{5}\right)\right) + sin\left(tan^{-1}\frac{3}{4}\right) + cos\left(cosec^{-1}\frac{5}{3}\right)$

(6) $sin\left(\frac{\pi}{2} - cos^{-1}\frac{3}{7}\right) + cos\left(\frac{3\pi}{2} - sin^{-1}\frac{2}{7}\right) + cot\left(tan^{-1}\frac{7}{6}\right)$

(7) $sin^{-1}\left(sin\frac{5\pi}{6}\right) + cos^{-1}\left(cos\frac{5\pi}{3}\right) + tan^{-1}\left(tan\frac{7\pi}{3}\right)$

2. Prove :

(1) $tan^{-1}\frac{4}{5} + tan^{-1}\frac{2}{3} = tan^{-1}\frac{22}{7}$

(2) $tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{13} = tan^{-1}\frac{2}{9}$

(3) $tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

(4) $tan^{-1}\frac{1}{3} + \frac{1}{2}tan^{-1}\frac{1}{7} = \frac{\pi}{8}$

(5) $tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{3} - tan^{-1}\frac{1}{7} = tan^{-1}\frac{21}{53}$

(6) $tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

3. Prove :

(1) $cos^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} = tan^{-1}\left(\frac{56}{33}\right)$

(2) $sin^{-1}\frac{3}{5} + cos^{-1}\frac{4}{5} = cot^{-1}\left(\frac{7}{24}\right)$

(3) $2sin^{-1}\frac{5}{13} = cos^{-1}\frac{119}{169}$

(4) $2sin^{-1}\frac{3}{5} + cos^{-1}\frac{24}{25} = \frac{\pi}{2}$

(5) $2cot^{-1}2 + cosec^{-1}\frac{5}{3} = \frac{\pi}{2}$

(6) $sin^{-1}\frac{3}{5} + sin^{-1}\frac{8}{17} + sin^{-1}\frac{36}{85} = \frac{\pi}{2}$

4. Prove :

(1) $2cot^{-1}\frac{1}{3} + tan^{-1}\frac{3}{4} = \pi$

(2) $cot^{-1}1 + tan^{-1}2 + cot^{-1}\frac{1}{3} = \pi$

(3) $cot^{-1}\frac{1}{5} + \frac{1}{2}cot^{-1}\frac{12}{5} = \frac{\pi}{2}$

(4) $sin^{-1}\frac{12}{13} + cos^{-1}\frac{4}{5} + tan^{-1}\frac{63}{16} = \pi$

*

Miscelleneous Examples :

Example 12 : Prove : $cos^{-1} a + cos^{-1} b + cos^{-1} c = \pi \Rightarrow a^2 + b^2 + c^2 + 2abc = 1$, where $a, b, c \in [-1, 1]$.

Solution : Let $cos^{-1}a = \alpha$, $cos^{-1}b = \beta$, $cos^{-1}c = \gamma$ $\quad [\alpha, \beta, \gamma \in [0, \pi]]$

$\therefore \quad a = cos\alpha,\ b = cos\beta,\ c = cos\gamma$

Now, $cos^{-1}a + cos^{-1}b + cos^{-1}c = \pi$

$\therefore \quad \alpha + \beta + \gamma = \pi$

$\therefore \quad \alpha + \beta = \pi - \gamma$

$\therefore \quad cos(\alpha + \beta) = cos(\pi - \gamma)$

$\therefore \quad cos\alpha\, cos\beta - sin\alpha\, sin\beta = -cos\gamma$

$\therefore \quad cos\alpha\, cos\beta + cos\gamma = sin\alpha\, sin\beta$

$\therefore \quad (cos\alpha\, cos\beta + cos\gamma)^2 = sin^2\alpha\, sin^2\beta$

$\therefore \quad (ab + c)^2 = (1 - a^2)(1 - b^2)$

$\therefore \quad a^2b^2 + 2abc + c^2 = 1 - a^2 - b^2 + a^2b^2$

$\therefore \quad a^2 + b^2 + c^2 + 2abc = 1$

Example 13 : Prove that $cosec[tan^{-1}(cos(cot^{-1}(sec(sin^{-1} a))))] = \sqrt{3-a^2}$, where $0 < a < 1$.

Solution : L.H.S. $= cosec[tan^{-1}(cos(cot^{-1}(sec(sin^{-1} a))))]$

$$= cosec\left[tan^{-1}\left(cos\left(cot^{-1}\left(sec\left(sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)\right] \qquad \left(sin^{-1} a = cos^{-1}\sqrt{1-a^2}\right)$$

$$= cosec\left[tan^{-1}\left(cos\left(cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right]$$

$$= cosec\left[tan^{-1}\left(cos\left(tan^{-1}\sqrt{1-a^2}\right)\right)\right] \qquad \left(\sqrt{1-a^2} > 0\right)$$

$$= cosec\left[tan^{-1}\left(cos\left(cos^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right)\right] \qquad \left(tan^{-1}x = cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$= cosec\left(tan^{-1}\frac{1}{\sqrt{2-a^2}}\right)$$

$$= cosec\left(sin^{-1}\frac{\frac{1}{\sqrt{2-a^2}}}{\sqrt{1+\frac{1}{2-a^2}}}\right) \qquad \left(tan^{-1}x = sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)$$

$$= cosec\left(sin^{-1}\frac{1}{\sqrt{3-a^2}}\right)$$

$$= cosec\left(cosec^{-1}\sqrt{3-a^2}\right)$$

$$= \sqrt{3-a^2} = \text{R.H.S.}$$

Example 14 : Solve the following equations :

(1) $tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{5\pi}{6}$ $\qquad$ (2) $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

Solution : (1) $tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{5\pi}{6}$

$\therefore \quad \frac{\pi}{3} + 2tan^{-1}x = \frac{5\pi}{6}$

$\therefore \quad 2tan^{-1}x = \frac{5\pi}{6} - \frac{\pi}{3}$

$\therefore \quad 2tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}x = \frac{\pi}{4}$

$\therefore \quad x = tan\frac{\pi}{4}$

$\therefore \quad x = 1$

Equations involving inverse trigonometric functions can also be solved. However, as the domain and range of such functions are restricted, one must always verify the answer by substituting the solution in the original equation.

Verification : Putting $x = 1$ in the given equation,

L.H.S. $= tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{\pi}{3} + 2\left(\frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6} =$ R.H.S.

$\therefore$ The solution set is $\{1\}$.

(2) $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

We observe that if $x \geq 1$, then $2tan^{-1}x \geq 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$

that is $tan^{-1}2x \leq 0$ which is not possible. Since $x \geq 1$.

If $x < 0$, L.H.S. < 0, R.H.S. > 0. This is not possible.

$\therefore \quad 0 < x < 1.$

Now, $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}2x + tan^{-1}x + tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}2x + tan^{-1}\left(\frac{x+x}{1-x^2}\right) = \frac{\pi}{2}$ $\qquad (0 < x^2 < 1)$

$\therefore \quad tan^{-1}2x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{2}$

We know that, $xy = 1 \Leftrightarrow tan^{-1}x + tan^{-1}y = \frac{\pi}{2}$

$\therefore \quad 2x \cdot \frac{2x}{1-x^2} = 1$

$\therefore \quad 4x^2 = 1 - x^2$

$\therefore \quad 5x^2 = 1$

$\therefore \quad x^2 = \frac{1}{5}$

$\therefore \quad x = \pm\frac{1}{\sqrt{5}}$. But $x > 0$

$\therefore \quad x = \frac{1}{\sqrt{5}}$

Verification : Taking $x = \frac{1}{\sqrt{5}}$.

$$\text{L.H.S.} = tan^{-1}\frac{2}{\sqrt{5}} + 2tan^{-1}\frac{1}{\sqrt{5}}$$

$$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\frac{1}{\sqrt{5}} + tan^{-1}\frac{1}{\sqrt{5}}$$

$$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\left(\frac{\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}}{1-\frac{1}{5}}\right)$$

$$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\left(\frac{\sqrt{5}+\sqrt{5}}{5-1}\right)$$

$$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\frac{\sqrt{5}}{2}$$

$$= \frac{\pi}{2} = \text{R.H.S.}$$

The solution set is $\left\{\frac{1}{\sqrt{5}}\right\}$.

Example 15 : If $0 < x < 1$ and if $tan^{-1}(1 - x)$, $tan^{-1}x$ and $tan^{-1}(1 - x)$ are in arithmetic progression, prove that $x^3 + x^2 = 1$.

Solution : As $tan^{-1}(1 - x)$, $tan^{-1}x$ and $tan^{-1}(1 - x)$ are in A.P.

$$2tan^{-1}x = tan^{-1}(1 - x) + tan^{-1}(1 + x)$$

$$\therefore \quad tan^{-1}x + tan^{-1}x = tan^{-1}\frac{1-x+1+x}{1-(1-x^2)} \qquad (1 - x > 0,\ 1 + x > 0,\ 0 < 1 - x^2 < 1)$$

$$\therefore \quad tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{2}{x^2}\right) \qquad (0 < x^2 < 1)$$

$$\therefore \quad \frac{2x}{1-x^2} = \frac{2}{x^2} \qquad (tan^{-1} \text{ is one-one})$$

$$\therefore \quad x^3 = 1 - x^2$$

$$\therefore \quad x^3 + x^2 = 1$$

Example 16 : Solve $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

Solution : $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

Let $cos^{-1}x = \alpha$, $\alpha \in [0, \pi]$. Then, $x = cos\alpha$.

$$\therefore \quad sin\alpha = \sqrt{1-cos^2\alpha} = \sqrt{1-x^2} \qquad (sin\alpha \geq 0 \text{ as } \alpha \in [0, \pi])$$

Let $sin^{-1}2x = \beta$, $\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, $2x = sin\beta$.

$$\therefore \quad cos\beta = \sqrt{1-4x^2} \qquad \left(cos\beta \geq 0 \text{ as } \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

Now, $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

$$\therefore \quad \alpha + \beta = \frac{\pi}{6}$$

$$\therefore \quad sin(\alpha + \beta) = sin\frac{\pi}{6}$$

$$\therefore \quad sin\alpha\, cos\beta + cos\alpha\, sin\beta = \frac{1}{2}$$

$$\therefore \quad \sqrt{1-x^2}\,\sqrt{1-4x^2} + x(2x) = \frac{1}{2}$$

$\therefore \quad \sqrt{1-x^2}\ \sqrt{1-4x^2} = \frac{1}{2} - 2x^2$

$\therefore \quad \sqrt{1-5x^2+4x^4} = \frac{1}{2} - 2x^2$

$\therefore \quad 1 - 5x^2 + 4x^4 = \left(\frac{1}{2} - 2x^2\right)^2$

$\therefore \quad 1 - 5x^2 + 4x^4 = \frac{1}{4} - 2x^2 + 4x^4$

$\therefore \quad 3x^2 = \frac{3}{4}$

$\therefore \quad x^2 = \frac{1}{4}$

$\therefore \quad x = \pm\frac{1}{2}$

Verification : For $x = \frac{1}{2}$,

L.H.S. $= cos^{-1}\frac{1}{2} + sin^{-1}1 = \frac{\pi}{3} + \frac{\pi}{2} \neq \frac{\pi}{6} \neq$ R.H.S.

For $x = -\frac{1}{2}$,

L.H.S. $= cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}(-1)$

$= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6} =$ R.H.S.

$\therefore$ The solution set is $\left\{-\frac{1}{2}\right\}$.

Exercise 2

1. Prove :

(1) $sin^{-1}(2x\sqrt{1-x^2}) = 2sin^{-1}x, \quad |x| < \frac{1}{\sqrt{2}}$

(2) $cos^{-1}(2x^2 - 1) = 2cos^{-1}x, \quad 0 < x < 1$

(3) $cos^{-1}(4x^3 - 3x) = 3cos^{-1}x, \quad \frac{1}{2} < x < 1$

(4) $cot^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\pi}{2} - \frac{1}{2}tan^{-1}x$

(5) $sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2tan^{-1}x, \ |x| \leq 1$

(6) $tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3tan^{-1}x, \ 0 < x < \frac{1}{\sqrt{3}}$.

(7) $cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right) = \frac{x}{2}, \quad 0 < x < \frac{\pi}{2}$.

(8) $tan^{-1}\left(\frac{\sqrt{1+cosx}+\sqrt{1-cosx}}{\sqrt{1+cosx}-\sqrt{1-cosx}}\right) = \frac{\pi}{4} - \frac{x}{2}, \quad \pi < x < \frac{3\pi}{2}$.

(9) $tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}cos^{-1}x^2, \quad -1 < x < 1, \quad x \neq 0$

(10) $tan^{-1}\left(\frac{acosx - bsinx}{bcosx + asinx}\right) = tan^{-1}\left(\frac{a}{b}\right) - x, -\frac{\pi}{2} < x < \frac{\pi}{2}, \frac{a}{b}\ tanx > -1$

(11) $sin^{-1}\left(\frac{sinx + cosx}{\sqrt{2}}\right) = \frac{\pi}{4} + x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$

2. (1) If $tan^{-1}x + tan^{-1}y + tan^{-1}z = \pi$, then prove that $x + y + z = xyz$

(2) If $cot^{-1}\frac{1}{x} + cot^{-1}\frac{1}{y} + cot^{-1}\frac{1}{z} = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$

(3) If $cot^{-1}a + cot^{-1}b + cot^{-1}c = \pi$, then prove that $ab + bc + ca = 1$

(4) If $a > b > c > 0$, then prove that $cot^{-1}\left(\frac{ab+1}{a-b}\right) + cot^{-1}\left(\frac{bc+1}{b-c}\right) + cot^{-1}\left(\frac{ca+1}{c-a}\right) = \pi$.

(5) If $tan^{-1}\frac{yz}{xr} + tan^{-1}\frac{zx}{yr} + tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$, then prove that $x^2 + y^2 + z^2 = r^2$.

(6) If $tan^{-1}\sqrt{\frac{ar}{bc}} + tan^{-1}\sqrt{\frac{br}{ca}} + tan^{-1}\sqrt{\frac{cr}{ab}} = \pi$, then prove that $a + b + c = r$. $(a, b, c, r > 0)$

(7) If $sin^{-1}x + sin^{-1}y + sin^{-1}z = \pi$, then prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

(8) Prove that $tan\left(\frac{\pi}{4} + \frac{1}{2}cos^{-1}\frac{a}{b}\right) + tan\left(\frac{\pi}{4} - \frac{1}{2}cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$

(9) Prove : $\sum_{r=1}^{n} tan^{-1}\left(\frac{1}{1+r(r+1)}\right) = tan^{-1}(n+1) - \frac{\pi}{4}$.

(10) $tan^{-1}\left(\frac{1}{2}tan2A\right) + tan^{-1}(cotA) + tan^{-1}(cot^3A) = \begin{cases} 0, \text{ if } \frac{\pi}{4} < A < \frac{\pi}{2} \\ \pi, \text{ if } 0 < A < \frac{\pi}{4} \end{cases}$

3. **Solve the following equations :**

(1) $tan^{-1}\frac{x-1}{x-2} + tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$.

(2) $tan^{-1}2x + tan^{-1}3x = \frac{\pi}{4}$.

(3) $2tan^{-1}(cosx) = tan^{-1}(2cosecx)$

(4) $sin^{-1}x + cos^{-1}2x = \frac{\pi}{6}$

(5) $sin^{-1}\frac{5}{x} + sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

(6) $tan^{-1}(x+1) + tan^{-1}(x-1) = tan^{-1}\frac{8}{31}$

(7) $tan^{-1}2x + tan^{-1}\left(\frac{1}{x+4}\right) = \frac{\pi}{2}$

4. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

Section A (1 mark)

(1) $sin\left(3sin^{-1}\frac{1}{3}\right) = ...$ ☐

(a) $\frac{23}{27}$ (b) $\frac{1}{3}$ (c) $\frac{27}{23}$ (d) $\frac{2\sqrt{3}}{9}$

(2) If $sin^{-1}x = \frac{\pi}{7}$ for some $x \in (-1, 1)$, then the value of $cos^{-1}x$ = ...

(a) $\frac{3\pi}{14}$ (b) $\frac{5\pi}{14}$ (c) $\frac{\pi}{14}$ (d) $\frac{6\pi}{7}$

(3) $sec^2(tan^{-1}2) + cosec^2(cot^{-1}3)$ = ...

(a) 15 (b) 6 (c) 13 (d) 25

(4) $cos^{-1}\left(cos\frac{7\pi}{6}\right)$ = ...

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $-\frac{\pi}{6}$ (d) $\frac{7\pi}{6}$

(5) The domain of cos^{-1} is ..... .

(a) $(-\infty, \infty)$ (b) $[0, 1]$ (c) $[0, \pi]$ (d) $[-1, 1]$

(6) The range of tan^{-1} is ..... .

(a) $(-\pi, \pi)$ (b) R (c) $(0, \pi)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(7) The value of $cos^{-1}\left(cos\left(-\frac{\pi}{3}\right)\right)$ is ..... .

(a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{2\pi}{3}$

(8) $sin^{-1}\left(cos\frac{\pi}{6}\right)$ is equal to ..... .

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

(9) The value of $sin^{-1}\left(sin\frac{5\pi}{3}\right)$ is ..... .

(a) $-\frac{\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

(10) $cos\left(cos^{-1}\left(-\frac{1}{5}\right) + sin^{-1}\left(-\frac{1}{5}\right)\right)$ is ..... .

(a) $\frac{4}{9}$ (b) $\frac{1}{3}$ (c) 0 (d) $-\frac{1}{3}$

(11) $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is ..... .

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{4\pi}{3}$ (d) $\frac{4\pi}{6}$

(12) $sin^{-1}\left(sin\frac{7\pi}{6}\right)$ is ..... .

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{-\pi}{6}$ (d) $\frac{7\pi}{6}$

(13) $sin\left\{\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right\}$ is ..... .

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

(14) Value of $sin\left(cos^{-1}\frac{4}{5}\right)$ is ..... .

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

(15) Value of $cos\left(tan^{-1}\frac{4}{3}\right)$ is ..... .

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{3}{5}$

Section B (2 marks)

(16) $2tan^{-1}5 + tan^{-1}\frac{5}{12} = ...$

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$ (c) π (d) $\frac{\pi}{2}$

(17) If $sin^{-1}x + sin^{-1}x = \frac{2\pi}{3}$, then $cos^{-1}x + cos^{-1}y = ...$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) π

(18) If $4sin^{-1}x + cos^{-1}x = \pi$, then $x = ...$

(a) $-\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

(19) $sin\left(tan^{-1}\left(tan\frac{7\pi}{6}\right)\right) + cos\left(cos^{-1}\left(cos\frac{7\pi}{3}\right)\right) = ...$

(a) -1 (b) 0 (c) 1 (d) $\frac{\sqrt{3}}{2}$

(20) If $cos(2sin^{-1}x) = \frac{1}{9}$, then the value of $x = ...$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 1

(21) The value of $sin[2sin^{-1}(cosA)]$ is .....

(a) $sinA$ (b) $cosA$ (c) $cos2A$ (d) $sin2A$

(22) The value of $sin\left[3sin^{-1}\left(\frac{1}{5}\right)\right]$ is .....

(a) $-\frac{3}{5}$ (b) $\frac{79}{12}$ (c) $-\frac{71}{125}$ (d) $\frac{71}{125}$

(23) $tan^{-1}\left(-tan\frac{13\pi}{8}\right)$ is .....

(a) $-\frac{5\pi}{8}$ (b) $\frac{3\pi}{8}$ (c) $-\frac{3\pi}{8}$ (d) $\frac{13\pi}{8}$

(24) $sin^{-1}\left(sin\frac{32\pi}{7}\right)$ is .....

(a) $\frac{3\pi}{7}$ (b) $\frac{4\pi}{7}$ (c) $\frac{18\pi}{7}$ (d) $\frac{32\pi}{7}$

(25) Value of $cos\left[\frac{\pi}{6} + cos^{-1}\left(-\frac{1}{2}\right)\right]$ is .....

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{5}-1}{4}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(26) $tan^{-1}2 + tan^{-1}3$ is .....

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{3\pi}{2}$

(27) The value of $sin\left[tan^{-1}(-\sqrt{3}) + cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is = ...

(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

(28) $sin^{-1}\frac{3}{5} + tan^{-1}\frac{1}{7}$ is .....

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $sin^{-1}\frac{4}{5}$

(29) The value of $tan\left(cos^{-1}\frac{3}{4} + sin^{-1}\frac{3}{4} - sec^{-1}3\right)$ is .....

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2\sqrt{3}}$ (d) $\frac{1}{2\sqrt{2}}$

(30) The value of $sec\left[tan^{-1}\left(\frac{b+a}{b-a}\right) - tan^{-1}\left(\frac{a}{b}\right)\right]$ is .....

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

Section C (3 marks)

(31) The value of $cot\left[\frac{\pi}{4} - 2cot^{-1}3\right]$ is .....

(a) 3 (b) 7 (c) 9 (d) $\frac{3}{4}$

(32) $tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\left(\frac{x-y}{x+y}\right) = ...\ \left(\frac{x}{y} \geq 0\right)$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(33) If $x = \frac{1}{3}$, then the value of $cos(2cos^{-1}x + sin^{-1}x) = ...$

(a) $-\sqrt{\frac{8}{9}}$ (b) $-\sqrt{\frac{1}{3}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

(34) $cos^{-1}\left(\frac{x}{5}\right) + cosec^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is .....

(a) 1 (b) 3 (c) 5 (d) 4

(35) The value of $cot\left(cosec^{-1}\frac{5}{3} + tan^{-1}\frac{2}{3}\right)$ is .....

(a) $\frac{3}{17}$ (b) $\frac{4}{17}$ (c) $\frac{5}{17}$ (d) $\frac{6}{17}$

(36) $tan\left(2cos^{-1}\frac{3}{5}\right)$ is .....

(a) $\frac{8}{3}$ (b) $\frac{24}{25}$ (c) $\frac{7}{25}$ (d) $-\frac{24}{7}$

(37) The value of $tan\left[\frac{1}{2}cos^{-1}\frac{\sqrt{5}}{3}\right]$ is .....

(a) $\frac{2+\sqrt{3}}{\sqrt{2}}$ (b) $\frac{3-\sqrt{5}}{2}$ (c) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (d) $\frac{\sqrt{5}+1}{4}$

(38) If $0 < x < 1$, then $tan^{-1}\left(\frac{\sqrt{1-x^2}}{1+x}\right)$ is .....

(a) $\frac{1}{2}sin^{-1}\sqrt{\frac{1-x}{2}}$ (b) $\frac{1}{2}cos^{-1}x$ (c) $\frac{1}{2}cot^{-1}\left(\frac{1-x}{1+x}\right)$ (d) $\frac{1}{2}tan^{-1}\left(\frac{x}{2}\right)$

(39) If $cos(2tan^{-1}x) = \frac{1}{2}$, then the value of x is .....

(a) $\frac{1}{\sqrt{3}}$ (b) $1 - \sqrt{3}$ (c) $1 - \frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

(40) The value of $tan\left\{sin^{-1}\left(\frac{3}{5}\right) + cos^{-1}\left(\frac{5}{13}\right)\right\}$ is .....

(a) $-\frac{24}{5}$ (b) $-\frac{22}{15}$ (c) $-\frac{63}{16}$ (d) $-\frac{47}{12}$

(41) If $sin^{-1}\frac{x}{5} + cosec^{-1}\frac{5}{4} = \frac{\pi}{2}$, then x is .....

(a) 1 (b) 2 (c) 3 (d) 4

(42) $sin^{-1}(cos(sin^{-1}x)) + cos^{-1}(sin(cos^{-1}x))$ is .....

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

Section D (4 marks)

(43) $cot^{-1}\left(\frac{\sqrt{1-sinx}+\sqrt{1+sinx}}{\sqrt{1-sinx}-\sqrt{1+sinx}}\right) = ...$ $\left(0 < x < \frac{\pi}{2}\right)$

(a) $\frac{x}{2}$ (b) $\frac{\pi}{2} - 2x$ (c) $2\pi - x$ (d) $\pi - \frac{x}{2}$

(44) If $sin^{-1}\frac{1}{x} = 2tan^{-1}\frac{1}{7} + cos^{-1}\frac{3}{5}$, then $x = ...$

(a) $\frac{24}{117}$ (b) $\frac{7}{3}$ (c) $\frac{125}{117}$ (d) $-\frac{117}{44}$

(45) If $\alpha = cos^{-1}\left(\frac{4}{5}\right)$, $\beta = tan^{-1}\left(\frac{2}{3}\right)$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, then $\alpha - \beta = ...$

(a) $sin^{-1}\frac{2}{\sqrt{13}}$ (b) $tan^{-1}\left(\frac{1}{18}\right)$ (c) $cos^{-1}\left(\frac{1}{5\sqrt{3}}\right)$ (d) $sin^{-1}\left(\frac{6}{5\sqrt{13}}\right)$

(46) **Match the following :**

Column (A)	Column (B)
(1) $tan^{-1}\left(\frac{1}{3}\right) + tan^{-1}\left(\frac{1}{4}\right)$	(a) $\frac{\pi}{2}$
(2) $sin^{-1}\left(\frac{3}{5}\right) + sin^{-1}\left(\frac{8}{17}\right) + sin^{-1}\left(\frac{36}{85}\right)$	(b) π
(3) $tan^{-1}(1) + cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right)$	(c) $tan^{-1}\left(\frac{7}{11}\right)$
(4) $2tan^{-1}(5) + tan^{-1}\left(\frac{5}{12}\right)$	(d) $\frac{3\pi}{4}$

(a) 1 - c, 2- b, 3 - d, 4 - a (b) 1 - c, 2 - a, 3 - d, 4 - b

(c) 1 - c, 2 - a, 3 - b, 4 - d (d) 1 - a, 2 - b, 3 - d, 4 - c

(47) $tan\left(2tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = ...$

(a) $\frac{14}{33}$ (b) $\frac{-7}{17}$ (c) $\frac{17}{7}$ (d) $\frac{24}{25}$

(48) If $sin^{-1}(1 - x) - 2sin^{-1}x = \frac{\pi}{2}$, then x is .....

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

(49) The number of values of x satisfying the equation

$tan^{-1}(x + 1) + tan^{-1}x + tan^{-1}(x - 1) = tan^{-1}3x$ is .....

(a) 2 (b) 3 (c) 4 (d) infinite

(50) If $cot^{-1}x + cot^{-1}y + cot^{-1}z = \frac{\pi}{2}$, then $x + y + z = ...$

(a) $xy + yz + zx$ (b) xyz (c) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ (d) $\frac{xy + yz + zx}{3}$

(51) If $sin^{-1}\left(\frac{2a}{1+a^2}\right) + sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2tan^{-1}x$, then x is ..... $(0 < a, b < 1)$

(a) $\frac{a-b}{1+ab}$ (b) $\frac{a+b}{1-ab}$ (c) $\frac{b}{1-ab}$ (d) $\frac{b}{1+ab}$

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Summary

We have studied the following points in this chapter :

1. Definition of inverse trigonometric functions.
2. Graphs of inverse trigonometric functions.
3. (1) $sin^{-1}(-x) = -sin^{-1}x, \quad |x| \leq 1$

   (2) $cos^{-1}(-x) = \pi - cos^{-1}x, \quad |x| \leq 1$

   (3) $tan^{-1}(-x) = -tan^{-1}x, \quad x \in R$

   (4) $cot^{-1}(-x) = \pi - cot^{-1}x, \quad x \in R$

   (5) $cosec^{-1}(-x) = -cosec^{-1}x, \quad |x| \geq 1$

   (6) $sec^{-1}(-x) = \pi - sec^{-1}x, \quad |x| \geq 1$
4. (1) $cosec^{-1}x = sin^{-1}\frac{1}{x}, \quad |x| \geq 1$

   (2) $sec^{-1}x = cos^{-1}\frac{1}{x}, \quad |x| \geq 1$

   (3) $cot^{-1}x = tan^{-1}\frac{1}{x}, \quad x > 0$

   $= \pi + tan^{-1}\frac{1}{x}, \quad x < 0$

5. (1) $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}, \quad |x| \leq 1$

(2) $cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}, \quad |x| \geq 1$

(3) $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

6. If $x > 0$, $y > 0$, then

(1) $tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(2) $tan^{-1}x + tan^{-1}y = \pi + tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy > 1$

(3) $tan^{-1}x + tan^{-1}y = \frac{\pi}{2}$, if $xy = 1$

(4) $tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

7. (1) $sin^{-1}x = cos^{-1}\sqrt{1-x^2} = tan^{-1}\frac{x}{\sqrt{1-x^2}}$, if $0 < x < 1$

(2) $cos^{-1}x = sin^{-1}\sqrt{1-x^2} = tan^{-1}\frac{\sqrt{1-x^2}}{x}$, if $0 < x < 1$

(3) $tan^{-1}x = cos^{-1}\frac{1}{\sqrt{1+x^2}} = sin^{-1}\frac{x}{\sqrt{1+x^2}}$

Srinivasa Ramanujan : Adulthood in India

On 14 July 1909, Ramanujan was married to a nine-year old bride, Janaki Ammal. In the branch of Hinduism to which Ramanujan belonged, marriage was a formal engagement that was consummated only after the bride turned 17 or 18, as per the traditional calendar.

After the marriage, Ramanujan developed a hydrocele testis, an abnormal swelling of the tunica vaginalis, an internal membrane in the testicle. The condition could be treated with a routine surgical operation that would release the blocked fluid in the scrotal sac. His family did not have the money for the operation, but in January 1910, a doctor volunteered to do the surgery for free.

After his successful surgery, Ramanujan searched for a job. He stayed at friends' houses while he went door to door around the city of Madras (now Chennai) looking for a clerical position. To make some money, he tutored some students at Presidency College who were preparing for their F.A. exam.

In late 1910, Ramanujan was sick again, possibly as a result of the surgery earlier in the year. He feared for his health, and even told his friend, R. Radakrishna Iyer, to "hand these [my mathematical notebooks] over to Professor Singaravelu Mudaliar [mathematics professor at Pachaiyappa's College] or to the British professor Edward B. Ross, of the Madras Christian College." After Ramanujan recovered and got back his notebooks from Iyer, he took a northbound train from Kumbakonam to Villupuram, a coastal city under French control.

DETERMINANTS 3

In mathematics, the art of proposing a question must be held of higher value than solving it.

– Georg Cantor

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Mathematics is the cheapest science. Unlike Physics and Chemistry, it does not require any expensive equipment. All one needs is a pencil and paper.

– George Polya

3.1 Introduction

The expression $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a 2×2 determinant and its value is defined to be $ad - bc$. This is another symbolic way to present real number $ad - bc$. Do you remember the method of solving a pair of simultaneous linear equations in R^2 by the method of cross multiplication ? Do you realise the connection now ?

3.2 Second Order Determinants

Symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, where a, b, c, d are real numbers, is called a second order determinant. These real numbers are called the **elements** or **entries** of the determinant. $a \quad b$ is the **first row**, $c \quad d$ is the **second row**, $\begin{matrix} a \\ c \end{matrix}$ is the **first column** $\begin{matrix} b \\ d \end{matrix}$ is the **second column**, $\begin{matrix} a & \\ & d \end{matrix}$ is the **principal diagonal**, $\begin{matrix} & b \\ c & \end{matrix}$ is the **secondary diagonal** of the determinant. **The expression $ad - bc$ is called the value of the given determinant.** We write $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Thus, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = the product of the elements on the principal diagonal minus the product of the elements on the secondary diagonal.

So, for instance $\begin{vmatrix} 1 & 5 \\ 4 & 6 \end{vmatrix} = (1)(6) - (4)(5) = 6 - 20 = -14$

3.3 Third Order Determinants

If we want to solve a system of three simultaneous linear equations in three variables, we shall have to deal with a **'third order determinant'**.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a third order determinant.

Here a_i, b_i, c_i are real numbers (i = 1, 2, 3). These real numbers are the elements or entries of the determinant. $a_1 \quad b_1 \quad c_1$; $a_2 \quad b_2 \quad c_2$; $a_3 \quad b_3 \quad c_3$ are the first, second and third rows respectively. $\begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}$; $\begin{matrix} b_1 \\ b_2 \\ b_2 \end{matrix}$; $\begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix}$ are the first, second and third columns respectively.

The value of a third order determinant is written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$ is the expansion or the value of the third order determinant.

Now, further we shall use D for both, a determinant as well as for its value.

Note : In obtaining the value of a third order determinant, the multiplier of a_1 is the second order determinant obtained by removing the row and the column of the given determinant containing a_1 and keeping the other entries in the same position. We get it as $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$. The same applies for the multiplier of b_1 and c_1.

3.4 Some Symbols

When we work with determinants, it is convenient to convert one determinant into another by performing certain operations on rows and on columns. To denote them in a precise form, we shall use some symbols. We introduce them now.

(1) $R_i \to C_i$: We shall use this symbol, if we want to convert every row into the corresponding column (or every column into the corresponding row). Performing the operation $R_i \to C_i$ on D, the given determinant D will become $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

(2) R_{ij} (C_{ij}) $(i \neq j)$: If we want to interchange ith and jth rows (columns), we shall use symbol R_{ij} (C_{ij}).

e.g. C_{23} is the process to interchanging second and third columns.

So, if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then after performing C_{23} on D,

we get a new determinant

$$D' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

(3) $R_i(k)$ $[C_i(k)]$: Denotes the symbol of multiplying all elements of ith row (column) by k. $k \in R - \{0\}$ (called multiply ith row (column) by k.)

If, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ on performing $R_1(3)$ on the given determinant, D changes to

$$D' = \begin{vmatrix} 3a_1 & 3b_1 & 3c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(4) $R_{ij}(k)$ $[C_{ij}(k)]$ $(i \neq j)$: This symbol is used for the operation of multiplying each elements of the ith row (column) by a non-zero real number k and adding its elements to the corresponding elements of jth row (column).

Thus, performing $R_{31}(-2)$ on the determinant, D changes to $D' = \begin{vmatrix} a_1 - 2a_3 & b_1 - 2b_3 & c_1 - 2c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Example 1 : Evaluate : **(1)** $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$ **(2)** $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}$.

Solution : (1) We have $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = 1 \cdot (-4) - (-2) \cdot 3 = -4 + 6 = 2$

(2) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix} = (x + 1)(x - 1) - (x)(x) = x^2 - 1 - x^2 = -1$

Example 2 : Evaluate $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$.

Solution : $D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -2 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$

$= 2(-6 - 3) - 3(-3 + 6) - 2(1 + 4)$

$= -18 - 9 - 10 = -37$

Exercise 3.1

1. Evaluate :

(1) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix}$ **(2)** $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ **(3)** $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix}$

2. Solve :

(1) $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$ **(2)** $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$.

3. Evaluate :

(1) $\begin{vmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 9 & -5 & 4 \end{vmatrix}$ **(2)** $\begin{vmatrix} 2 & 1 & -2 \\ -3 & 7 & 1 \\ 5 & -3 & 4 \end{vmatrix}$

4. Prove : $\begin{vmatrix} -\cos\alpha & \sin\beta & 0 \\ 0 & -\sin\alpha & \cos\alpha \\ \sin\alpha & 0 & -\sin\beta \end{vmatrix} = 0$

*

3.5 Properties of Determinants

We shall now prove some properties of determinants. We shall consider third order determinants. They are also true for second order determinants.

Theorem 3.1 : If all the rows of a determinant are converted into the corresponding columns, the value of the determinant remains same.

[**Note :** If $R_i \rightarrow C_i$ is performed on D, then the value of determinant does not change.]

Proof : Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Converting all the rows into the corresponding columns (i.e. performing the operation $R_i \rightarrow C_i$), the new determinant D' obtained is,

$$D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= D$$

So, converting all the rows into corresponding columns, the value of a determinant does not change.

Example 3 : Evaluate $D = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 5 & -3 & 4 \end{vmatrix}$. Verify that the value of D does not change while performing $R_i \rightarrow C_i$ on D.

Solution : $D = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 5 & -3 & 4 \end{vmatrix} = 2(16 + 6) - 1(12 - 10) + (-1)(-9 - 20)$

$$= 44 - 2 + 29 = 71$$

Now performing $R_i \rightarrow C_i$ on D gives

$$D' = \begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & -3 \\ -1 & 2 & 4 \end{vmatrix} = 2(16 + 6) - 3(4 - 3) + 5(2 + 4)$$

$$= 44 - 3 + 30 = 71$$

So, D = D'.

Theorem 3.2 : If two rows (columns) of a determinant are interchanged, the value of the new determinant is the additive inverse of the value of the given determinant.

[**Note :** If R_{ij} (C_{ij}) is performed on D and we get new determinant D', then D' = −D]

Proof : Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Interchanging, the first and the second rows (i.e. performing R_{12}), we get a new determinant

$$D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1)$$

$$= a_2b_1c_3 - a_2b_3c_1 - b_2a_1c_3 + b_2a_3c_1 + c_2a_1b_3 - c_2a_3b_1$$

$$= -[a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1]$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -D$$

Thus, the value of the determinant obtained by interchanging any two rows is the additive inverse of the value of the given determinant. We can have the same result for columns also. (Try it !)

Example 4 : Evaluate $D = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix}$. Verify that the value of the new determinant obtained by performing the operation C_{23} is the additive inverse of the value of D.

Solution : $D = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 1(16 + 1) - 2(-12 + 2) + 3(-3 - 8)$

$$= 17 + 20 - 33 = 4$$

Now, performing operation C_{23}, we get

$$D' = \begin{vmatrix} 1 & 3 & 2 \\ -3 & -1 & 4 \\ 2 & 4 & 1 \end{vmatrix} = 1(-1 - 16) - 3(-3 - 8) + 2(-12 + 2)$$

$$= -17 + 33 - 20 = -4$$

So, $D' = -D$.

Theorem 3.3 : The value of a determinant gets multiplied by k, if every entry in any of its row (column) is multiplied by $k (k \neq 0)$.

[**Note :** If the result of $R_i(k)$ or $C_i(k)$ on D is D', then $D' = kD$. $(k \neq 0)$]

Proof : Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Multiplying all the elements of its first row by k (i.e. performing $R_1(k)$); we get

$$D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$$

$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= kD$$

Hence, $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. $k \in \mathbf{R}$

Note : If all the elements of a row or a column of a determinant are multiplied by a non-zero real number k and the value of the determinant is divided by k, we get the value of the given determinant, i.e. $\frac{1}{k}\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(k \neq 0)$

Example 5 : Evaluate $D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 1 & -3 & 2 \end{vmatrix}$. Verify that the value of D becomes 3 times its value after performing $C_1(3)$.

Solution : $D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 1 & -3 & 2 \end{vmatrix} = 3(10 + 18) - 2(8 - 6) + 1(-12 - 5)$

$$= 84 - 4 - 17 = 63$$

Now, performing operation $C_1(3)$, we get

$$D' = \begin{vmatrix} 9 & 2 & 1 \\ 12 & 5 & 6 \\ 3 & -3 & 2 \end{vmatrix} = 9(10 + 18) - 2(24 - 18) + 1(-36 - 15)$$

$$= 252 - 12 - 51$$

$$= 189$$

$$= 3(63)$$

$\therefore \quad D' = 3(D)$.

Theorem 3.4 : $\begin{vmatrix} a_1 + d_1 & b_1 + e_1 & c_1 + f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Proof : L.H.S. $= (a_1 + d_1)(b_2c_3 - b_3c_2) - (b_1 + e_1)(a_2c_3 - a_3c_2) + (c_1 + f_1)(a_2b_3 - a_3b_2)$

$$= a_1(b_2c_3 - b_3c_2) + d_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) - e_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) + f_1(a_2b_3 - a_3b_2)$$

$$= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] + [d_1(b_2c_3 - b_3c_2) - e_1(a_2c_3 - a_3c_2) + f_1(a_2b_3 - a_3b_2)]$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{R.H.S.}$$

[**Note :** Thus a determinant can be written as a sum of two determinants. This applies to any row or column. Theorem 3.3 and 3.4 suggest that a determinant is a linear (multilinear) function of its rows or columns. Thus a determinant is a 'multilinear' form.]

Theorem 3.5 : If the corresponding entries in any two rows (columns) are identical, the value of the determinant is zero.

Proof : Suppose the first and second rows of a determinant are identical.

$$\text{i.e. } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By interchanging the first and the second row (i.e. performing R_{12}) we should get $-D$.

(Theorem 3.2)

But we see that D remains unchanged, since the first two rows are identical. So D' = D.

Hence D' = $-$D and D' = D, So D = $-$D.

$\therefore$ 2D = 0. Hence D = 0

Thus, if two rows or columns are identical, then determinant has value 0.

Theorem 3.6 : The value of a determinant does not change if any of its rows (columns) is multiplied by non-zero real number k and added to another row (column). ($k \neq 0$)

Proof : Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let D' be the determinant obtained from D by performing $R_{21}(k)$ on it. Then we get

$$D' = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \textbf{(Theorem 3.4)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \textbf{(Theorem 3.3)}$$

$$= D + k(0) \qquad \textbf{(Theorem 3.5)}$$

$$= D$$

Thus, a determinant does not change if all elements of any row (column) are multiplied by a non-zero real number k and added to the corresponding elements of another row (column).

Example 6 : Prove that $\begin{vmatrix} 1 & a & b \\ 1 & a+b & b \\ 1 & a & a+b \end{vmatrix} = ab$

Solution : Let $D = \begin{vmatrix} 1 & a & b \\ 1 & a+b & b \\ 1 & a & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -b & 0 \\ 1 & a+b & b \\ 1 & a & a+b \end{vmatrix} \qquad [R_{21}(-1)]$$

$$= b(a + b - b) \qquad \textbf{(terms with multiplier 0 are 0)}$$

$$= ab$$

Example 7 : Without expanding, prove that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution : Let $D = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix} \qquad (R_i \to C_i)$$

$$= (-1)^3 \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \qquad (R_i(-1),\ i = 1, 2, 3)$$

$$= -D$$

Thus $D = -D$, giving $2D = 0$ or $D = 0$.

Example 8 : Prove that the value of $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is independent of θ.

Solution : Let $D = \begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix} \qquad (R_{21}(\sin\phi),\ R_{31}(\cos\phi))$$

$$= (\cos 2\phi + 1)(\sin^2\theta + \cos^2\theta)$$

$= 1 + \cos 2\phi$, which is independent of θ.

Example 9 : Without expanding, show that 11 divides $\begin{vmatrix} 2 & 6 & 4 \\ 5 & 0 & 6 \\ 3 & 5 & 2 \end{vmatrix}$.

Solution : Performing $C_{13}(100)$ and $C_{23}(10)$, we get

$$\begin{vmatrix} 2 & 6 & 4 \\ 5 & 0 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 264 \\ 5 & 0 & 506 \\ 3 & 5 & 352 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 6 & 11\times 24 \\ 5 & 0 & 11\times 46 \\ 3 & 5 & 11\times 32 \end{vmatrix}$$

$$= 11\begin{vmatrix} 2 & 6 & 24 \\ 5 & 0 & 46 \\ 3 & 5 & 32 \end{vmatrix} \qquad \left(C_3\left(\frac{1}{11}\right)\right)$$

$$= 11 \cdot n \qquad (n \in Z)$$

$\therefore$ 11 divides the given determinant.

Example 10 : By using theorems, prove that $\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = -(a-b)(b-c)(c-a)$.

$(a \neq b,\ b \neq c,\ c \neq a)$

Solution : $\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a-b & b-c & c \\ 0 & 0 & 1 \end{vmatrix}$ (First $C_{21}(-1)$ and then $C_{32}(-1)$)

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ 1 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} \qquad \left(C_1\left(\frac{1}{a-b}\right),\ C_2\left(\frac{1}{b-c}\right),\ a \neq b,\ b \neq c\right)$$

$$= (a-b)(b-c)\begin{vmatrix} a-c & b+c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} \qquad [C_{21}(-1)]$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 1 & b+c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} \qquad \left(C_1\left(\frac{1}{a-c}\right),\ (a \neq c)\right)$$

$$= (a-b)(b-c)(a-c) \cdot [1]$$

$$= -(a-b)(b-c)(c-a)$$

[**Note :** If $a = b$ or $b = c$ or $c = a$, D = 0. Hence the result is true in this case also.]

Example 11 : Without expanding the determinant, prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 0$

Solution : Let $D = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \qquad [C_{23}(1)]$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \qquad \left(C_3\left(\frac{1}{x+y+z}\right),\ x+y+z \neq 0\right)$$

$$= (x+y+z)\ (0) \qquad [R_1 = R_3]$$

$$= 0$$

[**Note :** Even if $x + y + z = 0$, $R_3 = 0$ and the expansion gives D = 0.]

Example 12 : Solve $\begin{vmatrix} 2x+3 & 3x+4 & 4x+5 \\ x+2 & 2x+3 & 3x+4 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

Solution : $\begin{vmatrix} 2x+3 & 3x+4 & 4x+5 \\ x+2 & 2x+3 & 3x+4 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

$\therefore \begin{vmatrix} 2x+3 & 3x+4 & 4x+5 \\ x+2 & 2x+3 & 3x+4 \\ x+2 & 2x+4 & 6x+12 \end{vmatrix} = 0$ **[$R_{13}(-1)$]**

$\therefore \begin{vmatrix} x+1 & x+1 & x+1 \\ x+2 & 2x+3 & 3x+4 \\ x+2 & 2(x+2) & 6(x+2) \end{vmatrix} = 0$ **[$R_{21}(-1)$]**

$\therefore (x+1)(x+2) \begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2x+3 & 3x+4 \\ 1 & 2 & 6 \end{vmatrix} = 0$

$\therefore (x+1)(x+2) \begin{vmatrix} 1 & 0 & 0 \\ x+2 & x+1 & x+1 \\ 1 & 1 & 4 \end{vmatrix} = 0$ **($C_{23}(-1)$ and $C_{12}(-1)$)**

$\therefore (x+1)(x+2)[(x+1)4 - 1(x+1)] = 0$

$\therefore 3(x+1)(x+2) \cdot (x+1) = 0$

$\therefore x = -1$ or $x = -2$

$\therefore$ The solution set is $\{-1, -2\}$.

Example 13 : Solve : $\begin{vmatrix} x & 4 & 6 \\ 2 & 3 & -9 \\ 5 & 6 & 1 \end{vmatrix} + \begin{vmatrix} 5 & 6 & 1 \\ 6 & 4 & 5 \\ 2 & 3 & -9 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -9 \\ 2x-1 & -8 & -11 \\ 5 & 6 & 1 \end{vmatrix}$.

Solution : $(-1)\begin{vmatrix} 2 & 3 & -9 \\ x & 4 & 6 \\ 5 & 6 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 3 & -9 \\ 6 & 4 & 5 \\ 5 & 6 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 3 & -9 \\ 1-2x & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix}$

(Performing R_{12} in the first determinant, R_{13} in the second determinant and R_2 (-1) in the third determinant.)

$\therefore \begin{vmatrix} 2 & 3 & -9 \\ x & 4 & 6 \\ 5 & 6 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & -9 \\ 6 & 4 & 5 \\ 5 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -9 \\ 1-2x & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix}$

$\therefore \begin{vmatrix} 2 & 3 & -9 \\ x+6 & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -9 \\ 1-2x & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix}$ **(Theorem 3.4)**

$\therefore \begin{vmatrix} 2 & 3 & -9 \\ x+6 & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 3 & -9 \\ 1-2x & 8 & 11 \\ 5 & 6 & 1 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} 2 & 3 & -9 \\ 3x+5 & 0 & 0 \\ 5 & 6 & 1 \end{vmatrix} = 0$$

$$\therefore (3x+5)\begin{vmatrix} 2 & 3 & -9 \\ 1 & 0 & 0 \\ 5 & 6 & 1 \end{vmatrix} = 0$$

$$\therefore 3x+5=0 \text{ as } 1\begin{vmatrix} 3 & -9 \\ 6 & 1 \end{vmatrix} \neq 0$$

$$\therefore x = -\frac{5}{3}$$

$\therefore$ The solution set is $\left\{-\frac{5}{3}\right\}$.

Exercise 3.2

1. Prove using theorems,

(1) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x).$

(2) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$

(3) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ (x+1)^2 & (y+1)^2 & (z+1)^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

(4) $\begin{vmatrix} x & y & z \\ x-y & y-z & z-x \\ y+z & z+x & x+y \end{vmatrix} = x^3+y^3+z^3-3xyz.$

(5) $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3.$

2. Without expanding, prove that $\begin{vmatrix} \cos\alpha & \sin\alpha & \cos(\alpha+\delta) \\ \cos\beta & \sin\beta & \cos(\beta+\delta) \\ \cos\gamma & \sin\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

3. Find the solution set of $\begin{vmatrix} x & 5 & 9 \\ 16 & 3x+8 & 36 \\ 3 & 1 & 7 \end{vmatrix} = 0.$

4. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix} = -a^3.$

5. Solve $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0;\ 0 < \theta < \frac{\pi}{2}.$

6. Prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

7. Find the solution set of $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

8. Prove that the roots of the equation $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ are $x = 0$ and $x = 3a$.

9. Prove that $\begin{vmatrix} x^2 & yz & zx+z^2 \\ x^2+xy & y^2 & zx \\ xy & y^2+yz & z^2 \end{vmatrix} = 4x^2y^2z^2$.

10. Prove that $\begin{vmatrix} a+bx & d+ex & p+qx \\ ax+b & dx+e & px+q \\ c & f & r \end{vmatrix} = (1-x^2)\begin{vmatrix} a & d & p \\ b & e & q \\ c & f & r \end{vmatrix}$.

11. Prove that $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (c+a)^2 & bc \\ ca & bc & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$.

*

3.6 Minor and Cofactor

Minor : Removing entries of the column and the row containing a given element of a determinant and keeping the surviving entries as they are, yields a determinant called the minor of the given element.

For example, to get minor of c_3 in the determinant

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we shall remove the entries of the column and the row of D containing c_3 and we shall keep the remaining entries in the same position. We get the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This determinant is the minor of c_3.

Cofactor : If we multiply the minor of an element by $(-1)^{i+j}$, where i is the number of the row and j is the number of the column containing the element, then we get the cofactor of that element.

In $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of a_2 is $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ and multiplying this by $(-1)^{2+1}$, we get cofactor of a_2. (a_2 is in second row and first column.)

Thus, the cofactor of a_2 is $(-1)^{2+1} \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1)$.

The cofactors of a_i, b_i, c_i in D are denoted by A_i, B_i, C_i respectively, where $i = 1, 2, 3$.

In advanced mathematics, we write D as

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Here, a_{ij} means the element in the ith row and jth column.

Cofactor of $a_{ij} = (-1)^{i+j}$ (minor of a_{ij})

Simply remember following symbolic determinant format $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

Element in any diagonal have the minor prefixed by + and other elements have the minor prefixed by − to get the corresponding cofactor.

[**Note :** The cofactor of an element is the factor by which that element gets multiplied in the expansion of the determinant or the multiplier of that element in the expansion of the determinant.]

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then

$A_1 = b_2c_3 - b_3c_2$ $\quad B_1 = -(a_2c_3 - a_3c_2)$ $\quad C_1 = a_2b_3 - a_3b_2$

$A_2 = -(b_1c_3 - b_3c_1)$ $\quad B_2 = a_1c_3 - a_3c_1$ $\quad C_2 = -(a_1b_3 - a_3b_1)$

$A_3 = b_1c_2 - b_2c_1$ $\quad B_3 = -(a_1c_2 - a_2c_1)$ $\quad C_3 = a_1b_2 - a_2b_1$

Example 14 : Find the minors and cofactors of 2 and −1 in $\begin{vmatrix} 1 & 2 & 7 \\ 3 & 7 & -5 \\ -1 & 4 & 3 \end{vmatrix}$.

Solution : The minor of 2 = $\begin{vmatrix} 3 & -5 \\ -1 & 3 \end{vmatrix}$

The cofactor of 2 = $(-1)^{1+2} \begin{vmatrix} 3 & -5 \\ -1 & 3 \end{vmatrix} = -1 \times 4 = -4$

The minor of −1 = $\begin{vmatrix} 2 & 7 \\ 7 & -5 \end{vmatrix}$

The cofactor of −1 = $(-1)^{3+1} \begin{vmatrix} 2 & 7 \\ 7 & -5 \end{vmatrix} = 1 \times (-59) = -59$

Example 15 : If $D = \begin{vmatrix} 1 & 4 & 0 \\ -4 & 2 & 1 \\ 0 & -1 & 3 \end{vmatrix}$, then find the value of the determinant formed by the cofactors of the elements of D.

Solution : $D = \begin{vmatrix} 1 & 4 & 0 \\ -4 & 2 & 1 \\ 0 & -1 & 3 \end{vmatrix}$

The cofactor of 1 is $A_1 = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 1 \times (6 + 1) = 7$

The cofactor of 4 is $B_1 = (-1)^{1+2}\begin{vmatrix} -4 & 1 \\ 0 & 3 \end{vmatrix} = (-1)(-12) = 12$

The cofactor of 0 is $C_1 = (-1)^{1+3}\begin{vmatrix} -4 & 2 \\ 0 & -1 \end{vmatrix} = (1)(4) = 4$

The cofactor of -4 is $A_2 = (-1)^{2+1}\begin{vmatrix} 4 & 0 \\ -1 & 3 \end{vmatrix} = (-1)(12) = -12$

The cofactor of 2 is $B_2 = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = (1)(3) = 3$

Similarly $C_2 = 1$, $A_3 = 4$, $B_3 = -1$, $C_3 = 18$. (Find by yourself !)

$\therefore$ The determinant formed by the cofactors of the elements of D is $\begin{vmatrix} 7 & 12 & 4 \\ -12 & 3 & 1 \\ 4 & -1 & 18 \end{vmatrix}$.

$$\begin{aligned} \text{Its value} &= 7(54 + 1) - 12(-216 - 4) + 4(12 - 12) \\ &= 385 + 2640 + 0 \\ &= 3025 \end{aligned}$$

[**Note :** The value of D is 55. The value of the determinant formed by cofactors is $3025 = (55)^2$. This is true in general.]

Theorem 3.7 : The value of any third order determinant can be obtained by adding the products of the elements of any of its rows or columns by their corresponding cofactors.

Proof : If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and cofactors of a_2, b_2, c_2 are A_2, B_2, C_2 respectively, then

$$A_2 = (-1)^{2+1}\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1)$$

$$B_2 = (-1)^{2+2}\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$$

$$C_2 = (-1)^{2+3}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -(a_1b_3 - a_3b_1).$$

$$\begin{aligned} \text{Now, } a_2A_2 + b_2B_2 + c_2C_2 &= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1) \\ &= -a_2b_1c_3 + a_2b_3c_1 + b_2a_1c_3 - b_2a_3c_1 - c_2a_1b_3 + c_2a_3b_1 \\ &= a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 + a_2b_3c_1 - a_3b_2c_1 \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D \end{aligned}$$

$\therefore \quad a_2A_2 + b_2B_2 + c_2C_2 = D$ (i)

Similarly $a_1A_1 + b_1B_1 + c_1C_1 = D$ (ii)

$a_3A_3 + b_3B_3 + c_3C_3 = D$ (iii)

$a_1A_1 + a_2A_2 + a_3A_3 = D$ (iv)

$b_1B_1 + b_2B_2 + b_3B_3 = D$ (v)

$c_1C_1 + c_2C_2 + c_3C_3 = D$ (vi)

Here, (i), (ii) and (iii) are called expansions of the determinant by the second, the first and the third row respectively. Similarly (iv), (v), (vi) are expansions according to the first, the second and the third column respectively.

Theorem 3.8 : If we multiply all the elements of any row (column) of a third order determinant by the cofactors of the corresponding elements of another row (column) and the products are added, then the sum is zero.

Proof : Let us multiply the elements a_1, b_1, c_1 of the first row of $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, by the cofactors of the corresponding elements of the second row i.e. with A_2, B_2, C_2 respectively and add. That is to say we evaluate $a_1A_2 + b_1B_2 + c_1C_2$.

Now, $A_2 = (-1)^{2+1} \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1)$

$B_2 = (-1)^{2+2} \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$

$C_2 = (-1)^{2+3} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -(a_1b_3 - a_3b_1)$.

Thus, $a_1A_2 + b_1B_2 + c_1C_2 = -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$

$= -a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1b_3c_1 + a_3b_1c_1$

$= 0$

Similarly, $a_1A_3 + b_1B_3 + c_1C_3 = 0$

$a_2A_1 + b_2B_1 + c_2C_1 = 0$

$a_2A_3 + b_2B_3 + c_2C_3 = 0$

$a_3A_1 + b_3B_1 + c_3C_1 = 0$

$a_3A_2 + b_3B_2 + c_3C_2 = 0$

[**Note :** Similar results hold for columns also. Infact $a_1A_2 + b_1B_2 + c_1C_2$ = the value of the determinant where first row and second row are both a_1 b_1 c_1 i.e. identical.

$\therefore \quad a_1A_2 + b_1B_2 + c_1C_2 = 0$]

3.7 Solution of Two Simultaneous Linear Equations

Suppose, we wish to solve simultaneous linear equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ in R^2. Here $a_i, b_i, c_i \in R$ and $a_i^2 + b_i^2 \neq 0$. $(i = 1, 2)$.

We shall consider only those equations in which none of a_1, a_2, b_1, b_2 is zero. (If some of these are zero, then it is easy to solve the equations).

We have already studied the method of solving two simultaneous linear equations known as **Method of Cross-Multiplication**. The method is

$$\frac{x}{\begin{matrix} b_1 & c_1 \\ b_2 & c_2 \end{matrix}} = \frac{y}{\begin{matrix} c_1 & a_1 \\ c_2 & a_2 \end{matrix}} = \frac{1}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}}, \text{ i.e. } \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad \frac{x}{\begin{matrix} b_1 & c_1 \\ b_2 & c_2 \end{matrix}} = -\frac{y}{\begin{matrix} a_1 & c_1 \\ a_2 & c_2 \end{matrix}} = \frac{1}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}}$$

Now, in this chapter we shall use determinant notation.

$$\text{So, } x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ where } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

This form of the solution is known as **Cramer's Rule**.

Note : (1) If $a_1b_2 - a_2b_1 = 0$ but $a_1c_2 - a_2c_1 \neq 0$ or $b_1c_2 - b_2c_1 \neq 0$, then we get solution set of the above equations as empty set.

(2) If $a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = a_1c_2 - a_2c_1 = 0$, then solution is not unique it is the infinite set, $\{(x, y) \mid a_1x + b_1y + c_1 = 0, x, y \in R]$

Consistent Equations : If the solution set of a system of equations is not empty, then the equations are called consistent equations.

Equations which are not consistent are called inconsistent equations.

Equivalent Equations : For the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, if there is a non-zero real number k, such that $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$, then the two equations are said to be equivalent. If they are not equivalent, then they are called distinct.

Example 16 : Solve : $2x + 3y - 8 = 0$ and $5x - 4y + 3 = 0$, using Cramer's rule.

Solution : Here $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & -4 \end{vmatrix} = -8 - 15 = -23 \neq 0.$

So we shall have a unique solution.

$$\text{Cramer's rule gives } x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 3 & -8 \\ -4 & 3 \end{vmatrix}}{-23} = \frac{9 - 32}{-23} = \frac{-23}{-23} = 1$$

$$\text{and } y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = -\frac{\begin{vmatrix} 2 & -8 \\ 5 & 3 \end{vmatrix}}{-23} = -\frac{6 + 40}{-23} = \frac{-46}{-23} = 2$$

Hence, $(x, y) = (1, 2)$.

$\therefore$ The solution set is $\{(1, 2)\}$.

Example 17 : Solve : $2x + 3y = 13xy$ and $5x - 2y = 4xy$.

Solution : These equations are not linear equations. They are quadratic equations in x and y. So they have two solutions.

One solution is $x = 0$ and $y = 0$ $\qquad (x = 0 \Rightarrow 0 + 3y = 0 \Rightarrow y = 0)$

If $x \neq 0$, then obviously $y \neq 0$. $\qquad (y = 0 \Rightarrow x = 0)$

$\therefore$ **We can convert these equations in a linear form taking $xy \neq 0$ and dividing them by $xy \neq 0$, we have linear equations in the form of**

$$\frac{2}{y} + \frac{3}{x} = 13 \text{ and } \frac{5}{y} - \frac{2}{x} = 4.$$

Now, let us write $\frac{1}{y} = m$ and $\frac{1}{x} = n$. Then, the system is $2m + 3n - 13 = 0$, $5m - 2n - 4 = 0$ which is a system of linear equations.

Here, $D = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0.$

So, we shall have a unique solution for m, n.

$$(m, n) = \left(\frac{\begin{vmatrix} 3 & -13 \\ -2 & -4 \end{vmatrix}}{D}, -\frac{\begin{vmatrix} 2 & -13 \\ 5 & -4 \end{vmatrix}}{D} \right) = \left(\frac{-12 - 26}{-19}, -\frac{-8 + 65}{-19} \right) = \left(\frac{-38}{-19}, \frac{-57}{-19} \right) = (2, 3)$$

$\therefore \quad (m, n) = (2, 3).$

but $\quad m = \frac{1}{y}$ and $n = \frac{1}{x}$

$\therefore \quad \left(\frac{1}{y}, \frac{1}{x} \right) = (2, 3)$, i.e. $\left(\frac{1}{x}, \frac{1}{y} \right) = (3, 2)$

So, $(x, y) = \left(\frac{1}{3}, \frac{1}{2} \right)$

$\therefore$ The solution set is $\left\{ (0, 0), \left(\frac{1}{3}, \frac{1}{2} \right) \right\}$.

3.8 Area of a Triangle

In eleventh standard, we have found out area of a triangle, if coordinates of vertices of a triangle are given.

If the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then we have applied the expression $\frac{1}{2}\,[(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ to find the area of a given triangle.

We shall write this expression in the form of determinant as $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Since area is always a positive quantity, so we shall take absolute value of the above determinant. We denote area of a triangle by Δ.

Here $\Delta = \frac{1}{2}|\,D\,|$ where $D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Shifting of origin does not effect the area of a triangle.

If we shift origin to (h, k), then the vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) of the triangle will change to $(x_1 - h, y_1 - k)$, $(x_2 - h, y_2 - k)$ and $(x_3 - h, y_3 - k)$ respectively.

Now, area of the triangle after shifting the origin is $\Delta = \frac{1}{2}|\,D'\,|$.

where $D' = \begin{vmatrix} x_1 - h & y_1 - k & 1 \\ x_2 - h & y_2 - k & 1 \\ x_3 - h & y_3 - k & 1 \end{vmatrix}$

$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ **(applying $C_{31}(h)$ and $C_{32}(k)$)**

$= D$

$\therefore \quad \Delta = \frac{1}{2}|D'| = \frac{1}{2}|D|$

Thus, the area remains same.

As an example find the area of the triangle having vertices (2, 3), (5, 1), (7, –2).

$$D = \begin{vmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 7 & -2 & 1 \end{vmatrix} = 2(3) - 3(-2) + 1(-17)$$

$$= 6 + 6 - 17 = -5$$

$\therefore \quad \Delta = \frac{1}{2}|D| = \frac{1}{2}|-5| = \frac{5}{2}$

If we shift the origin to (2, 3), new coordinates of A(2, 3) are (0, 0). B(5, 1) changes to (5 – 2, 1 – 3) = (3, –2).

C(7, –2) changes to (7 – 2, –2 – 3) = (5, –5)

$$D' = \begin{vmatrix} 0 & 0 & 1 \\ 3 & -2 & 1 \\ 5 & -5 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 5 & -5 \end{vmatrix} = -15 + 10 = -5$$

$\therefore \quad \Delta = \frac{1}{2}|D'| = \frac{1}{2}|-5| = \frac{5}{2}$

Thus, the area remains same.

Example 18 : Find the area of the triangle whose vertices are (5, 4), (2, 5), (2, 3).

Solution : $D = \begin{vmatrix} 5 & 4 & 1 \\ 2 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 5(5 - 3) - 4(2 - 2) + 1(6 - 10)$

$= 10 - 0 - 4$

$= 6$

$\therefore \quad \Delta = \frac{1}{2}|D| = \frac{1}{2}|6| = 3$

$\therefore$ The area is 3.

Example 19 : If area of a triangle whose vertices are (8, 2), (k, 4) and (6, 7) is 13, find k.

Solution : $D = \begin{vmatrix} 8 & 2 & 1 \\ k & 4 & 1 \\ 6 & 7 & 1 \end{vmatrix} = 8(4 - 7) - 2(k - 6) + 1(7k - 24)$

$= -24 - 2k + 12 + 7k - 24$

$= 5k - 36$

Now, $\Delta = \frac{1}{2}|D|$

$\therefore \quad 13 = \frac{1}{2}|5k - 36|$

$\therefore \quad 5k - 36 = 26 \quad \text{or} \quad 5k - 36 = -26$

$\therefore \quad 5k = 62 \quad \text{or} \quad 5k = 10$

$\therefore \quad k = \frac{62}{5} \quad \text{or} \quad k = 2$

$\therefore \quad k = 2 \text{ or } \frac{62}{5}.$

Cartesian Equation of a line :

In standard XI, we have studied the equation of a line when two distinct points on the line are given.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points of $\overleftrightarrow{AB}$, then the equation of $\overleftrightarrow{AB}$ is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$. **($\overleftrightarrow{AB}$ is not perpendicular to any axis)**

The determinant form of the above expression is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

Example 20 : Find equation of the line passing through (7, 8) and (5, –2) using determinant form as well as by using two point form.

Solution : The equation of the line is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$

where $(x_1, y_1) = (7, 8)$ and $(x_2, y_2) = (5, -2)$

$\therefore \quad \begin{vmatrix} x & y & 1 \\ 7 & 8 & 1 \\ 5 & -2 & 1 \end{vmatrix} = 0$

$\therefore \quad x(8 + 2) - y(7 - 5) + 1(-14 - 40) = 0$

$\therefore \quad 10x - 2y - 54 = 0$

$\therefore \quad 5x - y - 27 = 0$

The two point form of the equation of the line is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

$\therefore \quad \frac{y - 8}{-2 - 8} = \frac{x - 7}{5 - 7}$

$\therefore \quad \frac{y - 8}{-10} = \frac{x - 7}{-2}$

$\therefore \quad -2y + 16 = -10x + 70$

$\therefore \quad 10x - 2y - 54 = 0$

$\therefore \quad 5x - y - 27 = 0$

Exercise 3.3

1. Solve the following systems of equations using Cramer's rule :

(1) $4x + 10y = 2xy$
$5x + 16y = 3xy$

(2) $x - 2y = 17$
$5x - 3y = 6$

(3) $\frac{2}{y} + \frac{5}{x} = 9$
$\frac{4}{y} + \frac{3}{x} = 11, (xy \neq 0)$

2. Using cofactors of the elements of third row, evaluate $\begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix}$.

3. Using cofactors of the elements of second column, evaluate $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$.

4. Using determinant, find the area of the triangle whose vertices are
   (1) (11, 8), (3, 2), (8, 12) (2) (7, 9), (10, 8), (12, 10)

5. Find k, if the area of the triangle whose vertices are (2, 2), (6, 6) and (5, k) is 4.

6. Find a, if area of the triangle whose vertices are (5, a), (−2, 5) and (−2, 3) is 7.

7. Using determinant, obtain the equation of the line pasing through the points
   (1) (3, −2), (−1, 4) (2) (5, −1), (5, 3) (3) (1, −3), (5, −2)

8. Find the value of the determinant formed by the cofactors of the elements of $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{vmatrix}$.

*

Miscellaneous Examples :

Example 21 : Prove that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

Solution : $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = \begin{vmatrix} b+c & a+c & a+b \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$ **($R_{21}(1)$ and $R_{31}(1)$)**

$= \begin{vmatrix} a+b+2c & 2a+b+c & a+b \\ a-b & c-b & b+c \\ a+b+2c & b-c & -2c \end{vmatrix}$ **(First $C_{21}(1)$ and then $C_{32}(1)$)**

$= \begin{vmatrix} 0 & 2(a+c) & a+b+2c \\ 2(a+c) & 0 & b-c \\ a+b+2c & b-c & -2c \end{vmatrix}$ **($R_{31}(-1)$ and $R_{32}(1)$)**

$= -2(a+c)\,[-4c(a+c) - (b-c)(a+b+2c)] + (a+b+2c) \cdot 2(a+c)(b-c)$

$= 8c(a+c)^2 + 2(a+c)(b-c)(a+b+2c) + 2(a+c)(b-c)(a+b+2c)$

$= 8c(a+c)^2 + 4(a+c)(b-c)(a+b+2c)$

$= 4(a+c)\,[2c(a+c) + (b-c)(a+b+2c)]$

$= 4(a+c)\,(2ac + 2c^2 + ab + b^2 + 2bc - ac - bc - 2c^2)$

$= 4(a+c)\,(ac + ab + b^2 + bc)$

$= 4(a+c)\,(a+b)(b+c)$

Example 22 : Prove that, $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = 2xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

$\left(xyz \neq 0,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \neq 0\right)$

Solution : L.H.S. $= xyz \begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$ $\left(R_1\left(\frac{1}{x}\right),\ R_2\left(\frac{1}{y}\right),\ R_3\left(\frac{1}{z}\right),\ xyz \neq 0\right)$

$= xyz \begin{vmatrix} 3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$ $(R_{21}(1)$ and $R_{31}(1))$

$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$ $\left(R_1\left(\frac{1}{3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right)\right)$

$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{y} & 1 & -1 \\ 1+\frac{1}{z} & 0 & 2 \end{vmatrix}$ $(C_{12}(-1)$ and $C_{13}(-1))$

$= 2xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Example 23 : Prove $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Solution : L.H.S. $= \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$ $(C_{31}(-b)$ and $C_{32}(a))$

$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$ $\left(C_1\left(\frac{1}{1+a^2+b^2}\right),\ C_2\left(\frac{1}{1+a^2+b^2}\right)\right)$

$= (1+a^2+b^2)^2\,[(1-a^2-b^2+2a^2)-2b(-b)]$

$= (1+a^2+b^2)^2\,(1+a^2+b^2)$

$= (1+a^2+b^2)^3$

$=$ R.H.S.

Example 24 : If $\begin{vmatrix} x-3 & x-4 & x-a \\ x-2 & x-3 & x-b \\ x-1 & x-2 & x-c \end{vmatrix} = 0$, then prove that a, b, c are in arithmetic progression.

Solution : $\begin{vmatrix} x-3 & x-4 & x-a \\ x-2 & x-3 & x-b \\ x-1 & x-2 & x-c \end{vmatrix} = 0$

$\therefore \begin{vmatrix} -1 & -1 & b-a \\ -1 & -1 & c-b \\ x-1 & x-2 & x-c \end{vmatrix} = 0$ **($R_{21}(-1)$ and then $R_{32}(-1)$)**

$\therefore \begin{vmatrix} 0 & 0 & 2b-a-c \\ -1 & -1 & c-b \\ x-1 & x-2 & x-c \end{vmatrix} = 0$ **[$R_{21}(-1)$]**

$\therefore \quad (2b - a - c)\,[-1(x - 2) - (x - 1)(-1)] = 0$

$\therefore \quad (2b - a - c)\,(-x + 2 + x - 1) = 0$

$\therefore \quad 2b - a - c = 0$

$\therefore \quad 2b = a + c$

$\therefore \quad a, b, c$ are in A.P.

Example 25 : If two rows of a determinant are identical, then the value of the determinant is zero. Using this fact prove that if two rows are interchanged, then the value of determinant so obtained is additive inverse of the value of the original determinant.

Solution : $\begin{vmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ a_1+a_2 & b_1+b_2 & c_1+c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Exercise 3

1. Solve $\begin{vmatrix} x+5 & x \\ x+9 & x-2 \end{vmatrix} = 0$

2. Solve $\begin{vmatrix} 3x+4 & x+2 & 2x+3 \\ 4x+5 & 2x+3 & 3x+4 \\ 10x+17 & 3x+5 & 5x+8 \end{vmatrix} = 0$

3. Solve $\begin{vmatrix} x & 2 & 2 \\ 7 & -2 & -6 \\ 5 & 4 & 3 \end{vmatrix} + \begin{vmatrix} 7 & -2 & -6 \\ 5 & 4 & 3 \\ 1 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 5 & 3 & 7 \\ 4 & 7 & -2 \\ 3 & 8 & -6 \end{vmatrix}$

4. If $\begin{vmatrix} 5 & 4 & 8 \\ x-3 & -8 & -16 \\ 3 & 9 & 4 \end{vmatrix} = 0$, determine the value of x.

5. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

6. Prove that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$.

7. Show that $\begin{vmatrix} x^2 & (y-z)^2-x^2 & yz \\ y^2 & (z-x)^2-y^2 & zx \\ z^2 & (x-y)^2-z^2 & xy \end{vmatrix} = -(x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$.

8. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$.

9. Prove that $\begin{vmatrix} x^2 & y^2 & z^2 \\ (x+1)^2 & (y+1)^2 & (z+1)^2 \\ (x-1)^2 & (y-1)^2 & (z-1)^2 \end{vmatrix} = -4(x-y)(y-z)(z-x)$.

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 Mark)

(1) $\begin{vmatrix} 2 & 3 & 4 \\ 4x & 6x & 8x \\ 5 & 7 & 8 \end{vmatrix} = \ldots\ldots$ ☐

(a) $18x$ (b) 0 (c) 1 (d) $18x^3$

(2) The value of $\begin{vmatrix} 2008 & 2009 \\ 2010 & 2011 \end{vmatrix}$ is ..... ☐

(a) -1 (b) 1 (c) -2 (d) 2

(3) $\begin{vmatrix} x & 1 & y+z \\ y & 1 & z+x \\ z & 1 & x+y \end{vmatrix} = \ldots\ldots$ ☐

(a) $x+y+z$ (b) $(x+y)(y+z)(z+x)$

(c) 3 (d) 0

(4) $\begin{vmatrix} sin40^o & -cos40^o \\ sin50^o & cos50^o \end{vmatrix}$ = ...

(a) 0 (b) 1 (c) −1 (d) not exist.

(5) If D = $\begin{vmatrix} 2 & 3 & 1 \\ 5 & -1 & 2 \\ 7 & 4 & -1 \end{vmatrix}$, performing $R_{12}(-1)$ on D; then D will become .....

(a) $\begin{vmatrix} -1 & 3 & 1 \\ 6 & -1 & 2 \\ 3 & 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & 1 \\ 5 & -6 & 2 \\ 7 & -3 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} -3 & 4 & -1 \\ 5 & -1 & 2 \\ 7 & 4 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 3 & 1 \\ 3 & -4 & 1 \\ 7 & 4 & -1 \end{vmatrix}$

Section B (2 Marks)

(6) $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^1 & 5^2 & 5^3 \\ 5^3 & 5^4 & 5^5 \end{vmatrix}$ = ...

(a) 5^9 (b) 5^{12} (c) 5^0 (d) 0

(7) If $D_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$ and $D_2 = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$, then .....

(a) $D_1 + 2D_2 = 0$ (b) $2D_1 + D_2 = 0$ (c) $D_1 + D_2 = 0$ (d) $D_1 = D_2$

(8) If $a \neq 0$, $b \neq 0$, $c \neq 0$, $\begin{vmatrix} 0 & x^2+a & x^4+b \\ x^2-a & 0 & x-c \\ x^3-b & x^2+c & 0 \end{vmatrix} = 0$, then x = ...

(a) 1 (b) 0 (c) $a + b + c$ (d) $-(a + b + c)$

(9) If $x, y, z \in$ R, $x > y > z$ and D = $\begin{vmatrix} (x+1)^2 & (y+1)^2 & (z+1)^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then D is .....

(a) negative (b) positive (c) zero (d) not real

(10) If D = $\begin{vmatrix} 1 & cos\theta & 1 \\ -cos\theta & 1 & cos\theta \\ -1 & -cos\theta & 1 \end{vmatrix}$, then value of D lies in the interval ..... .

(a) $(2, \infty)$ (b) (2, 4) (c) [2, 4] (d) [−2, 2]

Section C (3 Marks)

(11) If $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = 0$ and $ax^2 + 2bxy + cy^2 \neq 0$, then .....

(a) a, b, c are in A.P. (b) a, b, c are in G.P.

(c) a, b, c are in A.P. and G.P. both (d) a, b, c are neither in A.P. nor in G.P.

(12) If $\begin{vmatrix} 1 & 2 & 5 \\ 1 & x & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$, then $x =$ .....

(a) 2 (b) -2 (c) 5 (d) -5

(13) The roots of $\begin{vmatrix} 1+x & 1-x & 1-x \\ 1-x & 1+x & 1-x \\ 1-x & 1-x & 1+x \end{vmatrix} = 0$ are ..... .

(a) 0, 1 (b) 0, -1 (c) 0, -3 (d) 0, 3

Section D (4 Marks)

(14) If $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, then $x =$ ..... .

(a) $-3, -2, 1$ (b) $-3, 2, -1$ (c) $-3, 2, 1$ (d) $3, 2, 1$

(15) $\begin{vmatrix} \sqrt{14}+\sqrt{3} & \sqrt{20} & \sqrt{5} \\ \sqrt{15}+\sqrt{28} & \sqrt{25} & \sqrt{10} \\ 3+\sqrt{70} & \sqrt{15} & \sqrt{25} \end{vmatrix} =$ ..... .

(a) $25\sqrt{3} - 15\sqrt{2}$ (b) $15\sqrt{2} + 25\sqrt{3}$ (c) $-25\sqrt{3} - 15\sqrt{2}$ (d) $15\sqrt{2} - 25\sqrt{3}$

(16) If $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix} = 0$, then a, b, c are in ..... .

(a) A.P. (b) G.P.

(c) an increasing sequence (d) a decreasing sequence

*

Summary

We have studied the following points in this chapter :

1. Second order determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and its value is $ad - bc$.

2. Third order determinant $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

3. **Some symbols :**

(1) R_{ij} (C_{ij}) $(i \neq j)$: Process to interchange ith and jth row (column).

(2) $R_i \rightarrow C_i$: Process of converting every row into respective column.

(3) $R_i(k)$ $[C_i(k)]$: Operation of multiplying ith row (column) by k.

(4) $R_{ij}(k)$ $[C_{ij}(k)]$ $(i \neq j)$: Operation of multiplying ith row (column) by k and adding its elements to the corresponding elements of jth row (column).

4. **Properties of determinant :**

(1) If we interchange rows to respective columns, then value of determinant remains same.

(2) If we interchange any two rows (columns), then we get the value of new determinant as additive inverse of the given determinant.

(3) If we multiply every entry of a row (column) by k $(\neq 0)$, then the value of determinant so obtained is k times the value of original determinant.

(4) $$\begin{vmatrix} a_1+d_1 & b_1+e_1 & c_1+f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) If the corresponding elements in any two rows (columns) are identical, the value of the determinant is zero.

(6) If any row (column) is multiplied by k and added to another row (column), the value of determinant remains same.

5. **Minor :** Minor of an element is the determinant obtained by removing the elements of the column and row containing that element.

6. **Cofactor :** If the value of a minor is multiplied by $(-1)^{i+j}$, we get the cofactor of that element, where i is the number of the row and j is the number of the column containing that element.

7. On adding the products of the elements of any of its rows (columns) by their corresponding cofactors gives the value of the determinant.

8. If all the elements of any row (column) are multiplied by the cofactors of the corresponding elements of other row (column) and added, then the sum will be zero.

9. System of two simultaneous linear equations in two unknown can be solved by using Cramer's Rule.

10. Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}|D| \text{ where } D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

11. On shifting of origin, the area of a triangle remains same.

12. Cartesian equation of a line passing through two distinct points (x_1, y_1), (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Determinant as a function

We have studied about vectors in standard XI.

$\overline{x} = (x_1, x_2)$ is a vector in R^2 and $\overline{x} = (x_1, x_2, x_3)$ is a vector in R^3.

Now we shall define one function $D : R^3 \times R^3 \times R^3 \rightarrow R$

Let $\overline{x} = (a_1, b_1, c_1)$, $\overline{y} = (a_2, b_2, c_2)$ and $\overline{z} = (a_3, b_3, c_3)$

Let $D(\overline{x}, \overline{y}, \overline{z}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Similarly if $\overline{x} = (a_1, b_1)$, $\overline{y} = (a_2, b_2)$, then we can define

$D : R^2 \times R^2 \rightarrow R$, $D(\overline{x}, \overline{y}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Thus, a determinant is a real function, whose domain is an ordered triplet of vectors in R^3 or an ordered pair of vectors in R^2.

Thus, if $k \in R$, then $D(k\overline{x}, \overline{y}, \overline{z}) = k \cdot D(\overline{x}, \overline{y}, \overline{z})$

We can write this for all variables $\overline{x}, \overline{y}, \overline{z}$.

If $\overline{u} = (u_1, u_2, u_3)$, then $D(\overline{x} + \overline{u}, \overline{y}, \overline{z}) = D(\overline{x}, \overline{y}, \overline{z}) + D(\overline{u}, \overline{y}, \overline{z})$

This result is true for all variables.

Thus, determinant is a linear function in every variable. (Other variables being kept constant.) Thus, determinant is a multi-variable linear function.

Also, $D(\overline{x}, \overline{y}, \overline{z}) = -D(\overline{y}, \overline{x}, \overline{z})$

This function is called an **alternating function.**

From this, we can get $D(\overline{x}, \overline{x}, \overline{z}) = 0$

We can explain the example 25 like this,

$D(\overline{x} + \overline{y}, \overline{x} + \overline{y}, \overline{z}) = 0$

$\therefore \quad D(\overline{x} + \overline{y}, \overline{x}, \overline{z}) + D(\overline{x} + \overline{y}, \overline{y}, \overline{z}) = 0$

$\therefore \quad D(\overline{x}, \overline{x}, \overline{z}) + D(\overline{y}, \overline{x}, \overline{z}) + D(\overline{x}, \overline{y}, \overline{z}) + D(\overline{y}, \overline{y}, \overline{z}) = 0$

$\therefore \quad D(\overline{y}, \overline{x}, \overline{z}) = -D(\overline{x}, \overline{y}, \overline{z})$

MATRICES 4

It is easier to square the circle than to get round a mathematician.
– Augustus De Morgan

Our notion of symmetry is derived from the human face.
Hence we demand symmetry horizontally and in breadth only not vertically nor in depth.
– Blaise Pascal

4.1 Introduction

If you are asked about your weight in *kg*, you can use a real number such as 55 to answer the question. Again if you are asked for your height in *cm*, your answer is another real number say 135. One way to organise these data is to use an order pair. You can represent your weight and height with the order pair (55, 135). The elements of this order pair indicate the information such as weight and height respectively. If we want to include your age in years say 16, then we have order triplet (55, 135, 16). The elements of this triple indicate the information such as weight, height and age in the sequence for an individual. We can write them in a row, like $[55 \quad 135 \quad 16]$ or in a column, like $\begin{bmatrix} 55 \\ 135 \\ 16 \end{bmatrix}$.

If the above questions are asked to three or four individuals named Rita, Raman, Rahim and John, then the informations can be collected in the order triples as (55, 135, 16), (58.5, 140, 18), (59, 138, 17) and (60.5, 155, 20) respectively. However, it will be nice if we can combine all these triples together in one set of data. If we consider each triple as one column, then we will have all our data in one arrangement. If we organise them in an array form as :

$$\begin{array}{c} \\ \text{Weight} \\ \text{Height} \\ \text{Age} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Rita} & \text{Raman} & \text{Rahim} & \text{John} \end{array} \\ \begin{bmatrix} 55 & 58.5 & 59 & 60.5 \\ 135 & 140 & 138 & 155 \\ 16 & 18 & 17 & 20 \end{bmatrix} \end{array}$$

If there is a selection of soldiers for the Army wing, then they have to collect above data from so many individuals. If the data so collected can be arranged in the precise form as shown above, then it is easy to interpret them. Also, it is easy to make selection of individuals.

The above arrangement of real numbers in a rectangular array is known as a **Matrix** (Plural is matrices). The real numbers are the elements or entries of the matrix.

Matrix is a latin word. The origin of matrices lie with the study of systems of simultaneous linear equations. An important Chinese Text between 300 BC and 200 AD, nine chapters of Mathematical art (***Chiu Chang Suan Shu***), give the use of matrix methods to solve simultaneous equations. **Carl Friedrich Gauss** (1777-1855) also gave the method to solve simultaneous linear equations by matrix method.

Matrix operations are used in electronic physics. They are used in computers, budgeting, cost estimation, analysis and experiments. They are also used in cryptography, modern psychology, genetics, industrial management etc.

4.2 Matrix

Any rectangular arrangement or an array of numbers enclosed in brackets such as [ ] or ( ) is called a matrix. We shall consider only real matrices, i.e. elements or entries of the matrices will be real numbers only.

The matrix, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ has two rows and three columns. So we say that it is a 2×3 matrix. 2×3 is also known the **order** of the matrix.

In general, an $m \times n$ matrix is a matrix having m rows and n columns. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

Here 'a_{ij}' is the element of the matrix in '*ith*' row and '*jth*' column. In a compact form, we can write this matrix as $[a_{ij}]_{m \times n}$. If there is no confusion, we write it as $[a_{ij}]$ also. We denote matrices by A, B, C etc. In the notation of the order $m \times n$ of a matrix, m denotes the number of rows of the matrix and n denotes the number of columns of the matrix. An $m \times n$ matrix is called a rectangular matrix.

Example 1 : Construct 4×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = i - j$

Solution : We have matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

Here $a_{ij} = i - j$, so we have $a_{11} = 1 - 1 = 0$, $a_{12} = 1 - 2 = -1$, $a_{13} = 1 - 3 = -2$,

$a_{21} = 2 - 1 = 1$ etc. Thus, we have $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Difference Between a Determinant and a Matrix :

(1) A determinant has a real value where as a matrix has no real value as it is an arragement of real numbers only.

(2) In a determinant, number of rows is equal to the number of columns where as in a matrix, number of rows may or may not be equal to the number of columns.

Equality of Matrices :

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal if they have the same order and $a_{ij} = b_{ij}$, for all i and j. We denote equal matrices A and B as A = B.

Here, $A = B \Leftrightarrow [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n} \Leftrightarrow a_{ij} = b_{ij} \;\; \forall\; i = 1, 2, 3,..., m;\; j = 1, 2, 3,..., n$

Example 2 : Find x and y, if $\begin{bmatrix} x-1 & 2y \\ x+y & 4 \end{bmatrix} = \begin{bmatrix} 3x-7 & y^2-3 \\ 6 & 4 \end{bmatrix}$.

Solution : Corresponding elements of two matrices must be equal.

$\therefore \quad x - 1 = 3x - 7, \qquad 2y = y^2 - 3$ and $x + y = 6$ and $4 = 4$.

$\therefore \quad 2x = 6, \qquad y^2 - 2y - 3 = 0$

$\therefore \quad x = 3, \qquad (y - 3)(y + 1) = 0$

$y = 3$ or $y = -1$

Here, $x = 3$ and $y = 3$ satisfy the equation $x + y = 6$ and $x = 3$, $y = -1$ do not satisfy $x + y = 6$. Hence, $x = 3$ and $y = 3$.

Types of Matrices :

Row Matrix : A $1 \times n$ matrix $[a_{11} \quad a_{12} \quad a_{13} \ldots a_{1n}]$ is called a row matrix.

A row matrix has only one row (and any number of columns).

e.g. A = $[3 \quad 5 \quad -1 \quad 4 \quad 0]$ is a 1×5 row matrix.

Column Matrix : An $m \times 1$ matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$ is called a column matrix.

A column matrix has only one column (and any number of rows).

e.g. A = $\begin{bmatrix} 15 \\ 7 \\ 10 \\ -8 \end{bmatrix}$ is a 4×1 column matrix.

Square Matrix : An $n \times n$ matrix is called a square matrix.

A square matrix has the number of columns equal to the number of rows.

For instance $\begin{bmatrix} 5 & -1 & 3 \\ 11 & 2 & 9 \\ -4 & 0 & -7 \end{bmatrix}$ is 3×3 square matrix.

(**Note :** $[a_{ij}]_{1 \times 1}$ matrix is a row matrix, is a column matrix and a square matrix also.)

Diagonal Matrix : If in a square matrix A = $[a_{ij}]_{n \times n}$, we have $a_{ij} = 0$ whenever $i \neq j$, then A is called a diagonal matrix. This is a square matrix in which all entries are zero except possibly those on the diagonal from top left corner to bottom right corner (principal diagonal).

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ is a diagonal matrix.}$$

A diagonal matrix is also denoted as *diag* $[a_{11} \quad a_{22} \quad a_{33} \ldots a_{nn}]$.

e.g. A = $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix, i.e. *diag* $[5 \ 0 \ 3]$.

Here, 5, 0, 3 are the elements of the principal diagonal of the matrix A.

Zero Matrix : If all elements of a matrix are zero, then that matrix is known as zero matrix. We denote zero matrix by $[0]_{m \times n}$ or $O_{m \times n}$. $O_{m \times n}$ is also written as O.

Thus, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix. It is a $O_{2 \times 3}$ zero matrix.

4.3 Operations on Matrices

Sum of Two Matrices : If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then their sum is defined as $A + B = [a_{ij} + b_{ij}]_{m \times n}$, i.e. a matrix obtained by taking sum of the corresponding elements of A and B.

For the sum of two matrices, they must have the same number of rows and the same number of columns, otherwise it is not possible to add the matrices. If A and B are both $m \times n$ they are called compatible for sum. In notation $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$.

For instant, if $A = \begin{bmatrix} 1 & 5 \\ 2 & -3 \\ 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 1 & 2 \\ -5 & -4 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1-3 & 5+2 \\ 2+1 & -3+2 \\ 4-5 & -7-4 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & -1 \\ -1 & -11 \end{bmatrix}$.

Properties of Matrix Addition :

(1) Commutative Law for Addition :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then $A + B = B + A$.

$$\begin{aligned} \text{Now, } A + B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] && \text{(Commutativity of addition in R)} \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A \end{aligned}$$

$\therefore \quad A + B = B + A$

(2) Associative Law for Addition :

For $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$,

$(A + B) + C = A + (B + C)$.

$$\begin{aligned} \text{Now, } (A + B) + C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] && \text{(Associative law of addition in R)} \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + ([b_{ij}] + [c_{ij}]) \\ &= A + (B + C) \end{aligned}$$

$\therefore \quad (A + B) + C = A + (B + C)$

(3) The Identity for Addition of Matrices :

Let $A = [a_{ij}]_{m \times n}$ and $O = [0]_{m \times n}$ be the zero matrix. Then $A + O = O + A = A$

$$\begin{aligned} A + O &= [a_{ij}] + [0] \\ &= [a_{ij} + 0] \\ &= [a_{ij}] = A && \text{(0 is the additive identity in R)} \end{aligned}$$

$\therefore \quad A + O = [a_{ij}]$

By commutative law A + O = O + A

$\therefore$ A + O = O + A = A

Thus, **O is the identity matrix for addition.**

(4) Existence of Additive Inverse :

Let A = $[a_{ij}]_{m \times n}$ be any matrix. Then we have another matrix $[-a_{ij}]_{m \times n}$, so that A + $[-a_{ij}]$ = $O_{m \times n}$.

$$\begin{aligned} A + [-a_{ij}] &= [a_{ij}] + [-a_{ij}] \\ &= [a_{ij} - a_{ij}] \\ &= [\,0\,] \\ &= O_{m \times n} \end{aligned}$$

We denote $[-a_{ij}]$ as −A.

By commutative law A + (−A) = O = (−A) + A.

Thus, **−A = $[-a_{ij}]$ is called the additive inverse of A = $[a_{ij}]$.**

Difference of Matrices : If A = $[a_{ij}]$ and B = $[b_{ij}]$ are both $m \times n$ matrices, then the difference of A and B is defined as A − B = A + (−B) = $[a_{ij}]$ + $[-b_{ij}]$ = $[a_{ij} - b_{ij}]$.

Example 3 : If $A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$, then find A + B and A − B.

Solution : $A + B = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2+5 & -3+4 & 4-2 \\ 5+3 & 2+1 & 8+2 \end{bmatrix} = \begin{bmatrix} 7 & 1 & 2 \\ 8 & 3 & 10 \end{bmatrix}$$

$$A - B = A + (-B) = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix} + \begin{bmatrix} -5 & -4 & 2 \\ -3 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-5 & -3-4 & 4+2 \\ 5-3 & 2-1 & 8-2 \end{bmatrix} = \begin{bmatrix} -3 & -7 & 6 \\ 2 & 1 & 6 \end{bmatrix}$$

Example 4 : Can we add $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$? Give reason.

Solution : Here A is a 3 × 2 matrix and B is a 2 × 2 matrix. They do not have same number of rows. They are not compatible for addition. So we cannot add A and B.

Product of a Matrix with a Scalar and Properties :

If A = $[a_{ij}]$ is an $m \times n$ matrix and k is any real number, then the matrix $[ka_{ij}]$ is called the product of the matrix A by the scalar k. It is denoted by kA. Thus, for A = $[a_{ij}]$, kA = $[ka_{ij}]$.

In kA every element of A gets multiplied by k. (Compare corresponding result for a determinant !)

Properties of Addition of Matrices and of Multiplication of a Matrix by a Scalar :

Suppose, A = $[a_{ij}]$ and B = $[b_{ij}]$ are $m \times n$ matrices and $k, l \in$ R, then

(1) $k(A + B) = kA + kB$ **(2)** $(k + l)A = kA + lA$ **(3)** $(kl)A = k(lA)$

(4) $1A = A$ **(5)** $(-1)A = -A$

Proof : (1) $k(A + B) = k[a_{ij} + b_{ij}]$

$= [k(a_{ij} + b_{ij})]$

$= [ka_{ij} + kb_{ij}]$

$= [ka_{ij}] + [kb_{ij}]$

$= k[a_{ij}] + k[b_{ij}]$

$= kA + kB$

(2) $(k + l)A = (k + l)[a_{ij}]$

$= [(k + l)\, a_{ij}]$

$= [ka_{ij} + la_{ij}]$

$= [ka_{ij}] + [la_{ij}]$

$= k[a_{ij}] + l[a_{ij}]$

$= kA + lA$

(3) $(kl)A = (kl)[a_{ij}]$

$= [(kl)\, a_{ij}]$

$= [k(la_{ij})]$

$= k[la_{ij}]$

$= k[l(a_{ij})]$

$= k(lA)$

(4) $1A = [1 \cdot a_{ij}]$

$= [a_{ij}]$

$= A$

(5) $(-1)A = (-1)[a_{ij}] = [(-1)a_{ij}] = [-a_{ij}] = -A$

Thus, $(-1)\,A = -A$

Example 5 : If $A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ -3 & 1 & -5 & 7 \\ 2 & -9 & -8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 4 & 0 & 1 & -6 \\ -2 & 3 & 6 & -7 \end{bmatrix}$, then obtain $3A - 2B$.

Proof : $3A - 2B = 3A + (-2)B$

$$= 3\begin{bmatrix} 4 & 2 & 1 & 0 \\ -3 & 1 & -5 & 7 \\ 2 & -9 & -8 & 5 \end{bmatrix} + (-2)\begin{bmatrix} 1 & 2 & -3 & 5 \\ 4 & 0 & 1 & -6 \\ -2 & 3 & 6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 6 & 3 & 0 \\ -9 & 3 & -15 & 21 \\ 6 & -27 & -24 & 15 \end{bmatrix} + \begin{bmatrix} -2 & -4 & 6 & -10 \\ -8 & 0 & -2 & 12 \\ 4 & -6 & -12 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 12-2 & 6-4 & 3+6 & 0-10 \\ -9-8 & 3+0 & -15-2 & 21+12 \\ 6+4 & -27-6 & -24-12 & 15+14 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 2 & 9 & -10 \\ -17 & 3 & -17 & 33 \\ 10 & -33 & -36 & 29 \end{bmatrix}$$

Example 6 : If $A = \begin{bmatrix} 5 & 4 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 6 & -5 \end{bmatrix}$, then find the matrix X, such that $3A + 2X = 4B$.

Solution : We wish to find matrix X such that $3A + 2X = 4B$

$\therefore$ $(-3A) + (3A + 2X) = (-3A) + 4B$ **(adding additive inverse of 3A)**

$\therefore$ $(-3A + 3A) + 2X = (-3A) + 4B$

$\therefore$ $O + 2X = 4B - 3A$

$\therefore$ $2X = 4B - 3A$ **(O is the identity for addition)**

$\therefore \quad X = \frac{1}{2}(4B - 3A)$

$$\therefore \quad X = \frac{1}{2}\left(4\begin{bmatrix}1 & 2\\ 3 & -4\\ 6 & -5\end{bmatrix} + (-3)\begin{bmatrix}5 & 4\\ 0 & -2\\ 3 & 6\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}4 & 8\\ 12 & -16\\ 24 & -20\end{bmatrix} + \begin{bmatrix}-15 & -12\\ 0 & 6\\ -9 & -18\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}4-15 & 8-12\\ 12+0 & -16+6\\ 24-9 & -20-18\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-11 & -4\\ 12 & -10\\ 15 & -38\end{bmatrix} = \begin{bmatrix}-\frac{11}{2} & -2\\ 6 & -5\\ \frac{15}{2} & -19\end{bmatrix}$$

Transpose of a Matrix and its Properties :

Transpose of a Matrix : If all the rows of matrix $A = [a_{ij}]_{m \times n}$ are converted into corresponding columns, the matrix so obtained is called the tranpose of A.

If $A = [a_{ij}]_{m \times n}$ is a matrix, then its transpose is $[a_{ji}]_{n \times m}$ is denoted by A^T or A'.

If $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$.

For example, if $A = \begin{bmatrix}3 & \sqrt{5} & 2\\ \sqrt{2} & -1 & 0\end{bmatrix}$, then $A^T = \begin{bmatrix}3 & \sqrt{2}\\ \sqrt{5} & -1\\ 2 & 0\end{bmatrix}$

Symmetric Matrix : For a square matrix A, if $A^T = A$, then A is called a symmetric matrix. If $A = [a_{ij}]_{n \times n}$, then $A^T = [a_{ji}]_{n \times n}$. As $A^T = A$, so $a_{ij} = a_{ji}$ for all i and j.

Thus, if $A = \begin{bmatrix}1 & 3 & -5\\ 3 & 0 & 2\\ -5 & 2 & -7\end{bmatrix}$, then $A^T = \begin{bmatrix}1 & 3 & -5\\ 3 & 0 & 2\\ -5 & 2 & -7\end{bmatrix}$.

We have $A^T = A$, so A is a symmetric matrix.

Skew-Symmetric Matrix : For a square matrix A, if $A^T = -A$, then A is called a skew-symmetric matrix. In such a matrix $A^T = [a_{ji}]_{n \times n}$, $a_{ji} = -a_{ij}$ for all i and j.

Now, when $i = j$, then we have $a_{ii} = -a_{ii}$, for all i.

$\therefore \quad 2a_{ii} = 0$

$\therefore \quad a_{ii} = 0, \forall i.$

This means that all elements on the principal diagonal of a skew-symmetric matrix are zero. Here, $a_{11} = a_{22} = ... = a_{nn} = 0$.

For example, the matrix $A = \begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -5\\ -1 & 5 & 0\end{bmatrix}$, then

$$A^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix} = -1\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = (-1)A = -A$$

$\therefore$ A is a skew-symmetric matrix.

Some properties of Addition and Multiplication Regarding Transpose of Matrix :

(1) $(A + B)^T = A^T + B^T$, **(2)** $(A^T)^T = A$, **(3)** $(kA)^T = kA^T$, $k \in R$

Proof : (1) For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$,

$A^T = [a_{ji}]$ and $B^T = [b_{ji}]$, are $n \times m$ matrices.

Now, $A + B = [a_{ij} + b_{ij}] = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

$$\begin{aligned} \therefore \quad (A + B)^T &= [c_{ji}] \\ &= [a_{ji} + b_{ji}] \\ &= [a_{ji}] + [b_{ji}] \end{aligned}$$

$\therefore$ $(A + B)^T = A^T + B^T$

(2) Let $A = [a_{ij}]$

$\therefore$ $A^T = [a_{ji}]$ and hence $(A^T)^T = [a_{ij}] = A$

$\therefore$ $(A^T)^T = A$

(3) Suppose $A = [a_{ij}]$

$\therefore$ $kA = [ka_{ij}] = [c_{ij}]$ where $c_{ij} = ka_{ij}$.

$$\begin{aligned} \therefore \quad (kA)^T &= [c_{ji}] \\ &= [ka_{ji}] \\ &= k[a_{ji}] \\ &= kA^T \end{aligned}$$

Example 7 : If $A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix}$, obtain $A + A^T$ and $A - A^T$.

What can you say about the matrices $A + A^T$ and $A - A^T$?

Solution : $A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix}$. Hence $A^T = \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$

$$\text{Now } A + A^T = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -1 \\ -1 & -1 & 16 \end{bmatrix}$$

If $B = A + A^T$

Then $B^T = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -1 \\ -1 & -1 & 16 \end{bmatrix} = B$

Thus, $(A + A^T)^T = A + A^T$. Hence $A + A^T$ is a symmetric matrix.

Again, $A - A^T = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 & 11 \\ 4 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

Let $C = A - A^T$

$\therefore \quad C^T = \begin{bmatrix} 0 & 4 & -11 \\ -4 & 0 & 7 \\ 11 & -7 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 11 \\ 4 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

$\therefore \quad C^T = -C$

$\therefore \quad (A - A^T)^T = -(A - A^T)$. Hence $A - A^T$ is a skew-symmetric matrix.

Example 8 : Simplify $cosec\theta \begin{bmatrix} cosec\theta & -cot\theta \\ cot\theta & -cosec\theta \end{bmatrix} + cot\theta \begin{bmatrix} -cot\theta & cosec\theta \\ -cosec\theta & cot\theta \end{bmatrix}$

Solution : $cosec\theta \begin{bmatrix} cosec\theta & -cot\theta \\ cot\theta & -cosec\theta \end{bmatrix} + cot\theta \begin{bmatrix} -cot\theta & cosec\theta \\ -cosec\theta & cot\theta \end{bmatrix}$

$= \begin{bmatrix} cosec^2\theta & -cosec\theta\, cot\theta \\ cosec\theta\, cot\theta & -cosec^2\theta \end{bmatrix} + \begin{bmatrix} -cot^2\theta & cot\theta\, cosec\theta \\ -cot\theta\, cosec\theta & cot^2\theta \end{bmatrix}$

$= \begin{bmatrix} cosec^2\theta - cot^2\theta & -cosec\theta\, cot\theta + cot\theta\, cosec\theta \\ cot\theta\, cosec\theta - cot\theta\, cosec\theta & -cosec^2\theta + cot^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Example 9 : Prove that if A is a square matrix, then $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix and every matrix A can be uniquely written as a sum $A = B + C$ where B is a symmetric matrix and C is a skew-symmetric matrix.

Solution : If $B = A + A^T$, then $B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$

$\therefore \quad B = A + A^T$ is a symmetric matrix.

Let $C = A - A^T$

Then $C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

$\therefore \quad C = A - A^T$ is a skew-symmetric matrix.

Also $A = \frac{1}{2}(A + A^T + A - A^T) = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}B + \frac{1}{2}C$.

$\therefore$ A is a sum of a symmetric matrix and a skew-symmetric matrix as $\frac{1}{2}B$ and $\frac{1}{2}C$ are symmetric and skew symmetric matrices respectively.

Conversely let A = B + C where B is a symmetric matrix and C is a skew-symmetric matrix.

$\therefore$ $B^T = B$ and $C^T = -C$

Now $A^T = B^T + C^T = B - C$

$\therefore$ $A + A^T = 2B$ $A - A^T = 2C$

$\therefore$ $B = \frac{A + A^T}{2}$, $C = \frac{A - A^T}{2}$

$\therefore$ The expression for A as a sum of a symmetric matrix and a skew-symmetric matrix is unique.

Exercise 4.1

1. If $A = \begin{bmatrix} 2 & -4 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 0 & 5 \\ 4 & -2 \end{bmatrix}$, then find A + B, A − B, 2A + B, A − 2B.

2. If $A = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$, then obtain $A + A^T$ and $A - A^T$.

3. If A = *diag* [1 −1 2] and B = *diag* [3 2 1], find B − A, 2A + 3B.

4. Solve the matrix equation $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 4\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \end{bmatrix}$.

5. If $a_{ij} = \frac{(i - 2j)^2}{3}$, obtain $[a_{ij}]_{2 \times 2}$.

6. If $A = \begin{bmatrix} 1 & 2 & 5 \\ 5 & 1 & 1 \\ 3 & 0 & 4 \end{bmatrix}$, find $A - 2A^T$.

7. If $\begin{bmatrix} x+y & xy \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ -8 & 3 \end{bmatrix}$, then find x and y.

8. Obtain a, b, c, d, if $\begin{bmatrix} a-2b & c+d \\ 2a-b & 3a-c \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 7 & 10 \end{bmatrix}$.

9. Find matrix A and B, if $A + B = \begin{bmatrix} 2 & 5 \\ 9 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 6 & 3 \\ -1 & 0 \end{bmatrix}$.

10. Find matrix X, if 5A − 3X = 2B, where $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$.

11. Suppose $A = \begin{bmatrix} 3 & 1 & 1 \\ -12 & -3 & 0 \\ -9 & -1 & -12 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 2 & -4 \\ 5 & 1 & 9 \end{bmatrix}$ and 3A + 4B − X = O, then find matrix X.

12. Find a and b, if $2\begin{bmatrix} 5 & a \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 0 \end{bmatrix}$.

*

Multiplication of Matrices :

The product AB of two matrices A and B is defined only if the number of columns of A is equal to the number of rows of B.

Suppose, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices. Then their product $AB = [c_{ij}]_{m \times p}$ is defined by, $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + \ldots + a_{in} \cdot b_{nj}$.

To obtain the entry in i th row and jth column of matrix AB, we multiply elements of the ith row of the matrix A with corresponding elements of the jth column of the matrix B and then we take the sum of all these products. Thus, for $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, the product $AB = \left[\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right]_{m \times p}$.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ we say A and B are compatible for multiplication.

Example 10 : If $A = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, find AB and BA and also show that $AB \neq BA$.

Solution :
$$AB = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2(1)+3(3) & 2(-2)+3(4) \\ -4(1)+5(3) & -4(-2)+5(4) \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 8 \\ 11 & 28 \end{bmatrix} \quad \text{(i)}$$
$$BA = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1(2)+(-2)(-4) & 1(3)+(-2)5 \\ 3(2)+4(-4) & 3(3)+4(5) \end{bmatrix}$$
$$= \begin{bmatrix} 10 & -7 \\ -10 & 29 \end{bmatrix} \quad \text{(ii)}$$

Observing results (i) and (ii), we can say that $AB \neq BA$.

Example 11 : If $A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & 4 \\ 0 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -2 \\ 2 & -3 \end{bmatrix}$, then find AB. Is BA defined ? Why ?

Solution :
$$AB = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & 4 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & -2 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+(-1)4+1(2) & 2(1)+(-1)(-2)+1(-3) \\ -3(1)+2(4)+4(2) & -3(1)+2(-2)+4(-3) \\ 0(1)+3(4)+(-5)2 & 0(1)+3(-2)+(-5)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 13 & -19 \\ 2 & 9 \end{bmatrix}$$

BA is not defined because, B has two columns and A has three rows.

Example 12 : If $A = \begin{bmatrix} \cos^2\alpha & \cos\alpha \sin\alpha \\ \cos\alpha \sin\alpha & \sin^2\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos^2\beta & \cos\beta \sin\beta \\ \cos\beta \sin\beta & \sin^2\beta \end{bmatrix}$ and

$\alpha - \beta = (2n-1)\frac{\pi}{2}$, $n \in Z$, then prove that AB is zero matrix.

Solution : $AB = \begin{bmatrix} \cos^2\alpha & \cos\alpha \sin\alpha \\ \cos\alpha \sin\alpha & \sin^2\alpha \end{bmatrix} \begin{bmatrix} \cos^2\beta & \cos\beta \sin\beta \\ \cos\beta \sin\beta & \sin^2\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha \cos^2\beta + \cos\alpha \sin\alpha \cos\beta \sin\beta & \cos^2\alpha \cos\beta \sin\beta + \cos\alpha \sin\alpha \sin^2\beta \\ \cos\alpha \sin\alpha \cos^2\beta + \sin^2\alpha \cos\beta \sin\beta & \cos\alpha \sin\alpha \cos\beta \sin\beta + \sin^2\alpha \sin^2\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cos\beta (\cos\alpha \cos\beta + \sin\alpha \sin\beta) & \cos\alpha \sin\beta (\cos\alpha \cos\beta + \sin\alpha \sin\beta) \\ \cos\beta \sin\alpha (\cos\alpha \cos\beta + \sin\alpha \sin\beta) & \sin\alpha \sin\beta (\cos\alpha \cos\beta + \sin\alpha \sin\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cos\beta \cos(\alpha-\beta) & \cos\alpha \sin\beta \cos(\alpha-\beta) \\ \cos\beta \sin\alpha \cos(\alpha-\beta) & \sin\alpha \sin\beta \cos(\alpha-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \left(\cos(\alpha - \beta) = \cos(2n-1)\frac{\pi}{2} = 0\right)$$

Example 13 : If $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}$, prove that $(A + B)^2 \neq A^2 + 2AB + B^2$.

Solution : We have, $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & -3-12 \\ 2+8 & -6+16 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -15 \\ 10 & 10 \end{bmatrix}$$

$$\therefore \quad B^2 = BB = \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+20 & -4-8 \\ -5-10 & 20+4 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -12 \\ -15 & 24 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} -1-15 & 4+6 \\ -2+20 & 8-8 \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 18 & 0 \end{bmatrix}$$

$$\therefore \quad 2AB = \begin{bmatrix} -32 & 20 \\ 36 & 0 \end{bmatrix}$$

$$\therefore \quad A^2 + 2AB + B^2 = \begin{bmatrix} -5 & -15 \\ 10 & 10 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 36 & 0 \end{bmatrix} + \begin{bmatrix} 21 & -12 \\ -15 & 24 \end{bmatrix}$$

$$\therefore \quad A^2 + 2AB + B^2 = \begin{bmatrix} -16 & -7 \\ 31 & 34 \end{bmatrix} \qquad \text{(i)}$$

$$\therefore \quad A + B = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad (A + B)^2 = (A + B)(A + B)$$

$$= \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0+7 & 0+2 \\ 0+14 & 7+4 \end{bmatrix}$$

$$\therefore \quad (A + B)^2 = \begin{bmatrix} 7 & 2 \\ 14 & 11 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii), we can see that $(A + B)^2 \neq A^2 + 2AB + B^2$.

[**Note :** For the matrix A, $A^2 = AA$ and we do not take simply squares of entries of A.]

Properties of Matrix Multiplication :

Matrix multiplication has the following properties. We shall assume them without proof.

(1) Distributive Laws :

(i) For $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{n \times p}$

$A(B + C) = AB + AC$

(ii) For matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{n \times p}$

$(A + B)C = AC + BC$

(2) Associative Laws :

(ii) For matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{p \times q}$

$A(BC) = (AB)C$

Identity Matrix (Unit Matrix) : A square matrix in which all elements on principal diagonal are 1 and the rest of them are 0 is called an identity or a unit matrix. Identity matrix is denoted by I.

Thus, $I = [a_{ij}]_{n \times n}$ where $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

I is also represented as I_n or $I_{n \times n}$.

i.e. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

As this identity matrix is a 3×3 matrix, it is denoted by $I_{3 \times 3}$ or simply by I_3.

If $A = [a_{ij}]_{n \times n}$, then for the identity matrix I_n we have $AI_n = I_nA = A$.

(Note : A symbol δ_{ij} called Kronecker delta is used to define I.

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Thus, $I = [\delta_{ij}]$ **)**

Scalar Matrix : If $k \in R$, then kI_n is called a scalar matrix.

Thus, $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a scalar matrix.

Here $k = 4$ and $A = 4I_3$.

Example 14 : If $A = [x \quad y \quad z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then find (AB)C.

Solution : Now, $AB = [x \quad y \quad z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$$= [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]$$

$$\therefore \quad (AB)C = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [(ax + hy + gz)x + (hx + by + fz)y + (gx + fy + cz)z]$$

$$= [ax^2 + hxy + gzx + hxy + by^2 + fzy + gxz + fyz + cz^2]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$$

Example 15 : If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, find a 2×2 matrix X such that $BX - AC = O$

Solution : Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $BX - AC = O$

$$\therefore \quad \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2a+5c & -2b+5d \\ 6a+c & 6b+d \end{bmatrix} - \begin{bmatrix} -9 & -6 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2a+5c+9 & -2b+5d+6 \\ 6a+c-43 & 6b+d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad -2a + 5c + 9 = 0,$

$\quad 6a + c - 43 = 0$

$\therefore \quad -6a + 15c = -27, \quad$ (i)

$\quad 6a + c = 43 \quad$ (ii)

$\therefore$ Adding (i) and (ii),

$16c = 16 \Rightarrow c = 1$ and $a = 7$

$-2b + 5d + 6 = 0$

$6b + d - 22 = 0$

$-6b + 15d = -18 \quad$ (iii)

$6b + d = 22 \quad$ (iv)

$\therefore$ Adding (iii) and (iv),

$16d = 4 \Rightarrow d = \frac{1}{4}$ and $b = \frac{29}{8}$

Hence $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & \frac{29}{8} \\ 1 & \frac{1}{4} \end{bmatrix}$.

Example 16 : Prove that if $A(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then $A(\alpha)\,A(\beta) = A(\alpha + \beta)$ and deduce that $A(\alpha)\,A(\beta)$ is the identity matrix I_2, where $\alpha + \beta = 2n\pi$, $n \in Z$.

Solution : $A(\alpha)\,A(\beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A(\alpha + \beta)$$

If $\alpha + \beta = 2n\pi$, $A(\alpha)\,A(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ ($\cos 2n\pi = 1$ and $\sin 2n\pi = 0$)

$$= I_2$$

Exercise 4.2

1. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, then prove that $A(B + C) = AB + AC$.

2. Find a, b, c, d, if $\begin{bmatrix} a+b & 4 \\ 3 & c+d \end{bmatrix} + \begin{bmatrix} 6 & a \\ 2d & -1 \end{bmatrix} = \begin{bmatrix} 3b & 3a \\ 3d & 3c \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, then prove that $A(B - C) = AB - AC$.

4. If $A = [1 \;\; -1 \;\; 2]$, $B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, obtain AB and BA, if possible.

5. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A$.

6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find $A^2 - 5A$.

7. If $A = \begin{bmatrix} 0 & -tan\frac{\theta}{2} \\ tan\frac{\theta}{2} & 0 \end{bmatrix}$, then prove that $(I_2 - A)\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} = I_2 + A$.

8. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then obtain A^2.

9. Obtain X and Y if $X + Y = A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 9 \\ 3 & 4 & 8 \end{bmatrix}$, where X is a symmetric and Y is a skew-symmetric matrix.

10. Find a 2×2 matrix X such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

11. Find real numbers x and y such that $(xI + yA)^2 = A$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

12. Find x if $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$

13. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove by the principle of mathematical induction that

$A^n = \begin{bmatrix} 2n+1 & -4n \\ n & 1-2n \end{bmatrix}$, $n \in N$.

*

4.4 The Determinant of a Square Matrix :

If all the entries of a square matrix are kept in their respective places and the determinant of this array is taken, then the determinant so obtained is called the determinant of the given square matrix. If A is a square matrix, then determinant of A is denoted by $|A|$ or $det A$.

For instant, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant is $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

If $A = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 2 - 15 = -13$.

Theorem 4.1 : For square matrices A and B, $|AB| = |A||B|$.

We will accept this theorem without proof.

Example 17 : Find $|A|$, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & sin\,\theta & -cos\,\theta \\ 0 & cos\,\theta & sin\,\theta \end{bmatrix}$.

Solution : $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & sin\,\theta & -cos\,\theta \\ 0 & cos\,\theta & sin\,\theta \end{vmatrix} = sin^2\theta + cos^2\theta = 1$.

Adjoint of a Matrix : For a given square matrix A, if we replace every entry in A by its cofactor as in | A | and then the transpose of this matrix is taken, then the matrix so obtained is called the adjoint of A and is denoted by *adj*A.

If $A = [a_{ij}]_{n \times n}$, then $adjA = [A_{ji}]_{n \times n}$ where A_{ji} is the cofactor of the element a_{ji}.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

Example 18 : For $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$, find *adj*A.

Solution : We take $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

then $A_{11} = a_{22} = 5$, $A_{12} = -a_{21} = -1$, $A_{21} = -a_{12} = -2$ and $A_{22} = a_{11} = 4$

$\therefore \quad adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$.

[**Note :** To obtain the adjoint of 2×2 matrix, interchange the elements on the principlal diagonal and change the sign of the elements on the secondary diagonal. e.g. if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.]

Example 19 : Find *adj*A for $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We have $A_{11} = -1$ $\quad A_{12} = -8$ $\quad A_{13} = -10$

$A_{21} = -5$ $\quad A_{22} = -6$ $\quad A_{23} = 1$

$A_{31} = -1$ $\quad A_{32} = 9$ $\quad A_{33} = 7$

$\therefore \quad adjA = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$.

4.5 Inverse of a Matrix :

For an $n \times n$ square matrix A, if there exists another $n \times n$ square matrix B, such that $AB = I_n = BA$ (I is an identity matrix), then B is called an inverse matrix of A. Inverse of A is denoted by A^{-1}.

It is clear that if B is an inverse of A, then A is an inverse of B.

Theorem 4.2 : If inverse of matrix A exists, then it is unique.

Proof : If possible suppose B and C both are inverses of A.

$\therefore \quad AB = I = BA$ and $AC = I = CA$.

Now $AB = I$

$\therefore \quad C(AB) = CI$

$\therefore \quad (CA)B = C$

$\therefore \quad IB = C$

$\therefore \quad B = C$

This shows that A has a unique inverse matrix.

Note : Remember in chapter 1, we had seen that for an associative binary operation with identity, inverse is unique. Matrix multiplication of $n \times n$ matrices is associative and has identity I_n.

Theorem 4.3 : For a square matrix A, A(*adj*A) = (*adj*A)A = | A | I.

Proof : We will prove this result for a 3×3 square matrix A.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

Now, $A(adjA) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$ **(by the theorems on determinant)**

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A| I_3$$

Similarly, we can prove that $(adjA)A = |A| I_3$.

Non-singular Matrix : A square matrix is said to be non-singular, if it has an inverse matrix.

[**Note :** If A is a non-singular matrix, then A^{-1} is also non-singular matrix and $(A^{-1})^{-1} = A$.]

Singular Matrix : A matrix which is not non-singular is called a singular matrix.

Theorem 4.4 : A square matrix A is non-singular if and only if $|A| \neq 0$.

Proof : Suppose A is a non-singular matrix and let B be the inverse of A.

$\therefore \quad AB = I$

$\therefore \quad |AB| = |I|$

$\therefore \quad |A||B| = 1 \neq 0$

$\therefore \quad |A| \neq 0$

Conversely, let $|A| \neq 0$. So $\frac{1}{|A|}$ exists.

Let $B = \frac{1}{|A|} adj A$

Then $AB = A\left(\frac{1}{|A|} adj\, A\right) = \frac{1}{|A|}(A adj A) = \frac{1}{|A|}|A|I.$

$\therefore \quad AB = I$

Similarly, we can prove that $BA = I$.

$\therefore$ B is the inverse of A.

$\therefore$ A is a non-singular matrix.

(Note : Inverse of matrix A is $A^{-1} = \frac{1}{|A|} adj A$, if it exists.)

Example 20 : Find the inverse of $A = \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$, if it exists.

Solution : Here $|A| = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 8 + 15 = 23 \neq 0.$

$\therefore \quad A^{-1}$ exists.

Now, $adj A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|} adj A$

$$= \frac{1}{23}\begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{4}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{2}{23} \end{bmatrix}$$

Example 21 : Find A^{-1}, if $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$.

Solution : $|A| = \begin{vmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{vmatrix} = 5(-2-3) - 8(0-4) + 1(0-8)$

$$= -25 + 32 - 8$$
$$= -1 \neq 0$$

$\therefore \quad A^{-1}$ exists.

$$adj A = \begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$$

$\therefore \quad A^{-1} = \frac{1}{|A|} adj A$

$$= \frac{1}{-1}\begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}$$

Some Important Results :

(1) For a square non-singular matrix A, the value of the reciprocal of the determinant of A is the same as the value of the determinant of the inverse of A.

This means $|A^{-1}| = |A|^{-1}$.

Proof : A is a non-singular matrix. Hence $|A| \neq 0$ and A^{-1} exists.

So, $AA^{-1} = I$

So, $|AA^{-1}| = |I|$

$\therefore$ $|A||A^{-1}| = 1$

$\therefore$ $|A^{-1}| = \frac{1}{|A|}$ $(|A| \neq 0)$

$\therefore$ $|A^{-1}| = |A|^{-1}$

(2) If A and B are non-singular matrices, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : A and B are non-singular, so A^{-1} and B^{-1} exist and $|A| \neq 0$, $|B| \neq 0$.

$\therefore$ $|A||B| \neq 0$

$\therefore$ $|AB| \neq 0$

$\therefore$ AB is a non-singular matrix.

Again, $(AB)(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1}))$

$= A((BB^{-1})A^{-1})$

$= A(IA^{-1})$

$= AA^{-1}$

$= I$

Similarly, we can prove $(B^{-1}A^{-1})(AB) = I$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

(3) For $m \times n$ matrices A and B, $(AB)^T = B^TA^T$.

We shall accept this result without giving proof.

(4) A^T is non-singular if and only if A is non-singular and $(A^T)^{-1} = (A^{-1})^T$.

Proof : A is a non-singular matrix $\Leftrightarrow |A| \neq 0$

$\Leftrightarrow |A^T| \neq 0$ $(|A| = |A^T|)$

$\Leftrightarrow A^T$ is non-singular.

Again, $AA^{-1} = A^{-1}A = I$

So, $(AA^{-1})^T = (A^{-1}A)^T = I^T$

$\therefore$ $(A^{-1})^T A^T = A^T (A^{-1})^T = I$ $(I^T = I)$

$\therefore$ $(A^T)^{-1} = (A^{-1})^T$

(5) $adj A^T = (adj A)^T$

Proof : Let $A = [a_{ij}]$

$\therefore \quad A^T = [a_{ji}]$

$\therefore \quad adj A^T = [A_{ij}]$ (i)

But $adj A = [A_{ji}]$

$\therefore \quad (adj A)^T = [A_{ij}]$ (ii)

From (i) and (ii), we get $adj A^T = (adj A)^T$

4.6 Row Reduced Echelon Form

We have seen some operations like R_{ij}, $R_i(k)$ and $R_{ij}(k)$ as applied to a determinant. Similar operations for columns also can be applied.

The application being similar, we will consider row operations.

(1) If the operation R_{ij} is applied to identity matrix I_n, the resulting matrix is called an elementary matrix E_{ij}.

(2) If the operation $R_i(k)$ is applied to identity matrix I_n, the resulting matrix is called an elementary matrix $E_i(k)$.

(3) If the operation $R_{ij}(k)$ is applied to identity matrix I_n, the resulting matrix is called an elementary matrix $E_{ij}(k)$.

Applying R_{12} to matrix A is the same as finding product E_{12} A for any matrix A.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & 1 & 6 \end{bmatrix}$$

$$R_{12} \text{ gives } \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix} \quad \text{(i)}$$

$$\text{Also } E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{12} A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix} \quad \text{(ii)}$$

(i) and (ii) prove our assertion.

Similarly any elementary operation R_{ij}, $R_i(k)$ or $R_{ij}(k)$ on matrix A is equivalent to premultiplying A by E_{ij}, $E_i(k)$ or $E_{ij}(k)$ respectively.

For column operations post-multiplication has to be carried out.

Now we define a reduced row echelon matrix. A matrix is in reduced row echelon form if

(1) **The first non-zero entry of each row called the leading entry is 1.**

(2) **Each leading entry is in a column to the right of the leading entry of the previous row.**

(3) **A row with all entries zero is called a zero row. All zero rows occur below rows with at least one entry non-zero (called a non-zero row).**

(4) The leading entry is the only non-zero entry in its column

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **are in reduced row echelon form.**

A result : The row reduced form of a non singular matrix is I_n.

We can obtain inverse of a non-singular matrix as follows :

Write A = IA.

Apply elementary row operations on A and I so that A on left-hand side is converted to its reduced row echelon form namely I_n (being non-singular).

Then, we will have an equation like this I = PA.

where I gets converted to P by elementary row operations same as on left-hand side matrix A.

Then $P = A^{-1}$.

How to get row reduced echelon form of a matrix A ?

(1) (a) Find the pivot, the first non-zero entry in the first column.

For, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 2 \\ 3 & 4 & 5 \end{bmatrix}$. 1 is the pivot.

(b) If necessary interchange rows so that the leading entry in the first row is non-zero.

$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$. To have pivot in the first row, we will apply R_{12} or R_{13}.

For instant, if we apply R_{13}, then in $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, we will get 1 3 3 as a first row with 1 as a pivot.

(c) Multiply each element in the pivot row by inverse (reciprocal) of the leading entry, so that leading entry becomes 1.

In $\begin{bmatrix} 3 & 5 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ leading entry is 3. So we multiply each element of the first row by $\frac{1}{3}$ to get $1 \ \frac{5}{3} \ \frac{1}{3}$ as the first row.

So the matrix will be $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$.

(d) Add multiples of the pivot row to each of lower rows so that every element in the pivot column of lower rows becomes 0.

We apply $R_{12}(-2)$, $R_{13}(-4)$ to the matrix which we have at the end of (c). The matrix will become $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{7}{3} \\ 0 & \frac{-17}{3} & \frac{2}{3} \end{bmatrix}$ with first column $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$.

(2) (a) Repeat the above procedure from step (1) ignoring previous pivot row.

(b) Continue till there are no more leading entries to be processed.

(c) Now the matrix becomes a triangular matrix having zeroes below principal diagonal. After performing some operations on the matrix obtained in (1)(d), we have matrix as $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(3) (a) Identify the last row having leading entry equal to 1. Call it the pivot row now.

(b) Add multiples of this pivot row to each of the upper rows untill every element above the pivot becomes 0.

(c) Moving up the matrix repeat this process for each row.

Now performing $R_{31}\left(-\frac{1}{3}\right)$ and $R_{32}(1)$ we have,

$$\begin{bmatrix} 1 & \frac{5}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now applying $R_{21}\left(-\frac{5}{3}\right)$, we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ i.e., we get I_3.

Thus performing operations on $A = IA$, we get $I = PA$. Here $P = A^{-1}$.

Let us understand by an example.

Example 22 : Find inverse of $\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ by elementary row operations.

Solution : $\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & -3 & 4 \\ 3 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$ **(R_{31}) (To bring leading entry non-zero)**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 3 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_1\left(\frac{1}{2}\right)$ (To make leading entry 1)**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & \frac{3}{2} & -2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_{12}(-3)$**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_2\left(\frac{2}{3}\right)$ (Leading element of second row is made 1)**

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ 1 & \frac{2}{3} & -1 \end{bmatrix} A \qquad R_{23}(1)$$

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_3(-3)$$ **(Leading element of third row is made 1)**

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -\frac{11}{2} \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_{32}\left(\frac{4}{3}\right),\ R_{31}(-2)$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_{21}\left(\frac{3}{2}\right)$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix}$$

Example 23 : By using elementary operations, find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Solution : We take A = IA.

We shall use elementary row operations on this matrix equation.

$$\therefore \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\therefore \begin{bmatrix} 1 & 4 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A \qquad R_{12}(-3)$$

$$\therefore \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{10} & \frac{-1}{10} \end{bmatrix} A \qquad R_2\left(-\frac{1}{10}\right)$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix} A \qquad R_{21}(-4)$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$

Example 24 : Obtain the inverse of matrix $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$ by reduced row echelon method.

Solution : We write, A = IA

$$\begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (R_{31}(-1))$$

$$\therefore \begin{bmatrix} 1 & 5 & 2 \\ 0 & 2 & 1 \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad (R_{13}(-4))$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 1 \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad \left(R_{21}\left(-\frac{5}{2}\right)\right)$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & \frac{1}{2} & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad \left(R_{2}\left(\frac{1}{2}\right)\right)$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & \frac{1}{2} & 0 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{23}(17))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ 0 & \frac{1}{2} & 0 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{31}(-1))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{32}(1))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix} A \qquad (R_{3}(-2))$$

$$\therefore \quad I = A^{-1} A, \text{ where } A^{-1} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}.$$

Example 25 : Find inverse of $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$ by reduced row echelon method.

Solution : We write $\begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$$\therefore \quad \begin{bmatrix} 1 & 5 & 2 \\ 0 & -4 & 5 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_{12}(-1)$$

$$\therefore \quad \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -\frac{5}{4} \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2\left(-\frac{1}{4}\right)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & \frac{33}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{5}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{3}{4} & -\frac{3}{4} & 1 \end{bmatrix} \qquad R_{21}(-5),\ R_{23}(3)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & \frac{33}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{5}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 3 & -3 & 4 \end{bmatrix} \qquad R_3(4)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix} A \qquad R_{32}\left(\frac{5}{4}\right),\ R_{31}\left(\frac{-33}{4}\right)$$

$$\therefore \quad I_3 = PA$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}$$

Unique Solution of a System of Linear Equations Using Inverse of a Matrix :

Suppose, $a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

is a system of three linear equations in x, y, z.

If we take, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

then the system of equations can be written as, $AX = B$.

If A is a non-singular matrix, then A^{-1} exists.

Now, $AX = B$

$\therefore \quad A^{-1}(AX) = A^{-1}B$

$\therefore \quad (A^{-1}A)X = A^{-1}B$

$\therefore \quad IX = A^{-1}B$

$\therefore \quad X = A^{-1}B$

Suppose, $A^{-1}B = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.

Thus, $x = p_1$, $y = p_2$, $z = p_3$, is the unique solution of the given system of linear equations.

[**Note :** This result is also true for a system of two linear equations in two unknowns.]

Example 26 : Using matrix method, solve : $x - 2y = 4$ and $-3x + 5y = -7$.

Solution : The system can be expressed as $\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

or $AX = B$, where $A = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$

$\therefore$ A^{-1} exists.

Hence, the system has a unique solution given by $A^{-1}B = X$.

Now, $adjA = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|} adjA$

$= \frac{1}{-1} \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

Now, $X = A^{-1}B$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -20+14 \\ -12+7 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$\therefore$ $x = -6$, $y = -5$ is the required solution.

Example 27 : If the system of equations $x + y + z = 3$, $2x - y - z = 3$, $x - y + z = 9$ has unique solution, then find it.

Solution : The system of equations can be expressed in the matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$, then the system of equations is $AX = B$.

Now $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(-1-1) - 1(2+1) + 1(-2+1)$

$= -2 - 3 - 1 = -6 \neq 0$

$\therefore$ A^{-1} exists and hence the given system has a unique solution.

Now, $adj A = \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|} adj A$

$= \frac{1}{-6} \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix}$

Now, $X = A^{-1}B$

$\therefore$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$

$= \frac{1}{-6} \begin{bmatrix} -6+(-6)+0 \\ -9+0+27 \\ -3+6-27 \end{bmatrix}$

$= \frac{1}{-6} \begin{bmatrix} -12 \\ 18 \\ -24 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

So, $x = 2$, $y = -3$ and $z = 4$.

Exercise 4.3

1. Find the adjoint for the following matrices :

(1) $\begin{bmatrix} 5 & -2 \\ 1 & -3 \end{bmatrix}$ (2) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ (3) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1} if it exists.

3. If $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, prove that $A^{-1} = A^T$.

4. If $A = \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, verify $(AB)^{-1} = B^{-1}A^{-1}$.

5. For $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, show that $adj(adj A) = A$.

6. If $A = \begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 9 \\ 7 & 8 \end{bmatrix}$, then verify $(AB)^{-1} = B^{-1}A^{-1}$.

7. Find $x \in R$ if $A = \begin{bmatrix} 5x & 10 \\ 8 & 7 \end{bmatrix}$ and $|A| = 25$.

8. **By using reduced row echelon method, find the inverse of the following matrices :**

(1) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

9. **Solve the system of equations by matrix method :**

(1) $3x + 4y + 5 = 0$
$11x - 2y = 15$

(2) $5x - 7y = 2$
$7x - 5y = 3$

10. **Use matrix method to solve the following system of equations :**

(1) $4x - 3y + 2z = 4$
$3x - 2y + 3z = 8$
$4x + 2y - 2z = 2$

(2) $x + 2y + z = 4$
$x - y - z = 0$
$-x + 3y - z = -2$

*

Miscellaneous Examples :

Example 28 : For $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 - 4A + 7I_2 = O$ and hence obtain A^{-1}.

Solution : Now, $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 - 4A + 7I_2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O$$

Here, $|A| = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$

$\therefore$ A is a non-singular matrix. Hence A^{-1} exists.

Now, multiplying $A^2 - 4A + 7I_2 = O$ by A^{-1} on both the sides, we get,

$$A^{-1}(A^2 - 4A + 7I_2) = A^{-1}O$$

$$\therefore \quad A^{-1}(A^2) - 4(A^{-1}A) + 7(A^{-1}I_2) = O$$

$$\therefore \quad (A^{-1}A)A - 4I + 7A^{-1} = O$$

$$\therefore \quad IA - 4I + 7A^{-1} = O$$

$$\therefore \quad 7A^{-1} = 4I - A$$

$$\therefore \quad A^{-1} = \frac{1}{7}(4I - A)$$

$$= \frac{1}{7}\left\{4\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}2 & 3\\-1 & 2\end{bmatrix}\right\}$$

$$= \frac{1}{7}\left\{\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} + \begin{bmatrix}-2 & -3\\1 & -2\end{bmatrix}\right\}$$

$$= \begin{bmatrix}\frac{2}{7} & -\frac{3}{7}\\\frac{1}{7} & \frac{2}{7}\end{bmatrix}$$

Example 29 : If $A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$, then prove that $A^2 - 4A - 5I_3 = O$ and hence obtain A^{-1}.

Solution : $A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$

$$\therefore \quad A^2 - 4A - 5I_3 = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - 4\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - 5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} + \begin{bmatrix}-4 & -8 & -8\\-8 & -4 & -8\\-8 & -8 & -4\end{bmatrix} + \begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$= O$$

Now, $\begin{vmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{vmatrix} = 1(-3) - 2(-2) + 2(2)$

$$= -3 + 4 + 4$$

$$= 5 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Now, multiplying $A^2 - 4A - 5I_3 = O$ by A^{-1} on both the sides, we have,

$\therefore \quad A^{-1}(A^2) - 4(A^{-1}A) - 5(A^{-1}I_3) = A^{-1}O$

$\therefore \quad (A^{-1}A)A - 4I_3 - 5A^{-1} = O$

$\therefore \quad I_3 A - 4I_3 = 5A^{-1}$

$\therefore \quad A - 4I_3 = 5A^{-1}$

$$\therefore \quad A^{-1} = \frac{1}{5}(A - 4I_3)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1-4 & 2+0 & 2+0 \\ 2+0 & 1-4 & 2+0 \\ 2+0 & 2+0 & 1-4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Example 30 : Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\therefore \quad |A| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} = -\cos^2\alpha - \sin^2\alpha = -1 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Cofactors of the elements of A are,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0$$

Similarly, $A_{21} = 0$, $A_{22} = -\cos\alpha$, $A_{23} = -\sin\alpha$

$A_{31} = 0$, $A_{32} = -\sin\alpha$, $A_{33} = \cos\alpha$

$$\therefore \quad adj A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj A$$

$$= \frac{1}{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

[**Note :** If $A^{-1} = A$, then such a matrix said to be an idempotent matrix.]

Example 31 : Find the equations of lines passing through (2, −1) (4, 0) and (−1, −2), (4, 1) using determinant method. Find the point of intersection (if it exists) using matrix method.

Solution : The equation of the line passing through (2, −1) and (4, 0) is $\begin{vmatrix} x & y & 1 \\ 2 & -1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0.$

$\therefore \quad x(-1) - y(-2) + 4 = 0$

$\therefore \quad -x + 2y + 4 = 0$

$\therefore \quad x - 2y = 4$

The equation of the line passing through (−1, −2) and (4, 1) is $\begin{vmatrix} x & y & 1 \\ -1 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 0$

$\therefore \quad x(-3) - y(-5) + 7 = 0$

$\therefore \quad -3x + 5y = -7$

$\therefore \quad 3x - 5y = 7$

$\therefore$ The equations of lines are $x - 2y = 4$

$3x - 5y = 7$

The system of equations can be written in the matrix form as,

$$\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

or AX = B, where $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} = -5 + 6 = 1 \neq 0$

$\therefore$ A^{-1} exists.

$adj A = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$. Hence $A^{-1} = \frac{1}{|A|} adj A = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$

Now, $X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -20+14 \\ -12+7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

$\therefore$ $x = -6$ and $y = -5$.

$\therefore$ The point of intersection of the two lines is $(-6, -5)$.

Example 32 : Does the system of simultaneous linear equations,

$x + 3y + 4z = 8$, $2x + y + 2z = 5$, $5x + y + z = 7$ have unique solution ?

If so, find it using matrix method.

Solution : Writing $x + 3y + 4z = 8$

$2x + y + 2z = 5$

$5x + y + z = 7$ in the matrix form as

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

The system is AX = B.

Now, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 1(-1) - 3(-8) + 4(-3)$

$= -1 + 24 - 12$

$= 11 \neq 0$

$\therefore$ A^{-1} exists.

$\therefore$ The system has a unique solution.

Now, taking the matrix $A = [a_{ij}]_{3 \times 3}$, we have cofactors of the entries of A as,

$A_{11} = -1$, $A_{12} = 8$, $A_{13} = -3$

$A_{21} = 1$, $A_{22} = -19$, $A_{23} = 14$

$A_{31} = 2$, $A_{32} = 6$, $A_{33} = -5$

$$\therefore \quad adj A = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} adj A$

$$\therefore \quad A^{-1} = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

As, $X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 1, y = 1, z = 1.$

Exercise 4

1. If $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, prove that $A^{-1} = A^T$. Also find AA^T.

2. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, prove that $A^{-1} = \frac{1}{19}A$

3. If $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$, find $(AB)^{-1}$.

4. If $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$, find $(AB)^{-1}$.

5. If $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, find $B^{-1}AB$.

6. Prove that If $A^2 - 6A + 17I_2 = O$, where $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ and hence find A^{-1}.

7. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, prove that $A^{-1} = A^2$.

8. For $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, prove that If $A^3 - 6A^2 + 5A + 11I_3 = O$. Using this matrix relation, obtain A^{-1}.

9. If $A = \begin{bmatrix} 3 & 0 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ -4 & 3 \end{bmatrix}$, then obtain $A^2 + AB + 6B$ without multiplying the given matrices.

10. Solve the system of equations by matrix method (if unique solution exists).

(1) $3x - 5y = 1,\ x + 2y = 4$ (2) $3x + 4y - 5 = 0,\ y - x - 3 = 0$

11. **If the following system of equations has unique solution, then find the solution set :**

(1) $2x + y + z = 2$
$x + 3y - z = 5$
$3x + y - 2z = 6$

(2) $\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10$
$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$
$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13\ (xyz \neq 0)$

12. For $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$, find $(a^2 + bc + 1)I_2 - aA^{-1}$.

13. Two intersecting lines have slopes m_1 and m_2 and their y-intercepts are c_1 and c_2 $(m_1 \neq m_2)$ respectively. Using matrix, find their point of intersection.

14. Find $x \in R$, if $A = \begin{bmatrix} 2x & 9 \\ -3 & -2 \end{bmatrix}$ and $|A| = 3$.

15. Find $x \in R$, if $[x \ \ -5 \ \ -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$.

16. Express $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix}$ as a sum of a symmetric matrix and a skew-symmetric matrix.

17. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, prove $AA^T = I$. Deduce $A^{-1} = A^T$.

18. If for square matrices A and B, AB = A and BA = B, prove $A^2 = A$ and $B^2 = B$.

19. If B is a square matrix and $B^2 = B$, then prove that $A = I - B$ satisfies $A^2 = A$ and $AB = BA = O$.

20. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, prove $A^3 = O$. (See that $A^3 = O$, even though $A \neq O$)

21. A is a 3×3 square matrix, prove that, $|\, adj A \,| = |\, A \,|^2$.

22. Find matrix A and B such that $A \neq O$, $B \neq O$ but $AB = O$.

23. If $A(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, prove $A(\alpha)\, A(-\alpha) = I$.

24. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

Section A (1 Mark)

(1) A is a 3×3 matrix, then $|\,3A\,| = \ldots\ldots\ |\,A\,|$ ☐

(a) 3 (b) 6 (c) 9 (d) 27

(2) If $A = [a_{ij}]_{n \times n}$ such that $a_{ij} = 0$ for $i \neq j$ then A is ..... $(a_{ii} \neq a_{jj})$ $(n > 1)$ ☐

(a) a column matrix (b) a row matrix (c) a diagonal matrix (d) a scalar matrix

(3) $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, the correct statement is ..... . ☐

(a) A^{-1} does not exist (b) $A = (-1)\, I_3$

(c) $A^2 = I$ (d) A is a diagonal matrix

(4) A is 3×4 matrix, if A^TB and BA^T are defined then, B is a ..... matrix. ☐

(a) 4×3 (b) 3×3 (c) 4×4 (d) 3×4

(5) If A is skew-symmetric 3×3 matrix, $|\,A\,| = \ldots\ldots$. ☐

(a) 1 (b) 0 (c) -1 (d) 3

Section B (2 Marks)

(6) The system of equations $ax + y + z = a - 1$, $x + ay + z = a - 1$ and $x + y + az = a - 1$ does not have unique solution if a = ...

(a) 1 or -2 (b) 3 (c) 2 (d) -1

(7) If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$, then x = ..., y = ...

(a) $x = a^2 + b^2$, $y = a^2 - b^2$ (b) $x = 2ab$, $y = a^2 + b^2$

(c) $x = a^2 + b^2$, $y = ab$ (d) $x = a^2 + b^2$, $y = 2ab$

(8) If α and β are not the multiple of $\frac{\pi}{2}$ and

$\begin{bmatrix} cos^2\alpha & cos\alpha\, sin\alpha \\ cos\alpha\, sin\alpha & sin^2\alpha \end{bmatrix} \times \begin{bmatrix} cos^2\beta & sin\beta\, cos\beta \\ sin\beta\, cos\beta & sin^2\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\alpha - \beta$ is ..... .

(a) any multiple of π (b) odd multiple of $\frac{\pi}{2}$

(c) 0 (d) odd multiple of π

(9) If $\begin{bmatrix} x & 0 \\ 1 & y \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$, then x = ..., y = ...

(a) $x = 3$, $y = 2$ (b) $x = 3$, $y = -2$ (c) $x = -3$, $y = -2$ (d) $x = -3$, $y = 2$

(10) If inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ is $\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$, then α = ...

(a) 5 (b) -5 (c) 2 (d) -2

Section C (3 Marks)

(11) If AB = BA and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then B = ...

(a) $\begin{bmatrix} x & x \\ y & 0 \end{bmatrix}$ (b) $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ (c) $\begin{bmatrix} x & y \\ 0 & y \end{bmatrix}$ (d) $\begin{bmatrix} x & x \\ 1 & x \end{bmatrix}$

(12) If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $A^2 - kA - 5I = O$, then k = ...

(a) 3 (b) 7 (c) 5 (d) 9

(13) If $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$, then x = ...

(a) $\frac{-9 \pm \sqrt{35}}{2}$ (b) $\frac{-7 \pm \sqrt{53}}{2}$ (c) $\frac{-9 \pm \sqrt{53}}{2}$ (d) $\frac{-7 \pm \sqrt{35}}{2}$

(14) Matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ if $AA^T = I$, then $(x, y, z) = (...., ..., ...)$. $(x, y, z > 0)$

(a) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right)$ (d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

Section D (4 Marks)

(15) If $A \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then A = ...

(a) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -2 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$

(16) If $A = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$, then A^3 = ...

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

(17) Check, whether $\frac{1}{11}\begin{bmatrix} -1 & 8 & \alpha \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, if so, then α = ...

(a) -3 (b) 2 (c) -5 (d) not exists.

*

Summary

We have studied the following points in this chapter :

1. **Matrix** : Any rectangular arrangement or an array of numbers enclosed in brackets such as [ ] or ( ) is called a matrix. The numbers are the elements of the matrix.
2. If two matrices have same order and corresponding elements are same in both the matrices, then they are equal matrices. $A = B \Rightarrow [a_{ij}] = [b_{ij}] \Leftrightarrow a_{ij} = b_{ij}\ \forall i, j$
3. **Types of matrices** : Row matrix, Column matrix, Square matrix, Diagonal matrix, Zero matrix.
4. **Sum of two matrices** : Two matrices must have the same number of rows and the same number of columns, otherwise it is not possible to add the matrices.

   $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

5. **Properties of Matrix Addition :**

(1) Commutative Law for Addition

(2) Associative Law for Addition

(3) The Identity for Addition of Matrices

(4) Existence of Additive Inverse

6. **Product of a Matrix with a Scalar and Properties :**

(1) If $A = [a_{ij}]_{m \times n}$ and $k \in R$, then for $k \in R$, $kA = [ka_{ij}]_{m \times n}$.

(2) $k(A + B) = kA + kB$ where A, B are matrices and $k, l \in R$

(3) $(kl)A = k(lA)$

(4) $1A = A$

(5) $(-1)A = -A$

7. **Transpose of a Matrix :** $A = [a_{ij}]_{m \times n}$, then transpose of A is $A^T = A' = [a_{ji}]_{n \times m}$.

8. **Symmetric Matrix :** For a square matrix A, if $A^T = A$, then A is called a symmetric matrix.

9. **Skew-Symmetric Matrix :** For a square matrix A, if $A^T = -A$, then A is called a skew-symmetric matrix.

10. (1) $(A + B)^T = A^T + B^T$, (2) $(A^T)^T = A$, (3) $(kA)^T = kA^T$

11. **Multiplication of two matrics :** If the number of columns of A = the number of rows of B, then the product AB is possible.

12. **Identity (unit) matrix :** In a square matrix, if all elements on principal diagonal are 1 and the rest are 0, then the matrix is called an identity matrix, denoted by I.

13. Determinant of a square matrix A is deonted by $|A|$.

14. $|AB| = |A||B|$ where A and B are square matrices.

15. **Adjoint of a matrix :** If we replace every entry of a square matrix A by its cofactor and then tranpose of this is taken, then the matrix so obtained is the adjoint of A denoted by *adj*A.

16. **Inverse of a matrix :** For two square matrices A and B; if AB = BA = I, then they are inverse of each other.

17. **Non-singular matrix :** If the inverse matrix of a square matrix exists, then that matrix is called a non-singular matrix. Determinant of a non-singular matrix is a non-zero real number.

18. Inverse of A is $A^{-1} = \frac{1}{|A|}(adjA)$; $|A| \neq 0$

19. A^{-1} can be obtained by elementary rows (or column) operations on the matrix A. (Symbols of the operations are as determinant.)

20. **Echelon Method of finding inverse of a matrix :** Take matrix equation A = IA, now apply a sequence of elementary row (or column) operations on A on L.H.S. and same to I, then A of L.H.S. will be converted into I and I on R.H.S. will become A^{-1} as $I = A^{-1}A$. This method of finding inverse of matrix is called reduced row echelon method.

21. Solution of a system of simultaneous linear equations can be obtained by matrix.

CONTINUITY AND DIFFERENTIABILITY 5

Do not worry about your difficulties in mathematics.
I assure you that mine are greater.

– ***Albert Einstein***

The last thing one knows when writing a book is what to put first.

– ***Blaise Pascal***

5.1 Introduction

We introduced the idea of limit in standard XI. An intuitive approach and graphical understanding helped us to grasp the idea of limit. At several places, we mentioned the word 'continuous'. What is a 'continuous function' ? We will now try to learn the concept of continuity which is very useful to study limits and it links limits and differentiability. Look at the graph of $f(x) = [x]$, $x \in$ R.

We cannot draw the graph of the function without lifting the pencil from the plane of the paper. At every point on the graph, with integer x-coordinate, this situation arises. The same is the situation with the graph of signum function

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

or

$$f(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

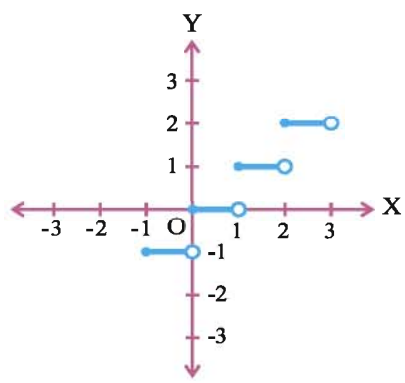

Figure 5.1

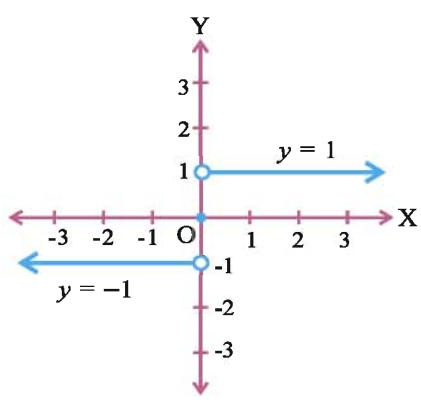

Figure 5.2

At $x = 0$, the graph 'jumps'.

Here $\lim\limits_{x \to 0-} f(x) = -1$ and $\lim\limits_{x \to 0+} f(x) = 1$.

So, $\lim\limits_{x \to 0} f(x)$ does not exist. In the example of $f(x) = [x]$ also, we infer from the graph, $\lim\limits_{x \to 1-} [x] = 0$, $\lim\limits_{x \to 1+} [x] = 1$.

$\therefore$ $\lim\limits_{x \to 1} [x]$ does not exist.

5.2 Continuity

Consider the function $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases}$

Hence, $f(x) = \begin{cases} x+2 & x \neq 2 \\ 5 & x = 2 \end{cases}$

Here, the graph of the function consists of $(\overleftrightarrow{AB} - \{P\}) \cup \{Q\}$

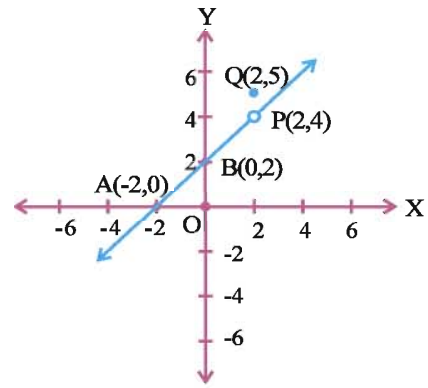

Figure 5.3

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = 4$$

$$\therefore \quad \lim_{x \to 2} f(x) = 4$$

But $f(2) = 5$

$$\therefore \quad \lim_{x \to 2} f(x) \neq f(2)$$

Here also the graph of $f(x)$ cannot be drawn without lifting the pencil from the plane of the paper. This is the idea of continuity. The graph 'breaks' or is 'not continuous'.

Let us now give a formal definition.

Continuity : Let f be a function defined on an interval (a, b) containing c. $c \in R$.

If $\lim_{x \to c} f(x)$ exists and is equal to $f(c)$, then we say f is continuous at $x = c$.

In other words, if $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and are equal to $f(c)$, we say f is continuous at $x = c$.

$\therefore$ f is continuous at $x = c \Leftrightarrow \lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and

$$\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = f(c).$$

If f is not continuous at $x = c$, we say f is discontinuous at $x = c$.

That f is discontinuous at $x = c$ in a domain may occur in one of the following situations.

(1) $\lim_{x \to c+} f(x)$ or $\lim_{x \to c-} f(x)$ does not exist.

(2) $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist but are unequal.

(3) $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and are equal.

i.e. $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = \lim_{x \to c} f(x)$

but f is not defined for $x = c$ or $\lim_{x \to c} f(x) \neq f(c)$

If f is defined at an isolated point, we say it is continuous at that point. Consequently a function defined on a finite set $\{x_1, x_2, x_3, \ldots, x_n\}$ is continuous.

We say f is continuous in a domain, if it is continuous at all points of the domain.

If f is defined on $[a, b]$, then f is continuous on $[a, b]$ if

(1) f is continuous at every point of (a, b)

(2) $\lim_{x \to a+} f(x) = f(a)$ **(f is not defined for $x < a$)**

(3) $\lim_{x \to b-} f(x) = f(b)$ **(f is not defined for $x > b$)**

Example 1 : Examine the continuity of $f : R \to R$, $f(x) = 2x - 4$ at $x = 3$.

Solution : $f(x) = 2x - 4$ is a polynomial in x.

$\therefore \quad \lim_{x \to 3} f(x) = \lim_{x \to 3} (2x - 4) = 2 \cdot 3 - 4 = 2$

$f(3) = 2 \cdot 3 - 4 = 2$

$\therefore \quad \lim_{x \to 3} f(x) = f(3)$

$\therefore$ f is continuous at $x = 3$.

The graph is a straight line and it is 'unbroken'.

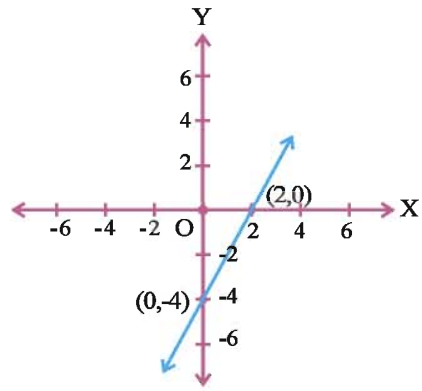

Figure 5.4

Example 2 : Examine continuity of $f : R \to R$, $f(x) = x^2$ at $x = 2$.

Solution : $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$, $f(2) = 4$

($f(x) = x^2$ is a polynomial)

$\therefore \quad \lim_{x \to 2} f(x) = f(2)$

$\therefore$ $f(x) = x^2$ is continuous at $x = 2$.

The graph is 'continuous'.

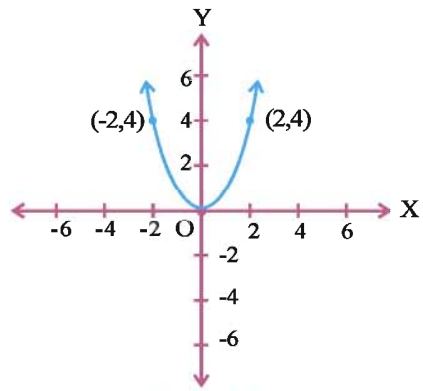

Figure 5.5

Example 3 : Is $f : R \to R$, $f(x) = |x|$ continuous on R?

Solution : Here, we have to examine continuity of $|x|$ on the domain.

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Let $c > 0$. For some $\delta > 0$,

we can have $c - \delta > 0$ **(let $\delta = \frac{c}{2}$)**

$f(x) = |x| = x$ in $(c - \delta, c + \delta)$ **$(c - \delta > 0)$**

$\therefore \quad \lim_{x \to c} f(x) = \lim_{x \to c} x = c$, $f(c) = |c| = c$ **$(c > 0)$**

$\therefore \quad \lim_{x \to c} f(x) = f(c)$

$\therefore$ f is continuous for all $c > 0$

Let $c < 0$. There exists some $\delta > 0$ such that $c + \delta < 0$.

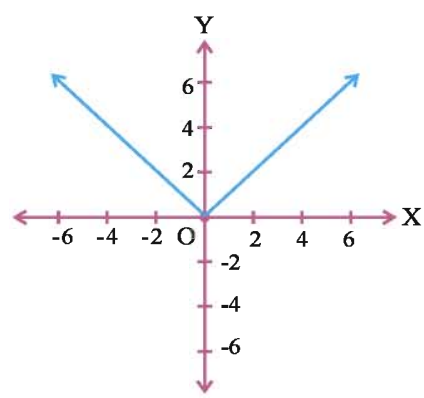

Figure 5.6

$\therefore \quad f(x) = |x| = -x$ in $(c - \delta, c + \delta)$ $\qquad (c + \delta < 0)$

$\therefore \quad \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-x) = -c, f(c) = |c| = -c$ $\qquad (c < 0)$

$\therefore \quad \lim\limits_{x \to c} f(x) = f(c)$

$\therefore$ f is continuous for all $c < 0$.

$\therefore \quad \lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} |x| = \lim\limits_{x \to 0+} x = 0$ $\qquad (x > 0)$

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} |x| = \lim\limits_{x \to 0-} -x = 0$ $\qquad (x < 0)$

$f(0) = |0| = 0$

$\therefore \quad \lim\limits_{x \to 0} f(x) = f(0) = 0$

$\therefore$ f is continuous at $x = 0$.

$\therefore$ f is continuous for all $x \in$ R.

Example 4 : Discuss the continuity of constant function $f(x) = k$ on R.

Solution : For $c \in$ R, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} k = k = f(c)$ $\qquad (\lim\limits_{x \to c} k = k)$

$\therefore$ A constant function is continuous on its domain.

Example 5 : Discuss the continuity at $x = 0$.

$$f(x) = \begin{cases} x^3 + x^2 + x + 1 & x \neq 0 \\ 5 & x = 0 \end{cases}$$

Solution : $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (x^3 + x^2 + x + 1) = 1$ $\qquad$ **(limit of a polynomial)**

$f(0) = 5$

$\therefore \quad \lim\limits_{x \to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$

Example 6 : Examine the continuity of the identity function on R.

Solution : Here $f(x) = x$.

Let $a \in$ R.

$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} x = a = f(a)$

$\therefore$ The identity function is continuous on R.

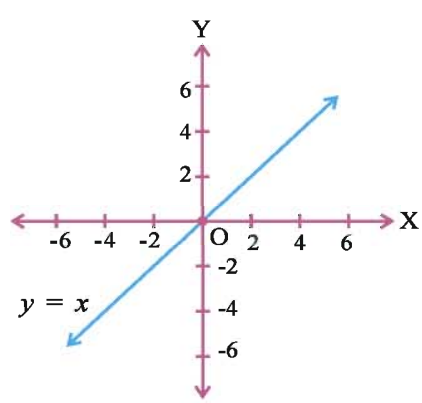

Figure 5.7

Example 7 : Discuss the continuity of $f(x) = \frac{1}{x}$, $x \in R - \{0\}$.

Solution : $f(x) = \frac{1}{x}$ is a rational function.

Let $c \neq 0$.

$$\lim_{x \to c} f(x) = \frac{\lim_{x \to c} 1}{\lim_{x \to c} x} = \frac{1}{c}$$

$$f(c) = \frac{1}{c}$$

$$\lim_{x \to c} f(x) = \frac{1}{c} = f(c)$$

$\therefore$ f is continuous for all $c \in R - \{0\}$.

Note : For $x = 0$, $f(x) = \frac{1}{x}$ is not defined. Let us study behaviour of $f(x)$ near 0.

Let $x > 0$.

x	0.1	0.01	0.001	10^{-n}
$f(x)$	10	$100 = 10^2$	$1000 = 10^3$	10^n

As $x \to 0+$, $f(x)$ increases unboundedly.

In such a case we say $f(x) \to \infty$ as $x \to 0+$. We do not write $\lim_{x \to 0+} f(x) = \infty$.

$\lim_{x \to 0+} f(x)$ does not exit.

Limit of a function is a **real number**. ∞ is not a real number or it is a member of extended real number system.

Let $x < 0$.

x	-0.1	-0.01	-0.001	-10^{-n}
$f(x)$	-10	$-100 = -10^2$	$-1000 = -10^3$	-10^n

$\therefore$ Here as x decreases $f(x)$ decreases and as $x \to 0-$, $f(x) \to -\infty$.

Again $\lim_{x \to 0-} f(x) = -\infty$ is **not** to be written.

$\lim_{x \to 0-} f(x)$ **does not exist.**

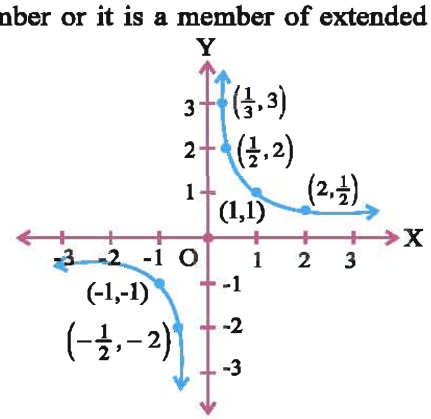

Figure 5.8

Example 8 : $f(x) = \frac{1}{x^2}$, $x \neq 0$. Discuss continuity for $x \in R - \{0\}$.

Solution : Let $c \neq 0$. $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x^2} = \frac{\lim_{x \to c} 1}{\lim_{x \to c} x^2} = \frac{1}{c^2}$

$\therefore$ f is continuous for $x \in R - \{0\}$

Note : For $x = 0$, $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

$\frac{1}{x^2} \to \infty$ as $x \to 0$.

x	-0.1	0.1	-0.01	0.01	$\pm 10^{-n}$
$f(x)$	100	100	10000	10000	10^{2n}

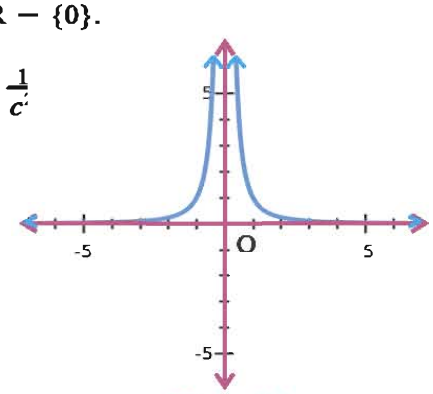

Figure 5.9

Example 9 : Examine the continuity of

$$f(x) = \begin{cases} x + 3 & x < 2 \\ 3 - x & x \geq 2 \end{cases} \quad \text{at } x \in \text{R}.$$

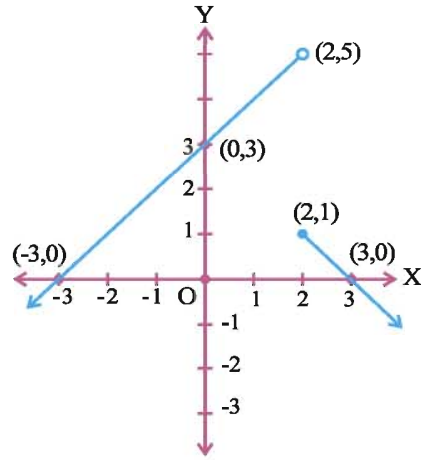

Figure 5.10

Solution : Let $a < 2$. So $f(x) = x + 3$ in some interval around a.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x + 3) = a + 3 = f(a)$$

$\therefore$ f is continuous for all $x \in$ R, with $x < 2$.

Let $a > 2$. So $f(x) = 3 - x$ in some interval around a.

$\therefore$ $f(a) = 3 - a$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (3 - x) = 3 - a = f(a)$$

$\therefore$ f is continuous for all $x \in$ R, with $x > 2$.

Let $a = 2$. $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-} (x + 3) = 5$ **$(x < 2)$**

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3 - x) = 1$$

$\therefore$ $\lim\limits_{x \to 2} f(x)$ does not exist.

$\therefore$ f is continuous for all $x \in$ R except at $x = 2$.

[**Note :** Generally, f is continuous at all points where possibly formula for $f(x)$ changes or its graph is in transition stage.]

Example 10 : Find points of discontinuity of

$$f(x) = \begin{cases} x + 1 & x > 2 \\ 0 & x = 2 \\ 1 - x & x < 2 \end{cases}$$

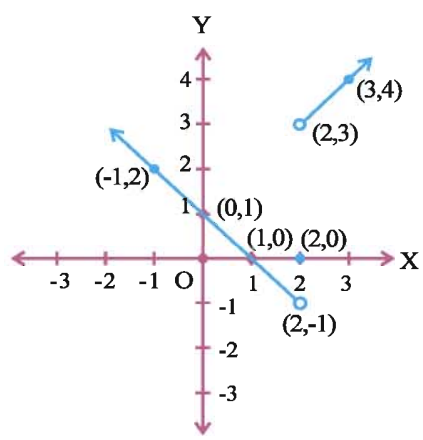

Figure 5.11

Solution : As per above note and a look at the graph of $y = f(x)$, it is clear that f is continuous at all $x \in$ R except at $x = 2$ possibly.

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (1 - x) = 1 - 2 = -1$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x + 1) = 2 + 1 = 3$$

$\therefore$ $\lim\limits_{x \to 2} f(x)$ does not exist.

$\therefore$ f is discontinuous at $x = 2$.

Example 11 : Prove that $f(x) = \begin{cases} x - 1 & x < 1 \\ 1 - x & x > 1 \end{cases}$ is continuous on R $- \{1\}$.

Solution : Let $a < 1$. So $f(a) = a - 1$.

For some $\delta > 0$, we can have $a + \delta < 1$.

Let $x \in (a - \delta, a + \delta)$. $f(x) = x - 1$

$\therefore \quad \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (x - 1) = a - 1 = f(a)$

$\therefore$ f is continuous for $a \in$ R with $a < 1$.

Let $a > 1$. So $f(a) = 1 - a$

For some $\delta > 0$, we can have $a - \delta > 1$

Let $x \in (a - \delta, a + \delta)$. Hence $x > 1$.

$\therefore \quad f(x) = 1 - x$

$\therefore \quad \lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} (1 - x) = 1 - a = f(a)$

$\therefore$ f is continuous for all $a \in$ R such that $a > 1$.

$\therefore$ f is continuous on its domain.

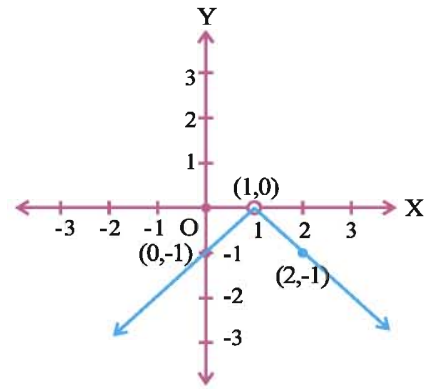

Figure 5.12

Example 12 : If $f(x) = \begin{cases} x - 1 & x < 1 \\ 0 & x = 1 \\ 1 - x & x > 1 \end{cases}$

Examine continuity of f.

Solution : As seen in example 11, f is continuous for all $x \in$ R, $x \neq 1$.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x - 1) = 0, \quad \lim_{x \to 1+} f(x) = \lim_{x \to 1+} (1 - x) = 0$$

$\therefore \quad f(1) = 0$

$\therefore$ f is continuous for $x = 1$.

$\therefore$ f is continuous on R.

Note : Is not $f : \text{R} \to \text{R}, f(x) = -|x - 1|$?

Example 13 : If $f(x) = \begin{cases} x + 2 & x < 0 \\ 2 - x & x > 0 \\ k & x = 0 \end{cases}$

determine k so that f is continuous on R.

Solution : Looking at the graph and since $f(x) = 2 - x$ for $x > 0$ and $f(x) = x + 2$ for $x < 0$ are linear polynomials, f is continuous for all $x \in \text{R} - \{0\}$.

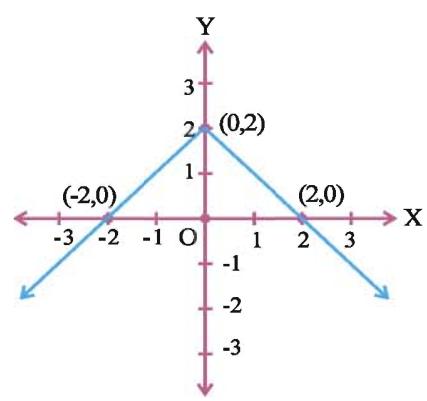

Figure 5.13

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x + 2) = 2$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (2 - x) = 2$$

$$\therefore \quad \lim_{x \to 0} f(x) = 2$$

In order that f is continuous at $x = 0$ also, $\lim\limits_{x \to 0} f(x) = 2 = f(0)$ is necessary.

$\therefore \quad f(0) = k = 2$

$\therefore$ If $k = 2$, f is continuous for all $x \in$ R.

Example 14 : Prove that a polynomial function is continuous.

Solution : $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, $a_i \in$ R $(i = 0, 1, 2, \ldots, n)$ $a_n \neq 0$ is a polynomial.

We know $\lim\limits_{x \to a} x^n = a^n$

$\lim\limits_{x \to a} a_i = a_i$ **(limit of a constant function)**

Also $\lim\limits_{x \to a} (f_1(x) + f_2(x) + \ldots + f_n(x)) = \lim\limits_{x \to a} f_1(x) + \lim\limits_{x \to a} f_2(x) + \ldots + \lim\limits_{x \to a} f_n(x)$

Now $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0)$

$$= \lim_{x \to a} a_n \lim_{x \to a} x^n + \lim_{x \to a} a_{n-1} \lim_{x \to a} x^{n-1} + \ldots + \lim_{x \to a} a_0$$

$$= a_n a^n + a_{n-1} a^{n-1} + \ldots + a_0$$

$$= f(a)$$

$\therefore$ A polynomial function is continuous for all $x \in$ R.

Example 15 : Prove $f(x) = [x]$ is continuous at all $x \in$ R except at all integers.

Solution : $f(x) = \begin{cases} \ldots & \ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ \ldots & \ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots \end{cases}$

$\therefore$ f is a constant function in any interval $(n, n + 1)$ where $n \in$ Z.

$\therefore$ f is continuous in all intervals $(n, n + 1)$ i.e. at all $x \in$ R $-$ Z.

Now $f(x) = \begin{cases} n - 1 & n - 1 \le x < n \\ n & n \le x < n + 1 \end{cases}$

Let $x = n$, $n \in$ Z

We can choose $\delta > 0$ such that $n - 1 < n - \delta < n$. **(In fact $0 < \delta < 1$)**

$\therefore$ $\lim\limits_{x \to n-} f(x) = \lim\limits_{x \to n-} n - 1 = n - 1$ **($x \in (n - \delta, n)$)**

Choose $\delta > 0$ so that $n < n + \delta < n + 1$. **($0 < \delta < 1$)**

$\therefore$ $\lim\limits_{x \to n+} f(x) = \lim\limits_{x \to n+} n = n$ **($x \in (n, n + \delta)$)**

$\therefore$ $\lim\limits_{x \to n} f(x)$ does not exist. (See figure 5.1)

$\therefore$ f is discontinuous for all integers.

$\therefore$ $f(x) = [x]$ is continuous on R $-$ Z and discontinuous for all $n \in$ Z.

Example 16 : Find k, if the following function is continuous at $x = 2$

$$f(x) = \begin{cases} kx + 3 & x \le 2 \\ 7 & x > 2 \end{cases}$$

Solution : $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-} (kx + 3) = 2k + 3$

$\lim\limits_{x \to 2+} f(x) = \lim\limits_{x \to 2+} 7 = 7$

$\therefore \quad \lim_{x \to 2} f(x)$ exists if $2k + 3 = 7$ i.e. $k = 2$

For $k = 2$, $f(2) = 2 \cdot 2 + 3 = 7$

$\therefore \quad \lim_{x \to 2} f(x) = 7 = f(2)$

$\therefore \quad f$ is continuous at $x = 2$, if $k = 2$.

Example 17 : Find a and b so that the following function is continuous.

$$f(x) = \begin{cases} 3 & x \le 1 \\ ax + b & 1 < x < 3 \\ 7 & x \ge 3 \end{cases}$$

Solution : f is a constant function except for $x \in (1, 3)$

f is a linear polynomial in $(1, 3)$. So it is continuous function.

Hence, f is continuous for $x \in \mathrm{R} - \{1, 3\}$ and in $(1, 3)$ except for possibly $x = 1$ and 3.

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (ax + b) = a + b, \quad \lim_{x \to 1-} f(x) = \lim_{x \to 1-} 3 = 3$

Since f is required to be continuous at $x = 1$, $\lim_{x \to 1} f(x)$ must exist.

$\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$

$\therefore \quad a + b = 3$ **(i)**

$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (ax + b) = 3a + b, \quad \lim_{x \to 3+} f(x) = \lim_{x \to 3+} 7 = 7$

Since f is required to be continuous at $x = 3$, $\lim_{x \to 3} f(x)$ must exist.

$\lim_{x \to 3+} f(x) = \lim_{x \to 3-} f(x)$

$\therefore \quad 3a + b = 7$ **(ii)**

Solving (i) and (ii), $a = 2$, $b = 1$. Also $\lim_{x \to 1} f(x) = 3$, $\lim_{x \to 3} f(x) = 7$.

Now, $f(1) = 3$, $\lim_{x \to 1} f(x) = 3 = f(1)$

$f(3) = 7$, $\lim_{x \to 3} f(x) = 7 = f(3)$

$\therefore$ If $a = 2$ and $b = 1$, f is continuous on R.

Example 18 : Find a and b, if following function is continuous at $x = 0$ and 1.

$$f(x) = \begin{cases} x + a & x < 0 \\ 2 & 0 \le x < 1 \\ bx - 1 & 1 \le x < 2 \end{cases}$$

Solution : $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x + a) = a$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2 = 2.$

Since f is continuous at $x = 0$, $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x)$

$\therefore \quad a = 2$. Also $f(0) = 2$

$\therefore \quad \lim_{x \to 0} f(x) = 2 = f(0)$

$\therefore$ Taking $a = 2$, f is continuous at $x = 0$.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 2 = 2$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (bx - 1) = b - 1$$

Since, f is continuous at $x = 1$, $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$

$\therefore$ $b - 1 = 2$

$\therefore$ $b = 3$

Also, $f(1) = b - 1 = 3 - 1 = 2$

$\therefore$ $\lim_{x \to 1} f(x) = 2 = f(1)$

$\therefore$ Taking $a = 2$ and $b = 3$, f is continuous at $x = 0$ and $x = 1$.

5.3 Algebra of continuous functions

The concept of continuity is formulated in terms of limit. Hence, just like working rules of limit, we can have working rules for continuity of $f \pm g$, $f \times g$, $\frac{f}{g}$, etc.

Theorem 5.1 : Let f and g be continuous at $x = c$ and $c \in (a, b)$ for some interval (a, b).

Then (1) $f + g$ is continuous at $x = c$.

(2) kf is continuous at $x = c$. $k \in \mathbf{R}$

(3) $f - g$ is continuous at $x = c$.

(4) $f \times g$ is continuous at $x = c$.

(5) $\frac{k}{g}$ is continuous at $x = c$ if $g(c) \neq 0$. $k \in \mathbf{R}$

(6) $\frac{f}{g}$ is continuous at $x = c$ if $g(c) \neq 0$

$\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$ as f, g are continuous at $x = c$.

(1) $$\lim_{x \to c} (f + g)(x) = \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$= f(c) + g(c)$$

$$= (f + g)(c)$$

$\therefore$ $f + g$ is continuous at $x = c$.

(2) $$\lim_{x \to c} (kf)(x) = \lim_{x \to c} kf(x)$$

$$= \lim_{x \to c} k \lim_{x \to c} f(x)$$

$$= kf(c)$$

$$= (kf)(c)$$

$\therefore$ kf is continuous at $x = c$.

(3) If $k = -1$, $-g$ is continuous at $x = c$ as g is continuous.

$\therefore$ $f + (-g) = f - g$ is continuous at $x = c$.

(4) $$\lim_{x \to c} (f \times g)(x) = \lim_{x \to c} f(x)g(x)$$

$$= \lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

$$= f(c)\, g(c)$$

$$= (f \times g)(c)$$

$\therefore$ $f \times g$ is continuous at $x = c$.

(5) $$\lim_{x \to c} \left(\frac{k}{g}\right)(x) = \frac{\lim\limits_{x \to c} k}{\lim\limits_{x \to c} g(x)} \qquad (g(x) \neq 0)$$

$$= \frac{k}{g(c)} \qquad (g(c) \neq 0)$$

$\therefore$ $\frac{k}{g}$ is continuous at $x = c$.

(6) $$\left(\frac{f}{g}\right)(x) = \left(f \times \frac{1}{g}\right)(x)$$

Taking $k = 1$ in (5), $\frac{1}{g}$ is continuous at $x = c$.

$\left(f \times \frac{1}{g}\right) = \frac{f}{g}$ is continuous at $x = c$.

or

$$\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \lim_{x \to c} \frac{f(x)}{g(x)}$$

$$= \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$$

$$= \frac{f(c)}{g(c)} \qquad (g(c) \neq 0)$$

$$= \left(\frac{f}{g}\right)(c)$$

$\therefore$ $\frac{f}{g}$ is continuous at $x = c$.

Some Important Results :

(1) A rational function is continuous on its domain.

$h(x) = \frac{p(x)}{q(x)}$ is a rational function, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)}$$

$$= \frac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)}$$

$$= \frac{p(a)}{q(a)} \qquad (q(a) \neq 0)$$

$$= h(a)$$

$\therefore$ h is a continuous on its domain.

(2) *sine* function is continuous on R.

We assume following results studied last year

$$\lim_{x \to 0} sinx = 0, \quad \lim_{x \to 0} cosx = 1$$

Let $a \in$ R. Let $x = a + h$, so that as $x \to a$, $h \to 0$

$$\lim_{x \to a} sinx = \lim_{h \to 0} sin\,(a + h)$$

$$= \lim_{h \to 0} (sina\ cosh + cosa\ sinh)$$

$$= sina \lim_{h \to 0} cosh + cosa \lim_{h \to 0} sinh$$

$$= sina \cdot 1 + cosa \cdot 0$$

$$= sina$$

$\therefore$ $\lim_{x \to a} sinx = sina$

$\therefore$ *sine* function is continuous for all $x \in$ R.

(3) *cosine* function is continuous on R.

Let $a \in$ R. Let $x = a + h$. As $x \to a$, $h \to 0$

$$\lim_{x \to a} cosx = \lim_{h \to 0} cos(a + h)$$

$$= \lim_{h \to 0} (cosa\ cosh - sina\ sinh)$$

$$= cosa \lim_{h \to 0} cosh - sina \lim_{h \to 0} sinh$$

$$= cosa \cdot 1 - sina \cdot 0$$

$$= cosa$$

$\therefore$ $\lim_{x \to a} cosx = cosa$

$\therefore$ *cosine* function is continuous for all $x \in$ R.

(4) *tan* function is continuous :

$$tanx = \frac{sinx}{cosx},\ x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \;\middle|\; k \in \text{Z}\right\}$$

sine is continuous for $x \in$ R.

cosine is continuous for $x \in$ R.

$$cosx = 0 \Leftrightarrow x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \;\middle|\; k \in \text{Z}\right\}$$

$\therefore$ By working rule of $\frac{f}{g}$ for continuous functions f and g, *tan* function is continuous on its domain.

(5) Continuity of Composite Function :

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be two functions, so that gof is defined.

If f is continuous at $x_1 \in (a, b)$ and g is continuous at $f(x_1) \in (c, d)$, then gof is continuous at $x_1 \in (a, b)$.

According to the rule of limit of composite functions (std XI, semester II).

$$\lim_{x \to x_1} g(f(x)) = g(\lim_{x \to x_1} (f(x)) = g(f(x_1))$$

$\therefore$ gof is continuous at $x = x_1$.

Example 19 : Prove that $x - [x]$ is discontinuous for all $n \in Z$.

Solution : $f(x) = \begin{cases} \ldots & \ldots \\ \ldots & \ldots \\ x & 0 \le x < 1 \\ x - 1 & 1 \le x < 2 \\ x - 2 & 2 \le x < 3 \\ \ldots & \ldots \\ \ldots & \ldots \end{cases}$

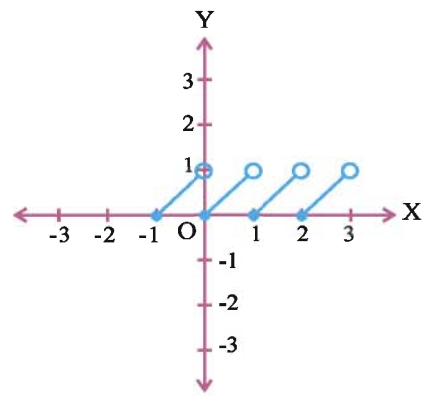

Figure 5.14

For any $n \in Z$

$$\lim_{x \to n-} f(x) = \lim_{x \to n-} (x - [x])$$

$$= \lim_{x \to n-} (x - (n - 1)) \quad \text{(For } 0 < \delta < 1, x \in (n - \delta, n)\text{)}$$

$$= n - (n - 1)$$

$$= 1$$

and $f(n) = n - [n] = n - n = 0$

$\therefore$ $\lim_{x \to n-} f(x) \neq f(n) \quad \forall n \in Z$

$\therefore$ $f(x) = x - [x]$ is not continuous for $n \in Z$.

Note : On intervals $(0, 1)$, $(1, 2)$,... etc. $f(x) = x - [x]$ is continuous. Let if possible, $x - [x]$ be continuous for $n \in Z$. $g(x) = x$ is continuous on R.

$\therefore$ $f(x) = x - [x]$ and $g(x) = x$ both are continuous on R.

$\therefore$ $g(x) - f(x) = x - (x - [x]) = [x]$ is also continuous on R. But $[x]$ is discontinuous for $n \in Z$. So $f(x) = x - [x]$ is not continuous for $n \in Z$.

Example 20 : Prove $sin\,|x|$ is continuous on R.

Solution : $f : R \to R$, $f(x) = |x|$ and $g : R \to R$, $g(x) = sinx$ are continuous.

$\therefore$ $gof : R \to R$, $(gof)(x) = g(f(x)) = g(|x|) = sin\,|x|$ is continuous for all $x \in R$.

Example 21 : Prove $f : R \to R$, $f(x) = |\,1 - x + |x|\,|$ is continuous.

Solution : $g(x) = 1 - x$ and $h(x) = |x|$ are continuous on R.

$\therefore$ $g(x) + h(x) = 1 - x + |x|$ is continuous.

$\therefore$ $f(x) = ho((g + h)(x)) = h((g + h)(x)) = |\,1 - x + |x|\,|$ is continuous as h, g are continuous on R.

Example 22 : Prove $cos(x^3)$ is continuous on R.

Solution : $f : R \to R$, $f(x) = x^3$, $g : R \to R$, $g(x) = cosx$ are continuous.

$\therefore$ $gof : R \to R$, $(gof)(x) = g(f(x)) = g(x^3) = cosx^3$ is continuous.

Example 23 : $f(x) = \begin{cases} \dfrac{kcosx}{\pi - 2x} & x \neq \dfrac{\pi}{2} \\ k^2 & x = \dfrac{\pi}{2} \end{cases}$

Can you find k so that f is continuous at $x = \frac{\pi}{2}$?

Solution : $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{kcosx}{2\left(\frac{\pi}{2} - x\right)} = \lim\limits_{\alpha \to 0} \dfrac{k \sin \alpha}{2\alpha} = \dfrac{k}{2}$ $\quad (\alpha = \frac{\pi}{2} - x)$

$f\left(\frac{\pi}{2}\right) = k^2$

Since f is continuous at $x = \frac{\pi}{2}$, $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore$ $\dfrac{k}{2} = k^2$

$\therefore$ $k = \frac{1}{2}$ or 0

[**Note :** For $k = 0$, $f(x) = 0$ for all $x \in$ R.]

Example 24 : $f(x) = \begin{cases} \dfrac{sinx}{|x|} & x \neq 0 \\ k & x = 0 \end{cases}$

Can you find k so that f is continuous at $x = 0$?

Solution : $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} \dfrac{sinx}{|x|} = \lim\limits_{x \to 0+} \dfrac{sinx}{x} = 1$

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} \dfrac{sinx}{|x|} = \lim\limits_{x \to 0-} \dfrac{sinx}{-x} = -1$

$\therefore$ $\lim\limits_{x \to 0} f(x)$ does not exist.

$\therefore$ f cannot be continuous for $x = 0$, for any value of $k \in$ R.

Example 25 : $f(x) = \begin{cases} \dfrac{sin4x}{9x} & x \neq 0 \\ k^2 & x = 0 \end{cases}$

Find k, if f is continuous for $x = 0$.

Solution : $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{sin4x}{9x}$

$= \lim\limits_{x \to 0} \dfrac{sin4x}{4x} \dfrac{4}{9}$

$= \dfrac{4}{9}$

$\therefore \quad f(0) = k^2$

$\therefore \quad k^2 = \frac{4}{9}$ for f to be continuous at $x = 0$.

$\therefore \quad k = \pm \frac{2}{3}$ for f to be continuous at $x = 0$.

Exercise 5.1

1. Prove *cot*, *cosec* and *sec* are continuous on their domains.
2. Prove ceiling function $f(x) = \lceil x \rceil$ is discontinuous for all $n \in Z$.
3. Prove signum function is discontinuous at $x = 0$.

Discuss continuity of following functions : (4 to 12)

4. $f(x) = \begin{cases} x + 3 & x \geq 2 \\ 3 - x & x < 2 \end{cases}$

5. $f(x) = \begin{cases} x^2 & x \geq 0 \\ x & x < 0 \end{cases}$

6. $f(x) = \begin{cases} 2x + 3 & x < 1 \\ 5 & x = 1 \\ 3x + 2 & x > 1 \end{cases}$

7. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$

8. $f(x) = \begin{cases} \frac{\tan x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

9. $f(x) = \begin{cases} 2x - 3 & x < 0 \\ 2 & x = 0 \\ 3x - 2 & x > 0 \end{cases}$

10. $f(x) = \begin{cases} \frac{\sin x}{3x} & x \neq 0 \\ \frac{2}{3} & x = 0 \end{cases}$

11. $f(x) = \begin{cases} \frac{2x+3}{3x+2} & x > 0 \\ \frac{\sin 3x}{2x} & x < 0 \\ \frac{3}{2} & x = 0 \end{cases}$

12. $f(x) = \begin{cases} \frac{x^2-1}{x^2+1} & x > 0 \\ \frac{\sin x}{|x|} & x < 0 \\ -1 & x = 0 \end{cases}$

Determine k, if following functions are continuous at given values of x : (13 to 16)

13. $f(x) = \begin{cases} \frac{\tan kx}{3x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ **(at $x = 0$)**

14. $f(x) = \begin{cases} \frac{\sin 5x}{kx} & x \neq 0 \\ 1 & x = 0 \end{cases}$ **(at $x = 0$)**

15. $f(x) = \begin{cases} \frac{(x+1)\tan(x-1)}{\sin(x^2-1)} & x \neq 1 \\ k & x = 1 \end{cases}$ **(at $x = 1$)**

16. $f(x) = \begin{cases} 2x^2 + k & x < 0 \\ x^2 - 2k & x \geq 0 \end{cases}$ **(at $x = 0$)**

Find a and b if f is continuous :

17. $f(x) = \begin{cases} 2x + 3 & 1 < x < 2 \\ ax + b & 2 \le x < 3 \\ 3x + 2 & 3 \le x \le 4 \end{cases}$ **(at $x = 2$ and $x = 3$)**

18. Prove $sin^2x - cos^2x$ is continuous on R.

19. Prove $sin2x\ cos3x$ is continuous on R.

20. Prove $sin\,|x|$ is continuous on R.

21. Prove $|\,sinx\,|$ is continuous on R.

22. Prove sin^3x and $sinx^3$ are continuous on R.

23. Prove $cosx^n$ is continuous on R. ($n \in$ N)

24. Prove cos^nx is continuous on R. ($n \in$ N)

25. $f(x) = \begin{cases} sinx - cosx & x \ne 0 \\ -1 & x = 0 \end{cases}$

Prove f is continuous at $x = 0$.

26. $f(x) = \begin{cases} |\,sinx - cosx\,| & x \ne 0 \\ -1 & x = 0 \end{cases}$

Is f is continuous at $x = 0$?

27. $f(x) = \begin{cases} \dfrac{sinx - cosx}{x - \frac{\pi}{4}} & x \ne \frac{\pi}{4} \\ k & x = \frac{\pi}{4} \end{cases}$

If f is continuous at $x = \frac{\pi}{4}$, find k.

28. $f(x) = \begin{cases} \dfrac{x^n - 2^n}{x - 2} & x \ne 2 \\ 80 & x = 2 \end{cases}$

If f is continuous at $x = 2$, find n.

*

5.4 Exponential and Logarithmic Functions

The function $f(x) = x^n$ is used in polynomial functions and rational functions.

Let $f_n(x) = x^n$.

$f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$,..... etc.

Let us draw the graphs.

For $f_2(x)$,

x	1	2	3	4	5	–1	–2	–3
$f_2(x)$	1	4	9	16	25	1	4	9

For $f_3(x)$,

x	1	2	3	4	5	–1	–2	–3
$f_3(x)$	1	8	27	64	125	–1	–8	–27

As x increasees, $f_n(x)$ increases. For a fixed increment in x, where $x > 1$, the increment in $f_n(x)$ increases as n increases. For example if x increases from 2 to 3, $f_{10}(2) = 2^{10}$, $f_{10}(3) = 3^{10}$, $f_{20}(2) = 2^{20}$, $f_{20}(3) = 3^{20}$.

Obviously $3^{20} - 2^{20} > 3^{10} - 2^{10}$.

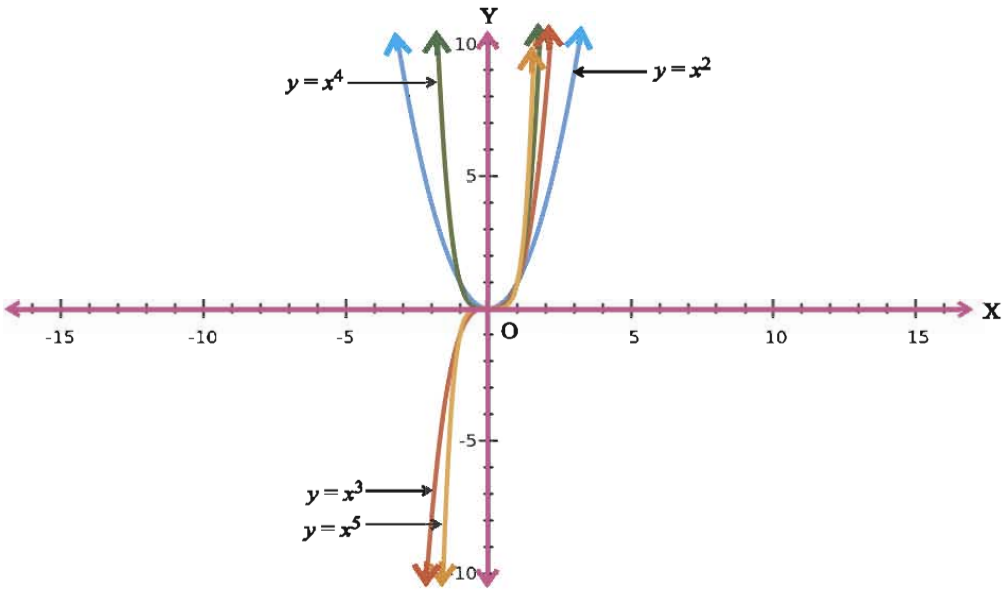

Figure 5.15

Now we consider 'common exponential' function $f(x) = 10^x$. This function increases faster than any $f_n(x)$. Let $x = 10^2$.

Now, $f_{100}(x) = x^{100} = (10^2)^{100} = 10^{200}$, $f(x) = 10^{10^2} = 10^{100}$

For $x = 10^3$, $f_{100}(x) = x^{100} = 10^{300}$, $f(x) = 10^{10^3} = 10^{1000}$

For $x = 10^4$, $f_{100}(x) = (10^4)^{100} = 10^{400}$, $f(x) = 10^{10^4} = 10^{10000}$

Obviously, if $x > 10^3$, $f(x)$ increases much faster than $f_{100}(x)$.

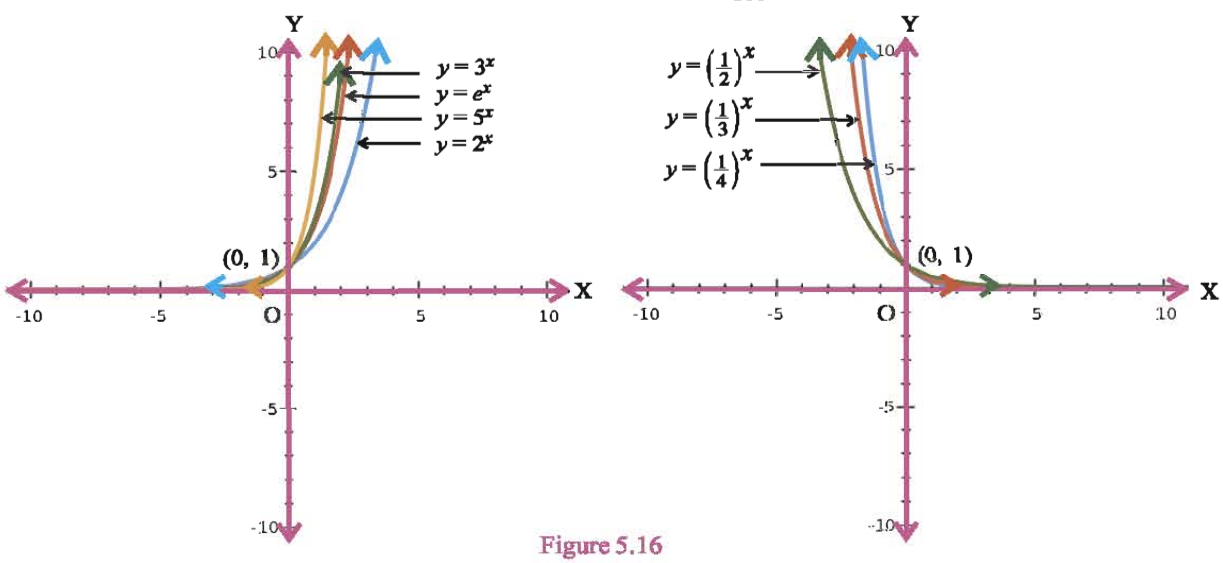

Figure 5.16

Exponential Function : $f(x) = a^x$, $a \in R^+$, $x \in R$ is called an exponential function.

(1) If $a > 1$, $f(x)$ increases as x increases.

If $a < 1$, $f(x)$ decreases as x increases.

(2) The graph of $f(x)$ passes through (0, 1) for any $a \in R^+$.

(3) If $a \neq 1$, the function is one-one and onto.

(4) Its range is R^+.

(5) If a becomes larger, the graph of $f(x)$ leans towards Y-axis for $a > 1$.

(6) As x becomes negative and decreases, the graph of $f(x)$ approaches X-axis but does not intersect X-axis.

Laws of indices for real numbers :

(1) $a^x a^y = a^{x+y}$ (2) $\dfrac{a^x}{a^y} = a^{x-y}$

(3) $(a^x)^y = a^{xy}$ (4) $(ab)^x = a^x b^x$ $a, b \in R^+$ $x, y \in R$

(This content is only for link to the discussion that follows and this is not from examination view point).

The constant e : Limit of a sequence : Just like functions, some sequences also approach a 'limit'.

The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}, \frac{1}{n} \ldots$ has terms nearing 0.

We say $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

We do not formally define limit of a sequence. We accept following results.

(1) $\lim\limits_{n \to \infty} \frac{1}{n} = 0$. $(n \in N)$ We also assume $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ $(x \in R)$

(2) $\lim\limits_{n \to \infty} r^n = 0$ $|r| < 1$

For example if $r = \frac{1}{2}$, we have the sequence, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ and $\left(\frac{1}{2}\right)^n$ approaches 0 as n becomes larger and larger.

Consider the sequence

$$\left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \ldots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + \ldots + \frac{n(n-1)\ldots\ldots 1}{n!\,n^n}$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \ldots + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)}{n!}$$

$1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}$ are all less than 1 and hence their products wherever occuring are less than 1.

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} \qquad (n > 1)$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} \qquad (2^{n-1} < n!)$$

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \qquad \text{(Geometric Progression)}$$

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + 2\left(1-\left(\frac{1}{2}\right)^n\right) = 3 - 2\left(\frac{1}{2}\right)^n < 3 \qquad \text{(i)}$$

Obviously $\left(1+\frac{1}{n}\right)^n > 2$ $\qquad (n > 1)$ (ii)

We assume sequence $\left(1+\frac{1}{n}\right)^n$ has a limit called e and by (i) and (ii) above $2 < e < 3$.

Thus e is a definite constant satisfying $2 < e < 3$. It is called Napier's constant.

Approximatly $e = 2.71828183$

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e$$

We can prove but we will not prove $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$ or replacing $\frac{1}{x}$ by x, $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

Logarithmic Function :

We know **exponential function $f : \mathrm{R} \to \mathrm{R}^+$, $f(x) = a^x$ $(a \in \mathrm{R}^+ - \{1\})$ is one-one and onto.**

Its inverse function $g : \mathrm{R}^+ \to \mathrm{R}$ is called logarithmic function. So if $y = f(x) = a^x$, then $x = g(y) = \log_a y$

This function is denoted as $g = \log_a$

If $y = a^x$, then $x = \log_a y$

We know for inverse functions, $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{A}$, $(fog)(y) = y$, $y \in \mathrm{B}$ and $(gof)(x) = x$, $x \in \mathrm{A}$

Now $f(g(y)) = y$

$\therefore \quad f(\log_a y) = y$

$\therefore \quad a^{\log_a y} = y$

or in other words, **$a^{\log_a x} = x$ for $x \in \mathrm{R}^+$**

If $a = 10$, we get what is called common logarithm. i.e. $\log_{10} x$

Thus, $f : \mathrm{R} \to \mathrm{R}^+$, $f(x) = 10^x$ has inverse $\log_{10} : \mathrm{R}^+ \to \mathrm{R}$, $g(x) = \log_{10} x$

If $a = e$, we get natural logarithm and it is denoted by $\ln_e x$. But unless otherwise stated, we will write $\ln x$ as $\log_e x$ or simply $\log x$.

(1) log has domain R^+ and range R. Hence, logarithm of only positive number can be obtained and log x is a real number if $x \in R^+$.

(2) $a^0 = 1$. Hence $\log_a 1 = 0$

Hence $\log_e 1 = 0$, $\log_{10} 1 = 0$

(3) $a^1 = a$. Hence $\log_a a = 1$

$\log_e e = 1$, $\log_{10} 10 = 1$

$e^{\log_e x} = x$ as $a^{\log_a x} = x$ for $a \in R^+ - \{1\}$

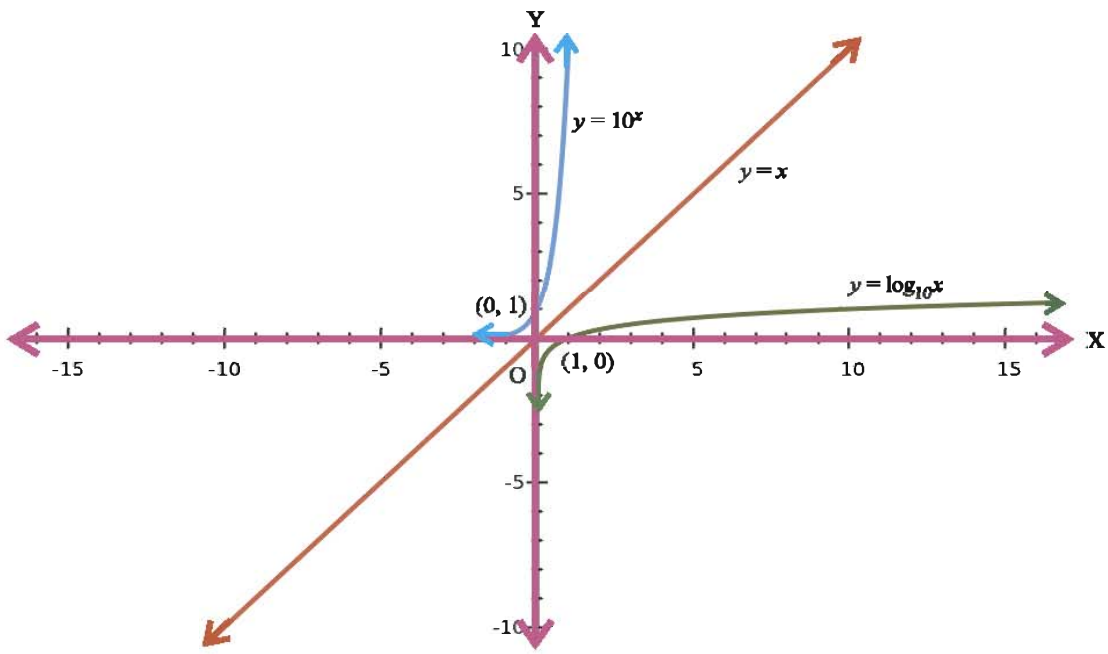

Figure 5.17

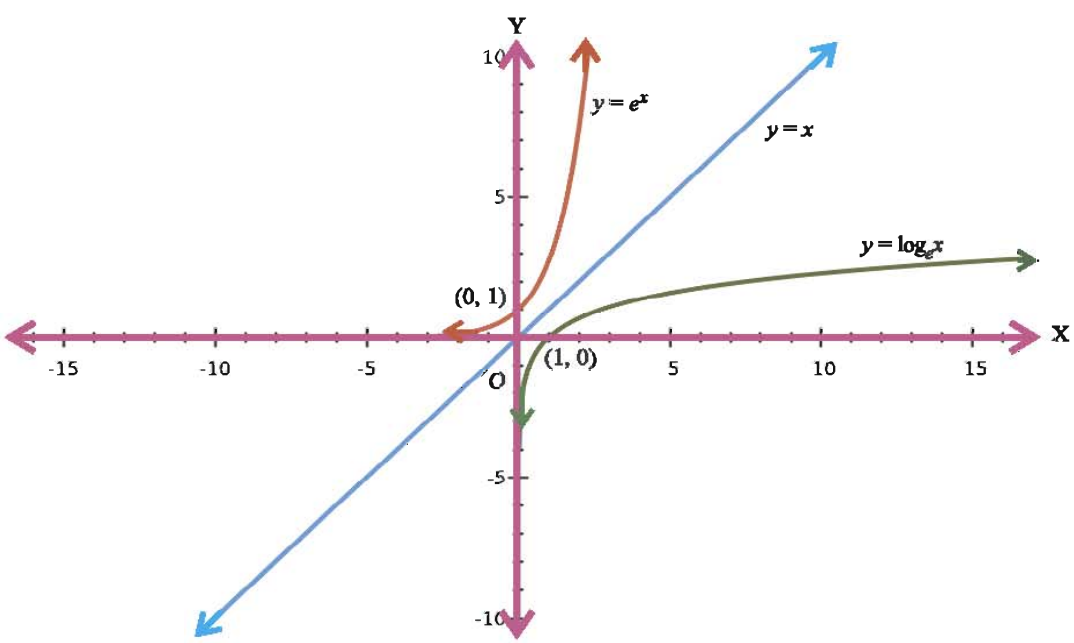

Figure 5.18

We can see that graphs of $f(x) = \log_e x$ and $f(x) = e^x$ are mirror images of each other in the line $y = x$.

(1) (1, 0) is on the graph of log function.

(2) For $a > 1$, it is increasing.

For $0 < a < 1$, it is decreasing.

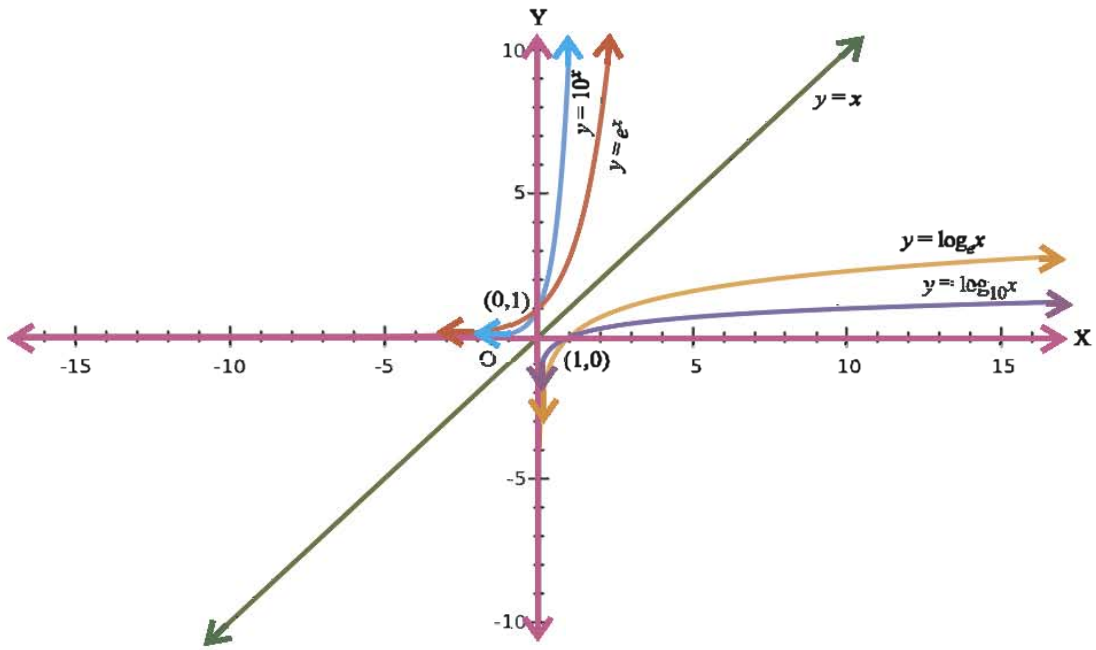

Figure 5.19

Some rules for logarithm :

(1) $\log_a mn = \log_a m + \log_a n$ $\quad (m, n \in \mathbb{R}^+, a \in \mathbb{R}^+ - \{1\})$

Let $\log_a m = x$, $\log_a n = y$

$\therefore \quad m = a^x, n = a^y$

$\therefore \quad mn = a^x a^y = a^{x+y}$

$\therefore \quad \log_a mn = x + y = \log_a m + \log_a n$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$ $\quad (m, n \in \mathbb{R}^+, a \in \mathbb{R}^+ - \{1\})$

Proof is similar as in (1)

(3) $\log_a x^n = n\log_a x$ $\quad (x \in \mathbb{R}^+, n \in \mathbb{Z}, a \in \mathbb{R}^+ - \{1\})$

Let $\log_a x = y$

$\therefore \quad x = a^y$

$\therefore \quad x^n = (a^y)^n = a^{ny}$

$\therefore \quad \log_a x^n = ny$

$\therefore \quad \log_a x^n = n\log_a x$

(4) Change of Basis Rule : $\log_a b = \dfrac{\log_c b}{\log_c a}$ $\quad$ **$(b \in R^+,\ a,\ c \in R^+ - \{1\})$**

Let $\log_a b = x,\ \log_c a = y$

$\therefore\quad b = a^x,\ a = c^y$

$\therefore\quad b = (c^y)^x = c^{xy}$

$\therefore\quad \log_c b = xy = \log_a b \times \log_c a$

$\therefore\quad \log_a b = \dfrac{\log_c b}{\log_c a}$ $\quad$ **(since $a \neq 1,\ \log_c a \neq 0$)**

Also $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{1}{x} \log(1+x)$

$= \lim\limits_{x \to 0} \log(1+x)^{\frac{1}{x}}$

$= \log\left(\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}\right)$ $\quad$ **(log is continuous)**

$= \log_e e$

$= 1$

$\therefore\quad \lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$

5.5 Differentiation

We have learnt the concept of differentiation last year. Let us remember.

If $f : (a, b) \to R$ is a function and if $c \in (a, b)$ and h is so small that $c + h \in (a, b)$, then $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$, if it exists, is called the derivative of f at c and is denoted by $f'(c)$ or $\left[\dfrac{d}{dx} f(x)\right]_{x=c}$ or $\left(\dfrac{dy}{dx}\right)_{x=c}$ where $y = f(x)$. If the derivative of f exists at $x = c$, we say f is differentiable at $x = c$. $\dfrac{dy}{dx}$ is also denoted by y_1.

If f is differentiable for all x in a set A, $(A \neq \emptyset)$, we say f is differentiable in A.

f is differentiable at $c \in (a, b)$ means $\lim\limits_{h \to 0+} \dfrac{f(c+h) - f(c)}{h}$ and $\lim\limits_{h \to 0-} \dfrac{f(c+h) - f(c)}{h}$ both exist and are equal.

Let f be defined on $[a, b]$. f is differentiable in $[a, b]$ means

(1) f is differentiable in (a, b)

(2) $\lim\limits_{h \to 0+} \dfrac{f(a+h) - f(a)}{h}$ exists.

We call this limit right-hand derivative of f at $x = a$ and write $f'(a+)$.

(3) $\lim\limits_{h \to 0-} \dfrac{f(b+h) - f(b)}{h}$ exists,

We call this left-hand derivative of f at $x = b$ and denote it by $f'(b-)$.

We also assume following working rules and standard forms.

If f and g are differentiable at x,

(1) $f \pm g$ is differentiable at x and $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

(2) $f \times g$ is differentiable at x and $\frac{d}{dx}f(x)\,g(x)) = f(x)\,\frac{d}{dx}g(x) + g(x)\,\frac{d}{dx}f(x)$

(3) $\frac{f}{g}$ is differentiable at x if $g(x) \neq 0$ and $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

(4) $\frac{d}{dx}x^n = nx^{n-1}$ $\qquad n \in \mathbf{R},\ x \in \mathbf{R}^+$

(5) $\frac{d}{dx}\sin x = \cos x$ $\qquad x \in \mathbf{R}$

(6) $\frac{d}{dx}\cos x = -\sin x$ $\qquad x \in \mathbf{R}$

(7) $\frac{d}{dx}\tan x = \sec^2 x$ $\qquad x \in \mathbf{R} - \left\{(2k-1)\frac{\pi}{2} \mid k \in \mathbf{Z}\right\}$

(8) $\frac{d}{dx}\sec x = \sec x \tan x$ $\qquad x \in \mathbf{R} - \left\{(2k-1)\frac{\pi}{2} \mid k \in \mathbf{Z}\right\}$

(9) $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$ $\qquad x \in \mathbf{R} - \{k\pi \mid k \in \mathbf{Z}\}$

(10) $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$ $\qquad x \in \mathbf{R} - \{k\pi \mid k \in \mathbf{Z}\}$

Now we prove a theorem.

Theorem 5.2 : If f is differentiable at $x = c$, it is continuous at $x = c$. $c \in (a, b)$

Proof : Let f be differentiable at $x = c$.

$\therefore \quad \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Now $f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c)$ for $x \neq c$.

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c}\frac{f(x) - f(c)}{x - c} \lim_{x \to c}(x - c)$$

(because f is differentiable, $\lim\limits_{x \to c}\frac{f(x) - f(c)}{x - c}$ exists)

$$= f'(c) \cdot 0 = 0$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(f(x) - f(c) + f(c))$$

$$= \lim_{x \to c}(f(x) - f(c)) + \lim_{x \to c} f(c)$$ **(both the limits exist)**

$$= 0 + f(c)$$

$$= f(c)$$

$\therefore$ f is continuous at $x = c$.

But a continuous function may not be differentiable.

Consider $f(x) = |x|$

$$\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0, \quad \lim_{x \to 0-} |x| = \lim_{x \to 0-} (-x) = 0, \; f(0) = |0| = 0$$

$\therefore$ f is continuous at $x = 0$.

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = 1$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

$\therefore$ $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist.

$\therefore$ $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Can we explain the situation ?

We had seen that $f'(c)$ is the slope of tangent to $y = f(x)$ at $x = c$.

See that the graph of $f(x) = |x|$ consists of two rays meeting at (0, 0) and does not have a tangent at (0, 0). It has a 'corner'.

When can a function fail to have a derivative ?

(1) It is discontinuous at that point. (Fig. 5.21)

(2) The tangent is vertical at $x = c$. (Fig. 5.22)

(3) There is no tangent at $x = c$. (Fig. 5.23)

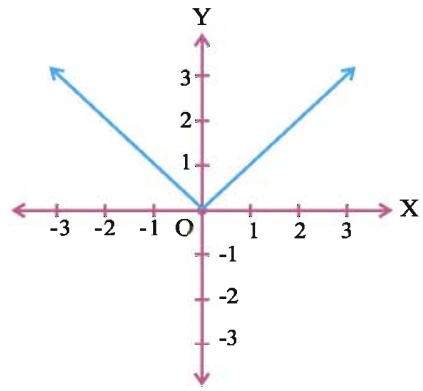

Figure 5.20

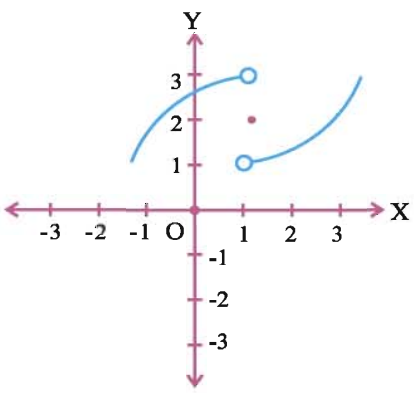

Figure 5.21

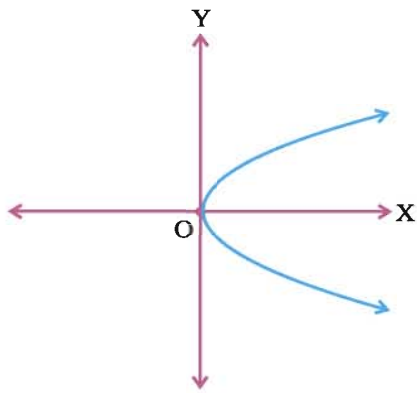

Figure 5.22

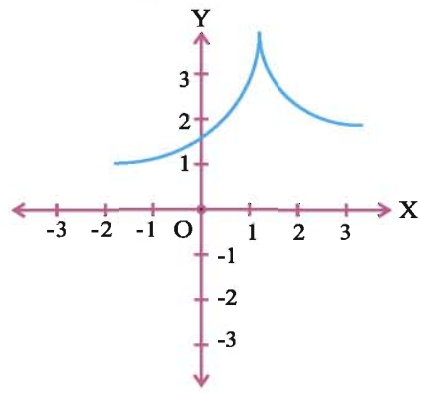

Figure 5.23

Exercise 5.2

1. Prove that $f(x) = |x - 1| + |x - 2| + |x - 3|$ is continuous on R but not differentiable at $x = 1$, 2 and 3 only.

2. Prove $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous but not differentiable at $x = 0$.

3. For $f(x) = \begin{cases} x^2 sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove $f'(0) = 0$. Deduce f is continuous at $x = 0$.

4. Find $f'(x)$ for (1) $f(x) = sin^2x$, (2) $f(x) = tan^2x$, (3) $f(x) = x^4$, (4) $f(x) = cos^4x$

*

5.6 Chain rule or Derivative of a Composite Function

We have seen how to find the derivative of sin^2x or tan^3x using product rule or the derivative of $sin2x$ or $cos2x$ using formulae from trigonometry like $sin2x = 2sinx\ cosx$, $cos2x = cos^2x - sin^2x$ along with product rule.

But they were simple cases. Suppose we want to find the derivative of $tan^5(x^2 - x + 1)$. It is not so easy.

Let us take an example.

Let $f(x) = (2x + 1)^4$

$= 16x^4 + 32x^3 + 24x^2 + 8x + 1$

$f'(x) = 64x^3 + 96x^2 + 48x + 8$

$= 8(8x^3 + 12x^2 + 6x + 1)$

$= 8(2x + 1)^3$

$= 2 \cdot 4\,(2x + 1)^3$

Let $g(t) = t^4$ and $t = h(x) = 2x + 1$. So, $g(h(x)) = g(2x + 1) = (2x + 1)^4 = f(x)$

$\therefore\ f(x) = g(h(x))$

Now $g'(t) = 4t^3$ and $\frac{dt}{dx} = h'(x) = 2$

$f'(x) = 8(2x + 1)^3 = 4(2x + 1)^3 \cdot 2$

$= 4t^3 \cdot 2 = g'(t)\frac{dt}{dx} = g'(t)\ h'(x)$

So, $\frac{d}{dx}f(x) = \frac{d}{dx}g(h(x)) = g'(t)\,h'(x) = g'(h(x))\ h'(x)$

Here, we have expressed $f(x)$ as a composite function of two functions $g(t) = t^4$ and $h(x) = 2x + 1$ whose derivative can be found out in a very simple manner and $f'(x)$ can be calculated in a simple way.

Let us make it formal.

Chain rule : $f : (a, b) \to (c, d)$ is differentiable at x and $g : (c, d) \to (e, f)$ is differentiable at $f(x)$ are two differentiable functions.

Now, $(gof)(x) = g(f(x))$

Then $(gof)'(x) = g'(f(x))\ f'(x)$

In other words let $h(x) = (gof)(x) = g(f(x))$. Let $f(x) = t$

Then $h'(x) = (gof)'(x) = g'(f(x))\ f'(x)$

$= g'(t)\ f'(x)$

$\therefore\ \frac{d}{dx}g(f(x)) = \frac{d}{dt}g(t)\frac{d}{dx}f(x)$. where $t = f(x)$

Thus, $\frac{d}{dx} g(f(x)) = \frac{du}{dt}\frac{dt}{dx}$. where $u = g(t)$ and $t = f(x)$.

Hence $\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx}$ where $u = g(t)$ and $t = f(x)$ and $u = g(f(x))$

Thus if u is a function of t and t is a function of x. Then u is a composite function of x and

$$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx}$$

This rule is called chain rule.

Continuing $\frac{du}{dx} = \frac{du}{dt}\frac{dt}{ds}\frac{ds}{dv}\frac{dv}{dx}$

Here u is a function of t, t is a function of s, s is a function of v and v is a function of x.

Example 26 : Find $f'(x)$ if $f(x) = sin(tan\ x)$

Solution : We have $g(t) = sin\ t$ and $t = h(x) = tanx$

$\therefore\ f(x) = (goh)(x) = g(h(x)) = sin(tanx)$

$\therefore\ f'(x) = g'(h(x))\ h'(x)$

$= g'(t)\ h'(x)$

$= cost\ h'(x)$

$= cos(tanx)\ sec^2x$ **($t = tanx$)**

$\therefore\ f'(x) = cos(tanx)\ sec^2x$

But we can make it simpler.

$f(u) = sinu$ where $u = tanx$

$\therefore\ f'(x) = \frac{df}{du}\frac{du}{dx} = cosu\ sec^2x = cos(tanx)\ sec^2x$

Generally, we make calculations orally.

Go on differentiating functions selecting the outermost function first and then proceeding to differentiate till we reach the variable and multiply all derivatives.

Let $f(x) = sin(cos(2x + 3))$

$\therefore\ f'(x) = cos\ (cos(2x + 3))\quad (-sin(2x + 3))\quad \cdot\quad 2$

Derivative of outer most function at its variable. (Proceed to 'inside') (Derivative of last function $2x + 3$)

$= -2sin\ (2x + 3)\ cos(cos(2x + 3))$ **(rearrange)**

Let $f(x) = sin(tan\ (cos\ (x^2 - 3x + 51)))$

$\therefore\ f'(x) = cos(tan\ (cos\ (x^2 - 3x + 51)))\ (sec^2\ (cos\ (x^2 - 3x + 51)))\ (-sin\ (x^2 - 3x + 51)) \times$

Stage 1 Stage 2 Stage 3

$(2x - 3)$

Stage 4

$= -(2x + 3)\ sin(x^2 - 3x + 51)\ sec^2\ (cos\ (x^2 - 3x + 51))\ cos(tan(cos(x^2 - 3x + 51)))$

(rearranging)

Example 27 : Find $\frac{dy}{dx}$, if $y = sin^3x\ cos^5x$

Solution : $\frac{dy}{dx} = sin^3x \frac{d}{dx} cos^5x + cos^5x \frac{d}{dx} sin^3x$

$= sin^3x \frac{d}{dx} (cosx)^5 + cos^5x \frac{d}{dx} (sinx)^3$

$= sin^3x \cdot 5cos^4x\ (-sinx) + cos^5x \cdot 3sin^2x\ cosx$

$= -5sin^4x \cdot cos^4x + 3sin^2x\ cos^6x$

[**Note :** In sin^nx, $sin^nx = (sinx)^n$; power is 'outermost' function.]

Example 28 : Find $\frac{d}{dx} sin^3(x^2 - x + 1)$

Solution : $\frac{d}{dx} sin^3(x^2 - x + 1) = \frac{d}{dx} [sin(x^2 - x + 1)]^3$

$= 3sin^2(x^2 - x + 1)\ cos(x^2 - x + 1)\ (2x - 1)$

$= 3(2x - 1)\ sin^2(x^2 - x + 1)\ cos(x^2 - x + 1)$

Example 29 : Find $\frac{d}{dx} \sqrt{sinx^3}$

Solution : $\frac{d}{dx} \sqrt{sinx^3} = \frac{d}{dx} (sinx^3)^{\frac{1}{2}}$

$= \frac{1}{2}(sinx^3)^{-\frac{1}{2}} \cdot cosx^3 \cdot 3x^2$ ($\sqrt{\ }$ is outermost function)

$= \frac{3}{2} \frac{x^2\ cosx^3}{\sqrt{sinx^3}}$

(**Note :** Remember $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$)

Example 30 : Find $\frac{d}{dx} \sqrt[4]{sin^3x}$

Solution : $\frac{d}{dx} \sqrt[4]{sin^3x} = \frac{d}{dx} [(sinx)^3]^{\frac{1}{4}} = \frac{d}{dx} (sinx)^{\frac{3}{4}}$

$= \frac{3}{4} sin^{-\frac{1}{4}}x \cdot cosx$

$= \frac{3cosx}{4\sqrt[4]{sinx}}$

Exercise 5.3

Find the derivative of the following functions defined on proper domains :

1. $sin^3(2x + 3)$
2. tan^3x
3. $sin^3x\ cos^5x$
4. $cos(sin(sec(2x + 3))$
5. $sec(cot(x^3 - x + 2))$
6. Differentiate the identity $sin3x = 3sinx - 4sin^3x$. What do you observe ?

7. Find $\frac{d}{dx}(2x+3)^m(3x+2)^n$

8. Find $\frac{d}{dx}(sin^n x - cos^n x)$

9. Find $\frac{d}{dx} sin^3 x \; cos^3 x$

10. Find $\frac{d}{dx} sin^3(4x-1) \; cos^3(2x+3)$

*

5.7 Derivative of Inverse Functions

We have studied inverse trigonometric functions in chapter 2. Now we would like to find their derivatives.

Derivative of Inverse Function : Let $f : (a, b) \to (c, d)$ be a one-one and onto function, so that its inverse function exists. Its inverse

$$g : (c, d) \to (a, b) \text{ satisfies } x = g(y) = f^{-1}(y) \text{ if } y = f(x)$$

We assume $f'(x) = \frac{dy}{dx} = \frac{1}{g'(y)} = \frac{1}{\frac{dx}{dy}}$ $\qquad \left(\frac{dx}{dy} \neq 0\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{or} \quad f'(x) = \frac{1}{\frac{d}{dy}f^{-1}(y)}$$

We have some standard forms :

(1) $\frac{d}{dx} sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $\qquad |x| < 1$

Let $y = sin^{-1}x$. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $x = siny$ $\qquad \left(y \neq \pm\frac{\pi}{2} \text{ as } x \neq \pm 1\right)$

$$\frac{dx}{dy} = cosy = \sqrt{1-sin^2 y} \qquad \left(cosy > 0 \text{ as } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$= \sqrt{1-x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{d}{dx} sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

(2) $\frac{d}{dx} cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$ $\qquad |x| < 1$

Let $y = cos^{-1}x$. $y \in (0, \pi)$. So $x = cosy$ $\qquad (y \neq 0, \pi \text{ as } x \neq \pm 1)$

$$\frac{dx}{dy} = -siny = -\sqrt{1-cos^2 y} \qquad (siny > 0 \text{ as } y \in (0, \pi))$$

$$= -\sqrt{1-x^2}$$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{d}{dx} cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

or

$sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

$\therefore \quad \frac{d}{dx} sin^{-1}x + \frac{d}{dx} cos^{-1}x = \frac{d}{dx}\frac{\pi}{2} = 0$

$\therefore \quad \frac{d}{dx} cos^{-1}x = -\frac{d}{dx} sin^{-1}x = -\frac{1}{\sqrt{1-x^2}}$ $\qquad |x| < 1$

(3) $\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2}$ $\qquad x \in \mathbf{R}$

Let $y = tan^{-1}x$. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $x = tany$.

$\therefore \quad \frac{dx}{dy} = sec^2y$

$\therefore \quad \frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$

$\therefore \quad \frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2}$

(4) $\frac{d}{dx} cot^{-1}x = -\frac{1}{1+x^2}$ $\qquad x \in \mathbf{R}$

We can prove as in (3) or $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$ will give the result.

(5) $\frac{d}{dx} sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad |x| > 1$

Let $y = sec^{-1}x$. $y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$. So, $x = secy$. $\qquad$ (Why $y \neq 0, y \neq \pi$?)

$\therefore \quad \frac{dx}{dy} = secy\ tany$

Now, $secy = x$, $\quad y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$

There are two cases. $y \in \left(0, \frac{\pi}{2}\right)$ or $y \in \left(\frac{\pi}{2}, \pi\right)$.

(i) $y \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad x = secy > 0$, $tany = \sqrt{x^2-1}$ as $tany > 0$

$\therefore \quad \frac{dx}{dy} = secy\ tany = x\sqrt{x^2-1} = |x|\sqrt{x^2-1}$. Since $x > 0$, so $|x| = x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

(ii) $y \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \quad x = secy < 0$. So $|x| = -x$

$tany = -\sqrt{x^2-1}$, since $tany < 0$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{secy\ tany} = \frac{1}{-x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}} \qquad \forall x \text{ such that } |x| > 1$$

(6) Similarly we can prove, $\frac{d}{dx} cosec^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$ $\qquad |x| > 1$

or since $sec^{-1}x + cosec^{-1}x = \frac{\pi}{2}$

$$\frac{d}{dx} sec^{-1}x + \frac{d}{dx} cosec^{-1}x = \frac{d}{dx}\frac{\pi}{2} = 0$$

$$\frac{d}{dx} cosec^{-1}x = -\frac{d}{dx} sec^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

We have introduced e in this chapter. $2 < e < 3$, e is the base of natural logarithm.

We assume $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$

We know $\lim\limits_{x \to 0} \frac{\log(1+x)}{x} = 1$ **(i)**

Let $\log_e(1 + x) = h$. So $x = e^h - 1$.

$\therefore$ Using (i), $\lim\limits_{h \to 0} \frac{h}{e^h - 1} = 1$ **(As $x \to 0$, $h = \log(1 + x) \to 0$)**

$$\therefore \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(7) $\frac{d}{dx} e^x = e^x$

$$\therefore \quad \frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = e^x \cdot 1 = e^x$$

$$\therefore \quad \frac{d}{dx} e^x = e^x$$

(8) $\frac{d}{dx} a^x = a^x \log_e a$

We know $a = e^{\log_e a}$

$\therefore \quad a^x = (e^{\log_e a})^x = e^{x \log_e a}$

$a^x = e^t$. Here $t = x \log_e a$

By chain rule $\frac{d}{dx} a^x = \frac{d}{dt} e^t \cdot \frac{dt}{dx}$

$= e^t \cdot \log_e a \qquad (\frac{d}{dx} kx = k)$

$= a^x \log_e a$

$$\frac{d}{dx} a^x = a^x \log_e a$$

Note : By using chain rule $\frac{d}{dx} e^{\sin x} = e^{\sin x} \cos x$.

It is like this $e^{\sin x} = \exp(\sin x)$ **(exponential ($\sin x$))**

Outermost function is exp. $\frac{d}{dx}(\exp x) = \frac{d}{dx} e^x = e^x = \exp x$

$\therefore \quad \frac{d}{dx} e^{\sin x} = \frac{d}{dx} \exp(\sin x) = \exp(\sin x) \frac{d}{dx} \sin x = e^{\sin x} \cos x$

$\therefore \quad \frac{d}{dx} e^{\tan 2x} = e^{\tan 2x} \frac{d}{dx} \tan 2x$

$= 2e^{\tan 2x} \sec^2 2x$

(9) $\frac{d}{dx} \log_e x = \frac{1}{x}$ $\qquad x \in \mathbf{R}^+$

Let, $y = \log_e x$

$\therefore \quad x = e^y$

$\therefore \quad \frac{dx}{dy} = e^y$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$

$\therefore \quad \frac{d}{dx} \log_e x = \frac{1}{x}$

Example 31 : Find $\frac{d}{dx} \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$ $\qquad |x| < \frac{1}{\sqrt{3}}$

Solution : Let $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$|x| < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\left(-\frac{\pi}{6}\right) < \tan\theta < \tan\frac{\pi}{6}$

$\Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6}$ $\qquad$ (since $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan$ is $\uparrow$)

$\Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$

Now, $y = \tan^{-1} \frac{3x - x^3}{1 - 3x^2} = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta$ $\qquad \left(3\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= 3\tan^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{1 + x^2}$

Example 32 : Find $\frac{d}{dx} sin^{-1} 2x\sqrt{1-x^2}$, $\quad |x| < \frac{1}{\sqrt{2}}$

Solution : Let $\theta = sin^{-1}x$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So $x = sin\theta$.

$|x| < \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Now, $sin\left(-\frac{\pi}{4}\right) < sin\theta < sin\frac{\pi}{4}$

$\therefore \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}$ $\qquad \left(sin \text{ is } \uparrow \text{ in } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right)$

$\therefore \quad -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

$\therefore \quad y = sin^{-1} 2x\sqrt{1-x^2}$

$= sin^{-1} (2sin\theta \ cos\theta)$ $\qquad \left(\sqrt{1-x^2} = \sqrt{1-sin^2\theta} = cos\theta \text{ as } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= sin^{-1} (sin2\theta)$

$= 2\theta$ $\qquad \left(2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\therefore \quad y = 2sin^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Example 33 : Find $\frac{d}{dx} sec^{-1} \frac{1}{2x^2-1}$, $\quad 0 < x < \frac{1}{\sqrt{2}}$

Solution : Let $\theta = cos^{-1}x$. $\theta \in (0, \pi)$. So $x = cos\theta$. $\qquad$ (Why $\theta \ne 0$ or π ?)

$\therefore \quad y = sec^{-1} \frac{1}{2x^2-1} = sec^{-1} \frac{1}{2cos^2\theta - 1} = sec^{-1} \frac{1}{cos\, 2\theta}$

$\therefore \quad y = sec^{-1} (sec2\theta)$

Now, $0 < x < \frac{1}{\sqrt{2}} \Rightarrow cos\frac{\pi}{2} < cos\theta < cos\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$ $\qquad (cos \text{ is } \downarrow)$

$\Rightarrow \frac{\pi}{2} < 2\theta < \pi$

$\therefore \quad y = sec^{-1} (sec2\theta) = 2\theta = 2cos^{-1}x$ $\qquad \left(2\theta \in \left(\frac{\pi}{2}, \pi\right) \subset [0, \pi] - \left\{\frac{\pi}{2}\right\}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$

Example 34 : Find $\frac{d}{dx} cos^{-1} (4x^3 - 3x)$ for (i) $\frac{1}{2} < x < 1$ (ii) $0 < x < \frac{1}{2}$

Solution : Let $\theta = cos^{-1}x$ so that $x = cos\theta$, $\quad 0 < \theta < \pi$

$\therefore \quad y = cos^{-1} (4x^3 - 3x) = cos^{-1} (4cos^3\theta - 3cos\theta)$

$y = cos^{-1} (cos3\theta)$

(i) $\frac{1}{2} < x < 1 \Rightarrow cos\frac{\pi}{3} < cos\theta < cos0$

$\Rightarrow 0 < \theta < \frac{\pi}{3}$ $\quad$ (cos is $\downarrow$)

$\Rightarrow 0 < 3\theta < \pi$

$\therefore \quad y = cos^{-1}(cos3\theta) = 3\theta = 3cos^{-1}x$ $\quad$ ($3\theta \in (0, \pi)$)

$\therefore \quad \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$

(ii) $0 < x < \frac{1}{2} \Rightarrow cos\frac{\pi}{2} < cos\theta < cos\frac{\pi}{3}$

$\Rightarrow \frac{\pi}{3} < \theta < \frac{\pi}{2}$ $\quad$ (cos is $\downarrow$)

$\Rightarrow \pi < 3\theta < \frac{3\pi}{2}$

$\Rightarrow 0 < 3\theta - \pi < \frac{\pi}{2}$

$\therefore \quad y = cos^{-1}(cos3\theta) = cos^{-1}(-cos(\pi - 3\theta))$

$= \pi - cos^{-1}(cos(\pi - 3\theta))$

$= \pi - cos^{-1}(cos(3\theta - \pi))$

$= \pi - (3\theta - \pi)$ $\quad$ $\left((3\theta - \pi) \in \left(0, \frac{\pi}{2}\right) \subset [0, \pi]\right)$

$= 2\pi - 3\theta$

$= 2\pi - 3cos^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

5.8 Derivative of an Implicit Function

Sometimes we encounter equations of type $f(x, y) = 0$ from which we may or may not get y as a function of x. Functions of type $y = sin^2x$ are called explicit functions of x. But $3y - sin2x = 0$ gives $y = \frac{1}{3}sin2x$.

This is an example of y being an implicit function of x.

Consider the circle $x^2 + y^2 = 1$.

It is not a graph of a function. But $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ two implicit functions can be defined from the relation $x^2 + y^2 - 1 = 0$.

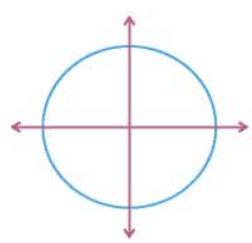

Figure 5.24

So we get two implicit functions. See that any vertical line meets the circle in two points but meets the semicircles in each semiplane of X-axis in only one point. So, each semicircle is a graph of an implicit function.

But some equations are not easy to solve.

$x^3 + y^3 = 3axy$ is such a relation. How to find $\frac{dy}{dx}$ for such implicit functions y ? We use the chain rule and differentiate the relation assuming that y is an implicit function of x.

For example $\frac{d}{dx}x^4 = 4x^3$

$$\frac{d}{dx} y^4 = \frac{d}{dy} y^4 \frac{dy}{dx} = 4y^3 \frac{dy}{dx}$$

So, when we differentiate a term involving variable y *w.r.t* x, we follow usual rules of differentiation and multiply the result by $\frac{dy}{dx}$.

Let us solve some examples.

Example 35 : Find $\frac{dy}{dx}$ from $x + y = sinxy$

Solution : Differentiating the equation,

$$\frac{d}{dx} x + \frac{d}{dx} y = \frac{d}{dx} sinxy$$

$$\therefore \quad 1 + \frac{dy}{dx} = cosxy \frac{d}{dx} (xy) \qquad \textbf{(chain rule)}$$

$$= cosxy \left(x \frac{d}{dx} y + y \cdot 1\right) \qquad \textbf{(product rule)}$$

$$\therefore \quad 1 + \frac{dy}{dx} = x\, cosxy \frac{dy}{dx} + ycosxy$$

$$\therefore \quad (1 - xcos\ xy) \frac{dy}{dx} = y\, cosxy - 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{ycosxy - 1}{1 - xcosxy}$$

Example 36 : Find $\frac{dy}{dx}$ for $x^3 + y^3 = 3axy$

Solution : $3x^2 + 3y^2 \frac{dy}{dx} = 3a \left(x\frac{dy}{dx} + y \cdot 1\right)$

$$\therefore \quad (y^2 - ax) \frac{dy}{dx} = ay - x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 37 : Find $\frac{dy}{dx}$ from $ax^2 + 2hxy + by^2 = 100$

Solution : $2ax + 2h\left(x \frac{dy}{dx} + y\right) + 2by \frac{dy}{dx} = 0$

$$\therefore \quad (hx + by) \frac{dy}{dx} = -(ax + hy)$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right)$$

Example 38 : Find $\frac{dy}{dx}$ from $sin^2x + sin^2y = 1$.

Solution : $2sinx\ cosx + 2siny\ cosy \frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-\sin 2x}{\sin 2y}$$

or

$$\sin^2 y = 1 - \sin^2 x = \cos^2 x$$

$$\therefore \quad \sin y = \pm \cos x$$ **(Two functions)**

$$\therefore \quad \cos y \frac{dy}{dx} = \mp \sin x$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{\sin x}{\cos y}$$

Note : If $\sin^2 x + \sin^2 y = 2$, then $\sin^2 x = \sin^2 y = 1$ as $|\sin x| \leq 1$, $|\sin y| \leq 1$. No such function exists. If $\sin^2 x + \sin^2 y = 3$, then can we write $\frac{dy}{dx} = \frac{-\sin 2x}{\sin 2y}$?

No. $\sin^2 x + \sin^2 y < 2$. No implict function exists if $\sin^2 x + \sin^2 y = 3$. We assume existence of implict function and differentiate. But an implict function may not exist.

Exercise 5.4

Find $\frac{dy}{dx}$: **(1 to 10)**

1. $x^2 + y^2 = 1$
2. $x + \sin x = \sin y$
3. $\sin(x + y) = x - y$
4. $2x^2 + 3xy + y^2 = 1$
5. $\sin x + \sin y = \tan xy$
6. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
7. $y^2 = 10x$
8. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
9. $x^2 + y^2 - 4x - 6y - 25 = 0$
10. $\sin x = \sin y$

Find the derivative : (11 to 16)

11. $y = \sin^{-1}(3x - 4x^3)$, $\quad 0 < x < \frac{1}{2}$
12. $y = \tan^{-1} \frac{2x}{1 - x^2}$, $\quad x \neq \pm 1$
13. $y = \cos^{-1} \frac{1 - x^2}{1 + x^2}$
14. $y = \sin^{-1} \frac{2x}{1 + x^2}$
15. $y = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$, $\quad |x| > \frac{1}{\sqrt{3}}$
16. $y = \sin^{-1} 2x\sqrt{1 - x^2}$, $\quad \frac{1}{\sqrt{2}} < x < 1$

*

5.9 Parametric Differentiation

Sometimes x and y are given as functions of another variable, say t, called a parameter.

Let $x = f(t)$ $\quad y = g(t)$

Assuming that we can obtain $t = f^{-1}(x)$ and substituting in $y = g(t)$, we get $y = g(f^{-1}(x))$. So, y is a function of x.

But this type of solving and differentiating would be cumbersome. We have the following rule :

Rule for differentiation of parametric functions :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)} \quad \text{where } f'(t) \neq 0$$

Example 39 : If $x = a\cos\theta$, $y = b\sin\theta$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b\cos\theta}{a\sin\theta} = \frac{-b}{a}\cot\theta$$

$$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta} = \frac{-b}{a}\left(\frac{\frac{x}{a}}{\frac{y}{b}}\right) = -\frac{b^2x}{a^2y}$$

or directly $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$

$$\therefore \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Example 40 : If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t} \quad (t \neq 0)$$

Example 41 : If $x = a\sin^3\theta$, $y = b\cos^3\theta$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$, $\frac{dy}{d\theta} = 3b\cos^2\theta(-\sin\theta)$

$$\therefore \quad \frac{dy}{dx} = \frac{-3b\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = \frac{-b}{a}\cot\theta$$

$$\cot^3\theta = \frac{\cos^3\theta}{\sin^3\theta} = \frac{ay}{bx}. \text{ So } \cot\theta = \left(\frac{ay}{bx}\right)^{\frac{1}{3}}.$$

$$\text{So } \frac{dy}{dx} = \frac{-b}{a}\left(\frac{ay}{bx}\right)^{\frac{1}{3}}$$

$$= -\frac{b^{\frac{2}{3}}y^{\frac{1}{3}}}{a^{\frac{2}{3}}x^{\frac{1}{3}}}$$

or $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta = 1$

$$\therefore \quad \frac{2}{3}\,\frac{x^{-\frac{1}{3}}}{a^{\frac{2}{3}}} + \frac{2}{3}\,\frac{y^{-\frac{1}{3}}}{b^{\frac{2}{3}}}\,\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^{\frac{2}{3}}\,y^{\frac{1}{3}}}{a^{\frac{2}{3}}\,x^{\frac{1}{3}}}$$

Exercise 5.5

Find $\frac{dy}{dx}$: (wherever y is defined as a function of x and $\frac{dx}{dt}$ or $\frac{dx}{d\theta} \neq 0$)

1. $x = a\sec\theta,\ y = b\tan\theta \qquad \theta \in R - \left[\left\{(2k-1)\frac{\pi}{2} \;\middle|\; k \in Z\right\} \cup \{k\pi \mid k \in Z\}\right]$
2. $x = \cos\theta - \cos 2\theta \qquad y = \sin\theta - \sin 2\theta \qquad \theta \in R - \{k\pi \mid k \in Z\},\ \cos\theta \neq \frac{1}{4}$
3. $x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$
4. $x = a(\cos t + \log \tan\frac{t}{2}), \qquad y = a\sin t$
5. $x = a(\cos\theta + \theta\sin\theta), \qquad y = a(\sin\theta - \theta\cos\theta)$
6. $x = \frac{a}{t^2} \qquad y = bt$
7. If $x = \sqrt{a^{\sin^{-1}t}}, \qquad y = \sqrt{a^{\cos^{-1}t}}$, prove $\frac{dy}{dx} = \frac{-y}{x} \qquad |t| < 1$

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5.10 Logarithmic Differentiation

Sometimes we have to differentiate a product of several functions or a complicated product or $[f(x)]^{g(x)}$ form.

In such a case, it is customary to find $\frac{dy}{dx}$ by taking logarithms.

Example 42 : Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{(2x+3)(3x-4)}{(4x+9)(x-8)}}$

Solution : $\log y = \frac{1}{2}\,[\log(2x+3) + \log(3x-4) - \log(4x+9) - \log(x-8)]$

$$\therefore \quad \frac{1}{y}\,\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{2x+3} + \frac{3}{3x-4} - \frac{4}{4x+9} - \frac{1}{x-8}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{2x+3} + \frac{3}{3x-4} - \frac{4}{4x+9} - \frac{1}{x-8}\right]$$

Example 43 : Find $\frac{dy}{dx}$ if $y = x^{\sin x}$

Solution : $\log y = \sin x \log x$

$$\therefore \quad \frac{1}{y}\,\frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \cos x \log x$$

$$\therefore \quad \frac{dy}{dx} = \left[\frac{\sin x}{x} + \cos x \log x\right] y$$

Example 44 : If $x^y + y^x + a^x + x^a = 1$, find $\frac{dy}{dx}$.

Solution : Let $u = x^y$, $v = y^x$, $w = a^x + x^a$

Now, $\log u = y \log x$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$$

$$\therefore \quad \frac{du}{dx} = \left(\frac{y}{x} + \log x \frac{dy}{dx}\right) x^y$$

Now, $v = y^x$

$$\therefore \quad \log v = x \log y$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\therefore \quad \frac{dv}{dx} = \left(\frac{x}{y}\frac{dy}{dx} + \log y\right) y^x$$

Now, $u + v + w = 1$

$$\therefore \quad \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

$$\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) x^y + \left(\frac{x}{y}\frac{dy}{dx} + \log y\right) y^x + a^x \log_e a + ax^{a-1} = 0$$

$$\left(x^y \log x + \frac{x}{y} y^x\right)\frac{dy}{dx} = -\left(\frac{x^y \cdot y}{x} + y^x \log y + a^x \log a + ax^{a-1}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(yx^{y-1} + y^x \log y + a^x \log a + ax^{a-1})}{xy^{x-1} + x^y \log x}$$

Example 45 : Find $\frac{dy}{dx}$ if $y = (sinx)^x + sinx^x$

Solution : Let $u = (sinx)^x = e^{x \log sinx}$

(since $a = e^{\log_e a}$, $sinx = e^{\log sinx}$)

$$\therefore \quad \frac{du}{dx} = e^{x \log sinx} \frac{d}{dx}(x \log sinx)$$

$$= e^{x \log sinx}\left(1 \cdot \log sinx + \frac{xcosx}{sinx}\right)$$

$$= (sinx)^x (\log sinx + xcot\, x)$$

$$\therefore \quad \frac{d}{dx} sinx^x = cosx^x \frac{d}{dx} x^x$$

$$= cosx^x \frac{d}{dx} e^{x \log x}$$

$$= cosx^x \cdot e^{x \log x}\left(x \frac{1}{x} + \log x\right)$$

$$= x^x cosx^x (1 + \log x)$$

$$\therefore \quad \frac{dy}{dx} = (sinx)^x (\log sinx + xcot\, x) + x^x cosx^x (1 + \log x)$$

(**Note :** $a = e^{\log_e a}$ helps to avoid taking logarithms.)

Exercise 5.6

Find $\frac{dy}{dx}$:

1. $y = \left(x+\frac{1}{x}\right)^x + \left(x+\frac{1}{x}\right)^{\frac{1}{x}}$
2. $y = cosx^x + sinx^x$
3. $y = \sqrt[3]{\frac{(2x+1)^3\,(4x+3)^5}{(7x-1)^6}}$
4. $y = (\log x)^{cosx}$
5. $y = (x+1)^2\,(x+2)^3\,(x+3)^4$
6. $y = (\log x)^x + \log x^x$
7. $y = x^{x\,sin\,x} + (sin\,x)^x$
8. $y = x^{\left(x+\frac{1}{x}\right)}$
9. $y = (sin\,x)^x + \left(\frac{1}{x}\right)^{cosx}$
10. $y = 3^{sinx} + 4^{cosx}$
11. $y^x = x^y$
12. $xy = e^{x-y}$
13. $x^y\,y^x = 1$
14. $y = (1+x)(1+x^2)(1+x^4)(1+x^8)$
15. If $y = (x^2 - 2x + 3)(x^2 - 3x + 15)$, find $\frac{dy}{dx}$

    by (1) Product rule

    (2) Multiply and using rule for polynomials.

    (3) Logarithmic differentiation

    and compare.

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5.11 Second Order Derivative

If f is a differentiable function of x on (a, b) and if $f'(x)$ is also a differentiable function of x on (a, b), its derivative is called second derivative of f and is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$ or y_2 where $y = f(x)$.

Thus $f''(x) = \frac{d}{dx}f'(x)$ or $\frac{d^2y}{dx^2}$, or y_2. Here y_1 denotes $f'(x)$ or $\frac{dy}{dx}$.

We can use chain rule as follows :

$$\frac{d}{dx}\,y^2 = \frac{d}{dy}\,y^2\,\frac{dy}{dx} = 2y\,\frac{dy}{dx} = 2yy_1$$

$$\frac{d}{dx}\,y_1^2 = \frac{d}{dy_1}\,y_1^2\,\frac{d}{dx}\,y_1 = 2y_1\,\frac{dy_1}{dx} = 2y_1y_2$$

Remember $\frac{d}{dx}\,y^2 = 2yy_1$, $\frac{d}{dx}\,y_1^2 = 2y_1y_2$

Example 46 : If $y = acosx + bsinx$, prove $\frac{d^2y}{dx^2} + y = 0$

Solution : $y = acosx + bsinx$

$\therefore \quad y_1 = -a\sin x + b\cos x$

$\therefore \quad y_2 = -a\cos x - b\sin x = -y$

$\therefore \quad \dfrac{d^2y}{dx^2} + y = 0$

Example 47 : $y = ae^{4x} + be^{5x}$, prove $y_2 - 9y_1 + 20y = 0$

Solution : $y = ae^{4x} + be^{5x}$

$y_1 = 4ae^{4x} + 5be^{5x}$

$\therefore \quad y_2 = 16ae^{4x} + 25be^{5x}$

$\therefore \quad y_2 - 9y_1 + 20y = (16ae^{4x} + 25be^{5x}) - 9(4ae^{4x} + 5be^{5x}) + 20(ae^{4x} + be^{5x})$

$= (16 - 36 + 20)\, ae^{4x} + (25 - 45 + 20)\, be^{5x} = 0$

$\therefore \quad y_2 - 9y_1 + 20y = 0$

Example 48 : $y = x^4 + \sin^3 x$. Find $\dfrac{d^2y}{dx^2}$.

Solution : $y = x^4 + \sin^3 x$

$\dfrac{dy}{dx} = 4x^3 + 3\sin^2 x \cos x$

$\therefore \quad \dfrac{d^2y}{dx^2} = 12x^2 + 6\sin x \cos^2 x + 3\sin^2 x\,(-\sin x)$

$= 12x^2 + 6\sin x \cos^2 x - 3\sin^3 x$

Example 49 : Find $\dfrac{d^2y}{dx^2}$ for $y = \log(\log x)$.

Solution : $y = \log(\log x)$

$\dfrac{d}{dx}\log(\log x) = \dfrac{1}{\log x}\,\dfrac{1}{x} = \dfrac{1}{x\log x}$

$\therefore \quad \dfrac{d^2}{dx^2}\log(\log x) = \dfrac{(x\log x)\,0 - 1\cdot(1\log x + x\cdot\frac{1}{x})}{(x\log x)^2}$

$= \dfrac{-(1+\log x)}{(x\log x)^2}$

Example 50 : If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.

Solution : $y = a\cos(\log x) + b\sin(\log x)$

$y_1 = \dfrac{-a\sin(\log x)}{x} + \dfrac{b\cos(\log x)}{x}$

$\therefore \quad xy_1 = -a\sin(\log x) + b\cos(\log x)$

$\therefore \quad \dfrac{d}{dx}(xy_1) = \dfrac{-a\cos(\log x)}{x} - \dfrac{b\sin(\log x)}{x}$

$\therefore \quad x(xy_2 + 1\cdot y_1) = -a\cos(\log x) - b\sin(\log x) = -y$

$\therefore \quad x^2y_2 + xy_1 + y = 0$

Example 51 : If $y = cos^{-1}x$, prove $(1 - x^2)y_2 - xy_1 = 0$.

Solution : $y = cos^{-1}x$

$\therefore \quad y_1 = \frac{-1}{\sqrt{1-x^2}}$

$\therefore \quad (1 - x^2)y_1{}^2 = 1$

$\therefore \quad \frac{d}{dx}(1 - x^2)y_1{}^2 = 0$

$\therefore \quad (1 - x^2)2y_1y_2 + (-2xy_1{}^2) = 0$

$\therefore \quad (1 - x^2)y_2 - xy_1 = 0$ $\qquad (y_1 \neq 0)$

Example 52 : If $y = tan^{-1}x$, prove $(1 + x^2)y_2 + 2xy_1 = 0$.

Solution : $y = tan^{-1}x$

$\therefore \quad y_1 = \frac{1}{1+x^2}$

$\therefore \quad (1 + x^2)y_1 = 1$

$\therefore \quad (1 + x^2)y_2 + 2xy_1 = 0$

Example 53 : If $y = ae^{px} + be^{qx}$, prove that $y_2 - (p + q)y_1 + pqy = 0$.

Solution : $y_1 = ape^{px} + bqe^{qx}$

$y_2 = ap^2e^{px} + bq^2e^{qx}$

$ape^{px} + bqe^{qx} - y_1 = 0$ $\qquad$ (i)

$ap^2e^{px} + bq^2e^{qx} - y_2 = 0$ $\qquad$ (ii)

Solving (i) and (ii) for e^{px} and e^{qx},

$e^{px} = \frac{-bqy_2 + bq^2y_1}{abpq^2 - abp^2q}$ $\qquad$ $e^{qx} = -\frac{-apy_2 + ap^2y_1}{abpq(q-p)}$

$\therefore \quad e^{px} = \frac{-y_2 + qy_1}{ap(q-p)}$ $\qquad$ $e^{qx} = -\frac{-y_2 + py_1}{bq(q-p)}$

$\therefore$ Substituting in $y = ae^{px} + be^{qx}$

$$y = \left(\frac{-y_2 + qy_1}{p(q-p)}\right) - \left(\frac{-y_2 + py_1}{q(q-p)}\right)$$

$\therefore \quad pq(q - p)y = -qy_2 + q^2y_1 + py_2 - p^2y_1$

$= (p - q)y_2 - (p^2 - q^2)y_1$

$\therefore \quad y_2 - (p + q)y_1 + pqy = 0$

5.12 Mean Value Theorems

There are some important theorems in differential calculus called mean value theorems.

Rolle's Theorem : If f is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b)$,

then there exists some $c \in (a, b)$ for which $f'(c) = 0$

Geometrical Interpretation : If the graph of $y = f(x)$ is continuous in $[a, b]$ and if it has a non-vertical tangent at all points $(x, f(x))$ where $x \in (a, b)$ and if $f(a) = f(b)$, there is some $c \in (a, b)$ such that tangent at $(c, f(c))$ to thecurve $y = f(x)$ is horizontal or we can say it is X-axis or parallel to X-axis.

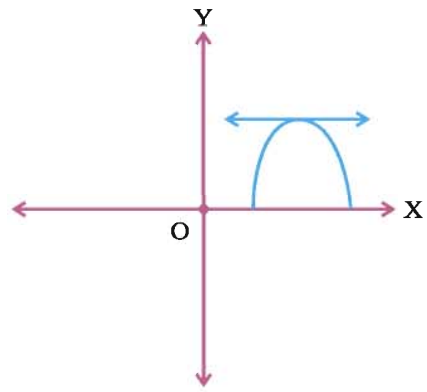

Figure 5.25

Mean-value Theorem (Lagrange) : If f is continuous in $[a, b]$ and differentiable in (a, b), then $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a, b)$.

Geometric Interpretation : If the graph of $y = f(x)$ is continuous in $[a, b]$ and if $y = f(x)$ has a non-vertical tangent at all points, $(x, f(x))$ where $x \in (a, b)$, then $\exists c \in (a, b)$ such that tangent at $(c, f(c))$ is parallel to the secant line joining $A(a, f(a))$ and $B(b, f(b))$.

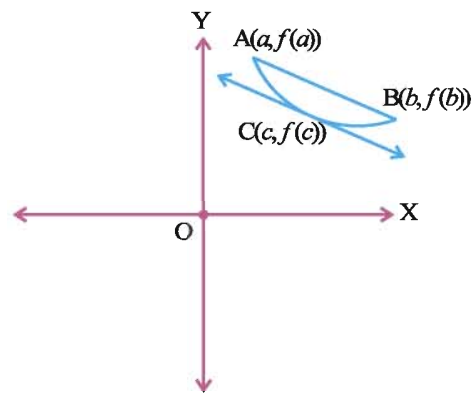

Figure 5.26

We know slope of $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b)-f(a)}{b-a}$

Slope of tangent at $(c, f(c)) = f'(c)$.

Hence the result.

Example 54 : Verify Rolle's theorem for $f(x) = x^2 - 4x + 3$ in $[1, 3]$.

Solution : f is continuous in $[1, 3]$ and differentiable in $(1, 3)$ as it is a polynomial in x.

$f(1) = 0, f(3) = 9 - 12 + 3 = 0$

$\therefore \quad \exists\, c \in (1, 3)$ such that $f'(c) = 0$

Now, $f'(c) = 2c - 4 = 0 \Rightarrow c = 2$ and $2 \in (1, 3)$

$\therefore \quad c = 2, c \in (1, 3)$

Example 55 : Verify Rolle's theorem for $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$.

Solution : f is continuous in $[1, 3]$ and differentiable in $(1, 3)$ and $f(1) = 0 = f(3)$

$$f'(x) = 3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{12 \pm \sqrt{144 - 132}}{6}$$

$\therefore \quad x = 2 \pm \frac{1}{\sqrt{3}} \in (1, 3)$

$\therefore$ The are two value of c namely $c = 2 \pm \frac{1}{\sqrt{3}}$. **$(c \in (1, 3))$**

Example 56 : Verify Rolle's theorem for $f(x) = sinx$ in $[0, \pi]$.

Solution : *sine* is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$ and $sin\, 0 = sin\pi = 0$

$$f'(x) = cosx = 0 \Rightarrow x = \frac{\pi}{2} \text{ in } [0, \pi].$$

$\therefore \quad c = \frac{\pi}{2}$ and $\frac{\pi}{2} \in (0, \pi)$ **$(c \in (0, \pi))$**

Example 57 : Apply the mean value theorem to $f(x) = cosx$ over $[0, \pi]$.

Solution : cos is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$

$a = 0,\ b = \pi$

$\frac{f(b) - f(a)}{b - a} = f'(c)$ gives, $\frac{cos\,\pi - cos\,0}{\pi - 0} = -sinc$

$\therefore \quad \frac{-1-1}{\pi} = -sinc$

$sinc = \frac{2}{\pi}$. Also $0 < \frac{2}{\pi} < 1$.

Since $\exists\, c,\ 0 < c < \pi$ such that $sinc = \frac{2}{\pi}$

[In fact, there will be two value of c in each of $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ such that $sinc = \frac{2}{\pi}$

If we take $c = sin^{-1}\frac{2}{\pi}$, we will get only one value of c in $\left(0, \frac{\pi}{2}\right)$.]

Example 58 : Apply the mean value theorem to $f(x) = e^x$ in $[0, 1]$.

Soloution : $f(x) = e^x$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$. $a = 0,\ b = 1$.

$\frac{f(b) - f(a)}{b - a} = f'(c)$ gives, $\frac{e-1}{1-0} = e^c$

$\therefore \quad e^c = e - 1$

$\therefore \quad c = \log_e(e - 1)$

Now, $2 < e < 3$

$\therefore \quad 1 < e - 1 < 2$

$\therefore \quad 0 < \log(e - 1) < \log_e 2 < \log_e e = 1$ $\qquad (e > 2)$

$\therefore \quad c \in (0, 1)$ and $c = \log_e(e - 1)$

Example 59 : Apply the mean-value theorem to $f(x) = \log x$ in $[1, e]$.

Solution : log function is continuous in $[1, e]$ and differentiable in $(1, e)$.

$a = 1,\ b = e,\ f'(x) = \frac{1}{x}$

$\therefore \quad \frac{\log e - \log 1}{e - 1} = \frac{1}{c}$

$\therefore \quad \frac{1}{c} = \frac{1}{e-1}$ $\qquad (\log 1 = 0,\ \log_e e = 1)$

$\therefore \quad c = e - 1$

Also $1 < e - 1 < e$ as $e > 2$

$\therefore \quad c = e - 1$ $\qquad (c \in (1, e))$

Example 60 : Can you apply the mean-value theorem and Rolle's theorem to $f(x) = [x]$ in $[-2, 2]$.

Solution : f is discontinuous at -1, 0, 1 and 2 (why not at -2 ?)

f is not differentiable at $-1, 0, 1$ in $(-2, 2)$.

$$f(x) = \begin{cases} -2 & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}$$

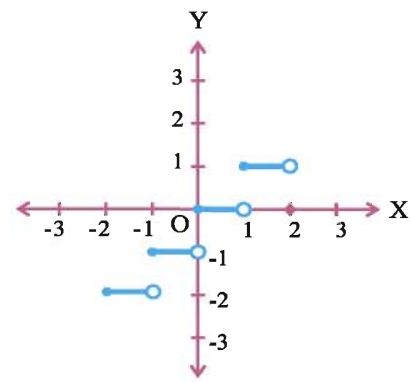

Figure 5.27

But $f'(x) = 0$, $x \in (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2)$

(Constant function)

$\therefore$ Conditions of Rolle's theorem are sufficient but not necessary.

Also $\dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{2 - (-2)}{4} = 1 \ne f'(c)$ for any c in $(-2, 2)$.

(Infact either $f'(c)$ does not exist or $f'(c) = 0$ for $c \in (-2, 2)$.)

In any interval $[a, b]$ not containing an integer, f is a constant function and Rolle's theorem and mean-value theorem can be verified but not otherwise.)

Exercise 5.7

Verify Rolle's theorem : (1 to 8)

1. $f(x) = x(x - 3)^2$ $\quad x \in [0, 3]$
2. $f(x) = x^3 - 6x^2 + 11x - 6$ $\quad x \in [1, 3]$
3. $f(x) = \sqrt{9 - x^2}$ $\quad x \in [-3, 3]$
4. $f(x) = \log\left(\dfrac{x^2 + ab}{x(a+b)}\right)$ $\quad x \in [a, b] \quad 0 < a < b$
5. $f(x) = sinx + cosx - 1$ $\quad x \in \left[0, \frac{\pi}{2}\right]$
6. $f(x) = e^x (sinx - cosx)$ $\quad x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
7. $f(x) = a^{sinx}$ $\quad x \in [0, \pi], a > 0$
8. $f(x) = e^x cosx$ $\quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Verify Mean Value Theorem : (9 - 10)

9. $f(x) = x - 2sinx, \quad x \in [-\pi, \pi]$
10. $f(x) = \log_e x, \quad x \in [1, 2]$

11. Prove $\frac{x-y}{x} < \log_e \frac{x}{y} < \frac{x-y}{y}$, $0 < y < x$ using Mean Value theorem and taking $f(x) = \log_e x$.

12. Apply Mean Value theorem and find c :

(1) $f(x) = x + \frac{1}{x}$ $\quad x \in [1, 3]$

(2) $f(x) = tan^{-1}x$ $\quad x \in [0, 1]$

13. Prove $sec^2 a < \frac{tanb - tana}{b-a} < sec^2 b$ $\qquad 0 < a < b < \frac{\pi}{2}$

14. Find a point on the graph of $y = (x-4)^2$ where tangent is parallel to the line joining A(4, 0), B(5, 1).

*

Miscellaneous Example :

Example 61 : Find $\frac{d}{dx} \log_7 (\log_7 x)$.

Solution : $y = \log_7 \left(\frac{\log x}{\log 7}\right) = \log_7(\log x) - \log_7(\log 7)$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \log_7 (\log x).$ $\qquad \left(\frac{d}{dx} \log_7 (\log 7) = 0\right)$

$= \frac{d}{dx} \frac{\log (\log x)}{\log 7}$

$= \frac{1}{\log 7} \frac{d}{dx} \log (\log x)$

$= \frac{1}{\log 7} \frac{1}{\log x} \frac{1}{x} = \frac{1}{x \log x \log 7}$

Example 62 : Find $\frac{d}{dx} tan^{-1} \left(\frac{sinx}{1+cosx}\right)$ $\qquad \pi < x < 2\pi$

Solution : $y = tan^{-1} \left(\frac{sinx}{1+cosx}\right)$

$= tan^{-1} \left(\frac{2 sinx \frac{x}{2} cos \frac{x}{2}}{2cos^2 \frac{x}{2}}\right)$

$= tan^{-1} \left(tan \frac{x}{2}\right)$ $\qquad \frac{\pi}{2} < \frac{x}{2} < \pi$

Now, $\frac{\pi}{2} < \frac{x}{2} < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} - \pi < 0$

Now, $y = tan^{-1} \left(tan\left(\frac{x}{2}\right)\right) = tan^{-1} \left(tan\left(\frac{x}{2} - \pi\right)\right) = \frac{x}{2} - \pi$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2}$

Example 63 : If $f(x) = cos^{-1} \frac{1-9^x}{1+9^x}$, find $f'(x)$, $x \in$ R

Solution : Let $t = 3^x$

$\therefore \quad f(t) = cos^{-1} \dfrac{1-t^2}{1+t^2}$

Let $\theta = tan^{-1}t$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So $t = tan\theta$

$3^x > 0$. So $t = tan\theta > 0$. So, $0 < \theta < \frac{\pi}{2}$

$\therefore \quad 0 < 2\theta < \pi$

$$\therefore \quad cos^{-1} \frac{1-t^2}{1+t^2} = cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$$

$$= cos^{-1}(cos2\theta)$$

$$= 2\theta \qquad (0 < 2\theta < \pi)$$

$$= 2tan^{-1}t$$

$$\therefore \quad cos^{-1} \frac{1-9^x}{1+9^x} = 2tan^{-1}3^x \qquad (\text{Taking } t = 3^x)$$

$$\therefore \quad f(x) = cos^{-1} \frac{1-9^x}{1+9^x} = 2tan^{-1}3^x$$

$$\therefore \quad f'(x) = \frac{2 \cdot 3^x \log_e 3}{1+(3^x)^2} = \frac{2 \cdot 3^x \log_e 3}{1+3^{2x}}$$

Example 64 : If $x = a\ (cost + tsint)$, $y = a(sint - tcost)$, find $\dfrac{d^2y}{dx^2}$.

Solution : $\dfrac{dx}{dt} = a(-sint + tcost + sint) = at\ cost$

$$\frac{dy}{dt} = a(cost - cost + tsint) = at\ sint$$

$$\therefore \quad \frac{dy}{dx} = tant$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(tant)$$

$$= \frac{d}{dt}(tant)\ \frac{dt}{dx}$$

$$= \frac{sec^2t}{\frac{dx}{dt}}$$

$$= \frac{sec^2t}{a\,t\cos t} = \frac{sec^3t}{at}$$

Example 65 : If $y = e^{asin^{-1}x}$, $|x| \le 1$ prove that $(1 - x^2)y_2 - xy_1 - a^2y = 0$.

Solution : $\dfrac{dy}{dx} = y_1 = e^{asin^{-1}x} \dfrac{a}{\sqrt{1-x^2}} = \dfrac{ay}{\sqrt{1-x^2}}$

$\therefore \quad (1 - x^2)y_1{}^2 = a^2y^2$

$\therefore \quad (1 - x^2)2y_1y_2 + (-2x)y_1{}^2 = a^2 2yy_1$ $\qquad (\frac{d}{dx}y^2 = 2yy_1,\ \frac{d}{dx}y_1{}^2 = 2y_1y_2$ etc.)

$\therefore \quad (1 - x^2)y_2 - xy_1 - a^2y = 0$ $\qquad (y_1 \neq 0)$

Example 66 : Does there exists a function continuous everywhere but not differentiable at exactly n real numbers ?

Solution : Let $f(x) = |x - 1| + |x - 2| + |x - 3| + ... + |x - n|$

$\therefore$ $|x|$ is continuous on R. So $|x - 1|$, $|x - 2|$,..., $|x - n|$ all are continuous on R, because composite function of continuous functions is continuous.

So, $f(x)$ is continuous on R, because it is a sum of continuous functions.

$|x - 1|$, $|x - 2|$,..., $|x - n|$ are differentiable except at $x = 1$, $x = 2$,..., $x = n$ respectively.

$|x - 2|$, $|x - 3|$,..., $|x - n|$ are differentiable at $x = 1$.

$\therefore$ $g(x) = |x - 2| + |x - 3| + ... + |x - n|$ is differentiable at $x = 1$.

If $f(x) = |x - 1| + |x - 2| + ... + |x - n|$ is differentiable at $x = 1$, then

$f(x) - g(x) = |x - 1|$ is differentiable at $x = 1$.

But $|x - 1|$ is not differentiable at $x = 1$.

$\therefore$ $f(x) = |x - 1| + |x - 2| + ... + |x - n|$ is not differentiable at $x = 1$.

Similarly $|x - 1| + |x - 2| + ... + |x - n|$ is not differentiable at $x = 2, 3,..., n$.

$\therefore$ f is continuous on R but not differentiable at $x = 1, 2, 3,..., n$.

Example 67 : $siny = xsin(a + y)$. Prove $\dfrac{dy}{dx} = \dfrac{sin^2(a + y)}{sina}$

Solution : $cosy\dfrac{dy}{dx} = sin(a + y) + xcos(a + y)\dfrac{dy}{dx}$

$\therefore \quad [cosy - xcos(a + y]\dfrac{dy}{dx} = sin(a + y)$

$$\therefore \quad \frac{dy}{dx} = \frac{sin(a + y)}{cosy - xcos(a + y)}$$

$$= \frac{sin(a + y)}{cosy - \frac{siny}{sin(a + y)}cos(a + y)}$$

$$= \frac{sin^2(a + y)}{sin(a + y)cosy - cos(a + y)siny}$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

$$(\sin(a+y)\cos y - \cos(a+y)\sin y = \sin(a+y-y) = \sin a)$$

or

$$x = \frac{\sin y}{\sin(a+y)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example 68 : If $(x-a)^2 + (y-b)^2 = r^2$, prove that $\left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right|$ is a constant.

Solution : $2(x-a) + 2(y-b)y_1 = 0$

$$\therefore \quad y_1 = -\frac{x-a}{y-b}$$

$$\therefore \quad y_2 = -\frac{(y-b)\cdot 1 - (x-a)y_1}{(y-b)^2}$$

$$= -\frac{(y-b) + \frac{(x-a)(x-a)}{y-b}}{(y-b)^2}$$

$$= -\frac{(x-a)^2 + (y-b)^2}{(y-b)^3}$$

$$= -\frac{r^2}{(y-b)^3}$$

$$\therefore \quad \left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right| = \left|\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-r^2}{(y-b)^3}}\right|$$

$$= \left|\frac{[(x-a)^2 + (y-b)^2]^{\frac{3}{2}}}{-r^2}\right|$$

$$= \left|-\frac{r^3}{r^2}\right| = |r| \text{ is a constant.}$$

($\left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right|$ is called the radius of curvature of curve $y = f(x)$ at any point $(x, f(x))$. Circle is a curve having 'uniform' radius of curvature at every point.)

Example 69 : Find $\frac{d}{dx}(\log x)^{\log x}$ wherever defined.

Solution : $y = (\log x)^{\log x}$

$\therefore \quad \log y = \log x\,(\log(\log x))$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\log(\log x) + \frac{\log x}{\log x}\frac{1}{x}$$

$$= \frac{\log(\log x) + 1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{1 + \log(\log x)}{x}\right)(\log x)^{\log x}$$

Example 70 : Find $\left[\frac{d}{dx} sec^{-1}x\right]_{x=-2}$ by definition. (First principle)

Solution :

$$\left[\frac{d}{dx} sec^{-1}x\right] = \lim_{x \to -2} \frac{sec^{-1}x - sec^{-1}(-2)}{x - (-2)}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - (\pi - sec^{-1}2)}{sect + 2} \qquad (t = sec^{-1}x)$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \left(\pi - \frac{\pi}{3}\right)}{sect + 2}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{sect + 2}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{2sect\left(cost - cos\frac{2\pi}{3}\right)}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{2sect\left(-2\,sin\,\frac{t + \frac{2\pi}{3}}{2}\,sin\,\frac{t - \frac{2\pi}{3}}{2}\right)}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{\left(t - \frac{2\pi}{3}\right)/2}{-2sect \cdot sin\,\frac{t + \frac{2\pi}{3}}{2}\,sin\,\frac{t - \frac{2\pi}{3}}{2}}$$

$$= \frac{-1}{2sec\,\frac{2\pi}{3}\,sin\,\frac{2\pi}{3}}$$

$$= \frac{-1}{2(-2)\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2\sqrt{3}}$$

Verify : $\frac{d}{dx} sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}} = \frac{1}{|-2|\sqrt{4 - 1}} = \frac{1}{2\sqrt{3}}$

Exercise 5

Find points of discontinuity, if any, for following functions (1 to 4)

1. $f(x) = \begin{cases} \dfrac{x^3 - 27}{x - 3} & x \neq 3 \\ 5 & x = 3 \end{cases}$

2. $f(x) = \begin{cases} \dfrac{\sin(x-1)}{|x-1|} & x \neq 1 \\ 2 & x = 1 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x + 1} & x \neq -1 \\ -1 & x = -1 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{e^{2x} - e^4}{e^x - e^2} & x \neq 2 \\ e^2 & x = 2 \end{cases}$

Find k, if following functions are continuous at given value of x : (5 to 8)

5. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3} & x \neq 3 \\ k & x = 3 \end{cases}$ at $x = 3$

6. $f(x) = \begin{cases} kx^2 & x < 1 \\ x^2 + 1 & x \geq 1, \end{cases}$ at $x = 1$

7. $f(x) = \begin{cases} 2x + 3 & x < 2 \\ k & x = 2 \\ 3x + 1 & x > 2 \end{cases}$ at $x = 2$

8. $f(x) = \begin{cases} \cos x & 0 < x < \frac{\pi}{2} \\ k^2 - 4 & x = \frac{\pi}{2} \\ \sin x - 1 & x > \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

Find a and b, if following functions are continuous (9 to 10) :

9. $f(x) = \begin{cases} a\sin x + b & 0 \leq x \leq \frac{\pi}{2} \\ \cos x & \frac{\pi}{2} < x \leq \pi \\ \tan x + b & \pi < x < \frac{3\pi}{2} \end{cases}$

10. $f(x) = \begin{cases} ax + b & 0 \leq x < 1 \\ 2x + 3 & 1 \leq x < 2 \\ x + a & x \geq 2 \end{cases}$

Find $\dfrac{dy}{dx}$ for following functions y where ever defined :

11. $y = \log_{10}(x^2 + 1)$

12. $y = \cot^{-1}\dfrac{2x}{1 - x^2} \quad x \neq \pm 1$

13. $y = \sin(\log(\cos x))$

14. $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a, \ |x| < 1, \ |y| < 1$

15. $y = (\sin x)^{\sin x}$

16. $y = (\sin x - \cos x)^{\sin x - \cos x}$

17. $y = x^x + \left(x + \frac{1}{x}\right)^x$

18. $y = x^{\left(x + \frac{1}{x}\right)}$

19. $y = \cos(x^x) + (\tan x)^x$

20. $y = \sin^{-1}x + \sin^{-1}\sqrt{1 - x^2}, \ |x| < 1$

21. $y = \tan^{-1}x + \cot^{-1}x \ x \in \mathrm{R}$

22. $x = (\cos t)^t \ \ y = (\sin t)^t \quad 0 < t < \frac{\pi}{2}$

23. Prove $\dfrac{d}{dx} e^{ax}\cos(bx + c) = re^{ax}\cos(bx + c + \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \dfrac{a}{r}$, $\sin\alpha = \dfrac{b}{r}$

and $\dfrac{d^2}{dx^2} e^{ax}\cos(bx + c) = r^2 e^{ax}\cos(bx + c + 2\alpha)$

24. Find $\frac{d}{dx}\, tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $x \in R - \{0\}$

25. Find $\frac{d}{dx}\, tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, $|x| < 1$

26. Find $\frac{d}{dx}\, tan^{-1}\sqrt{\frac{1+sinx}{1-sinx}}$ $\quad 0 < x < \frac{\pi}{2}$

27. If $y = (cos^{-1}x)^2$, prove $(1 - x^2)y_2 - xy_1 = 2$

28. If $y = sinpt$, $x = sint$ prove $(1 - x^2)y_2 - xy_1 + p^2y = 0$

29. If $y = e^{mtan^{-1}x}$, prove $(1 + x^2)y_2 + (2x - m)y_1 = 0$

30. If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$ $(x \geq 1)$, prove $(x^2 - 1)y_2 + xy_1 = m^2y$

31. If $y = (x + \sqrt{x^2-1})^m$, prove $(x^2 - 1)y_2 + xy_1 = m^2y$

32. If $x^y = e^{x-y}$, prove $\frac{dy}{dx} = \frac{\log x}{(\log x + 1)^2}$

33. If $y = e^{ax}\, sinbx$, prove $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

34. If $(a - bcosy)(a + bcosx) = a^2 - b^2$, prove $\frac{dy}{dx} = \frac{\sqrt{a^2-b^2}}{a+bcosx}$, $0 < x < \frac{\pi}{2}$

35. If $y = (tan^{-1}x)^2$, prove $(1 + x^2)^2y_2 + 2x(1 + x^2)y_1 = 2$

36. If $y = x\log\frac{x}{a+bx}$, prove $x^3y_2 = (xy_1 - y)^2$

37. If $x = asint - bcost$, $y = acost + bsint$, find y_2.

38. If $y = sin(sinx)$, prove $y_2 + tanx \cdot y_1 + ycos^2x = 0$

39. If $y = cos^{-1}\frac{3+5cosx}{5+3cosx}$, prove $\frac{dy}{dx} = \frac{4}{5+3cosx}$.

40. Find the derivative of $tan^{-1}\frac{x}{\sqrt{1-x^2}}$ *w.r.t.* $sin^{-1}(2x\sqrt{1-x^2})$. $\quad 0 < x < \frac{1}{\sqrt{2}}$

41. Find the derivative of $cos^{-1}\frac{1-x^2}{1+x^2}$ *w.r.t.* $sin^{-1}\frac{2x}{1+x^2}$.

42. Find $\left[\frac{d}{dx}(cosec^{-1}x)\right]_{x=-2}$ by defination.

43. Find $\frac{d}{dx}\left(sin^{-1}\frac{x}{\sqrt{1+x^2}} + cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$, $x > 0$

44. Find $\frac{d}{dx}\, tan^{-1}\frac{4x}{1+21x^2}$, $\quad x > 0$

45. Find $\frac{d}{dx} tan^{-1} \frac{a+bx}{b-ax}$

46. Find $\frac{d}{dx} tan^{-1} \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ $|x| < 1$

47. Find $\frac{d}{dx} tan^{-1}(secx - tanx)$.

48. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 mark)

(1) $\left[\frac{d}{dx} sec^{-1}x\right]_{x=-3} = ...$ ☐

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $-\frac{1}{\sqrt{x^2-1}}$ (c) $\frac{1}{6\sqrt{2}}$ (d) $\frac{-1}{6\sqrt{2}}$

(2) $\frac{d}{dx} x^x = ...$ $(x > 0)$ ☐

(a) x^{x-1} (b) x^x (c) 0 (d) $x^x(1 + \log x)$

(3) $\frac{d}{dx} (sin^{-1}x + cos^{-1}x) = ...$ $(|x| < 1)$ ☐

(a) 0 (b) $\frac{2}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1-x^2}}$ (d) does not exist

(4) $\frac{d}{dx} a^a = ...$ $(a > 0)$ ☐

(a) $a^a(1 + \log a)$ (b) 0 (c) a^a (d) does not exist

(5) $\frac{d}{dx} e^{5x} = ...$. ☐

(a) e^{5x} (b) $5e^{5x}$ (c) $5x\ e^{5x-1}$ (d) 0

(6) $\frac{d}{dx} \log|x| = ...$. $(x \neq 0)$ ☐

(a) $\frac{1}{|x|}$ (b) $\frac{1}{x}$ (c) does not exist (d) e^x

(7) $\frac{d}{dx} sin^3x = ...$. ☐

(a) $3sin^2x$ (b) $3cos^2x$ (c) $3sin^2x\ cosx$ (d) $-3cos^2x\ sinx$

(8) $\frac{d}{dx} tan^nx = ...$. ☐

(a) $ntan^{n-1}x$ (b) $ntan^{n-1}x\ sec^2x$ (c) $n\ sec^{2n}x$ (d) $ntan^{n-1}x\ sec^{n-1}x$

(9) If $f(x) = \begin{cases} ax + b & 1 \le x < 5 \\ 7x - 5 & 5 \le x < 10 \\ bx + 3a & x \ge 10 \end{cases}$

is continuous, $(a, b) =$ .....

(a) (5, 10) (b) (5, 5) (c) (10, 5) (d) (0, 0)

(10) If $f(x) = \begin{cases} \frac{x^2}{a} - a & x < a \\ 0 & x = a \\ a - \frac{x^2}{a} & x > a \end{cases}$ then...

(a) $\lim_{x \to a+} f(x) = a$ (b) $\lim_{x \to a} f(x) = -a$

(c) f is continuous at $x = a$ (d) f is differentiable at $x = a$

(11) If $f(x) = \begin{cases} x & x \in (0, 1) \\ 1 & x \ge 1 \end{cases}$

(a) f is continuous at $x = 1$ only (b) f is discontinuous at $x = 1$ only

(c) f is continuous on R^+ (d) f is not defined for $x = 1$

(12) $\frac{d}{dx} \frac{1}{\log|x|} =$ ..... .

(a) $\frac{1}{|x|}$ (b) $\frac{1}{(\log x)^2}$ (c) $-\frac{1}{x(\log|x|)^2}$ (d) e^x

(13) If $y = asinx + bcosx$, $y^2 + (y_1)^2 =$ ..... . $(a^2 + b^2 \neq 0)$

(a) $acosx - bsinx$ (b) $(asinx - bcosx)^2$ (c) $a^2 + b^2$ (d) 0

(14) $\frac{d}{dx} (x^2 + sin^2x)^3 =$ ..... .

(a) $3(x^2 + sin^2x)$ (b) $3(x^2 + sin^2x)^2 (2x + sin2x)$

(c) $2x + 2sinx\ cosx$ (d) 0

(15) $\frac{d}{dx} \sqrt{xsinx} =$ ..... . $0 < x < \pi$

(a) $\frac{xsinx + cosx}{\sqrt{xsinx}}$ (b) $\frac{xcosx}{2\sqrt{xsinx}}$ (c) $\frac{xcosx + sinx}{2\sqrt{xsinx}}$ (d) $\frac{1}{2\sqrt{xsinx}}$

Section B (2 marks)

(16) $\frac{d}{dx} tan^{-1} \frac{1-x}{1+x} =$ ..... .

(a) $-\frac{1}{1+x^2}$ (b) $\frac{1}{1+x^2}$ (c) $\frac{1+x}{1-x}$ (d) $\frac{2}{1+x^2}$

(17) $\frac{d}{dx}\ tan^{-1}\sqrt{\frac{1-cosx}{1+cosx}}$ = ... $\pi < x < 2\pi$

(a) $\frac{1}{1+cos^2x}$ (b) $-\frac{1}{1+cos^2x}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

(18) If $x = e^{tan^{-1}\frac{y-x^2}{x^2}}$, then $\frac{dy}{dx}$ = ...

(a) $2x\,(tan\,(\log x) + 1)$ (b) $2x\,(tan\,(\log x) + 1) + x^2\,sec\,(\log x)$

(c) $2x\,(tan\,(\log x) + 1) + x^2 sec\,(\log x)$ (d) 0

(19) $\frac{d}{dx}\ sin^{-1}\left(\frac{3x}{5}+\frac{4}{5}\sqrt{1-x^2}\right)$ = ... $(0 < x < \frac{3}{5})$

(a) $\frac{3}{5} + \frac{1}{\sqrt{1-x^2}}$ (b) $\frac{4}{5}\ \frac{1}{\sqrt{1-x^2}}$ (c) $-\frac{1}{\sqrt{1-x^2}}$ (d) $\frac{1}{\sqrt{1-x^2}}$

(20) $\frac{d}{dx}\ tan^{-1}\left(\frac{x+a}{1-xa}\right)$ = ... $(x,\ a \in R^+,\ xa > 1)$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1+a^2}$ (c) $\frac{1}{1+x^2} + \frac{1}{1+a^2}$ (d) $\frac{1}{1+x^2a^2}$

(21) If $f(x) = \log_7(\log_3 x)$, then $f'(x)$ = ...

(a) $\frac{1}{x\log 7\log 3}$ (b) $\frac{1}{\log 3\log x}$ (c) $\frac{1}{x\log x\log 7}$ (d) $\frac{1}{x\log x}$

(22) $\frac{d}{dx}\ x\,|x|$ = ... $(x < 0)$

(a) $2x$ (b) $-2x$ (c) $|x|$ (d) 0

(23) If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ = ...

(a) $\frac{2t^2}{1-t^2}$ (b) $\frac{2t}{1+t^2}$ (c) $2t$ (d) $\frac{-2t}{1-t^2}$

(24) $\frac{d}{dx}\ e^{x\log x}$ = ...

(a) $x^x\,(1 + \log x)$ (b) x^x (c) $1 + \log x$ (d) x^{x-1}

(25) $\frac{d}{dx}\ \frac{tan^{-1}x}{1+tan^{-1}x}$ w.r.t. $tan^{-1}x$ = ...

(a) $\frac{1}{1+tan^{-1}x}$ (b) $\frac{1}{(1+tan^{-1}x)^2}$ (c) $\frac{1}{1+x^2}$ (d) $\frac{-1}{1+x^2}$

Section C (3 marks)

(26) If $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2}$ = ...

(a) $\frac{-1}{t^2}$ (b) $\frac{1}{t^2}$ (c) $\frac{-1}{2at^3}$ (d) $\frac{1}{2at^3}$

(27) $\frac{d}{dx} \cot^{-1} \frac{\sqrt{1+x^2}-1}{x} = \ldots\ldots \quad (x \in R - \{0\})$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{2(1+x^2)}$ (c) $\frac{2}{1+x^2}$ (d) $-\frac{1}{1+x^2}$

(28) $\frac{d^2x}{dy^2} = \ldots\ldots$

(a) $\frac{1}{\frac{d^2y}{dx^2}}$ (b) $\frac{1}{\left(\frac{dy}{dx}\right)^2}$ (c) $-\frac{1}{\left(\frac{dy}{dx}\right)^2}$ (d) $-\frac{1}{\left(\frac{dy}{dx}\right)^3} \frac{d^2y}{dx^2}$

(29) For the curve $f(x) = (x-3)^2$, applying mean value theorem on [2, 4] the tangent at ..... is parallel to the chord joining A(2, 1) and B(4, 1).

(a) (1, 0) (b) (4, 3) (c) (2, 3) (d) (3, 0)

(30) The value of c for the mean-value theorem for $f(x) = x^3$ in $[-1, 1]$ is ..... .

(a) $\pm \frac{1}{\sqrt{3}}$ (b) $\pm \sqrt{3}$ (c) ± 1 (d) 0

(31) If we apply the Rolle's theorem to $f(x) = e^x \sin x \quad x \in [0, \pi]$, then $c =$ ..... .

(a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{4}$ (c) $\frac{\pi}{4}$ (d) $\frac{7\pi}{4}$

(32) If we apply the Rolle's theorem to $f(x) = x^3 - 4x$, $x \in [0, 2]$, then $c =$ ..... .

(a) $\sqrt{3}$ (b) 2 (c) $\frac{2}{\sqrt{3}}$ (d) -2

Section D (4 marks)

(33) If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$, then...

(a) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$ (b) $(x^2-4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2-4)$

(c) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = 1$ (d) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = y^2+4$

(34) $\frac{d}{dx} \tan^{-1} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^2}+\sqrt{1+x^2}} = \ldots\ldots \; |x| < 1$

(a) $\frac{1}{\sqrt{1-x^4}}$ (b) $\frac{-x}{\sqrt{1-x^4}}$ (c) $\frac{1}{2\sqrt{1-x^4}}$ (d) $\frac{x^2}{1-x^4}$

(35) $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right) = \ldots\ldots \; (a > 0)$

(a) $\frac{1}{\sqrt{a^2-x^2}}$ (b) $\sqrt{a^2-x^2}$ (c) $\sqrt{x^2-a^2}$ (d) $\sqrt{x^2+a^2}$

(36) Conditions of Mean Value Theorem are not applicable to ..... in $[-1, 1]$.

(a) $f(x) = |x|$ (b) $f(x) = x^3$ (c) $f(x) = \sin x$ (d) $f(x) = x^2$

(37) For $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$ the value of c for mean-value theorem and for $f(x) = x^2 - 4x + 3$ for Roll's theorem are ..... .

(a) $\sqrt{3}$, 1 (b) 2, 1 (c) $\sqrt{3}$, 2 (d) 2, $\sqrt{3}$

(38) If the tangent to the curve $y = x \log x$ at $(c, f(x))$ is parallel to the line-segment joining A(1, 0) and B(e, e), then c = ...

(a) $\frac{e-1}{e}$ (b) $\log \frac{e-1}{e}$ (c) $e^{\frac{1}{1-e}}$ (d) $e^{\frac{1}{e-1}}$

(39) If we apply the mean value theorem to $f(x) = 2sinx + sin\,2x$, then c = ...

(a) π (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

(40) If we apply the mean value theorem to $f(x) = \begin{cases} 2 + x^3 & x \le 1 \\ 3x & x > 1 \end{cases}$ $x \in [-1, 2]$

then c = ...

(a) 2 (b) 0 (c) 1 (d) $\frac{\sqrt{5}}{3}$

*

Summary

We have studied the following points in this chapter :

1. Continuous functions
2. Algebra of continuous functions
3. Differentiation and continuity
4. Chain rule
5. Rules for derivative of inverse function
6. Derivative of Implicit function
7. Derivative of parametric function
8. Logarithmic differentiation
9. Second order Derivative
10. Mean value theorems

Prehistory

Excavations at Harappa, Mohenjo-daro and other sites of the Indus Valley Civilization have uncovered evidence of the use of "practical mathematics". The people of the IVC manufactured bricks whose dimensions were in the proportion 4:2:1, considered favourable for the stability of a brick structure. They used a standardized system of weights based on the ratios: 1/20, 1/10, 1/5, 1/2, 1, 2, 5, 10, 20, 50, 100, 200, and 500, with the unit weight equal to approximately 28 grams (and approximately equal to the English ounce or Greek uncia). They mass produced weights in regular geometrical shapes, which included hexahedra, barrels, cones, and cylinders, thereby demonstrating knowledge of basic geometry.

The inhabitants of Indus civilization also tried to standardize measurement of length to a high degree of accuracy. They designed a ruler—the Mohenjo-daro ruler—whose unit of length (approximately 1.32 inches or 3.4 centimetres) was divided into ten equal parts. Bricks manufactured in ancient Mohenjo-daro often had dimensions that were integral multiples of this unit of length.

INDEFINITE INTEGRATION 6

What we know is not much, what we do not know is immense.
(Allegedly his last words)

– Laplace

A mathematics teacher is midwife to ideas.

– George Polya

6.1 Introduction

In the chapter on derivatives, we have already learnt about the differentiability of a function on some interval I. If a function is differentiable in an interval I, we know how to find its unique derivative f' at each point on I. Now, we shall study an operation which is 'inverse' to differentiation. For example we know that the derivative of x^3 with respect to x is $3x^2$. Now if we raise the question, derivative of which function or functions is $3x^2$? Then, it is difficult to find the answer. It is a question of an operation inverse to the operation of differentiation.

Let us frame a general question, "Is there a function whose derivative a given function can be and if there is such a function, how to find it ?" The process of finding answer to this question is called 'antiderivation'. It is possible that this question has no answer or it may have more than one answer. For example, (i) $\frac{d}{dx}(x^3) = 3x^2$, $\frac{d}{dx}(x^3 - 15) = 3x^2$ and in general $\frac{d}{dx}(x^3 + c) = 3x^2$, where c is any constant. (ii) $\frac{d}{dx}(sinx) = cosx$, $\frac{d}{dx}(sinx - 3) = cosx$. In general $\frac{d}{dx}(sinx + c) = cosx$.

Thus, antiderivatives of the above functions are not unique. Actually, there exist infinitely many antiderivatives of these functions which can be obtained by choosing c, from the set of real numbers. For this reason, such a constant is called an arbitrary constant.

6.2 Definition

If we can find a function g defined on an interval I such that $\frac{d}{dx}(g(x)) = f(x)$, $\forall x \in$ I, then $g(x)$ is called a **primitive** or **antiderivative** or **indefinite integral** of $f(x)$. It is denoted by $\int f(x)dx$. $\int f(x)dx$ is called an indefinite integral of $f(x)$ with respect to x. The process (operation) of finding $g(x)$, given $f(x)$ is called indefinite integration. This 'indefiniteness' is upto arbitrary constant.

Thus, the question whether we can find primitive of f is not easy to answer. There are some sufficient conditions such as continuous functions and monotonic functions have primitives. $\frac{sinx}{x}$ is continuous, $\int \frac{sinx}{x}\, dx$ is defined, but cannot be expressed in terms of known elementary functions. Similarly, $\int \sqrt{secx}\, dx$ and $\int \sqrt{x^3+1}\, dx$ cannot be expressed as a known function.

In $\int f(x)dx$, $\int ....dx$ indicates the process of integration with respect to x. $\int f(x)dx$ denotes, integral of $f(x)$ with respect to x and in $\int f(x)dx$, $f(x)$ is called integrand.

6.3 Some Theorems on Antiderivative :

Theorem 6.1 : If f and g are differentiable on (a, b) and if $f'(x) = g'(x)$, $\forall x \in (a, b)$, then $f(x) = g(x) + c$, where c is a constant.

Proof : Let $h(x) = f(x) - g(x)$, $x \in (a, b)$.

f and g are differentiable on (a, b) and hence f and g are continuous on (a, b).

$\therefore$ If $x_1, x_2 \in (a, b)$, $x_1 < x_2$, then h is continuous on $[x_1, x_2]$.

Now, h is differentiable on (x_1, x_2) as $[x_1, x_2] \subset (a, b)$.

$\therefore$ By mean value theorem,

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c) \text{ for some } c \in (x_1, x_2).$$

$\therefore$ $h(x_2) - h(x_1) = h'(c)(x_2 - x_1)$. **(i)**

Now $c \in (x_1, x_2) \Rightarrow c \in (a, b)$

But it is given that $\forall x \in (a, b)$, $f'(x) = g'(x)$.

$\therefore$ $f'(c) = g'(c)$

$\therefore$ $f'(c) - g'(c) = 0$

$\therefore$ $h'(c) = 0$ **$(h(x) = f(x) - g(x) \Rightarrow h'(x) = f'(x) - g'(x))$**

$\therefore$ $h(x_2) - h(x_1) = 0 \quad \forall x_1, x_2 \in (a, b)$ **(by (i))**

$\therefore$ $h(x_1) = h(x_2)$

$\therefore$ $f(x_1) - g(x_1) = f(x_2) - g(x_2), \quad \forall x_1, x_2 \in (a, b)$

$\therefore$ $f - g$ is a constant function on (a, b).

$\therefore$ $f(x) - g(x) = c$, where $c \in \mathrm{R}$ is a constant.

$\therefore$ $f(x) = g(x) + c, \quad \forall x \in (a, b)$

General Antiderivative : If $\frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x)) = h(x)$, then $\int h(x)dx = f(x)$ and $\int h(x)dx = g(x)$.

But $f(x) = g(x) + c$. So $\int h(x)dx = f(x) = g(x) + c$. Here $g(x)$ is a differentiable function on (a, b) with $\frac{d}{dx}(g(x)) = \frac{d}{dx}f(x) = h(x)$. Hence if one integral of $h(x)$ is $g(x)$, any other integral of $h(x)$ is $g(x) + c$. Also if $\frac{d}{dx}(g(x)) = h(x)$, then $\frac{d}{dx}[g(x) + c] = \frac{d}{dx}g(x) = h(x)$.

Thus $g(x) + c$ is also an integral of $f(x)$.

Thus, if one primitive of $h(x)$ is $g(x)$, then all its primitives are given by $g(x) + c$, where c is a constant. As c is any constant, it is called an arbitrary constant.

Let us perform the operation of differentiation and integration successively in any order.

By definition of antiderivative, we know that,

$$\frac{d}{dx}g(x) = f(x), \ \forall x \in \mathrm{I} \Leftrightarrow \int f(x)dx = g(x) + c.$$

Now, $\frac{d}{dx}\left[\int f(x)dx\right] = \frac{d}{dx}[g(x) + c] = f(x)$.

$\therefore$ If we first integrate $f(x)$ and then differentiate the integral, we get the same function $f(x)$ as a result.

But, $\int \left[\frac{d}{dx} g(x)\right] dx = \int f(x)dx = g(x) + c.$

If we first differentiate the function $g(x)$ and then integrate its derivative, we get $g(x) + c$.

Theorem 6.2 : If f and g are integrable on (a, b), then $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.

Proof : $\frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$

$$= f(x) + g(x)$$

$\therefore$ Using the definition of antiderivative,

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

In general if $f_1, f_2, f_3, \ldots, f_n$ are integrable over an interval, then

$$\int [f_1(x) + f_2(x) + \ldots + f_n(x)] = \int f_1(x)dx + \int f_2(x)dx + \ldots + \int f_n(x)dx.$$

Theorem 6.3 : If f is an integrable function on (a, b) and $k \in \mathbb{R}$, then $\int kf(x)dx = k\int f(x)dx$.

Proof : $\frac{d}{dx}[k\int f(x)dx] = k\frac{d}{dx}\int f(x)dx$

$$= kf(x)$$

$\therefore$ Using the definition of antiderivative,

$$\int kf(x)dx = k\int f(x)dx.$$

Corollary 1 : If f and g are integrable functions in (a, b), then

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$$

Proof : $\int (f(x) - g(x))dx = \int (f(x)dx + (-1)g(x))dx$

$$= \int f(x)dx + \int (-1)g(x)dx$$

$$= \int f(x)dx + (-1)\int g(x)dx$$

$$= \int f(x)dx - \int g(x)dx$$

Thus, $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$

In general, $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]dx$

$$= k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + \ldots + \int k_n f_n(x)dx$$

Theorem 6.2, 6.3 and corollary 1 are known as working rules for integration.

6.4 Standard Integrals

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \in \mathbb{R} - \{-1\},\ x \in \mathbb{R}^+.$

$\frac{x^{n+1}}{n+1}$ is differentiable for all $x \in \mathbb{R}^+$ and $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}[(n+1)x^n] = x^n$

$\therefore$ By the definition of antiderivative, $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ \forall x \in \mathbb{R}^+.$

(Also let us remember that if $g(x)$ is one primitive then $g(x) + c$ is the general primitive.)

Thus, for $n = 0$, $\int x^0\, dx = \frac{x^{0+1}}{0+1} + c = x + c$

$\therefore$ $\int dx = x + c$

(2) $\int \frac{1}{x} dx = \log|x| + c,\ x \in R - \{0\}.$

$\log|x|$ is a differentiable function, $\forall x \in R - \{0\}$ and if $x > 0$, $\frac{d}{dx}(\log|x|) = \frac{d}{dx}\log x = \frac{1}{x}$.

If $x < 0$, $\frac{d}{dx}\log|x| = \frac{d}{dx}\log(-x) = \frac{-1}{-x} = \frac{1}{x}$

$\therefore$ $\frac{d}{dx}\log|x| = \frac{1}{x} \quad \forall x \in R - \{0\}$

$\therefore$ By the definition of antiderivative,

$\int \frac{1}{x} dx = \log|x| + c, \quad \forall x \in R - \{0\}.$

We write $\int \frac{dx}{x} = \log|x| + c,\ x \neq 0.$

(3) $\int cosx\, dx = sinx + c,\ \forall x \in R$

sin is a differentiable function $\forall x \in R$ and $\frac{d}{dx}(sinx) = cosx, \quad \forall x \in R$

$\therefore$ By the definition of antiderivative,

$\int cosx\, dx = sinx + c, \quad \forall x \in R$

In the same way, we can prove that

(4) $\int sinx\, dx = -cosx + c,\ \forall x \in R$

(5) $\int sec^2x\, dx = tanx + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

tan is differentiable on any interval not containing $(2k-1)\frac{\pi}{2}$, $k \in Z$ and $\frac{d}{dx}(tanx) = sec^2x$.

$\therefore$ By the definition of antiderivative, $\int sec^2x\, dx = tanx + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

In the same way, we can prove that

(6) $\int cosec^2x\, dx = -cotx + c,\ x \neq k\pi,\ k \in Z$

(7) $\int secx\ tanx\, dx = secx + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

(8) $\int cosecx\ cotx\, dx = -cosecx + c,\ x \neq k\pi,\ k \in Z$

(9) $\int a^x\, dx = \frac{a^x}{\log_e a} + c,\ a \in R^+ - \{1\},\ x \in R$

$\frac{a^x}{\log_e a}$ is differentiable $\forall x \in R$ and $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = \frac{1}{\log_e a}(a^x \log_e a) = a^x, \quad \forall x \in R$

$\therefore$ By the definition of antiderivative, $\int a^x\, dx = \frac{a^x}{\log_e a} + c,\ a \in R^+ - \{1\}.$

Now, for $a = e$

$\int e^x\, dx = \frac{e^x}{\log_e e} + c$

$\therefore$ $\int e^x\, dx = e^x + c, \quad \forall x \in R.$

(10) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c, \quad a \in R - \{0\},\ x \in R$

$= -\frac{1}{a} cot^{-1}\left(\frac{x}{a}\right) + c', \quad a \in R - \{0\},\ x \in R$

$tan^{-1}\left(\frac{x}{a}\right)$ is differentiable for $\forall x \in R$ and for any non-zero constant a.

$\therefore \quad \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right)$ is differentiable for $\forall a \in R - \{0\}$ and

$$\frac{d}{dx}\left[\frac{1}{a} tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{a} \cdot \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} = \frac{1}{x^2+a^2}$$

$\therefore$ By the definition of antiderivative, $\int \frac{dx}{x^2+a^2} = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c, \forall x \in R$

Thus, $\frac{1}{a} tan^{-1} \frac{x}{a}$ and $-\frac{1}{a} cot^{-1} \frac{x}{a}$ both can be taken as integrals of $\frac{1}{x^2+a^2}$.

Let us try to understand the reason behind this.

Let $f(x) = \frac{1}{a} tan^{-1} \frac{x}{a}$ and $g(x) = -\frac{1}{a} cot^{-1} \frac{x}{a}$.

Now, we know that $tan^{-1} \frac{x}{a} + cot^{-1} \frac{x}{a} = \frac{\pi}{2}$

$\therefore \quad \frac{1}{a} tan^{-1} \frac{x}{a} + \frac{1}{a} cot^{-1} \frac{x}{a} = \frac{\pi}{2a}$

$\therefore \quad f(x) - g(x) = \frac{\pi}{2a}$

$\therefore \quad f(x) = g(x) + \frac{\pi}{2a}$.

$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx} g(x)$.

As antiderivative is not unique, $\int h(x)dx = g(x)$ and $\int h(x)dx = f(x)$ does not give $f(x) = g(x)$. We can say that there is a constant c such that $f(x) = g(x) + c$.

(11) $\int \frac{dx}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c, a \in R - \{0\}$ **(on any interval not containing $-a$ and a)**

On any interval not containing $-a$ and a, $\frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$ is differentiable and

$$\frac{d}{dx}\left(\frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|\right) = \frac{1}{2a}\frac{d}{dx}[\log|x-a| - \log|x+a|]$$

$$= \frac{1}{2a} \cdot \left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$= \frac{1}{2a} \cdot \left[\frac{x+a-x+a}{(x-a)(x+a)}\right]$$

$$= \frac{1}{2a} \cdot \left(\frac{2a}{x^2-a^2}\right)$$

$$= \frac{1}{x^2-a^2}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c, a \in R - \{0\}$

(12) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{x+a}{x-a}\right| + c, a \in R - \{0\}$ **(on any interval not containing $-a$ and a)**

We have, $\int \frac{1}{a^2-x^2} dx = -1 \int \frac{1}{x^2-a^2} dx$

$$= -\frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$$

$$= \frac{1}{2a} \log\left|\frac{x+a}{x-a}\right| + c$$

(13) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c, \quad x \in (-a, a),\ a > 0.$

$$= -\cos^{-1}\frac{x}{a} + c', \quad x \in (-a, a),\ a > 0.$$

$\sin^{-1}\left(\frac{x}{a}\right)$ is a differentiable function for $x \in (-a, a)$, $a > 0$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{|a|}{\sqrt{a^2-x^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2-x^2}} \qquad (a > 0,\ |a| = a)$$

$\therefore$ Using the definition of antiderivative. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c, \quad x \in (-a, a),\ a > 0.$

As shown in (10) we have $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}\frac{x}{a} + c', \quad x \in (-a, a)$

Also if $a < 0$, then $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\sin^{-1}\frac{x}{a} + c = \cos^{-1}\frac{x}{a} + c'.$ (as $|a| = -a$)

We shall usually use the formula for $a > 0$.

(14) $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c, \quad |x| > |a| > 0.$

$$= -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c', \quad |x| > |a| > 0.$$

If $a \in \mathrm{R} - \{0\}$ and $|x| > |a|$, $\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$ is differentiable and

$$\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{a} \cdot \frac{1}{\left|\frac{x}{a}\right|\sqrt{\frac{x^2}{a^2}-1}} \cdot \frac{1}{a}$$

$$= \frac{1}{a^2} \cdot \frac{|a|^2}{|x|\sqrt{x^2-a^2}}$$

$$= \frac{1}{a^2} \cdot \frac{a^2}{|x|\sqrt{x^2-a^2}}$$

$$= \frac{1}{|x|\sqrt{x^2-a^2}}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c, \quad (|x| > |a| > 0)$

As shown in (10) we can write $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c'$

(15) $\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \log |\, x + \sqrt{x^2 \pm a^2}\,| + c, \quad \forall x \in \mathbf{R}.$

$$\frac{d}{dx}\left(\log |\, x + \sqrt{x^2 \pm a^2}\,|\right) = \frac{1}{x + \sqrt{x^2 \pm a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$= \frac{1}{x + \sqrt{x^2 \pm a^2}} \left(1 + \frac{2x}{2\sqrt{x^2 \pm a^2}}\right)$$

$$= \frac{1}{x + \sqrt{x^2 \pm a^2}} \left(\frac{\sqrt{x^2 \pm a^2} + x}{\sqrt{x^2 \pm a^2}}\right)$$

$$= \frac{1}{\sqrt{x^2 \pm a^2}}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log |\, x + \sqrt{x^2 \pm a^2}\,| + c, \forall x \in \mathrm{R}$

(**Note :** For existence of $\frac{dx}{\sqrt{x^2 - a^2}}$, it is necessary that $|\, x\,| > |\, a\,|$.)

Generally, if $g(x)$ is any primitive of $f(x)$, we will not write $\int f(x)dx = g(x) + c$. But instead, we will write $\int f(x)dx = g(x)$ assuming that c is included in $g(x)$. According to this, in an equation like $\int f(x)dx = \int g(x)dx + \int h(x)dx$, there is no need to write c. It is included in the symbol $\int .....dx$. But it is necessary to write $\int x^2 dx = \frac{x^3}{3} + c$. Here $\frac{x^3}{3}$ is not the general integral. It is one integral.

Thus, we may introduce c when all symbols $\int .....dx$ are removed after carrying out integration.

Again, it is not necessary to write $\int x^2 dx + \int \frac{x^3}{3}\, dx = \frac{x^3}{3} + c_1 + \frac{x^4}{4} + c_2$ as $c_1 + c_2$ is also an arbitrary constant. Thus, we can write $\int x^2 dx + \int x^3 dx = \frac{x^3}{3} + \frac{x^4}{4} + c$.

For the following examples, we will assume that integral is defined on some appropriate domain of R. We use symbol I for an integral.

Example 1 : Obtain the integral of the following functions *w.r.t. x*.

(1) $x^{\frac{5}{2}} + 4 \cdot 3^x - \frac{1}{x}$ **(2)** $\frac{(2x+1)^3}{\sqrt{x}}, (x > 0)$ **(3)** $\frac{x}{a} + \frac{a}{x} + x^a + a^x$ **(4)** $\frac{1}{1 + \cos 2x}$

(5) $\frac{1}{9 - x^2}, x^2 \neq 9$ **(6)** $\frac{1}{\sqrt{x^2 - 4}}, |\, x\,| > 2$

Solution : (1) $\mathrm{I} = \int \left(x^{\frac{5}{2}} + 4 \cdot 3^x - \frac{1}{x}\right) dx$

$$= \int x^{\frac{5}{2}} dx + 4 \int 3^x dx - \int \frac{1}{x} dx$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 4 \cdot \frac{3^x}{\log_e 3} - \log |\, x\,| + c$$

$$= \frac{2}{7} x^{\frac{7}{2}} + \frac{4 \cdot 3^x}{\log_e 3} - \log |\, x\,| + c$$

(2) $I = \int \frac{(2x+1)^3}{\sqrt{x}}\, dx = \int \frac{8x^3 + 1 + 12x^2 + 6x}{\sqrt{x}}\, dx$

$$= \int \left(\frac{8x^3}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} + \frac{12x^2}{x^{\frac{1}{2}}} + \frac{6x}{x^{\frac{1}{2}}} \right) dx$$

$$= 8 \int x^{\frac{5}{2}} dx + \int x^{-\frac{1}{2}} dx + 12 \int x^{\frac{3}{2}} dx + 6 \int x^{\frac{1}{2}} dx$$

$$= 8 \cdot \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 12 \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 6 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{16}{7} x^{\frac{7}{2}} + 2x^{\frac{1}{2}} + \frac{24}{5} x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + c$$

(3) $I = \int \left(\frac{x}{a} + \frac{a}{x} + x^a + a^x \right) dx = \frac{1}{a} \int x\, dx + a \int \frac{1}{x}\, dx + \int x^a dx + \int a^x dx$

$$= \frac{1}{a} \frac{x^2}{2} + a \log |x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + c$$

$$= \frac{x^2}{2a} + a \log |x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + c$$

(4) $I = \int \frac{1}{1 + \cos 2x}\, dx = \int \frac{1}{2\cos^2 x}\, dx$

$$= \frac{1}{2} \int \sec^2 x\, dx$$

$$= \frac{1}{2} \tan x + c$$

(5) $I = \int \frac{1}{9 - x^2}\, dx = \int \frac{1}{(3)^2 - (x)^2}\, dx$

$$= \frac{1}{2(3)} \log \left| \frac{x+3}{x-3} \right| + c$$

$$= \frac{1}{6} \log \left| \frac{x+3}{x-3} \right| + c$$

(6) $I = \int \frac{1}{\sqrt{x^2 - 4}}\, dx = \int \frac{1}{\sqrt{x^2 - 2^2}}\, dx$

$$= \log \left| x + \sqrt{(x)^2 - (2)^2} \right| + c$$

$$= \log \left| x + \sqrt{x^2 - 4} \right| + c$$

Example 2 : Evaluate the following :

(1) $\int \frac{dx}{4x^2 + 9}$ (2) $\int \frac{dx}{9x^2 - 25}$, $x^2 \neq \frac{25}{9}$ (3) $\int \frac{(x^4 + x^2 + 3)dx}{2(x^2 + 1)}$ (4) $\int \frac{(x^2 + 5)dx}{x^2 - 5}$, $x^2 \neq 5$

(5) $\int \frac{\sin x\, dx}{1 + \sin x}$ (6) $\int \sec^2 x \cdot \operatorname{cosec}^2 x\, dx$

Solution : (1) $I = \int \frac{1}{4x^2 + 9}\, dx$

$$= \frac{1}{4} \int \frac{1}{x^2 + \frac{9}{4}}\, dx$$

$$= \frac{1}{4} \int \frac{1}{(x^2) + \left(\frac{3}{2}\right)^2} \, dx$$

$$= \frac{1}{4} \frac{1}{\left(\frac{3}{2}\right)} \tan^{-1}\left(\frac{x}{\frac{3}{2}}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

(2) $$I = \int \frac{1}{9x^2 - 25} \, dx$$

$$= \frac{1}{9} \int \frac{1}{x^2 - \frac{25}{9}} \, dx$$

$$= \frac{1}{9} \int \frac{1}{(x)^2 - \left(\frac{5}{3}\right)^2} \, dx$$

$$= \frac{1}{9} \frac{1}{2\left(\frac{5}{3}\right)} \log \left| \frac{x - \frac{5}{3}}{x + \frac{5}{3}} \right| + c$$

$$= \frac{1}{30} \log \left| \frac{3x - 5}{3x + 5} \right| + c$$

(3) $$I = \int \frac{x^4 + x^2 + 3}{2(x^2 + 1)} \, dx$$

$$= \int \frac{x^2 (x^2 + 1) + 3}{2(x^2 + 1)} \, dx$$

$$= \frac{1}{2} \int \left(x^2 + \frac{3}{x^2 + 1} \right) dx$$

$$= \frac{1}{2} \int x^2 dx + \frac{3}{2} \int \frac{1}{x^2 + 1^2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right] + \frac{3}{2} \tan^{-1} x + c$$

$$= \frac{x^3}{6} + \frac{3}{2} \tan^{-1} x + c$$

(4) $$I = \int \frac{x^2 + 5}{x^2 - 5} \, dx, \; x^2 \neq 5$$

$$= \int \frac{(x^2 - 5) + 10}{x^2 - 5} \, dx$$

$$= \int \left(1 + \frac{10}{x^2 - 5} \right) dx$$

$$= \int dx + 10 \int \frac{1}{(x)^2 - \left(\sqrt{5}\right)^2} \, dx$$

$$= x + \frac{10}{2\sqrt{5}} \log \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + c$$

$$= x + \sqrt{5} \log \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + c$$

(5) $$I = \int \frac{\sin x}{1 + \sin x} \, dx$$

$$= \int \frac{\sin x}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} \, dx$$

$$= \int \frac{\sin x - \sin^2 x}{1 - \sin^2 x} \, dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} \, dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int (\sec x \tan x - \tan^2 x) \, dx$$

$$= \int \sec x \tan x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \sec x \tan x \, dx - \int \sec^2 x \, dx + \int 1 \, dx$$

$$= \sec x - \tan x + x + c$$

(6) $$I = \int \sec^2 x \cdot \operatorname{cosec}^2 x$$

$$= \int \frac{1}{\cos^2 x \sin^2 x} \, dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} \, dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + c$$

Example 3 : Evaluate : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$

Solution : $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)}\, dx$$

$$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha}\, dx$$

$$= 2\int (\cos x + \cos \alpha)\, dx$$

$$= 2\int \cos x\, dx + 2\cos \alpha \int 1\, dx$$

$$= 2 \sin x + 2\cos \alpha \cdot x + c$$

$$= 2(\sin x + x\cos \alpha) + c$$

Example 4 : Evaluate : $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}}\, dx, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution : $I = \int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}}\, dx$

$$= \int \tan^{-1} \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$. So, $-\frac{\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$.

$\therefore \quad 0 < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{2}$.

$\therefore \quad I = \int \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) dx \qquad \left(0 < \left(\frac{\pi}{4} - \frac{x}{2}\right) < \frac{\pi}{2}\right)$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c$$

Example 5 : If $f'(x) = 3x^2 - \frac{2}{x^3}$ and $f(1) = 4$, find $f(x)$.

Solution : We have, $f'(x) = 3x^2 - \frac{2}{x^3}$

$\therefore \quad f(x) = \int (3x^2 - 2x^{-3})\, dx$

$\therefore \quad f(x) = 3\frac{x^3}{3} - 2\frac{x^{-2}}{-2} + c$

$\therefore \quad f(x) = x^3 + \frac{1}{x^2} + c$ **(i)**

Now, $f(1) = 1^3 + \frac{1}{1^2} + c$

$\therefore \quad 4 = 1 + 1 + c$

$\therefore \quad c = 2$ **($f(1) = 4$)**

$\therefore \quad f(x) = x^3 + \frac{1}{x^2} + 2$ **(Substituting $c = 2$ in (i))**

Exercise 6.1

Integrate the following functions *w.r.t.* x considering them well defined and integrable over proper domain :

1. $3x^2 + 5x - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}$
2. $\frac{5x^3 + x^2 + 2}{\sqrt{x}}$
3. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3$
4. $(ax^2 + bx + c)\sqrt{x}$
5. $x^e + e^x + e^e$
6. $e^{a \log x} + e^{x \log a}$
7. $\frac{x^3 - 8}{x^2 - 2x}$
8. $2^x + \frac{1}{\sqrt{x^2 - 9}}$
9. $\frac{2x^3 + 18x - 1}{x^2 + 9}$
10. $\frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x}$
11. $\frac{x^4 + x^2 + 1}{x^2 + x + 1}$
12. $\frac{x^6 + 2}{x^2 + 1}$
13. $\frac{x^4 + 1}{x^2 + 1}$
14. $3\sin x + 5\cos x + \frac{7}{\cos^2 x} - \frac{4}{\sin^2 x} + \tan^2 x$
15. $\frac{2 + 3\cos x}{\sin^2 x}$
16. $(2\tan x - 3\cot x)^2$
17. $\frac{\cos 2x}{\sin^2 2x}$
18. $\frac{\cos x}{\cos x - 1}$
19. $\frac{1}{1 + \cos x}$
20. $\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}$
21. $\frac{\cot x}{\operatorname{cosec} x - \cot x}$
22. $\frac{\tan x}{\sec x + \tan x}$
23. $(a\tan x + b\cot x)^2$
24. $\frac{x^2}{x^2 - 3}$
25. If $f'(x) = 8x^3 - 2x$, $f(2) = 8$, then find $f(x)$.

*

6.5 Method of Substitution for Integration

If the integrand $f(x)$ is in one of the standard forms or it can be put in one such form, it can be easily integrated. But if the integrand $f(x)$ is not in one of the standard forms or cannot be easily converted in one such form, then we may use a very useful method of substition.

In this method $\int f(x)dx$ is converted into $\int g(t)dt$ by a proper substitution $x = \phi(t)$, where $\int g(t)dt$ can be obtained by using standard forms or some known method. Now, let us prove the theorem which is called the rule of substitution for integration.

Theorem 6.4 : $g : [\alpha, \beta] \to R$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). $g'(t)$ is continuous on (α, β) and $g'(t) \neq 0$, $\forall t \in (\alpha, \beta)$. $R_g \subset [a, b]$ and $f : [a, b] \to R$ is continuous, then $x = g(t)$, gives

$$\int f(x)dx = \int f(g(t))\ g'(t)dt.$$

Proof : Since f is continuous on $[a, b]$, $\int f(x)dx$ exists. Now, $x = g(t)$ is continuous on $[\alpha, \beta]$ and $f(x)$ is continuous on $[a, b]$.

So $f(g(t))$ is also continuous on $[\alpha, \beta]$ and $g'(t)$ is given to be continuous. Hence $f(g(t))\ g'(t)$ is continuous. So,

$$\int f(g(t)) \cdot g'(t)dt \text{ also exists.}$$

Let $h(x) = \int f(x)dx$

$\therefore \quad h'(x) = f(x)$

Since $x = g(t)$

$\therefore \quad h'(g(t)) = f(g(t))$

As h is a differentiable function of x and x is a differentiable function of t, h is a differentiable function of t.

$$\therefore \quad \frac{d}{dt} h(g(t)) = \frac{d}{dt}((hog)(t))$$

$$= h'(g(t))\ g'(t)$$

$$= f(g(t))\ g'(t)$$

$$\therefore \quad \frac{d}{dt} h(g(t)) = f(g(t))\ g'(t)$$

$$\therefore \quad h(g(t)) = \int f(g(t))\ g'(t)dt$$

$$\therefore \quad h(x) = \int f(g(t))\ g'(t)dt$$

$$\therefore \quad \int f(x)dx = \int f(g(t))\ g'(t)dt$$

Here on the left hand side, we have a function of x. On the right hand side, we have a function of t. Since $g'(t)$ is continuous and non-zero, $x = g(t)$ is one-one function. Hence $t = g^{-1}(x)$ can convert the function on the right hand side into a function of x.

In this rule, a new variable is introduced replacing the variable x. Hence, it is called the method of change of variable also.

Note : (1) In the formula for the method of substitution, $g(t) = x$ converts the right hand side according to $\int f(x)dx = \int f(x) \frac{dx}{dt} dt$.

(2) According to the definition, for $y = f(x)$, $f'(x) = \frac{dy}{dx}$.

Here, $\frac{dy}{dx}$ is not ratio of dy and dx.

But $f'(x) = \frac{(dy)}{(dx)}$ where dx and dy are 'differentials' of x and y respectively. Thus, we can write $dy = f'(x)dx$. Hence, if $t = sinx$, then $dt = cosx\ dx$. (We will study this in the next semester.)

(3) Commonly used functions e^x, $sinx$, $cosx$, $secx$ satisfy the conditions of the theorem on some interval. Thus we will not verify these conditions every time.

Theorem 6.5 : If $\int f(x)dx = F(x)$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$ where $f : I \to R$ is continuous on some interval I. $(a \neq 0)$.

Proof : Let $t = ax + b$. So $x = \frac{t-b}{a}$.

Hence, $x = g(t)$ is continuous and differentiable and $\frac{dx}{dt} = g'(t) = \frac{1}{a} \neq 0$. Also $g'(t)$ is continuous.

$$\therefore \quad \int f(ax+b)dx = \int f(t)\frac{dx}{dt}dt$$

$$= \int f(t)\frac{1}{a}dt$$

$$= \frac{1}{a}\int f(t)dt$$

$$= \frac{1}{a}F(t)$$

$$= \frac{1}{a}F(ax+b)$$

Thus, (1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ gives $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(2) $\int \frac{1}{x}dx = \log|x| + c$ gives $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c$

(3) $\int \cos x\, dx = \sin x + c$ gives $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(4) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ gives $\int \frac{1}{(px+q)^2-(a)^2}dx = \frac{1}{p}\cdot\frac{1}{2a}\log\left|\frac{(px+q)-(a)}{(px+q)+(a)}\right| + c$

We can also use all standard forms stated earlier in this manner.

Theorem 6.6 : $[f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$, $(n \neq -1, f(x) > 0)$ where f, f' are continuous and $f'(x) \neq 0$.

Proof : Let $t = f(x)$. So $1 = f'(x)\frac{dx}{dt}$

Again $f'(x) \neq 0$ and is continuous implies $t = f(x)$ is one-one and

$$\int [f(x)]^n f'(x)dx = \int [f(x)]^n \left(f'(x)\frac{dx}{dt}\right)dt$$

$$= \int t^n \cdot 1\, dt$$

$$= \frac{t^{n+1}}{n+1} + c$$

$$\therefore \quad \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c \qquad (t = f(x))$$

Thus, (1) $\int \sin^2 x \cos x\, dx = \int (\sin x)^2 \left(\frac{d}{dx}\sin x\right)dx = \frac{(\sin x)^{2+1}}{2+1} + c = \frac{\sin^3 x}{3} + c$

(2) $\int \frac{\sqrt{\tan x}}{\cos^2 x}dx = \int (\tan x)^{\frac{1}{2}} \sec^2 x\, dx = \int (\tan x)^{\frac{1}{2}}\left(\frac{d}{dx}\tan x\right)dx = \frac{(\tan x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$$= \frac{2}{3}(\tan x)^{\frac{3}{2}} + c$$

(3) $\int \frac{x}{\sqrt{x^2+5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+5}}\,dx$

$$= \frac{1}{2}\int (x^2+5)^{-\frac{1}{2}}\,\frac{d}{dx}\,(x^2+5)dx = \frac{1}{2}\,\frac{(x^2+5)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \sqrt{x^2+5} + c$$

Theorem 6.7 : If f is continuous in $[a, b]$ and differentiable in (a, b) and f' is continuous and non-zero, $\forall x \in [a, b]$ and $f(x) \neq 0$, $\forall x \in [a, b]$, then $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$.

Proof : f' is continuous and non-zero. Hence, f is monotonic (increasing or decreasing) function.

Substitution $t = f(x)$ gives $x = f^{-1}(t)$

$\therefore \quad f'(x)\,\frac{dx}{dt} = 1$

Now, $\int \frac{f'(x)}{f(x)}dx = \int \frac{f'(x)}{f(x)} \cdot \frac{dx}{dt}\,dt$

$$= \int \frac{1}{t}\,dt$$

$$= \log|t| + c$$

$\therefore \quad \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

Thus,

(1) $\int \frac{x}{x^2-15}\,dx = \frac{1}{2}\int \frac{2x}{x^2-15}\,dx$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(x^2-15)}{x^2-15}\,dx = \frac{1}{2}\log|x^2-15| + c$$

(2) $\int \frac{2cosx - 3sinx}{6cosx + 4sinx}\,dx = \frac{1}{2}\int \frac{-6sinx + 4cosx}{6cosx + 4sinx}\,dx$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(6cosx + 4sinx)}{(6cosx + 4sinx)}\,dx$$

$$= \frac{1}{2}\log|6cosx + 4sinx| + c$$

6.6 Some More Standard Forms

(16) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int tanx\,dx = \log|secx| + c.$

Here, $\int tanx\,dx = \int \frac{secx\,tanx}{secx}\,dx$ $\qquad$ $(secx \neq 0)$

On given interval, $t = secx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = secx\,tanx$ is also continuous and non-zero.

Taking, $t = secx$, $dt = secx\,tanx\,dx$

$\therefore \quad \int tanx\,dx = \int \frac{secx\,tanx}{secx}\,dx$

$$= \int \frac{1}{t}\,dt$$

$= \log |t| + c$

$= \log |secx| + c$

(17) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int cotx\ dx = \log |sinx| + c.$

Here, $\int cotx\ \ dx = \int \frac{cosx}{sinx}\ dx$

On given interval, $t = sinx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = cosx$ is also continuous and non-zero.

Taking $t = sinx$, $dt = cosx\ dx$

$\therefore \quad \int cotx\ dx = \int \frac{cosx}{sinx}\ dx$

$= \int \frac{1}{t}\ dt$

$= \log |t| + c$

$= \log |sinx| + c$

(18) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int cosecx\ \ dx = \log |cosecx - cotx| + c,\ x \neq k\pi,\ k \in Z$

$= \log |tan\frac{x}{2}| + c$

On given interval, $1 - cosx \neq 0$ and $sinx \neq 0$

$\therefore \quad cosecx - cotx = \frac{1-cosx}{sinx} \neq 0$ in the domain.

Now, $I = \int cosecx\ dx = \int \frac{cosecx\ (cosecx - cotx)}{(cosecx - cotx)}\ dx$

$= \int \frac{cosec^2x - cosecx\ cotx}{cosecx - cotx}\ dx$

Now, $t = cosecx - cotx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = cosec^2x - cosecx\ cotx$ is continuous and non-zero on given interval.

$\therefore \quad I = \int \frac{1}{t}\ dt$

$= \log |t| + c$

$= \log |cosecx - cotx| + c$

Again, $\log |cosecx - cotx| = \log \left|\frac{1-cosx}{sinx}\right|$

$= \log \left|\frac{2sin^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}}\right|$

$= \log \left|tan\frac{x}{2}\right|$

Thus, $\int cosecx\ dx = \log |cosecx - cotx| + c$

$= \log \left|tan\frac{x}{2}\right| + c$

(19) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$$\int secx\ dx = \log|secx + tanx| + c$$

$$= \log\left|tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$$

$secx + tanx = \dfrac{1 + sinx}{cosx}$ is defined and non-zero as $x \neq (4k-1)\frac{\pi}{2}$, $k \in Z$

On given interval, $1 + sinx \neq 0$ and $cosx \neq 0$

Now, $I = \int secx\ dx = \int \dfrac{secx\ (secx + tanx)}{secx + tanx}\ dx$

Now, $t = secx + tanx$ is continuous and differentiable and non-zero and $\dfrac{dt}{dx} = secx\ tanx + sec^2x$ is continuous and non-zero on given interval.

$$\therefore \quad I = \int \frac{sec^2x + secx\,tanx}{secx + tanx}\ dt$$

$$= \int \frac{1}{t}\ dt$$

$$= \log|t| + c$$

$$= \log|secx + tanx| + c$$

Again, $\log|secx + tanx| = \log\left|\dfrac{1 + sinx}{cosx}\right|$

$$= \log\left|\frac{sin^2\frac{x}{2} + cos^2\frac{x}{2} + 2sin\frac{x}{2}cos\frac{x}{2}}{cos^2\frac{x}{2} - sin^2\frac{x}{2}}\right|$$

$$= \log\left|\frac{(cos\frac{x}{2} + sin\frac{x}{2})^2}{(cos^2\frac{x}{2} - sinx^2\frac{x}{2})}\right|$$

$$= \log\left|\frac{cos\frac{x}{2} + sin\frac{x}{2}}{cos\frac{x}{2} - sin\frac{x}{2}}\right|$$

$$= \log\left|\frac{1 + tan\frac{x}{2}}{1 - tan\frac{x}{2}}\right|$$

$$= \log\left|tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$$

Thus, $\int secx\ dx = \log|secx + tanx| + c$

$$= \log\left|tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$$

Example 6 : Evaluate : $\displaystyle\int \frac{2x^3 + 5x^2 + 3x + 1}{2x - 1}\ dx$

Solution : $I = \displaystyle\int \frac{2x^3 + 5x^2 + 3x + 1}{2x - 1}\ dx$

$$= \int \frac{(2x-1)(x^2 + 3x + 3) + 4}{2x - 1}\ dx$$

$$= \int \left(x^2 + 3x + 3 + \frac{4}{2x-1}\right) dx$$

$$= \int x^2 dx + 3 \int x\, dx + 3 \int dx + 4 \int \frac{1}{2x-1}\, dx$$

$$= \frac{x^3}{3} + 3\,\frac{x^2}{2} + 3x + 4 \times \frac{1}{2} \log |\, 2x - 1 \,| + c$$

$$= \frac{x^3}{3} + \frac{3}{2}x^2 + 3x + 2 \log |\, 2x - 1 \,| + c$$

Example 7 : Evaluate : $\int \left(\frac{1}{\sqrt{16-9x^2}} + \frac{1}{25-9x^2}\right) dx$

Solution : $\mathrm{I} = \int \left(\frac{1}{\sqrt{16-9x^2}} + \frac{1}{25-9x^2}\right) dx$

$$= \int \frac{1}{\sqrt{(4)^2 - (3x)^2}}\, dx + \int \frac{1}{(5)^2 - (3x)^2}\, dx$$

$$= \frac{1}{3}\, sin^{-1}\left(\frac{3x}{4}\right) + \frac{1}{2(5)} \times \frac{1}{3} \log \left|\frac{5+3x}{5-3x}\right| + c$$

$$= \frac{1}{3}\, sin^{-1}\, \frac{3x}{4} + \frac{1}{30} \log \left|\frac{5+3x}{5-3x}\right| + c$$

Example 8 : Evaluate : $\int (7x + 5)\sqrt{3x+2}\, dx$

Solution : We will find m and n such that

$7x + 5 = m(3x + 2) + n$

$7x + 5 = 3mx + 2m + n$

Comparing the coefficient of x and constant term on both sides,

$3m = 7$ and $2m + n = 5$

$\therefore \quad m = \frac{7}{3}$ and $\frac{14}{3} + n = 5$. Thus $n = 5 - \frac{14}{3} = \frac{1}{3}$

$\therefore \quad \mathrm{I} = \int [m(3x + 2) + n]\sqrt{3x+2}\, dx$

$$= \int \left[\frac{7}{3}(3x + 2) + \frac{1}{3}\right]\sqrt{3x+2}\, dx$$

$$= \int \left[\frac{7}{3}(3x + 2)^{\frac{3}{2}} + \frac{1}{3}(3x + 2)^{\frac{1}{2}}\right] dx$$

$$= \frac{7}{3} \int (3x + 2)^{\frac{3}{2}} dx + \frac{1}{3} \int (3x + 2)^{\frac{1}{2}}\, dx$$

$$= \frac{7}{3}\, \frac{(3x+2)^{\frac{5}{2}}}{3 \times \frac{5}{2}} + \frac{1}{3}\, \frac{(3x+2)^{\frac{3}{2}}}{3 \times \frac{3}{2}} + c$$

$$= \frac{14}{45}\, (3x + 2)^{\frac{5}{2}} + \frac{2}{27}\, (3x + 2)^{\frac{3}{2}} + c$$

Example 9 : Evaluate : $\int \frac{3x+4}{\sqrt{4x+5}}\, dx$

Solution : $I = \int \frac{3x+4}{\sqrt{4x+5}}\, dx$

$$= \int \frac{\frac{3}{4}(4x+5)+\frac{1}{4}}{\sqrt{4x+5}}\, dx$$

$$= \frac{3}{4}\int \frac{4x+5}{\sqrt{4x+5}}\, dx + \frac{1}{4}\int \frac{1}{\sqrt{4x+5}}\, dx$$

$$= \frac{3}{4}\int (4x+5)^{\frac{1}{2}}\, dx + \frac{1}{4}\int (4x+5)^{-\frac{1}{2}}\, dx$$

$$= \frac{3}{4}\,\frac{(4x+5)^{\frac{3}{2}}}{4\times\frac{3}{2}} + \frac{1}{4}\,\frac{(4x+5)^{\frac{1}{2}}}{4\times\frac{1}{2}} + c$$

$$= \frac{1}{8}\,(4x+5)^{\frac{3}{2}} + \frac{1}{8}\,(4x+5)^{\frac{1}{2}} + c$$

Example 10 : Evaluate $\int sin^4x\ cos^4x\ dx$.

Solution : $I = \int sin^4x\ cos^4x\ dx.$

$$= \frac{1}{16}\int (2sinx\ cosx)^4\, dx$$

$$= \frac{1}{16}\int (sin2x)^4\, dx$$

$$= \frac{1}{16}\int \left(\frac{1-cos4x}{2}\right)^2 dx$$

$$= \frac{1}{64}\int (1 - 2cos\ 4x + cos^2 4x)\, dx$$

$$= \frac{1}{64}\int \left(1 - 2cos\ 4x + \left(\frac{1+cos\ 8x}{2}\right)\right) dx$$

$$= \frac{1}{128}\int (3 - 4cos\ 4x + cos\ 8x)\, dx$$

$$= \frac{1}{128}\left[3x - \frac{4sin\,4x}{4} + \frac{sin\,8x}{8}\right] + c$$

$$= \frac{1}{128}\left[3x - sin\ 4x + \frac{1}{8}sin\ 8x\right] + c$$

Example 11 : Evaluate : $\int sinax\ cosbx\ dx,\ a \neq \pm b$

Solution : $I = \int (sinax\ cosbx)\, dx$

$$= \frac{1}{2}\int (2sinax\ cosbx)\, dx$$

$$= \frac{1}{2}\int [sin\ (ax+bx) + sin(ax-bx)]\, dx$$

$$= \frac{1}{2}\int [sin\ (a+b)x\ dx + \frac{1}{2}\int sin(a-b)x\ dx$$

$$= -\frac{1}{2}\,\frac{cos\,(a+b)\,x}{a+b} - \frac{1}{2}\,\frac{cos\,(a-b)\,x}{a-b} + c$$

$$= -\frac{1}{2}\left[\frac{cos\,(a+b)\,x}{a+b} + \frac{cos\,(a-b)\,x}{a-b}\right] + c$$

Example 12 : Evaluate : $\int sinx\ sin2x\ sin3x\ dx$

Solution : $I = \int sinx\ sin2x\ sin3x\ dx$

$$= \frac{1}{2}\int (2sin\ 2x \cdot sinx)\ sin3x\ dx$$

$$= \frac{1}{2}\int (cosx - cos3x)\ sin3x\ dx$$

$$= \frac{1}{4}\int (2sin\ 3x\ cosx - 2sin\ 3x\ cos3x)\ dx$$

$$= \frac{1}{4}\int (sin\ 4x + sin\ 2x - sin\ 6x)\ dx$$

$$= \frac{1}{4}\left[-\frac{cos4x}{4} - \frac{cos2x}{2} + \frac{cos6x}{6}\right] + c$$

$$= \frac{1}{24}\ cos6x - \frac{1}{16}\ cos4x - \frac{1}{8}\ cos2x + c$$

Example 13 : Evaluate : $\int \frac{1}{sin\,(x-a)\,cos\,(x-b)}\ dx$

Solution : $I = \int \frac{1}{sin\,(x-a)\,cos\,(x-b)}\ dx$

$$= \frac{1}{cos\,(a-b)}\int \frac{cos\,(a-b)}{sin\,(x-a)\,cos\,(x-b)}\ dx$$

$$= \frac{1}{cos\,(a-b)}\int \frac{cos\,[(x-a)-(x-b)]}{sin\,(x-a)\,cos\,(x-b)}\ dx \qquad (cos(b-a) = cos(a-b))$$

$$= \frac{1}{cos\,(a-b)}\int \frac{cos(x-a)\,cos(x-b) + sin\,(x-a)\,sin\,(x-b)}{sin(x-a)\cdot cos(x-b)}\ dx$$

$$= \frac{1}{cos\,(a-b)}\int [cot(x-a) + tan(x-b)]\ dx$$

$$= \frac{1}{cos\,(a-b)}\ [\log|\,sin(x-a)\,| - \log|\,cos(x-b)\,|] + c$$

$$= \frac{1}{cos\,(a-b)}\ \log\left|\frac{sin(x-a)}{cos(x-b)}\right| + c$$

Example 14 : Evaluate : $\int \frac{sinx\ cosx}{3sin^2x - 4cos^2x}\ dx$

Solution : $I = \int \frac{sinx\ cosx}{3sin^2x - 4cos^2x}\ dx$

Let $3sin^2x - 4cos^2x = t$

$\therefore\ [3(2sinx\ cosx) + 4(2cosx\ sinx)]dx = dt$

$\therefore\ 14sinx\ cosx\ dx = dt$

$\therefore\ sinx\ cosx\ dx = \frac{1}{14}\ dt$

$$\therefore\ I = \frac{1}{14}\int \frac{1}{t}\ dt$$

$$= \frac{1}{14}\ \log|\,t\,| + c$$

$$= \frac{1}{14}\ \log|\,3sin^2x - 4sin^2x\,| + c$$

Example 15 : Evalaute $\int \frac{1}{2-3\cos 2x}\,dx$

Solution : $I = \int \frac{1}{2-3\cos 2x}\,dx$

$$= \int \frac{1}{2-3\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)}\,dx$$

$$= \int \frac{\sec^2 x\,dx}{2(1+\tan^2 x)-3+3\tan^2 x}$$

$$= \int \frac{\sec^2 x\,dx}{5\tan^2 x-1}$$

Taking $\tan x = t$, $\sec^2 x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{5t^2-1}$$

$$= \int \frac{dt}{(\sqrt{5}t)^2-(1)^2}$$

$$= \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right| + c$$

$$= \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right| + c$$

Example 16 : Evaluate : $\int \frac{\cos x}{\sqrt[3]{1-9\sin x}}\,dx$ $\left(\sin x < \frac{1}{9}\right)$

Solution : $I = \int \frac{\cos x}{\sqrt[3]{1-9\sin x}}\,dx$

Taking $1-9\sin x = t^3$, $-9\cos x\,dx = 3t^2\,dt$

$$\therefore \quad \cos x\,dx = -\frac{1}{3}t^2\,dt$$

$$\therefore \quad I = \int \frac{-\frac{1}{3}t^2\,dt}{\sqrt[3]{t^3}}$$

$$= -\frac{1}{3}\int t\,dt$$

$$= -\frac{1}{3}\left(\frac{t^2}{2}\right) + c$$

$$= -\frac{1}{6}(1-9\sin x)^{\frac{2}{3}} + c$$

Example 17 : Evaluate $\int \frac{\cos^9 x}{\sin x}\,dx$

Solution : $I = \int \frac{\cos^9 x}{\sin x}\,dx$

Taking $\sin x = t$, $\cos x\,dx = dt$

$$\therefore \quad I = \int \frac{(\cos^2 x)^4}{\sin x}\cos x\,dx$$

$$= \int \frac{(1-\sin^2 x)^4}{\sin x}\cos x\,dx$$

$$= \int \frac{(1-t^2)^4}{t}\,dt$$

$$= \int \frac{1-4t^2+6t^4-4t^6+t^8}{t}\,dt$$

$$= \int \left(\frac{1}{t}-4t+6t^3-4t^5+t^7\right)dt$$

$$= \log|t| - 4\frac{t^2}{2} + \frac{6t^4}{4} - \frac{4t^6}{6} + \frac{t^8}{8} + c$$

$$= \log|\sin x| - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{1}{8}\sin^8 x + c$$

Example 18 : Evaluate $\int \frac{x^2 \sin^{-1}(x^3)}{\sqrt{1-x^6}}\,dx$

Solution : $I = \int \frac{x^2 \sin^{-1}(x^3)}{\sqrt{1-x^6}}\,dx$

Taking $\sin^{-1}x^3 = t$, $\frac{3x^2 dx}{\sqrt{1-x^6}} = dt$

i.e. $\frac{x^2\,dx}{\sqrt{1-x^6}} = \frac{1}{3}dt$

$$\therefore \quad I = \int \sin^{-1}(x^3)\cdot\frac{x^2\,dx}{\sqrt{1-x^6}}$$

$$= \int \frac{1}{3}t\cdot dt$$

$$= \frac{1}{3}\left[\frac{t^2}{2}\right] + c$$

$$= \frac{1}{6}[\sin^{-1}(x^3)]^2 + c$$

Exercise 6.2

Integrate the following functions defined on proper domain *w.r.t. x*.

1. $\frac{1}{5x-3}$
2. $e^{7x+4}+(5x-3)^8$
3. $\frac{7^{2x+3}\sin^2 2x+\cos^2 2x}{\sin^2 2x}$
4. $5^{4x+3}-3\sin(2x+3)$
5. $\frac{1}{\sqrt{5x^2-4}}$
6. $\frac{1}{\sqrt{16-9x^2}}$
7. $\frac{1}{\sqrt{5x^2+3}}+\frac{1}{9-4x^2}$
8. $\frac{1}{\sqrt{2x^2+3}}+\frac{1}{7x^2+3}$
9. $\frac{(2x+1)^2}{x-2}$
10. $\frac{x^5+2}{x+1}$
11. $\frac{1}{\sqrt{5-3x}}$
12. $3^{5x-2}+\frac{1}{(2x+1)^3}$
13. $\cot^2(3+5x)$
14. $\sin^2(3x+5)$
15. $\frac{1-\cos 3x}{\sin^2 3x}$
16. $\sqrt{1+\cos x}$, $0<x<\pi$
17. $\frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}$
18. $\frac{1}{\sqrt{5-2x}+\sqrt{3-2x}}$
19. $\frac{x+2}{(x+1)^2}$
20. $\frac{x^2+1}{(x+1)^2}$
21. $\frac{x^3+3x^2+2x+1}{x-1}$
22. $x\sqrt{x+3}$
23. $\frac{x}{\sqrt{x+1}}$
24. $\frac{x+1}{\sqrt{2x+1}}$
25. $\frac{8x+13}{\sqrt{4x+7}}$
26. $\cos^4 x$
27. $\sin^3 x\ \cos^3 x$
28. $\sin^3(2x-1)$
29. $\cos 2x\cdot\cos 4x$
30. $\frac{\sin 4x}{\sin x}$
31. $\cos 2x\cdot\cos 4x\cdot\cos 6x$
32. $\frac{1}{\sqrt{1-\cos x}}$
33. $\sqrt{\frac{1+\cos x}{1-\cos x}}$ $0<x<\pi$
34. $\sin mx\cdot\sin nx,\ m\neq n,\ m,\ n\in\mathbb{N}$
35. $\frac{\sin x}{\sin(x-a)}$
36. $\frac{1}{\sin(x-a)\sin(x-b)}$
37. $\frac{3x+2}{3-2x}$
38. $(3x^2-4x+5)^{\frac{3}{2}}(3x-2)$
39. $\frac{x+3}{\sqrt{x^2+6x+4}}$
40. $x^3\sqrt{5x^4+3}$
41. $\frac{\sin^2(\log x)}{x}$
42. $\frac{\sqrt{1+\log x}}{x}$
43. $\frac{\sin 2x}{(m+n\cos 2x)^2}$
44. $\frac{1-\tan x}{1+\tan x}$
45. $\frac{e^x(1+x)}{\cos^2(xe^x)}$
46. $e^{-x}\operatorname{cosec}^2(2e^{-x}+3)$
47. $\frac{x^{e-1}+e^{x-1}}{x^e+e^x}$
48. $\frac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}$
49. $\frac{\sin 2x}{(b\cos^2 x+a\sin^2 x)^2}$
50. $\frac{\tan x}{a^2+b^2\tan^2 x}$, $(a<b)$
51. $\frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}$
52. $\frac{(\tan^{-1}x)^{-\frac{3}{2}}}{1+x^2}$

53. $\dfrac{e^x \log(sine^x)}{tane^x}$ 54. $\dfrac{\log(x+1) - \log x}{x(x+1)}$ 55. tan^3x

56. $sec^4x \;\; tanx$ 57. tan^6x 58. $\dfrac{1}{\sqrt{x+1} + \sqrt[3]{x+1}}$

59. $\dfrac{x^2}{(x+2)^{\frac{1}{3}}}$ 60. $\dfrac{1}{a^2cos^2x + b^2sin^2x}$ 61. $\dfrac{1}{3 - 2sin^2x}$

62. $\dfrac{sinx}{sin\,3x}$ 63. $\dfrac{1}{8cos^2x + 3sin^2x + 1}$ 64. $\dfrac{1}{3sin^2x + cos\,2x}$

*

6.7 Trigonometric Substitutions

Sometimes using proper trigonometric substitutions, we can transform given integrand into a form whose integration can be easily obtained. Particularly, when expressions like $x^2 - a^2$, $a^2 - x^2$, $x^2 + a^2$ occur under square root in integrand, trigonometric substitutions are very useful.

Suppose our aim is to obtain $\int \dfrac{x^2}{\sqrt{4 - x^2}}\, dx$, $(x > 0)$

Let $x = 2sin\theta$. Then $dx = 2cos\theta\; d\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$$\therefore \quad I = \int \frac{x^2}{\sqrt{4 - x^2}}\, dx$$

$$= \int \frac{4sin^2\theta}{\sqrt{4 - 4sin^2\theta}} \cdot 2cos\theta\; d\theta$$

$$= \int \frac{4sin^2\theta \cdot 2cos\theta\, d\theta}{2cos\theta} \qquad \left(cos\theta > 0 \text{ as } \theta \in \left(0, \tfrac{\pi}{2}\right)\right)$$

$$= 4\int sin^2\theta\; d\theta$$

$$= 4\int \frac{1 - cos\,2\theta}{2}\, d\theta$$

$$= 2\left[\theta - \frac{sin\,2\theta}{2}\right] + c$$

$$= 2\theta - 2sin\theta\; cos\theta + c$$

Now, $x = 2sin\theta$. Hence $\theta = sin^{-1}\left(\frac{x}{2}\right)$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$$2sin\theta\; cos\theta = 2 \cdot \frac{x}{2}\sqrt{1 - \frac{x^2}{4}} = \frac{1}{2}x\sqrt{4 - x^2}$$

$$\therefore \quad I = 2sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4 - x^2} + c$$

Following is a list of some frequently used substitutions. Mostly they are used to remove radical sign from the integrand. Usually we will take $0 < \theta < \frac{\pi}{2}$.

Integrands	Substitution
$\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
$\sqrt{2ax-x^2}$	$x = 2a\sin^2\theta$
$\sqrt{2ax-x^2} = \sqrt{a^2-(x-a)^2}$	$x - a = a\sin\theta$ or $a\cos\theta$

Example 19 : Evaluate : $\int \frac{1}{x\sqrt{x^4-b^4}}\,dx$

Solution : Here, $I = \int \frac{1}{x\sqrt{x^4-b^4}}\,dx$

Let $x^2 = b^2\sec\theta$ $\quad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad 2x\,dx = b^2\sec\theta\,\tan\theta\,d\theta$

Now, $I = \int \frac{2x\,dx}{2x^2\sqrt{x^4-b^4}}$

$= \int \frac{b^2\sec\theta\tan\theta\,d\theta}{2b^2\sec\theta\sqrt{b^4\sec^2\theta-b^4}}$

$= \frac{1}{2b^2}\int d\theta$

$= \frac{1}{2b^2}(\theta) + c$

But, since $x^2 = b^2\sec\theta$, $\sec\theta = \frac{x^2}{b^2}$, $\theta = \sec^{-1}\frac{x^2}{b^2}$ $\quad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad I = \frac{1}{2b^2}\sec^{-1}\left(\frac{x^2}{b^2}\right) + c$

Example 20 : Evaluate : $\int \frac{\sqrt{3-x}}{x}\,dx,\ 0 < x < 3$

Solution : Here, $I = \int \frac{\sqrt{3-x}}{x}\,dx$

Let $x = 3\sin^2\theta$ $\quad (0 < \theta < \frac{\pi}{2})$

Then $dx = 3(2\sin\theta\cos\theta)\,d\theta$

$$\therefore \quad I = \int \frac{\sqrt{3-3\sin^2\theta}}{3\sin^2\theta}\, 6\sin\theta\, \cos\theta\, d\theta$$

$$= \int \frac{2\sqrt{3}\cos^2\theta}{\sin\theta}\, d\theta$$

$$= 2\sqrt{3} \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$$= 2\sqrt{3} \int (\operatorname{cosec}\theta - \sin\theta)\, d\theta$$

$$= 2\sqrt{3}\, [\log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta] + c$$

But, since $\sin^2\theta = \frac{x}{3}$, $\cos^2\theta = 1 - \frac{x}{3}$. So $\cos\theta = \sqrt{\frac{3-x}{3}}$

$\operatorname{cosec}^2\theta = \frac{3}{x}$. So $\operatorname{cosec}\theta = \sqrt{\frac{3}{x}}$

Also $1 + \cot^2\theta = \frac{3}{x}$. So $\cot\theta = \sqrt{\frac{3}{x} - 1} = \sqrt{\frac{3-x}{x}}$

$$\therefore \quad I = 2\sqrt{3}\left[\log\left|\left(\sqrt{\frac{3}{x}} - \sqrt{\frac{3-x}{x}}\right)\right| + \sqrt{\frac{3-x}{3}}\right] + c$$

Example 21 : Evaluate $\int \frac{\sqrt{x^2+1}}{x^4}\, dx$, $(x < 0)$

Solution : $I = \int \frac{\sqrt{x^2+1}}{x^4}\, dx$

Let $\theta = \tan^{-1}x$, $-\frac{\pi}{2} < \theta < 0$. So, $x = \tan\theta$.

$\therefore \quad dx = \sec^2\theta\, d\theta, \quad \theta \in \left(-\frac{\pi}{2}, 0\right)$

$$\therefore \quad I = \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^4\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int \frac{\sec\theta \cdot \sec^2\theta}{\tan^4\theta}\, d\theta \qquad \left(\sec\theta > 0 \text{ as } \theta \in \left(-\frac{\pi}{2}, 0\right)\right)$$

$$= \int \frac{\cos\theta}{\sin^4\theta}\, d\theta$$

$$= \int (\sin\theta)^{-4} \frac{d}{d\theta}(\sin\theta)\, d\theta$$

$$= \frac{(\sin\theta)^{-3}}{-3} + c$$

$$= -\frac{1}{3}\frac{1}{\sin^3\theta} + c$$

$$= -\frac{1}{3}\operatorname{cosec}^3\theta + c$$

Now, $\tan\theta = x$. So $\cot\theta = \frac{1}{x}$

and $\operatorname{cosec}\theta = -\sqrt{1+\cot^2\theta} = -\sqrt{1+\frac{1}{x^2}} = \frac{-\sqrt{x^2+1}}{|x|} = \frac{-\sqrt{x^2+1}}{-x} = \frac{\sqrt{x^2+1}}{x} \qquad \left(-\frac{\pi}{2} < \theta < 0\right)$

$$\therefore \quad I = -\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 + c$$

$$= -\frac{1}{3}\,\frac{(1+x^2)^{\frac{3}{2}}}{x^3} + c$$

Example 22 : Evaluate : $\int \frac{1}{(x-1)^{\frac{3}{2}}\,(x-2)^{\frac{1}{2}}}\,dx,\ (x > 2)$

Solution : $I = \int \frac{dx}{(x-1)^{\frac{3}{2}}\,(x-2)^{\frac{1}{2}}}$

Let $x - 1 = sec^2\theta$, $dx = 2sec\theta\ sec\theta\ tan\theta\ d\theta$ $\qquad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad dx = 2\ sec^2\theta\ tan\theta\ d\theta$

$$\therefore \quad I = \int \frac{2sec^2\theta\, tan\theta\, d\theta}{(sec^2\theta)^{\frac{3}{2}}\,(sec^2\theta - 1)^{\frac{1}{2}}}$$

$$= \int \frac{2sec^2\theta\, tan\theta\, d\theta}{sec^3\theta \cdot tan\theta}$$

$$= 2\int cos\theta\ d\theta$$

$$= 2sin\theta + c$$

Now, $sec^2\theta = x - 1$. So $cos^2\theta = \frac{1}{x-1}$

and $sin^2\theta = 1 - cos^2\theta = 1 - \frac{1}{x-1} = \frac{x-2}{x-1}$

$\therefore \quad sin\theta = \sqrt{\frac{x-2}{x-1}}$ $\qquad \left(0 < \theta < \frac{\pi}{2}\right)$

$$\therefore \quad I = 2\sqrt{\frac{x-2}{x-1}} + c$$

6.8 An Important Substitution

If the integrand is $\frac{1}{a+bsinx}$, $\frac{1}{a+bcosx}$ or $\frac{1}{a+bsinx+c\,cosx}$, then $tan\frac{x}{2} = t$ is a useful substitution. Using this substitution, we can transform integrand into a standard form of t.

Taking $tan\frac{x}{2} = t$, $sec^2\frac{x}{2} \cdot \frac{1}{2}\,dx = dt$

$$\therefore \quad dx = \frac{2dt}{sec^2\frac{x}{2}} = \frac{2dt}{1+tan^2\frac{x}{2}} = \frac{2dt}{1+t^2}$$

$$sinx = \frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{2t}{1+t^2} \text{ and } cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

This will transform the integrand into a function of t.

Example 23 : Evaluate : $\int \frac{1}{1-2sinx}\, dx$

Solution : Let $tan\frac{x}{2} = t$. So, $dx = \frac{2dt}{1+t^2}$ and $sinx = \frac{2t}{1+t^2}$

$$\therefore \quad I = \int \frac{1}{1-2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{t^2-4t+1}\, dt$$

$$= 2\int \frac{1}{t^2-4t+4-3}\, dt$$

$$= 2\int \frac{1}{(t-2)^2-(\sqrt{3})^2}\, dt$$

$$= 2 \times \frac{1}{2\sqrt{3}} \log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\frac{tan\frac{x}{2}-2-\sqrt{3}}{tan\frac{x}{2}-2+\sqrt{3}}\right| + c$$

Example 24 : Evaulate $\int \frac{dx}{cos\,\alpha + cosx}$, $\quad \alpha \in \left(0, \frac{\pi}{2}\right)$

Solution : $I = \int \frac{dx}{cos\,\alpha + cosx}$

Let $tan\frac{x}{2} = t$. So $dx = \frac{2dt}{1+t^2}$, $cosx = \frac{1-t^2}{1+t^2}$

$$\therefore \quad I = \int \frac{1}{cos\alpha + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2\,dt}{cos\,\alpha + t^2 \cdot cos\,\alpha + 1 - t^2}$$

$$= 2\int \frac{dt}{(1+cos\,\alpha)-(1-cos\,\alpha)t^2}$$

$$= 2\int \frac{dt}{2cos^2\frac{\alpha}{2} - 2sin^2\frac{\alpha}{2} \cdot t^2}$$

$$= \int \frac{dt}{\left(cos\,\frac{\alpha}{2}\right)^2 - \left(t\,sin\,\frac{\alpha}{2}\right)^2}$$

$$= \frac{1}{2sin\frac{\alpha}{2}\,cos\frac{\alpha}{2}} \log\left|\frac{cos\,\frac{\alpha}{2} + sin\,\frac{\alpha}{2}\,t}{cos\,\frac{\alpha}{2} - sin\,\frac{\alpha}{2}\,t}\right| + c$$

$$= \frac{1}{sin\,\alpha} \log\left|\frac{1+tan\,\frac{\alpha}{2}\cdot tan\,\frac{x}{2}}{1-tan\,\frac{\alpha}{2}\cdot tan\,\frac{x}{2}}\right| + c$$

6.9 Integrals of the type $\int sin^m x\ cos^n x\ dx$ $m, n \in N$

If $m, n \in$ N, the following cases may occur :

(1) m, n are odd

(2) m is odd and n is even.

(3) m is even and n is odd

(4) m and n both are even.

Let $I = \int sin^m x\ cos^n x\ dx$

Case 1 : m, n are odd.

We may take $sinx = t$ or $cosx = t$. Usually if $m > n$, $sinx = t$ and if $n > m$, $cosx = t$ will be convenient.

Case 2 : m is odd and n is even.

We take $cosx = t$.

Case 3 : m is even and n is odd.

We take $sinx = t$.

Case 4 : m and n both are even.

In this situation, we transform $sin^m x\ cos^n x$ using $sin^2 x = \frac{1 - cos2x}{2}$ and $cos^2 x = \frac{1 + cos2x}{2}$

For small values of m and n, these methods are simple. For larger and negative values of m and n, other methods are availables, but at this stage we will not study them.

Example 25 : Evaluate $\int cos^2 x\ sin^5 x\ dx$

Solution : Here, $m = 5$ is odd. $n = 2$ is even.

$\therefore$ Let $cosx = t$. So $-sinx\ dx = dt$

$\therefore$ $sinx\ dx = -dt$

$$I = \int cos^2 x\ sin^5 x\ dx$$

$$= \int sin^4 x \cdot cos^2 x \cdot sinx\ dx$$

$$= \int (1 - cos^2 x)^2 \cdot cos^2 x\ sinx\ dx$$

$$= \int (1 - t^2)^2\ t^2\ (-dt)$$

$$= \int (1 - 2t^2 + t^4)(-t^2)\ dt$$

$$= \int (2t^4 - t^6 - t^2)\ dt$$

$$= \frac{2t^5}{5} - \frac{t^7}{7} - \frac{t^3}{3} + c$$

$$= \frac{2}{5}cos^5 x - \frac{1}{7}cos^7 x - \frac{1}{3}cos^3 x + c$$

Example 26 : Evaluate $\int sin^{23} x \cdot cos^3 x\ dx$

Solution : $I = \int sin^{23} x \cdot cos^3 x\ dx$

Here, $m = 23$, $n = 3$. m and n both are odd.

But $m > n$. Let $sinx = t$, so $cosx\ dx = dt$

$$I = \int sin^{23} x\ cos^2 x\ cosx\ dx$$

$$= \int sin^{23} x\ (1 - sin^2 x)\ cosx\ dx$$

$$= \int t^{23}\ (1 - t^2)\ dt$$

$$= \int (t^{23} - t^{25})\ dt$$

$$= \frac{t^{24}}{24} - \frac{t^{26}}{26} + c$$

$$= \frac{sin^{24} x}{24} - \frac{sin^{26} x}{26} + c$$

Example 27 : Evaluate $\int sin^2 x\ cos^4 x\ dx$

Solution : $I = \int sin^2 x\ cos^4 x\ dx$

Here, m and n both are even.

$$\therefore \quad \sin^2 x \cos^4 x = \frac{1}{4}(4\sin^2 x \cos^2 x) \cos^2 x$$

$$= \frac{1}{4}\sin^2 2x \cdot \cos^2 x$$

$$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{16}(1 - \cos 4x + \cos 2x - \cos 4x \cos 2x)$$

$$= \frac{1}{16}\left[1 - \cos 4x + \cos 2x - \frac{(2\cos 4x \cos 2x)}{2}\right]$$

$$= \frac{1}{32}[2 - 2\cos 4x + 2\cos 2x - \cos 6x - \cos 2x]$$

$$= \frac{1}{32}(2 - 2\cos 4x + \cos 2x - \cos 6x)$$

$$\therefore \quad I = \frac{1}{32}\int [2 + \cos 2x - 2\cos 4x - \cos 6x]\, dx$$

$$= \frac{1}{32}\left[2x + \frac{\sin 2x}{2} - \frac{2\sin 4x}{4} - \frac{\sin 6x}{6}\right] + c$$

$$= \frac{1}{192}[12x + 3\sin 2x - 3\sin 4x - \sin 6x] + c$$

Exercise 6.3

Integrate the following functions defined on proper domains using trigonometric substition :

1. $\dfrac{1}{x^2\sqrt{1-x^2}} \quad (|x| < 1)$

2. $\dfrac{\sqrt{9-x^2}}{x^2}, \quad (0 < x < 3)$

3. $\dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$

4. $x^2\sqrt{a^6 - x^6}, \quad (0 < x < a)$

5. $\dfrac{1}{\sqrt{2ax - x^2}} \quad (0 < x < 2a)$

6. $\sqrt{\dfrac{2-x}{x}} \quad (0 < x < 2)$

7. $\sqrt{\dfrac{a-x}{a+x}} \quad (0 < x < a)$

8. $\dfrac{x^2}{\sqrt{a^6 - x^6}} \quad (0 < x < a)$

9. $\dfrac{1}{x^2(1+x^2)^2}$

10. $\dfrac{x}{(16-9x^2)^{\frac{3}{2}}} \quad \left(0 < x < \frac{4}{3}\right)$

11. $\dfrac{x^2}{(x^2-a^2)^{\frac{3}{2}}} \quad (|x| > |a|)$

12. $x\sqrt{\dfrac{a^2-x^2}{a^2+x^2}} \quad (0 < x < a)$

13. $\dfrac{1}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}} \quad (x > 2)$

14. $\dfrac{\sqrt{25-x^2}}{x^2} \quad (0 < x < 5)$

15. $\dfrac{1}{1+\sin x+\cos x}$

16. $\dfrac{1}{3+2\sin x+\cos x}$

17. $\dfrac{1}{5+4\cos x}$

18. $\dfrac{1}{1+\cos\alpha \cos x}$

19. $\dfrac{1}{2-\cos x}$

20. $\dfrac{1}{\cos x - \sin x} \quad \left(0 < x < \frac{\pi}{4}\right)$

21. $\sin^4 x \ \cos^3 x$

22. $\sin^3 x \ \cos^{10} x$

23. $\cos^3 x \ \sin^7 x$

24. $\sin^5 x \ \cos^4 x$

25. $\sin^5 x$

26. $\sin^4 x \ \cos^2 x$

*

6.10 Integration of the type (1) $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(2) $\int \frac{Ax+B}{ax^2+bx+c}\,dx$ and $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$

(1) To evaluate this type of integrals, we express $ax^2 + bx + c$ as the sum or difference of two squares.

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$$

$$= a\,[(x+\alpha)^2 - \beta^2], \text{ if } b^2 - 4ac > 0, \text{ where } \beta^2 = \frac{b^2-4ac}{4a^2}$$

$$= a\,[(x+\alpha)^2 + \beta^2], \text{ if } b^2 - 4ac < 0, \text{ where } \beta^2 = -\frac{b^2-4ac}{4a^2}$$

Thus, $ax^2 + bx + c = a\,[(x+\alpha)^2 \pm \beta^2]$. Hence $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ can be evaluated using previous standard forms. Now let us understand the method by the following examples :

$\left(\text{Note : If } b^2 = 4ac, \text{ then } ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2\right)$

Example 28 : Evaluate : $\int \frac{dx}{3x^2+13x-10}$

Solution : $I = \int \frac{dx}{3x^2+13x-10}$

$$= \frac{1}{3}\int \frac{dx}{x^2+\frac{13}{3}x-\frac{10}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{x^2+\frac{13}{3}x+\left(\frac{13}{6}\right)^2-\left(\frac{13}{6}\right)^2-\frac{10}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}$$

$$= \frac{1}{3}\times\frac{1}{2\left(\frac{17}{6}\right)}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + c$$

$$= \frac{1}{17} \log \left| \frac{x - \frac{2}{3}}{x+5} \right| + c$$

$$= \frac{1}{17} \log \left| \frac{3x-2}{3(x+5)} \right| + c$$

$$\left(\textbf{Note : } I = \frac{1}{17} \log \left| \frac{3x-2}{3(x+5)} \right| + c \right.$$

$$= \frac{1}{17} \left[\log |3x-2| - \log 3 - \log |x+5| \right] + c$$

$$\left. = \frac{1}{17} \log \left| \frac{3x-2}{x+5} \right| + c' \text{ where } c' = c - \frac{1}{17} \log 3 \right)$$

Example 29 : Evaluate : $\int \frac{1}{\sqrt{x(1-2x)}}\, dx$ $\left(0 < x < \frac{1}{2}\right)$

Solution : $I = \int \frac{1}{\sqrt{x(1-2x)}}\, dx$

$$= \int \frac{1}{\sqrt{x - 2x^2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{x}{2} - x^2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{1}{16} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right)}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x - \frac{1}{4}}{\frac{1}{4}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} (4x - 1) + c$$

(2) In order to evaluate this type of integrals, first we find constants m and n such that,

$Ax + B = m(\text{derivative of } ax^2 + bx + c) + n$

$Ax + B = m(2ax + b) + n$

$Ax + B = (2ma)x + (mb + n)$

Comparing the coefficient of x and constant term on both sides, we get

$A = 2ma$ and $mb + n = B$

$\therefore \quad m = \frac{A}{2a}$ and $n = B - mb$

Now, $\int \frac{Ax+B}{ax^2+bx+c}\,dx = \int \frac{m(2ax+b)+n}{ax^2+bx+c}\,dx$

$$= m\int \frac{2ax+b}{ax^2+bx+c}\,dx + n\int \frac{1}{ax^2+bx+c}\,dx$$

$$= m \log|\,ax^2+bx+c\,| + n\int \frac{1}{ax^2+bx+c}\,dx$$

For the first integral, we use $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$ and for the second integral, we have to use method (1) of making perfect square in the denominator.

Now, for $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{m(2ax+b)+n}{\sqrt{ax^2+bx+c}}\,dx$

$$= \int \frac{m(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx + n\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$= m\int (ax^2+bx+c)^{-\frac{1}{2}}(2ax+b)dx + n\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$= m\,\frac{(ax^2+bx+c)^{\frac{1}{2}}}{\frac{1}{2}} + n\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$= 2m\,(ax^2+bx+c)^{\frac{1}{2}} + n\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

For the first integral, we use $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$ and to evaluate the second integral, we have to use method (1) of making perfect square in the denominator.

Example 30 : Evaluate : $\int \frac{2x+3}{3x^2+4x+5}\,dx$

Solution : First, we will find constants m and n such that $2x+3 = m\frac{d}{dx}(3x^2+4x+5)+n$

$2x+3 = m(6x+4)+n$

$2x+3 = (6m)x+4m+n$

Comparing coefficient of x and constant term on both sides, we get $6m = 2$ and $4m+n = 3$.

$\therefore\quad m = \frac{1}{3}$ and $\frac{4}{3}+n = 3$. Thus, $n = \frac{5}{3}$

$\therefore\quad I = \int \frac{2x+3}{3x^2+4x+5}\,dx = \int \frac{\frac{1}{3}(6x+4)+\frac{5}{3}}{3x^2+4x+5}\,dx$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}\,dx + \frac{5}{3}\int \frac{1}{3x^2+4x+5}\,dx$$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}\,dx + 5\int \frac{1}{9x^2+12x+4+11}\,dx$$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}\,dx + 5\int \frac{1}{(3x+2)^2+(\sqrt{11})^2}\,dx$$

$$= \frac{1}{3}\log|3x^2+4x+5| + \frac{5}{3\sqrt{11}}\,tan^{-1}\frac{3x+2}{\sqrt{11}} + c$$

Example 31 : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Solution : Here, the derivative of denominator $x^2 + 4x + 1$ is $2x + 4$. Thus $2x + 3$ in the numerator can be written as $2x + 3 = (2x + 4) - 1$.

$$\therefore \quad I = \int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{(2x+4)-(1)}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{(2x+4)}{\sqrt{x^2+4x+1}}\,dx - \int \frac{1}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int (x^2+4x+1)^{-\frac{1}{2}}(2x+4)\,dx - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}\,dx$$

$$= \frac{(x^2+4x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \log|(x+2) + \sqrt{(x+2)^2-(\sqrt{3})^2}| + c$$

$$= 2\sqrt{x^2+4x+1} - \log|x+2+\sqrt{x^2+4x+1}| + c$$

Example 32 : Evaluate $\int \frac{x^2}{x^4+1}\,dx$

Solution : $I = \int \frac{x^2}{x^4+1}\,dx$

$$= \frac{1}{2}\int \frac{2x^2}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx + \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2} + \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$$

Let $x - \frac{1}{x} = u$ for the first integral and $x + \frac{1}{x} = v$ for the second integral.

So $\left(1+\frac{1}{x^2}\right) dx = du$ and $\left(1-\frac{1}{x^2}\right) dx = dv$

$$\therefore \quad I = \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} + \int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2} \times \frac{1}{2\sqrt{2}} \log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}} + \frac{1}{4\sqrt{2}} \log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{4\sqrt{2}} \log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + c$$

Exercise 6.4

Integrate the following *w.r.t. x*.

1. $\frac{1}{x^2+3x+3}$
2. $\frac{1}{4x^2-4x+3}$
3. $\frac{1}{1-6x-9x^2}$
4. $\frac{1}{3+2x-x^2}$
5. $\frac{1}{\sqrt{x^2-x+5}}$
6. $\frac{1}{\sqrt{2x^2+3x-2}}$
7. $\frac{1}{\sqrt{7-3x-2x^2}}$
8. $\frac{1}{\sqrt{3x^2+5x+7}}$
9. $\frac{1}{\sqrt{(x-1)(x-2)}}$
10. $\frac{1}{\sqrt{9+8x-x^2}}$
11. $\frac{4x+1}{x^2+3x+2}$
12. $\frac{3x+2}{2x^2+x+1}$
13. $\frac{2x+3}{\sqrt{x^2+4x+5}}$
14. $\frac{3x+1}{\sqrt{5-2x-x^2}}$
15. $\frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x}$
16. $\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}$
17. $\frac{x^2}{\sqrt{x^6+2x^3+3}}$
18. $\frac{2x}{\sqrt{1-x^2-x^4}}$
19. $\frac{x^2+1}{x^4+1}$
20. $\frac{x^2+4}{x^4+16}$
21. $\frac{x^2+1}{x^4+7x^2+1}$
22. $\frac{1}{x^4+1}$
23. $\frac{x^2-1}{x^4+x^2+1}$
24. $\frac{x^2}{x^4+x^2+1}$

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Miscellaneous Examples

Example 33 : Evaluate : $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}\, dx$

Solution : $I = \int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)}\, dx$

$$= \int \frac{\sin 2x}{\sin^2 x - \sin^2 \frac{\pi}{3}}\, dx$$

$$= \int \frac{\sin 2x}{\sin^2 x - \frac{3}{4}}\, dx$$

$$= \int \frac{\frac{d}{dx}\left(\sin^2 x - \frac{3}{4}\right)}{\sin^2 x - \frac{3}{4}}\, dx$$

$$= \log\left|\sin^2 x - \frac{3}{4}\right| + c$$

Example 34 : Evaluate : $\int \frac{1}{x(x^n + 1)}\, dx \qquad (x > 0)$

Solution : $I = \int \frac{1}{x(x^n + 1)}\, dx$

Let $x^n + 1 = t$. Then $nx^{n-1}\, dx = dt$

$$\therefore \quad I = \int \frac{nx^{n-1}\, dx}{nx^n\,(x^n + 1)}$$

$$= \frac{1}{n}\int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n}\int \frac{1}{t^2 - t + \frac{1}{4} - \frac{1}{4}}\, dt$$

$$= \frac{1}{n}\int \frac{1}{\left(t - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dt$$

$$= \frac{1}{n}\log\left|\frac{\left(t - \frac{1}{2}\right) - \left(\frac{1}{2}\right)}{\left(t - \frac{1}{2}\right) + \left(\frac{1}{2}\right)}\right| + c$$

$$= \frac{1}{n}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n + 1}\right) + c$$

Second Method :

$$I = \frac{1}{n}\int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n}\int \frac{[t - (t-1)]\, dt}{(t-1)t}$$

$$= \frac{1}{n}\left[\int \frac{dt}{t-1} - \int \frac{dt}{t}\right]$$

$$= \frac{1}{n}\,[\log|t-1| - \log|t|] + c$$

$$= \frac{1}{n}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n + 1}\right) + c$$

Example 35 : Evaluate : $\int \sqrt{\frac{\sin(x - \theta)}{\sin(x + \theta)}}\, dx \qquad \theta < x < \frac{\pi}{2} + \theta,\ 0 < x < \frac{\pi}{2}$

Solution : $I = \int \sqrt{\frac{\sin(x - \theta)}{\sin(x + \theta)}}\, dx$

$$= \int \sqrt{\frac{\sin(x - \theta)}{\sin(x + \theta)} \times \frac{\sin(x - \theta)}{\sin(x - \theta)}}\, dx \qquad (\sin(x - \theta) > 0)$$

$$= \int \frac{\sin(x - \theta)}{\sqrt{\sin^2 x - \sin^2 \theta}}\, dx$$

$$= \int \frac{\sin x \cos\theta - \cos x \sin\theta}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{1 - \cos^2 x - 1 + \cos^2\theta}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x\, dx}{\sqrt{\cos^2\theta - \cos^2 x}} - \sin\theta \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2\theta}}$$

Let $\cos x = u$ in the first integral and $\sin x = v$ in the second integral.

$\therefore \quad -\sin x\, dx = du$ and $\cos x\, dx = dv$

$$\therefore \quad I = \cos\theta \int \frac{-du}{\sqrt{\cos^2\theta - u^2}} - \sin\theta \int \frac{dv}{\sqrt{v^2 - \sin^2\theta}}$$

$$= -\cos\theta \sin^{-1}\left(\frac{u}{\cos\theta}\right) - \sin\theta \log\left| v + \sqrt{v^2 - \sin^2\theta} \right| + c$$

$$= -\cos\theta \sin^{-1}\left(\frac{\cos x}{\cos\theta}\right) - \sin\theta \log\left| \sin x + \sqrt{\sin^2 x - \sin^2\theta} \right| + c$$

Example 36 : Evaluate : $\int \frac{\sin x}{\sqrt{1 + \sin x}}\, dx \qquad 0 < x < \pi$

Solution : $I = \int \frac{(\sin x + 1) - 1}{\sqrt{1 + \sin x}}\, dx$

$$= \int \sqrt{1 + \sin x}\, dx - \int \frac{1}{\sqrt{1 + \sin x}}\, dx$$

$$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx - \int \frac{1}{\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}}\, dx$$

$$= \int \left| \sin\frac{x}{2} + \cos\frac{x}{2} \right| dx - \int \frac{1}{\left| \sin\frac{x}{2} + \cos\frac{x}{2} \right|}\, dx$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \int \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\frac{x}{2} + \frac{1}{\sqrt{2}}\sin\frac{x}{2}\right)}\, dx \qquad \left(0 < \frac{x}{2} < \frac{\pi}{2}\right)$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \int \frac{1}{\sqrt{2}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)}\, dx$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \frac{1}{\sqrt{2}} \int \sec\left(\frac{x}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{-\cos\frac{x}{2}}{\frac{1}{2}} + \frac{\sin\frac{x}{2}}{\frac{1}{2}} - \frac{1}{\sqrt{2}} \times \frac{1}{\left(\frac{1}{2}\right)} \log\left| \sec\left(\frac{x}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) \right| + c$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \sqrt{2} \log\left| \sec\left(\frac{x}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) \right| + c$$

Exercise 6

Integrate the following with respect to x.

1. $\dfrac{\sqrt{x}}{\sqrt{x}+\sqrt[3]{x}}$ $(x > 0)$

2. $\dfrac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$

3. $\dfrac{1}{(x+1)^2\sqrt{x^2+2x+2}}$

4. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$, $x \in (0, 1)$

5. $\sqrt{\dfrac{x+3}{x+2}}$ $(x > -2)$

6. $\dfrac{x^2+5x+3}{x^2+3x+2}$ $(x \neq -2, -1)$

7. $\dfrac{x^2}{x^2+7x+10}$ $(x \neq -5, -2)$

8. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

9. $\dfrac{\sin(x+a)}{\sin(x+b)}$

10. $x(1-x)^n$

11. $\sqrt{\tan x}$

12. $\dfrac{1}{\sin^4 x+\cos^4 x}$

13. $\dfrac{1}{1-2a\cos x+a^2}$, $0 < a < 1$

14. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

Section A

(1) If $\int f(x)dx = \dfrac{(\log x)^5}{5} + c$, then $f(x) = \ldots\ldots$ ☐

(a) $\dfrac{\log x}{4}$ (b) $\dfrac{(\log x)^5}{5}$ (c) $\dfrac{(\log x)^4}{x}$ (d) $\dfrac{(\log x)^6}{6}$

(2) $\int e^{x\log a}\, e^x\, dx = \ldots\ldots + c$ ☐

(a) $a^x \cdot e^x$ (b) $\dfrac{(ae)^x}{(1+\log a)}$ (c) $\dfrac{e^x}{\log(ae)}$ (d) $\dfrac{a^x}{1+\log_e a}$

(3) $\int \dfrac{(\log x)^3}{x}\, dx = \ldots\ldots + c$ ☐

(a) $(\log x)^2$ (b) $\dfrac{(\log x)^2}{2}$ (c) $\frac{1}{4}(\log x)^4$ (d) $\frac{2}{3}(\log x)^3$

(4) $\int \sec^2\left(5-\frac{x}{2}\right) dx = \ldots\ldots + c$ ☐

(a) $\tan\left(5-\frac{x}{2}\right)$ (b) $2\tan\left(5-\frac{x}{2}\right)$ (c) $-2\tan\left(5-\frac{x}{2}\right)$ (d) $-\frac{1}{2}\tan\left(5-\frac{x}{2}\right)$

(5) $\int \dfrac{1}{4x^2+9}\, dx = \ldots\ldots + c$ ☐

(a) $\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)$ (b) $\frac{1}{4}\tan^{-1}\left(\frac{2x}{3}\right)$ (c) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$ (d) $\frac{3}{2}\tan^{-1}\left(\frac{2x}{3}\right)$

(6) $\int \sqrt{1-cosx}\ dx = ... + c,\ 2\pi < x < 3\pi$

(a) $-2\sqrt{2}\ cos\ \frac{x}{2}$ (b) $-\sqrt{2}\ cos\ \frac{x}{2}$ (c) $2\sqrt{2}\ cos\ \frac{x}{2}$ (d) $-\frac{1}{2}\ cos\left(\frac{x}{2}\right)$

(7) $\int \frac{dx}{x\sqrt{3+\log x}} = ... + c$

(a) $2\sqrt{3+\log x}$ (b) $\frac{2}{\sqrt{3+\log x}}$ (c) $\sqrt{3+\log x}$ (d) $-2\sqrt{3+\log x}$

(8) $\int \frac{1}{\sqrt{4-3x}}\ dx = ... + c$

(a) $-\frac{2}{3}(4-3x)^{-\frac{1}{2}} + c$ (b) $-\frac{2}{3}(4+3x)^{\frac{1}{2}}$

(c) $-\frac{2}{3}(4-3x)^{\frac{1}{2}}$ (d) $\frac{2}{3}(4+3x)^{\frac{1}{2}}$

(9) $\int \frac{x-2}{x^2-4x+5}\ dx = ... + c$

(a) $\log|x^2-4x+5| + c$ (b) $\log\sqrt{x^2-4x+5}$

(c) $\frac{1}{2}(x^2-4x+5)^2$ (d) $\log\left(\frac{x-3}{x-1}\right)$

(10) $\int \frac{1}{3t^2+4}\ dt = ... + c$

(a) $\frac{1}{12}\ tan^{-1}\left(\frac{3t}{4}\right)$ (b) $\frac{1}{3}\log\left|\frac{t+2}{t-2}\right|$

(c) $\frac{1}{2\sqrt{3}}\ tan^{-1}\left(\frac{\sqrt{3}t}{2}\right)$ (d) $\frac{1}{2\sqrt{3}}\ tan^{-1}\left(\frac{3t}{4}\right)$

(11) $\int \frac{1}{1-cost}\ dt = ... + c$

(a) $cosec\ t + cot\ t$ (b) $-cot\ \frac{t}{2}$ (c) $-4cot\ \frac{t}{2}$ (d) $cosec\ t + cot\ t$

(12) $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}\ dx = ... + c$

(a) $e \cdot 3^{-3x}$ (b) $e^3 \log x$ (c) $\frac{x^3}{3}$ (d) $\frac{x^2}{3}$

(13) $\int sec^2x \cdot cosec^2x\ dx = ... + c$

(a) $tanx + cotx$ (b) $tanx - cotx$ (c) $sec^2x + cosec^2x$ (d) $cotx - tanx$

(14) $\int e^{3\log x} \cdot (x^4+1)^{-1}\ dx = ... + c$

(a) $\log(x^4+1)$ (b) $-\log(x^4+1)$ (c) $\frac{1}{4}\log(x^4+1)$ (d) $\frac{-3}{(x^4+1)^2}$

(15) $\int \frac{(\log x)^4}{x}\ dx = ... + c$

(a) $\frac{(\log x)^5}{5}$ (b) $\frac{(\log x)^2}{2}$ (c) $\frac{\log x^5}{5x}$ (d) $\log x \cdot (\log x)^4 + \frac{(\log x)^5}{5x}$

(16) $\int \frac{dx}{\sqrt{1-x}} = ...$

(a) $sin^{-1}\sqrt{x} + c$ (b) $-2\sqrt{1-x} + c$ (c) $-sin^{-1}\sqrt{x} + c$ (d) $2\sqrt{1-x} + c$

(17) $\int \frac{(sinx)^{99}}{(cosx)^{101}} dx = ... + c$

(a) $\frac{(tanx)^{100}}{100}$ (b) $\frac{(tanx)^2}{2}$ (c) $\frac{(tanx)^{98}}{98}$ (d) $\frac{(tanx)^{97}}{97}$

(18) $\int \frac{\log x^2}{x} dx = ...$

(a) $\log |x^2| + c$ (b) $\log x + c$ (c) $(\log x)^2 + c$ (d) $\frac{1}{2}(\log x)^2 + c$

(19) $\int \frac{xsinx}{(xcosx - sinx + 5)} dx = ... + c$

(a) $\log | xcosx - sinx + 5 |$ (b) $-\log | xcosx - sinx + 5 |$

(c) $\log | xsinx - cosx + 5 |$ (d) $-\log | xsinx - cosx + 5 |$

(20) $\int (1 - cosx)cosec^2x \, dx = ... + c$

(a) $tan \frac{x}{2}$ (b) $cot \frac{x}{2}$ (c) $\frac{1}{2} tan \frac{x}{2}$ (d) $2 \, tan \frac{x}{2}$

Section B

(21) If $f'(x) = x^2 + 5$, then $\int f(x)dx = ...$. (c and k are arbitrary constants)

(a) $\frac{x^4}{12} + \frac{5x^2}{8} + cx + k$ (b) $-\frac{x^4}{12} - \frac{5x^2}{2} - cx + k$

(c) $\frac{x^4}{12} - \frac{5x^2}{12} + cx + k$ (d) $\frac{x^4}{12} + \frac{5x^2}{2} + cx + k$

(22) $\int \frac{10x^9 + 10^x \log 10}{10^x + x^{10}} dx = ... + c$

(a) $10^x - x^{10}$ (b) $10^x + x^{10}$ (c) $(10^x - x^{10})^{-1}$ (d) $\log | 10^x + x^{10} |$

(23) $\int cos^3x \cdot e^{\log sinx} dx = ... + c$

(a) $-\frac{sin^4x}{4}$ (b) $\frac{e^{sinx}}{4}$ (c) $\frac{e^{cosx}}{4}$ (d) $\frac{-cos^4x}{4}$

(24) $\int \frac{sinx}{1+4cosx} dx = ... + c$

(a) $\log | 1 + 4cosx |$ (b) $-4 \log | 1 + 4cosx |$

(c) $-\frac{1}{4} \log | 1 + 4cosx |$ (d) $-\log | 1 + 4cosx |$

(25) $\int \frac{1}{\sqrt{x+3}-\sqrt{x}}\, dx = \ldots\ldots + c$

(a) $\frac{3}{2}(x+2)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}}$ (b) $\frac{2}{9}(x+2)^{\frac{1}{2}} + \frac{2}{9}x^{\frac{1}{2}}$

(c) $\frac{2}{9}(x+3)^{\frac{3}{2}} + \frac{2}{9}x^{\frac{3}{2}}$ (d) $\frac{2}{9}(x+2)^{\frac{3}{2}} + \frac{2}{9}x$

(26) $\int sin2x\, cos\ 3x\ dx = A\ cosx + Bcos\ 5x + c$, then A + B = ...

(a) $\frac{1}{5}$ (b) $\frac{3}{10}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

(27) $\int \frac{cos\ 4x+1}{cotx - tanx}\, dx = A\ cos4x + c$, then A = ...

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{8}$

(28) $\int \frac{1+cosx}{sinx\ cosx}\, dx = \ldots\ldots + c$

(a) $\log|sinx| + \log|cosx|$ (b) $\log\left|tanx \cdot tan\frac{x}{2}\right|$

(c) $\log\left|1 + tan\frac{x}{2}\right|$ (d) $\log\left|sec\frac{x}{2} + tan\frac{x}{2}\right|$

(29) $\int \frac{sinx - cosx}{sinx + cosx}\, dx = \ldots\ldots + c$

(a) $\frac{1}{sinx + cosx}$ (b) $\frac{1}{sinx - cosx}$

(c) $\log|sinx + cosx|$ (d) $\log\left|\frac{1}{sinx + cosx}\right|$

(30) $\int \frac{1-cosx}{cosx\,(1+cosx)}\, dx = \ldots\ldots + c$

(a) $2\log|cosx| + tan\frac{x}{2}$ (b) $\log|secx + tanx| - 2tan\frac{x}{2}$

(c) $\log|tanx| + 2tan\frac{x}{2}$ (d) $\frac{1}{2}\log|secx| - tan\frac{x}{2}$

(31) $\int \frac{dx}{e^x + e^{-x}} = \ldots\ldots + c$

(a) $\log|e^x - e^{-x}|$ (b) $\log|e^x + e^{-x}|$ (c) $tan^{-1}(e^x)$ (d) $tan^{-1}(e^{2x})$

(32) $\int \frac{dx}{x + x\log x} = \ldots\ldots + c$

(a) $\log|x + x\log x|$ (b) $x\log|1 + \log x|$

(c) $\log|1 + \log x|$ (d) $\frac{1+\log x}{x^2}$

(33) $\int \frac{\sqrt{tanx}}{sinx\,cosx}\, dx = \ldots\ldots + c$

(a) $\frac{\sqrt{tanx}}{2}$ (b) $\frac{\sqrt{cotx}}{2}$ (c) $2\sqrt{cotx}$ (d) $2\sqrt{tanx}$

Section C

(34) $\int \frac{dx}{cosx - sinx} = ... + c$

(a) $\frac{1}{\sqrt{2}} \log \left| tan\left(\frac{x}{2} + \frac{3\pi}{8}\right) \right|$ (b) $\frac{1}{\sqrt{2}} \log \left| tan\left(\frac{\pi}{8} + \frac{x}{2}\right) \right|$

(c) $\frac{1}{\sqrt{2}} \log \left| tan\left(\frac{x}{2} - \frac{3\pi}{8}\right) \right|$ (d) $\log \left| cos \frac{x}{2} \right|$

(35) $\int \frac{dx}{(1 + sinx)^{\frac{1}{2}}} = ... + c$

(a) $\sqrt{2} \log \left| tan\left(\frac{3\pi}{8} - \frac{x}{4}\right) \right|$ (b) $\sqrt{2} \log \left| cosec\left(\frac{\pi}{8} + \frac{x}{2}\right) - cot\left(\frac{\pi}{8} + \frac{x}{2}\right) \right|$

(c) $\sqrt{2} \log \left| tan\left(\frac{\pi}{8} + \frac{x}{4}\right) \right|$ (d) $\sqrt{2} \log \left| sec\left(\frac{\pi}{2} + \frac{x}{4}\right) + tan\left(\frac{\pi}{2} + \frac{x}{4}\right) \right|$

(36) $\int \frac{dx}{5 - 4\, cosx} = ... + c$

(a) $\frac{1}{3}\, tan^{-1}\left(3 \tan \frac{x}{2}\right)$ (b) $\frac{1}{3}\, tan^{-1}\left(\frac{2}{3} \tan \frac{x}{2}\right)$

(c) $\frac{2}{3}\, tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right)$ (d) $\frac{2}{3}\, tan^{-1}\left(3 \tan \frac{x}{2}\right)$

(37) $\int \frac{sinx}{sin(x - a)}\, dx = ... + c$

(a) $x\, cosa + sin\, a \log |\, sin(x - a)\,|$ (b) $(x - a)\, cosa - sin\, a \log |\, sin(x - a)\,|$

(c) $sin\, a \log |\, sin(x - a)\,| + cosa\, x$ (d) $sina \cdot x + cosa \log |\, sin(x - a)\,|$

(38) $\int \frac{sin\, 2x}{pcos^2 x + q\, sin^2 x}\, dx = ... + c$

(a) $\frac{q}{p} \log |\, p\, sin2x + qcos\, 2x\,|$ (b) $(q - p) \log |\, p\, cos^2 x + q\, sin^2 x\,|$

(c) $\frac{1}{q - p} \log |\, pcos^2 x + qsin^2 x\,|$ (d) $\frac{1}{p^2 + q^2} \log |\, pcos^2 x + qsin^2 x\,|$

(39) $\int \frac{tanx}{4 + 9tan^2 x}\, dx = ... + c$

(a) $\frac{2}{3}\, tan^{-1}\left(\frac{2}{3} tanx\right)$ (b) $\frac{3}{2}\, tan^{-1}\left(\frac{1}{3} tanx\right)$

(c) $\frac{1}{10} \log |\, 4 + 9tan^2 x\,|$ (d) $\frac{1}{10} \log |\, 4cos^2 x + 9sin^2 x\,|$

(40) $\int \sqrt{\frac{a - x}{a + x}}\, dx = ... + c$

(a) $\frac{a}{2}\, sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$ (b) $\frac{1}{a}\, sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$

(c) $sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$ (d) $a\, sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$

(41) If $\int \frac{\cos^4 x}{\sin^2 x}\, dx = px + q\sin 2x + r\cot x + c$, then

(a) $p = -\frac{3}{2},\ q = -\frac{1}{4},\ r = -1$ (b) $p = -\frac{1}{4},\ q = -\frac{3}{2},\ r = -1$

(c) $p = 1,\ q = -\frac{1}{4},\ r = 1$ (d) $p = \frac{3}{2},\ q = -\frac{1}{4},\ r = 1$

(42) $\int \frac{e^x}{e^{2x} + e^x + 1}\, dx = \ldots\ldots + c$

(a) $\frac{1}{\sqrt{3}} \sec^{-1}\left(\frac{2e^x + 1}{\sqrt{3}}\right)$ (b) $\tan^{-1}(1 + e^x)$

(c) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2e^x + 1}{\sqrt{3}}\right)$ (d) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{e^x + 1}{\sqrt{3}}\right)$

(43) $\int \frac{1}{\sqrt{2 - 3x - x^2}}\, dx = \ldots\ldots + c$

(a) $\sin^{-1}\left(\frac{2 - 3x}{\sqrt{3}}\right)$ (b) $\sin^{-1}\left(\frac{2x - 1}{\sqrt{15}}\right)$ (c) $\sin^{-1}\left(\frac{2x + 3}{\sqrt{17}}\right)$ (d) $\sin^{-1}\left(\frac{3 + 2x}{3\sqrt{2}}\right)$

Section D

(44) $\int \frac{x^2 + 1}{x^4 - x^2 + 1}\, dx = \ldots\ldots + c$

(a) $x \tan^{-1}\left(\frac{x^2 + 1}{x}\right)$ (b) $\tan^{-1}\left(\frac{x^2 - 1}{x}\right)$

(c) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right)$ (d) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right)$

(45) $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \ldots\ldots + c$

(a) $\frac{\tan x}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x + 1}{\sqrt{2\tan x}}\right)$ (b) $\sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$

(c) $\sqrt{2} \tan^{-1}\left(\frac{\tan x + 1}{\sqrt{2\tan x}}\right)$ (b) $\frac{\tan x}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\tan x}}\right)$

(46) $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, \frac{dx}{x} = \ldots\ldots + c$

(a) $2 \log\left(\frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1}\right) - 2\sin^{-1}\sqrt{x}$ (b) $\log\left(\frac{\sqrt{1 - x} - 1}{\sqrt{1 + x} + 1}\right) + 2\sin^{-1}\left(\frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right)$

(c) $2\log\left(\frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}}\right) + \frac{1}{2}\cot^{-1}\sqrt{x + 1}$ (b) $\log\left(\frac{\sqrt{1 - x} - 1}{\sqrt{1 - x} + 1}\right) - 2\sin^{-1}\sqrt{x}$

(47) $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}} = \ldots\ldots + c$

(a) $\frac{x}{3\sqrt{9-x^2}}$ (b) $\frac{x}{9\sqrt{9+x^2}}$ (c) $\frac{x}{9\sqrt{9-x^2}}$ (b) $\frac{x}{(9-x^2)^{\frac{3}{2}}}$

(48) If $\int x^3 \sqrt{\frac{1+x^2}{1-x^2}}\, dx = p\cos^{-1}x^2 + q\sqrt{1-x^4} + rx^2\sqrt{1-x^4} + c$, then $p + q + r = \ldots\ldots$

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (b) -1

*

Summary

We have studied the following points in this chapter :

1. Definition of primitive or antiderivative or indefinite integral.
2. Working rules for integration.
3. Standard integrals :

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \in R - \{-1\},\ x \in R^+.$

(2) $\int \frac{1}{x} dx = \log|x| + c,\ x \in R - \{0\}$

(3) $\int \cos x\, dx = \sin x + c,\ \forall x \in R$

(4) $\int \sin x\, dx = -\cos x + c,\ \forall x \in R$

(5) $\int \sec^2 x\, dx = \tan x + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

(6) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c,\ x \neq k\pi,\ k \in Z$

(7) $\int \sec x \tan x\, dx = \sec x + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

(8) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c,\ x \neq k\pi,\ k \in Z$

(9) $\int a^x dx = \frac{a^x}{\log_e a} + c,\ a \in R^+ - \{1\},\ x \in R$

(10) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c,\quad a \in R - \{0\},\ x \in R$

(11) $\int \frac{dx}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c,\ a \in R - \{0\},\ x \neq \pm a$

(12) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right| + c,\ a \in R - \{0\},\ x \neq \pm a$

(13) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c,\ x \in (-a, a),\ a > 0.$

(14) $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\,sec^{-1}\frac{x}{a} + c, \quad |x| > |a| > 0.$

(15) $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \log|\,x + \sqrt{x^2 \pm a^2}\,| + c, \quad \forall x \in \mathrm{R}.$

4. Rule of substitution for integration.

5. If $\int f(x)dx = \mathrm{F}(x)$, then $\int f(ax+b)dx = \frac{1}{a}\mathrm{F}(ax+b)$ where $f : \mathrm{I} \to \mathrm{R}$ is continuous on some interval I. $(a \neq 0)$.

6. $\int f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$, $(n \neq -1, f(x) > 0)$ where f, f' are continuous and $f'(x) \neq 0$.

7. If f is continuous in $[a, b]$ and differentiable in (a, b) and f' is continuous and non-zero, $\forall x \in [a, b]$ and $f(x) \neq 0$, $\forall x \in [a, b]$, then $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c.$

(16) $\int tanx\ dx = \log|secx| + c,$

on any interval $\mathrm{I} = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in \mathrm{Z}$

(17) $\int cotx\ dx = \log|sinx| + c,$

on any interval $\mathrm{I} = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in \mathrm{Z}$

(18) $\int cosecx\ dx = \log|cosecx - cotx| + c,\ x \neq k\pi,\ k \in \mathrm{Z},$

on any interval $\mathrm{I} = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in \mathrm{Z}$

(19) $\int secx\ dx = \log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c,$

on any interval $\mathrm{I} = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in \mathrm{Z}$

Classical Period (400 – 1200)

This period is often known as the golden age of Indian Mathematics. This period saw mathematicians such as Aryabhata, Varahamihira, Brahmagupta, Bhaskara I, Mahavira, and Bhaskara II who gave broader and clearer shape to many branches of mathematics. Their contributions would spread to Asia, the Middle East, and eventually to Europe. Unlike Vedic mathematics, their works included both astronomical and mathematical contributions. In fact, mathematics of that period was included in the 'astral science' (jyotisha-shatra) and consisted of three sub-disciplines: mathematical sciences (ganita or tantra), horoscope astrology (hora or jataka) and divination (samhita). This tripartite division is seen in Varahamihira's 6th century compilation—Pancasiddhantika (literally panca, "five," siddhanta, "conclusion of deliberation", dated 575 CE)—of five earlier works, Surya Siddhanta, Romaka Siddhanta, Paulisa Siddhanta, Vasishtha Siddhanta and Paitamaha Siddhanta, which were adaptations of still earlier works of Mesopotamian, Greek, Egyptian, Roman and Indian astronomy. As explained earlier, the main texts were composed in Sanskrit verse, and were followed by prose commentaries.

PROBABILITY 7

The description of right lines and circles upon which geometry is founded belongs to mechanics. Geometry does not teach us to draw these lines but requires them to be drawn.

– Newton

7.1 Introduction

We have already started our study on probability. Recall that the set of all possible outcomes of a random experiment is the sample space and any subset of a sample space is an event. We know the axiomatic definition of probability and related theorems on it. We may also recall the classical definition of probability that if a finite sample space associated with a random experiment has n equally likely outcomes and of these $r\,(0 \le r \le n)$ outcomes are favourable for the occurrence of an event A, then probability of event A namely P(A), is given by $\frac{r}{n}$. Now we elaborate these ideas further.

7.2 Conditional Probability

As we have defined probability, it is meaningless to ask for the probability of an event without referring to a sample space. As an example, if we ask for the probability that an engineer earns at least ₹ 4,00,000 a year is meaningless. We must specify whether we are referring to all engineers in the India, all those in a particular industry, all those in academic field, all those working in a government department and so on. Thus, when we use the symbol P(A) for the probability of an event A, some sample space U must be associated with it. Now we introduce the symbol P(A | B), read as "the probability of A, given B".

The symbol P(A | B) is used to make it clear that the underlying sample space is B. Here, P(A | B) is called the conditional probability of A relative to B. Thus, every probability is a conditional probability. Of course we use the simplified notation P(A) whenever the underlying sample space is U. But whenever the sample space is reduced to some proper subset B, then we write the conditional probability of A, given B as P(A | B). Thus, a conditional probability is the probability of an event given that another event has occurred.

Let us illustrate some of the ideas connected with conditional probabilities. Let us consider the experiment of rolling a pair of balanced dice. We try to find the probability that the total of numbers appearing on the upper face of two dice is greater than 8, given that the number on the first die is 6. Let A be the event that total of the number on top face of two dice is greater than 8 and let B be the event that the first die has 6 on the top face. We wish to find P(A | B). This probability can be computed by considering only those outcomes for which the first die has a 6. Then, determine the favourable outcomes of these. All the possible outcomes for two dice are shown below :

$U = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

Die 1	Die 2	Total
1	1	2
1	2	3
1	3	4
1	4	5
1	5	6
1	6	7
2	1	3
2	2	4
2	3	5
2	4	6
2	5	7
2	6	8
3	1	4
3	2	5
3	3	6
3	4	7
3	5	8
3	6	9

Die 1	Die 2	Total
4	1	5
4	2	6
4	3	7
4	4	8
4	5	9
4	6	10
5	1	6
5	2	7
5	3	8
5	4	9
5	5	10
5	6	11
6	1	7
6	2	8
6	3	9
6	4	10
6	5	11
6	6	12

Fig. 7.1

There are 6 outcomes for which the first die shows 6, and out of these, there are four outcomes whose total on two dice is more than 8 (6, 3; 6, 4; 6, 5; 6, 6).

$\therefore \quad P(A \mid B) = \frac{4}{6} = \frac{2}{3}$ **(i)**

Let us look at this example in another way. Note that with respect to the sample space U,

we have $P(A \cap B) = \frac{4}{36}$ **($n = 36, r = 4$)**

and $P(B) = \frac{6}{36}$ **($n = 36, r = 6$)**

$\therefore \quad \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{36}}{\frac{6}{36}} = \frac{4}{6} = \frac{2}{3}$ **(ii)**

From (i) and (ii) we see that

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$.

Following these observations, let us now make the following definition :

Conditional Probability : If A and B are any events of S, where S = P(U) and P(B) ≠ 0, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Let us first prove that the set function $P(A \mid B)$ which is a function of A, is infact a probability function with respect to fixed event B.

Let us recall the axiomatic definition of probability.

Let U be a finite sample space and S be its power set. Suppose that set function $P : S \to R$ satisfies following axioms :

Axiom 1 : $P(A) \geq 0 \quad \forall A \in S$

Axiom 2 : $P(U) = 1$

Axiom 3 : Whenever $A_1, A_2 \in S$ and A_1, A_2 are mutually exclusive,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

Then P is called a probability function on S.

A result : For a fixed event B the set function $P(A|B)$ which is a function of A is a probability function where $P(B) > 0$.

(1) $P(A \cap B) \geq 0$ and $P(B) > 0$.

$$\therefore \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

Hence, for each $A \in S$ and for fixed event B of S, we have $P(A \mid B) \geq 0$. So, conditional probability satisfies the axiom 1 of the probability function.

(2) If A = U, then by the definition of $P(A \mid B)$,

We have $P(U \mid B) = \frac{P(U \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Thus, conditional probability satisfies the axiom 2 of the probability function.

(3) If A_1 and A_2 are mutually exclusive events, then by the definition of conditional probability,

We have $P((A_1 \cup A_2) \mid B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$ **(i)**

Now, $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$ **(Distributive law)**

Since A_1 and A_2 are mutually exclusive events, $A_1 \cap B$ and $A_2 \cap B$ are also mutually exclusive.

$\therefore \quad P[(A_1 \cup A_2) \cap B] = P(A_1 \cap B) + P(A_2 \cap B)$ **(Axiom 3) (ii)**

$$\therefore \quad P((A_1 \cup A_2) \mid B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \quad \textbf{(by (i) and (ii))}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B)$$

So, conditional probability satisfies axiom 3 of the probabilty function.

Thus, conditional probability satisfies all axioms of a probability function.

Properties of Conditional Probability :

(1) If A_1 and A_2 are any two events of a sample space and B is an event of U such that $P(B) \neq 0$, then $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P((A_1 \cap A_2) \mid B)$

We have $P((A_1 \cup A_2) \mid B) = \dfrac{P((A_1 \cup A_2) \cap B)}{P(B)}$

$$= \frac{P((A_1 \cap B) \cup P(A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) - P(A_1 \cap A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} - \frac{P(A_1 \cap A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B) - P((A_1 \cap A_2) \mid B)$$

(2) $P(A' \mid B) = 1 - P(A \mid B)$, where $P(B) \neq 0$

We have $P(U \mid B) = 1$

$\therefore \quad P((A \cup A') \mid B) = 1$ **($A \cup A' = U$)**

$\therefore \quad P(A \mid B) + P(A' \mid B) = 1$ **(A and A' are disjoint events)**

$\therefore \quad P(A' \mid B) = 1 - P(A \mid B)$

Example 1 : In a box of 100 memory cards of mobile phones, 10 cards have defects of type A, 5 cards have defects of type B and 2 cards have defects of both the types. Find the probabilities that

(1) a card to be drawn at random has a B type defect under the condition that it has an A type defect, and

(2) a card to be drawn at random has no B type defect under the condition that it has no A type defect.

Solution : Let us define the following events :

A : The memory card has A type defect.

B : The memory card has B type defect.

Then by given information

$$P(A) = \frac{10}{100} = 0.10,\ P(B) = \frac{5}{100} = 0.05,\ P(A \cap B) = \frac{2}{100} = 0.02$$

(1) The required probability is given by,

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.02}{0.10} = 0.2$$

(2) The required probability is given by

$$P(B' \mid A') = \frac{P(B' \cap A')}{P(A')} = \frac{P((A \cup B)')}{P(A')}$$

$$= \frac{1 - P(A \cup B)}{1 - P(A)}$$

$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - P(A)}$$

$$= \frac{1 - (0.10 + 0.05 - 0.02)}{1 - 0.10}$$

$$= \frac{1 - 0.13}{0.90} = \frac{0.87}{0.90} = \frac{87}{90} = \frac{29}{30}$$

Example 2 : The probability that a regularly scheduled flight departs on time is 0.83; the probability that it arrives on time is 0.82; and the probability that it departs and arrives on time is 0.78. Find the probability that a plane (1) arrives on time given that it departed on time, and (2) departed on time given that it has arrived on time.

Solution : Let D be the event that a plane departs on time and A be the event that a plane arrives on time. By the given information we have $P(D) = 0.83$, $P(A) = 0.82$, $P(D \cap A) = 0.78$

(1) The probability that a plane arrives on time, given that it departed on time is

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = \frac{78}{83}$$

(2) The probability that a plane departed on time given that it has arrived on time is

$$P(D \mid A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = \frac{78}{82} = \frac{39}{41}$$

Example 3 : For a biased die the probabilities for different integers to turn up on top face are given below :

Face	1	2	3	4	5	6
Probability	0.10	0.32	0.21	0.15	0.05	0.17

The die is tossed and 1 or 2 has turned upon top face. What is the probability that it is face 1 ?

Solution : Let A : The event that face 1 turns up

B : The event that face 2 turns up.

From the table, we see that $P(A) = 0.10$, $P(B) = 0.32$.

Now, $P(A \cup B) = P(A) + P(B)$ **(A and B are mutually exclusive events)**

$= 0.10 + 0.32 = 0.42$

We have to find $P(A \mid (A \cup B))$

$$P(A \mid (A \cup B)) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)} \quad \textbf{(Why ?)}$$

$$= \frac{0.10}{0.42} = \frac{10}{42} = \frac{5}{21}$$

Example 4 : A survey of 500 adults inquired about monthly expenses of their child. The survey asked questions about whether or not the person has a child studying in a college and about their monthly expenses. The probabilities are shown in the table below :

	Probability of monthly expenses		
	Expense too much	**Just right**	**Too low**
Child studying in college	0.30	0.13	0.01
Child not studying in college	0.20	0.25	0.11

Suppose a person is chosen at random. Given that the person has a child studying in a college, what is the probability that he or she ranks the expense as "too much" ?

Solution : Let B be the event that randomly chosen person's child studying in a college.

$\therefore \quad P(B) = 0.30 + 0.13 + 0.01 = 0.44$

Let A be the event that randomly chosen person's child's monthly expense is 'too much'.

From the table, we see that $P(A \cap B) = P(\text{expense too much} \cap \text{child in a college}) = 0.30$

Hence, required probability $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.44} = \frac{15}{22}$

Example 5 : A family has two children. What is the probability that both the children are girls given that at least one of them is a girl ?

Solution : Let b denote a boy and g denote a girl.

The sample space of the experiment is

$U = \{(b, b), (g, b), (b, g), (g, g)\}$.

Let A : The event that both the children are girls.

B : The event that at least one of the child is a girl.

Then $A = \{(g, g)\}$ and $B = \{(b, g), (g, b), (g, g)\}$

$\therefore \quad A \cap B = \{(g, g)\}$

Thus, $P(B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$

$\therefore$ The required probability is $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Exercise 7.1

1. If $P(A) = 0.35$, $P(B) = 0.45$ and $P(A \cup B) = 0.65$, then find $P(B \mid A)$.
2. If $P(A) = 0.40$, $P(B) = 0.35$ and $P(A \cup B) = 0.55$, then find $P(A \mid B)$.
3. If $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \mid B) = 0.4$, then find $P(A \cap B)$ and $P(B \mid A)$.
4. A balanced die is thrown twice and the sum of the numbers appearing on the top face is observed to be 7. What is the conditional probability that the number 2 has appeared at least once ?
5. A balanced die is rolled. If the outcome is an odd number, what is the probability that it is a prime ?
6. From the table of example 4, find (1) the probability that a person thinks monthly expense of his child is too low given that she is not in a college. (2) The probability that a person thinks monthly expense of his child is just right given that she is in a college.
7. 100 cards numbered 1 to 100 are placed in a box, shuffled thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is a perfect square, what is the probability that it is an odd perfect square ?
8. In a certain town, 40 % residents have computers, 25 % have internet connections and 15 % have both computer and internet connection. A resident is selected at random from the town. (1) If he has a computer, then what is the probability that he has internet connection also ? (2) If he has an internet connection, then determine the probability that he does not have a computer.
9. A balanced die is thrown three times. Let A be the event that 4 appears on the third toss and B be the event that 6 and 5 appears respectively on first two tosses. Find $P(A \mid B)$.

10. A fair coin is tossed three times. The events A, B, E, F, M, N are described as given (1) A : head on third toss, B : head on first toss. Find $P(A|B)$. (2) E : at least two heads, F : at most two heads. Find $P(E|F)$. (3) M : at the most two tails, N : at least one tail. Find $P(M|N)$.

*

7.3 Multiplication Theorem on Probability

We know that the conditional probability of event A given that event B has occured is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

From this result, we can write $P(A \cap B) = P(B) \cdot P(A|B)$ **(i)**

Also, we know that $P(B|A) = \frac{P(B \cap A)}{P(A)}, \; P(A) \neq 0$

$\therefore \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$ **$(A \cap B = B \cap A)$**

$\therefore \quad P(A \cap B) = P(A) \cdot P(B|A)$ **(ii)**

Combining (i) and (ii) we get,

$$P(A \cap B) = P(A) \cdot P(B|A) \quad \text{if } P(A) \neq 0$$
$$= P(B) \cdot P(A|B) \quad \text{if } P(B) \neq 0$$

The above result is known as the **Multiplication Rule of Probability**.

Multiplication rule of probability for more than two events : If A, B and C are three events of sample space, we have

$$P(A \cap B \cap C) = P((A \cap B) \cap C)$$
$$= P(A \cap B)\, P(C|(A \cap B)) \quad \textbf{(Multiplication rule of two events)}$$
$$= P(A)\, P(B|A)\, P(C|(A \cap B))$$

Theorem on total probability :

Theorem 7.1 : If B_1 and B_2 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, then for any event A of S,

$P(A) = P(B_1)\, P(A|B_1) + P(B_2)\, P(A|B_2)$

Proof : Since B_1 and B_2 are mutually exclusive and exhaustive events, we have

$B_1 \cup B_2 = U$ and $B_1 \cap B_2 = \emptyset$

$$\therefore \quad A = A \cap U$$
$$= A \cap (B_1 \cup B_2)$$
$$= (A \cap B_1) \cup (A \cap B_2) \quad \textbf{(Distributive law)} \quad \textbf{(i)}$$

Now, $(A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2)$

$$= A \cap \emptyset \quad \textbf{(}B_1 \cap B_2 = \emptyset\textbf{)}$$
$$= \emptyset$$

$\therefore$ $A \cap B_1$ and $A \cap B_2$ are mutually exclusive events

$\therefore$ By (i), $P(A) = P(A \cap B_1) + P(A \cap B_2)$

$P(A) = P(B_1)\, P(A|B_1) + P(B_2)\, P(A|B_2)$ **(Multiplication Rule of Probability)**

Similarly, if B_1, B_2, B_3 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, $P(B_3) \neq 0$, then for any event A of S,

$$P(A) = P(B_1)\, P(A|B_1) + P(B_2)\, P(A|B_2) + P(B_3)\, P(A|B_3)$$

Bayes' Theorem :

Bayes' theorem is a theorem of probability theory originally stated by the mathematician Reverend Thomas Bayes (1702 - 1761).

Theorem 7.2 : If B_1, B_2 and B_3 are mutually exclusive and exhaustive events and A is any event such that $P(A) \neq 0$, then

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}, \qquad i = 1, 2, 3$$

Proof : By the definition of conditional probability,

$$P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)} \qquad \text{(i)}$$

Now, using multiplication rule of probability and theorem on total probability we have

$$P(A \cap B_i) = P(A \mid B_i)\, P(B_i) \qquad \text{(ii)}$$

and $P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)$ (iii)

Hence by (i), (ii) and (iii) we get,

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}, \qquad i = 1, 2, 3$$

$$= \frac{P(A \mid B_i)\, P(B_i)}{\sum_{i=1}^{3} P(A \mid B_i)\, P(B_i)}, \qquad i = 1, 2, 3$$

Independent Events :

If the probability of occurrence or non-occurrence of event B does not affect the probability of occurrence of A i.e. if $P(A \mid B) = P(A)$ then A and B are said to be independent events.

As an example, the event of getting number 6 when a die is rolled first time and the event of getting number 6 when a die is rolled second time are independent events. By contrast, the event of getting number 6 when a die is rolled first time and the event that the sum of the numbers seen on the first and second trials is 8 are not independent.

Now, by the definition of conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

If A and B are independent events, then

$$P(A \mid B) = P(A)$$

$$\therefore \quad \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B)$$

Then $P(B) = \dfrac{P(A \cap B)}{P(A)}$ $\qquad (P(A) \neq 0)$

$$\therefore \quad P(B) = P(B \mid A)$$

Thus, **if events A and B are independent and $P(A) > 0$, $P(B) > 0$, then $P(A \cap B) = P(A) \cdot P(B)$ and $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.**

Also, if $P(A \cap B) = P(A) \cdot P(B)$, then we can say that A and B are independent events.

So, if A and B are independent events, $P(A \cap B) = P(A) \cdot P(B)$.

Theorem 7.3 : If A and B are independent events, then A and B', A' and B and A' and B' are also independent.

Proof : Events $A \cap B$ and $A \cap B'$ are mutually exclusive and $A = (A \cap B) \cup (A \cap B')$

$\therefore \quad P(A) = P(A \cap B) + P(A \cap B')$

$\therefore \quad P(A) = P(A) \cdot P(B) + P(A \cap B')$ **(A and B are independent)**

$\therefore \quad P(A \cap B') = P(A)\,(1 - P(B))$

$\therefore \quad P(A \cap B') = P(A)\,P(B')$

$\therefore$ A and B' are independent events. Similarly, we can prove that A' and B are independent events.

Now, $P(A' \cap B') = P[(A \cup B)']$ **(De Morgan's law)**

$= 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)\,P(B)$ **(A and B are independent)**

$= (1 - P(A)) - P(B)\,(1 - P(A))$

$= (1 - P(A))\,(1 - P(B))$

$P(A' \cap B') = P(A')\,P(B')$

$\therefore$ A' and B' are independent events.

Remark : (1) Three events A, B and C are said to be mutually independent, if

$P(A \cap B) = P(A)\,P(B)$

$P(B \cap C) = P(B)\,P(C)$

$P(A \cap C) = P(A)\,P(C)$

and $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

If at least one of the above is not true for three given events, we say that the events are not mutually independent.

(2) Three events A, B and C are said to be pairwise independent, if

$P(A \cap B) = P(A)\,P(B)$

$P(B \cap C) = P(B)\,P(C)$

and $P(A \cap C) = P(A)\,P(C)$

Example 6 : Three cards are drawn in succession, without replacement, from an ordinary deck of 52 playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a ten or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Solution : Here, events A_1 : the first card is a red ace, A_2 : the second card is a ten or a jack, A_3 : the third card is greater than 3 but less than 7.

Now, $P(A_1) = \frac{2}{52}$ **(Ace of heart and diamond)**

$P(A_2 \mid A_1) = \frac{8}{51}$ **(Without replacement, 4 cards of 10 and 4 jacks)**

$P(A_3 \mid (A_1 \cap A_2)) = \frac{12}{50}$ **(Why ?)**

$\therefore$ By multiplication rule of probability,

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid (A_1 \cap A_2))$$

$$= \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50}$$

$$= \frac{8}{5525}$$

Example 7 : A bag contains 8 red and 5 white balls. Two successive draws of 3 balls are made in such a way that **(1)** balls are replaced before the second trial, **(2)** the balls are not replaced before the second trial. Find the probability that the first draw will give 3 white balls and the second draw will give 3 red balls.

Solution : Let A denote the event of drawing 3 white balls in the first draw and B denote the event of drawing 3 red balls in the second draw. We have to find $P(A \cap B)$.

(1) Draw with replacement : If the balls drawn in the first draw are replaced back in the bag before the 2nd draw, then the events A and B are independent and the required probability is given by the expression :

$$P(A \cap B) = P(A) \cdot P(B)$$

1st draw : 3 balls can be drawn out of $8 + 5 = 13$ balls in $\binom{13}{3}$ ways.

$\therefore \quad n = \binom{13}{3}$

If all the 3 balls drawn are white, then $r = \binom{5}{3}$

$$\therefore \quad P(A) = \frac{r}{n} = \frac{\binom{5}{3}}{\binom{13}{3}}$$

2nd draw : When the balls drawn in the first draw are replaced before the 2nd draw, the bag again contain 13 balls. Now, if all the 3 drawn balls are red, then $r = \binom{8}{3}$

$$\therefore \quad P(B) = \frac{r}{n} = \frac{\binom{8}{3}}{\binom{13}{3}}$$

Hence, $P(A \cap B) = P(A) \cdot P(B)$

$$= \frac{\binom{5}{3}}{\binom{13}{3}} \cdot \frac{\binom{8}{3}}{\binom{13}{3}} = \frac{560}{(286)^2} = \frac{140}{20449}$$

(2) Draw without replacement : If the balls drawn are not replaced back before the second draw, then the events A and B are not independent and the required probability is given by :

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

As discussed in part (i) $P(A) = \dfrac{\binom{5}{3}}{\binom{13}{3}}$ **(i)**

If 3 white balls which were drawn in the first draw are not replaced back, then there are $13 - 3 = 10$ balls left in the bag. (8 red, 2 white)

Hence, $P(B \mid A) = \dfrac{\binom{8}{3}}{\binom{10}{3}}$ **(ii)**

Thus, from (i) and (ii)

$$P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{\binom{5}{3}}{\binom{13}{3}} \cdot \frac{\binom{8}{3}}{\binom{10}{3}} = \frac{7}{429}$$

Example 8 : A and B are two independent events such that $P(A \cup B) = 0.5$ and $P(A) = 0.2$, find P(B).

Solution : Since A and B are independent events, we have $P(A \cap B) = P(A) \cdot P(B)$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\,P(B)$

$= P(A) + P(B)\,(1 - P(A))$

$\therefore \quad 0.5 = 0.2 + P(B)\,(1 - 0.2)$

$\therefore \quad 0.3 = P(B) \times 0.8$

$\therefore \quad P(B) = \frac{3}{8}$

Example 9 : A machine manufactured by a firm consists of two parts A and B. Out of 100 A's manufactured, 9 are likely to be defective and out of 100 B's manufactured, 5 are likely to be defective. Find the probability that a machine manufactured by the firm is free of any defect.

Solution : Let event E : Part A of the machine is defective

and event F : Part B of the machine is defective.

By the given conditions,

$P(E) = \frac{9}{100}$, $P(F) = \frac{5}{100}$

Event E' : Part A is not defective and

Event F' : Part B is not defective.

$\therefore \quad P(E') = 1 - P(E) = 1 - \frac{9}{100} = \frac{91}{100}$

$P(F') = 1 - P(F) = 1 - \frac{5}{100} = \frac{95}{100}$

Since E and F are independent events, E' and F' are also independent.

Now, machine manufactured is free of any defect is the event $E' \cap F'$.

$\therefore \quad P(E' \cap F') = P(E') \cdot P(F')$

$= \frac{91}{100} \cdot \frac{95}{100} = \frac{8645}{10000} = 0.8645$

Example 10 : A purse contains 6 silver coins and 3 gold coins. Another purse contains 4 silver coins and 5 gold coins. A purse is selected at random and a coin is drawn from it. What is the probability that it is a silver coin ?

Solution : Let the event B_1 be the first purse is selected and the event B_2 be the second purse is selected.

$\therefore \quad P(B_1) = \frac{1}{2}$ and $P(B_2) = \frac{1}{2}$

Event A : Selected coin is a silver coin.

$\therefore \quad P(A \mid B_1) = \frac{6}{9} = \frac{2}{3}$ **(Total coins 9, silver coins 6)**

Similarly, $P(A \mid B_2) = \frac{4}{9}$

$\therefore$ Required probability

$$P(A) = P(B_1)\, P(A \mid B_1) + P(B_2)\, P(A \mid B_2)$$

$$= \frac{1}{2} \times \frac{6}{9} + \frac{1}{2} \times \frac{4}{9} = \frac{10}{18} = \frac{5}{9}$$

Example 11 : In a class of 75 students, 15 students have taken AB group. 45 have taken A group and the rest of them have taken B group. The probability that an AB group student fails in a KVPY (Kishor Vigyan Protsahak Yojana) examination is 0.005; an A group student failing has a probability 0.05 and the corresponding probability for a B group student is 0.15. If a student is known to have passed the KVPY examination, what is the probability that she is a student of B group ?

Solution : Let us define the following events :

B_1 : The student has taken AB group

B_2 : The student has taken A group

B_3 : The student is of B group

A : The student passes in the KVPY examination.

By the given information :

$P(B_1) = \frac{15}{75} = 0.2,\ P(B_2) = \frac{45}{75} = 0.6,\ P(B_3) = \frac{15}{75} = 0.2$

$P(A \mid B_1) = 1 - 0.005 = 0.995,\ P(A \mid B_2) = 1 - 0.05 = 0.950,\ P(A \mid B_3) = 1 - 0.15 = 0.850$

Now, $P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)$

$= (0.995)(0.2) + (0.95)(0.6) + (0.85)(0.2)$

$= 0.1990 + 0.570 + 0.170$

$= 0.939$ **(i)**

We have to find $P(B_3 \mid A)$.

By Bayes' theorem,

$$P(B_3 \mid A) = \frac{P(A \mid B_3)\, P(B_3)}{\sum_{i=1}^{3} P(A \mid B_i)\, P(B_i)}$$

$$= \frac{P(A \mid B_3)\, P(B_3)}{P(A)}$$

$$= \frac{(0.2)\,(0.850)}{0.939} \quad \textbf{(by (i))}$$

$$= \frac{0.170}{0.939} = \frac{170}{939}$$

Exercise 7.2

1. A card is drawn from a well shuffled pack of 52 cards. Events A and B are defined as follows :

   A : getting a card of spade

   B : getting an ace

   Determine whether the events A and B are independent or not.

2. If $P(B') = 0.65$, $P(A \cup B) = 0.85$ and A and B are independent events, then find P(A).

3. 10 boys and 5 girls study in a class. Three students are selected at random, one after the other. Find probability that,

   (1) First two are boys and the third is a girl,

   (2) First and third are boys and second is a girl,

   (3) First and third are of same sex and the second is of opposite sex.

4. Police plan to enforce speed limits by using radar system at 3 different locations within the city limits. The radar system at each of these locations are operated for 40 %, 30 % and 20 % of the time. If a person who is speeding on his way to work has probabilities of 0.2, 0.1 and 0.5 respectively of passing through these locations, what is the probability that he will be fined ?

5. Suppose coloured balls are distributed in three boxes are as follows :

Colour ↓	Box 1	Box 2	Box 3
Red	2	4	3
White	3	1	4
Blue	5	3	3
Total	**10**	**8**	**10**

   A box is selected at random from which a ball is selected at random. What is the probability that the ball selected of red colour ?

6. Three machines A, B and C produce respectively 50 %, 30 % and 20 % of the total number of items of a factory. The percentage of defective output of these machines are 3 %, 4 % and 5 % respectively. If an item is selected at random, find the probability that the item is non-defective.

7. In a certain college 25 % of boys and 10 % of girls are studying mathematics. The girls constitute 60 % of the student body.

   (1) What is the probability that mathematics is being studied ?

   (2) If a student is selected at random and is found to be studying mathematics, what is the probability that the student is a girl ?

8. There are two therapies B_1 and B_2 available for curing a patient suffering from a certain disease. The patient can choose any one of the two therapies. If he selects therapy B_1 the probability of his recovery from the disease is $\frac{7}{8}$ and if he selects therapy B_2 the the probability of his recovery from the disease is $\frac{9}{10}$ (i) what is the probability that the patient is cured from the disease ? (ii) Given that the patient is cured, what is the probability that he has selected therapy B_2 ?

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7.4 Random Variable and Probability Distribution

We have studied how we can determine probability of various events using probability function defined on the power-set S of a sample space associated with all possible outcomes of a random experiment. In many real situations we are not interested in studying the details of all outcomes of a random experiment. For instance, in a sample space with possible outcomes *bb*, *bg*, *gb*, *gg* of a random experiment of having two children in a family, we are interested in knowing the number of boys (or number of girls) rather than the outcomes themselves. Similarly, in case of a randomly selected electric bulb from a lot of electric bulbs produced in a factory, we are interested in determining the life in hours. Thus, we associate a real number, in one way or another, with an outcome of each of the random experiments described above. In other words, we define a real-valued function on a sample space associated with a random experiment and we shall call this real valued function a **'random variable'**. We shall study a random variable and its probability distribution in this section.

Let us understand the idea of a random variable by considering a simple example. Suppose we select a family having two children. *b* represents a boy. *g* represents a girl. The sample space associated with the random experiment is U = {*bb*, *bg*, *gb*, *gg*}.

If the outcomes of U are equally likely, we have by the classical definition of probability,

$$P(\{bb\}) = P(\{bg\}) = P(\{gb\}) = P(\{gg\}) = \frac{1}{4}$$

Suppose X : U $\longrightarrow$ R is a real valued function defined by, X(u) = number of boys in u.

If $u = bb$, then X(bb) = 2. If $u = gg$, then X(gg) = 0 and for $u = bg$ or gb, X(bg) = X(gb) = 1.

Hence, the range of function X : U $\longrightarrow$ R is the set {0, 1, 2}. We now take the subset {1} of the range of the function X. Pre-image set of {1} is $\{u \in U \mid X(u) = 1\} = \{bg, gb\}$.

Similarly, pre-image set of {2} is {*bb*} and pre-image set of {0} is {*gg*} and pre-image set of {0, 1, 2} is {*bb*, *bg*, *gb*, *gg*} = U.

Thus, corresponding to any value in the set {0, 1, 2} assumed by X there corresponds some event of sample space U.

As an example for X(u) = 0 for $u \in$ U there corresponds the event {*gg*}. Hence, the probability that X(u) = 0 is equal to the probability of the event {*gg*}. Therefore $P(X(u) = 0) = P(\{gg\}) = \frac{1}{4}$.

In the table below the values of probabilities associated with the elements of the range set of the function X are shown :

Element u of U	Probability of event {u}, P({u})	X(u) = x	P(X(u) = x)
bb	$P(\{bb\}) = \frac{1}{4}$	2	$\frac{1}{4}$
bg	$P(\{bg\}) = \frac{1}{4}$	1	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
gb	$P(\{gb\}) = \frac{1}{4}$		
gg	$P(\{gg\}) = \frac{1}{4}$	0	$\frac{1}{4}$

We shall call a real valued function on the sample space as a random variable, denoted by X and its value by x. The probability with which X assumes a value x shall be denoted by $p(x)$.

That is $p(x) = P(X = x) = P(X(u) = x)$

The various real values assumed by a random variable X and its corresponding probabilities, as shown in the table above, can be expressed as follows :

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Obviously, $\sum_{x=0}^{2} p(x) = p(0) + p(1) + p(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

This table gives the probability distribution of random variable X and $p(x)$ is called the probability function of random variable X.

Now, we shall define a random variable X and its probability distribution.

Random Variable : Let U be the sample space associated with a random experiment. A real valued function X defined on U i.e. X : U $\rightarrow$ R is called a random variable.

There are two kinds of random variables in the study of statistics, namely discrete random variable and continuous random variable. If the range of the real function X : U $\rightarrow$ R is a finite set or an infinite sequence of real numbers, then it is called a discrete random variable. If the range of X contains interval of R, then X is called a continuous random variable. We shall consider a discrete random variable with finite range and its probability distribution only. Thus, we shall assume the range of random variable X : U $\rightarrow$ R as $\{x_1, x_2, ..., x_n\}$.

Probability Distribution of Random Variable :

Let X : U $\rightarrow$ R be a random variable. Suppose X has range $\{x_1, x_2, ..., x_n\}$ which is a subset of R. Further suppose that X assumes a value x_i with probability $p(x_i) = P(X = x_i)$.

If (i) $p(x_i) \geq 0$, $i = 1, 2, ..., n$ and (ii) $\sum_{i=1}^{n} p(x_i) = 1$, then the set $\{p(x_1), p(x_2), ..., p(x_n)\}$ is called a probability distribution of the random variable X.

We can write probability distribution of the random variable X in tabular form as follows :

$X = x$	x_1	x_2	x_3	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_n)$

Example 12 : A random variable X : U $\rightarrow$ R, where U is a sample space associated with tossing of a fair coin three times, is defined as : For $u \in$ U , X(u) = number of heads in u. If the outcomes of U are equally likely, then obtain probability distribution of X.

Solution : The sample space associated with tossing of a fair coin three times is

U = {HHH, HHT, HTH, THH, THT, HTT, TTH, TTT}

If u = HHH, then according to the definition of the random variable, X(HHH) = 3.

If u = HHT or HTH or THH, then X(u) = 2

If u = THT or HTT or TTH, then X(u) = 1

If u = TTT, then X(u) = 0

Thus, the range of random variable X is the set {0, 1, 2, 3}. Since the elementary events of U are equally likely, we have

P({HHH}) = P({HHT}) = P({HTH}) = P({THH}) = P({THT}) = P({HTT}) = P({TTH}) = P({TTT}) = $\frac{1}{8}$

The probabilities associated with various values assumed by random variable X are given in the following table :

Element u of U	Probability P({u})	X(u) = x	P(X = x)
HHH	$\frac{1}{8}$	3	$\frac{1}{8}$
HHT	$\frac{1}{8}$		
HTH	$\frac{1}{8}$	2	$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$
THH	$\frac{1}{8}$		
TTH	$\frac{1}{8}$		
THT	$\frac{1}{8}$	1	$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$
HTT	$\frac{1}{8}$		
TTT	$\frac{1}{8}$	0	$\frac{1}{8}$

Thus, the probability distribution of the random variable X is as follows :

X = x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example 13 : Four raw mangoes are mixed accidently with 16 ripe mangoes. Find the probability distribution of the number of raw mangoes in a draw of two mangoes.

Solution : Let X denote the number of raw mangoes in a draw of 2 mangoes drawn from the group of 16 ripe mangoes and 4 raw mangoes. Since there are 4 raw mangoes in the group, X can take values 0, 1 and 2.

Now, P(X = 0) = Probability of getting 0 raw mango

$$= \frac{\binom{16}{2}}{\binom{20}{2}} = \frac{16 \times 15}{2} \times \frac{2}{20 \times 19} = \frac{12}{19}$$

P(X = 1) = Probability of getting one raw mango

$$= \frac{\binom{4}{1}\binom{16}{1}}{\binom{20}{2}} = \frac{4 \times 16 \times 2}{20 \times 19} = \frac{32}{95}$$

and P(X = 2) = Probability of getting two raw mangos

$$= \frac{\binom{4}{2}}{\binom{20}{2}}$$

$$= \frac{4 \times 3}{2} \times \frac{2}{20 \times 19}$$

$$= \frac{3}{95}$$

Thus, the probability distribution of X is given by

X = x	0	1	2
$p(x)$	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

Example 14 : Find the constant c for the probability distribution $p(x) = c\binom{5}{x}$, $x = 0, 1, 2, 3, 4, 5$

Solution : Here, $p(x) = c\binom{5}{x}$, $x = 0, 1, 2, 3, 4, 5$

Since, $p(x)$ represents probability distribution of X, we should have

$$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) = 1$$

$$\therefore \quad c\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right] = 1$$

$$\therefore \quad c(2^5) = 1$$

$$\therefore \quad 32c = 1$$

$$\therefore \quad c = \frac{1}{32}$$

Also, for each value of x, $p(x) > 0$.

$\therefore$ Required value of c is $\frac{1}{32}$

Example 15 : Probability distribution of a discrete random variable X is given in the following table :

X = x	–3	–2	–1	0	1	2	3
$p(x)$	0.08	0.14	0.19	0.27	0.17	0.09	0.06

(1) Find the probability of random variable X assuming negative values.

(2) Find the value of $P(0 \le x < 3)$.

Solution : (1) Probability that X assumes negative values is

$$p(-3) + p(-2) + p(-1) = 0.08 + 0.14 + 0.19 = 0.41$$

(2) $P(0 \le x < 3) = p(0) + p(1) + p(2)$

$$= 0.27 + 0.17 + 0.09$$

$$= 0.53$$

Exercise 7.3

1. Find the constant c for each of the following probability distribution :

(1) $p(x) = cx$, $x = 1, 2, 3, 4$

(2) $p(x) = cx^2$, $x = 1, 2, \ldots, 10$

(3) $p(x) = c \cdot 3^x$, $x = 0, 1, 2, 3$

(4) $p(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, 3$

(5) $p(x) = c\binom{4}{x}$, $x = 0, 1, 2, 3, 4$

2. Examine whether $p(x)$ defined for a random variable X as below is a probability distribution :

$p(x) = \frac{2x}{n(n+1)}$, $x = 1, 2, 3, \ldots, n$

3. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take values x, has the following form, where k is some unknown constant.

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx, & \text{if } x = 1 \text{ or } 2 \\ k(5 - x), & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases}$$

(1) Find the value of k.

What is the probability that you study.

(2) for at least two hours (3) for exactly two hours (4) for at most two hours ?

4. Two balanced dice are tossed once. A random variable X is defined on the sample space U associated with this random experiment as follows :

For $u \in$ U, X(u) = sum of integers in u.

Find the probability distribution of X.

5. A box contains 4 distinct balls of which 2 are white and 2 are black. Two balls are selected at random with replacement. If X denotes the number of black balls in the two balls selected from the box, then find the probability distribution of X.

6. From a lot of 10 bulbs, which includes 3 defectives bulbs, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs.

7. The probability distribution of a discrete random variable X is given in the following table :

X = x	0	1	2
$p(x)$	$3c^3$	$4c - 10c^2$	$5c - 1$

where $c > 0$. Find (1) c (2) $P(X < 2)$ (3) $P(1 < X \leq 2)$

8. We take 8 identical slips of paper, write the number 0 on one of them, the number 1 on three of the slips, the number 2 on three of the slips and the number 3 on one of the slips. The slips are folded. Put in a box and throughly mixed. One slip is drawn at random from the box. If X is the random variable denoting the number written on the drawn slip, find the probability distribution of X.

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7.5 Mathematical Expectation

Suppose that the following is the probability distribution of a random variable X :

X = x	x_1	x_2	x_3	...	x_{n-1}	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_{n-1})$	$p(x_n)$

where $p(x_i) \geq 0$ for each x_i and $\sum_{i=1}^{n} p(x_i) = 1$ (i)

Mean : X is a random variable with probability distribution given by (i). We denote mathematical expectation of X by E(X) and it is defined as :

$$E(X) = \sum_{i=1}^{n} x_i \, p(x_i) \qquad \text{(ii)}$$

Mathematical expectation of a random variable X is called the expected value of X or mean of X. E(X) is also denoted by the symbol μ. Mean of X is infact the weighted average of the possible values of X, each value being weighted by its probability with which it occurs.

Suppose Y = g(X) is a real function of a discrete random variable X. Then Y = g(X) will also be a discrete random variable and its mean is defined as

$$E(Y) = E[g(X)] = \sum_{i=1}^{n} g(x_i) \, p(x_i) \qquad \text{(iii)}$$

e.g. if $g(X) = X^2$, then

$$E[g(X)] = E(X^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i) \qquad \text{(iv)}$$

Variance of Random Variable :

The mean or expected value of a random variable X is of special importance in statistics because it describes where the probability distribution is centered. However, the only mean does not give adequate description of the shape of the distribution. We need to characterise the variability in the distribution. In figure 7.2 we have the histograms of two discrete probability distributions with the same mean $\mu = 2$ that differ considerably in the variability of their observations about the mean.

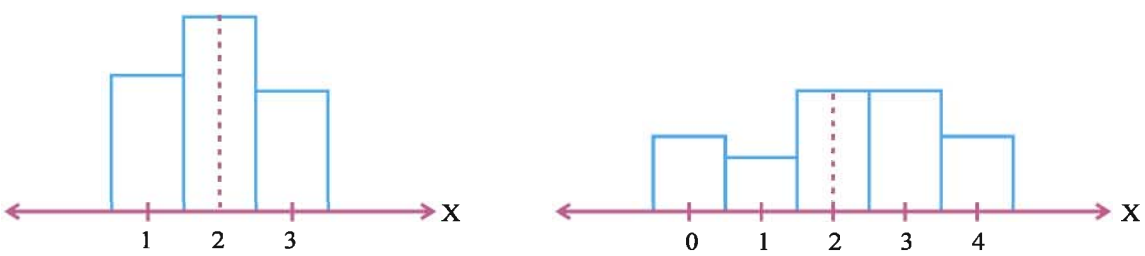

Fig. 7.2

The most important measure of variability of random variable X is referred to as the variance of the random variable X. We shall denote it by the symbol σ_X^2 or V(X). If the probability distribution of a random variable X is given by (i), then variance of X is defined by :

$$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

Using formula (ii) and (iv) the formula for σ_X^2 is written as

$$\sigma_X^2 = \sum_{i=1}^{n} x_i^2 \, p(x_i) - \left[\sum_{i=1}^{n} x_i \, p(x_i)\right]^2 \qquad \text{(v)}$$

Standard Deviation of Random Variables :

The positive square root of variance σ_X^2 of a random variable X is called the standard deviation of X. It is denoted by the symbol σ_X or $\sqrt{V(X)}$.

Some Results About Mathematical Expectation :

Suppose that the mathematical expectation and variance of a random variable X are E(X) and σ_X^2 respectively. For real constants a, b and c, let $Y = aX + b$ and $Z = aX^2 + bX + c$ be the new random variables. We shall assume the following results on expectation without proof.

$E(Y) = E(aX + b) = aE(X) + b$ **(vi)**

$\sigma_Y^2 = V(Y) = V(aX + b) = a^2\, V(X) = a^2 \sigma_X^2$ **(vii)**

$\sigma_Y = \sqrt{V(Y)} = |\, a \,|\, \sigma_X$ **(viii)**

$E(Z) = E(aX^2 + bX + c) = aE(X^2) + bE(X) + c$ **(ix)**

Example 16 : Probability distribution of a random variable X is as follows :

$X = x$	–2	–1	0	1	2	3
$p(x)$	0.05	0.14	0.23	0.31	0.16	0.11

Find E(X) and σ_X.

Solution : $E(X) = \Sigma x_i\, p(x_i)$

$= (-2)(0.05) + (-1)(0.14) + (0)(0.23) + (1)(0.31) + (2)(0.16) + (3)(0.11)$

$= -0.10 - 0.14 + 0 + 0.31 + 0.32 + 0.33$

$= 0.72$

$\therefore \quad E(X) = 0.72$

$\sigma_X^2 = \sum_i x_i^2\, p(x_i) - [E(X)]^2$

$= \{4(0.05) + 1(0.14) + 0(0.23) + 1(0.31) + 4(0.16) + 9(0.11)\} - (0.72)^2$

$= 2.28 - 0.5184 = 1.7616$

$\therefore \quad \sigma_X^2 = 1.7616$ and

$\sigma_X = \sqrt{1.7616} = 1.3272$

Example 17 : The mean and the standard deviation of a random variable X are given by E(X) = 5 and $\sigma_X = 3$ respectively. Find $E(X^2)$, $E((3X + 2)^2)$. Also find the standard deviation of $2 - 3X$.

Solution : Here, $E(X) = 5$ and $\sigma_X = 3$

We know that, $\sigma_X^2 = E(X^2) - [E(X)]^2$

$\therefore \quad E(X^2) = \sigma_X^2 + [E(X)]^2$

$= 9 + 25$

$E(X^2) = 34$

$E((3X + 2)^2) = E(9X^2 + 12X + 4)$

$= 9E(X^2) + 12E(X) + 4$

$= 9 \cdot 34 + 12 \cdot 5 + 4$

$= 306 + 60 + 4$

$E((3X + 2)^2) = 370$

Now, $V(2 - 3X) = 3^2V(X) = 9V(X) = 9\,\sigma_X^2 = 9 \cdot 9 = 81$

$\therefore$ The standard deviation of $2 - 3X$ is $\sqrt{81} = 9$.

If the expected gain of two players playing a game is zero, then the game is said to be fair. If the expected gain of any player is positive, the game is said to be in his favour. If the expected gain of a player is negative the game is said to be against him.

Example 18 : A player playing a game of tossing a balanced die receives ₹ 10 from his opponent if he throws an integer 3 or 4. If he throws 1 or 2 or 5 or 6, then how much should be pay to his opponent, so that the game becomes fair ?

Solution : Sample space associated in the game of tossing a die is U = {1, 2, 3, 4, 5, 6}. We define a random variable X on U as follows :

$$X(u) = \begin{cases} 10 & u = 3, 4 \\ a & u = 1, 2, 5, 6 \end{cases}$$

where a is the amount in rupees which the player has to pay to his opponent.

The probability distribution of X is as follows :

$X = x$	10	a
$p(x)$	$\frac{2}{6}$	$\frac{4}{6}$

Now, $E(X) = 10 \cdot \frac{2}{6} + a \cdot \frac{4}{6}$

$= \frac{4a + 20}{6}$

Since the game is to be fair, we must have E(X) = 0.

$\therefore \quad \frac{4a + 20}{6} = 0$

$\therefore \quad 4a + 20 = 0$

$\therefore \quad a = -5$

Hence, the player has to pay ₹ 5 to his opponent if u = 1, 2, 5 or 6.

Exercise 7.4

1. Determine the discrete probability distribution, mathematical expectation, variance, standard deviation of a discrete random variable X which denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once.
2. A player tosses 3 fair coins. He wins ₹ 500 if 3 heads occur, ₹ 300 if 2 heads occur, ₹ 100 if one head occurs. On the other hand, he loses ₹ 1500 if 3 tails occur. Find the expected value of the game for the player. Is it favourable to him ?
3. The probability distribution of a random variable X is as follows :

$X = x$	1	2	3	4	k
$p(x)$	0.1	k	0.2	$3k$	0.3

(1) Find the value of k.

(2) Find the mean and variance.

4. The probability distribution of a random variable X is as follows :

X = x	−1	0	1	2	3
$p(x)$	0.2	0.1	k	$2k$	0.1

(1) Find the value of k.

(2) Calculate the mean, variance and standard deviation.

5. Find the variance of the numbers obtained at the throw of an unbaised die.

6. Probability distribution of a random variable X is as follows :

X = x	−2	−1	0	1	2
$p(x)$	0.2	0.1	0.3	0.3	0.1

Find (1) E(X) (2) V(X) (3) E(3X + 2) (4) V(3X + 2)

7. A bakery owner finds from his past experience that sale of number of chocolate cakes produced in his bakery on any day is a random variable X having the following probability distribution :

No. of cakes sold X = n	0	1	2	3	4	5
$p(n)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

He gets a profit of ₹ 5 per each cake sold and incurs a loss of ₹ 2 per cake not being sold. If the bakery owner produces 3 cakes on a given day what is the value of his expected profit ?

8. The mean and standard deviation of a random variable X are 10 and 5 respectively. Find $E(X^2)$, $E[X(X + 1)]$, $E\left(\frac{X-10}{5}\right)$ and $E\left(\frac{X-10}{5}\right)^2$.

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7.6 Binomial Distribution

We have studied a random variable and its probability distributions in the earlier sections of this chapter. In this section we shall study a special distribution, a **binomial distribution**.

Binomial distribution is also known as the **'Bernolli distribution'** after the Swiss mathematician **James Bernoulli (1654-1705)** who discovered it in 1700.

Let us consider an experiment of tossing a coin. If we toss a coin, we get two outcomes namely, 'Head' or 'Tail'. For the sake of definiteness we shall call 'Head' **a success** and 'Tail' **a failure**. Hence sample space associated with the experiment is U = {S, F} where S denotes success and F denotes failure. Suppose that probability of getting a success is p and that of getting failure is q. That is P({S}) = p and P({F}) = q. Since there are two outcomes of the experiment we must have $p + q = 1$ and hence $q = 1 - p$.

Suppose a coin is tossed n times under identical conditions. Alternatively we can say that an experiment of tossing a coin is repeated n times under identical conditions. Since the experiment is performed under identical conditions, the probability of getting success 'S' at each of the n trials remains the same i.e., p. Trials of a random experiment possessing this property are called **Bernoulli trials**. We now define Bernoulli trials as follows :

Bernoulli trials : Suppose a random experiment has two possible outcomes namely success (S) and failure (F). If the probability p $(0 < p < 1)$ of getting success at each of the n trials of this experiment is constant, then the trials are called Bernoulli trials.

Bernoulli trials have following properties :

(1) **There is a constant probability of success (S) or failure (F) at each Bernoulli trial.**

(2) **Bernoulli trials are mutually independent**

(3) **If the constant probability of getting a success (S) at any Bernoulli trial is p $(0 < p < 1)$, then probability of getting a failure (F) is $q = 1 - p$.**

Suppose X denotes number of successes in a sequence of n Bernoulli trials of a random experiment having a constant probability p of success. Suppose that the probability distribution of random variable X is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, 2, \ldots n \qquad \text{(i)}$$

where $0 < p < 1$ and $q = 1 - p$

Probability distribution of random variable X given by (i) is called a Binomial distribution and random variable X is called a binomial random variable. The positive integer n and probability p of success 'S' are called the parameters of the binomial distribution.

The formula of $p(x)$ given by (i) for $x = 0, 1, 2, \ldots, n$ can be obtained from the binomial expansion of $(p + q)^n$. The general term of the binomial expansion of $(p + q)^n$ is $\binom{n}{x} p^x q^{n-x}$ which is equal to the formula (i). Hence, the probability distribution of random variable is called the binomial distribution. Also, sum of all probabilities is

$$\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p + q)^n = 1^n = 1$$

The binomial distribution occurs in games of chance (e.g. rolling a dice), quality inspection (e.g. count of number of defectives), opinion polls (e.g. number of empolyees favouring certain schedule changes), medicine (e.g. number of patients recovered by a new medication) and so on.

Result : The mean and variance of binomial distribution with parameters n and p are np and npq respectively.

Example 19 : It has been claimed that in 60 % of all solar-light installations, the utility bill is reduced by at least one third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one third in

(1) four of five installations;

(2) at least four of five installations.

Solution : Let X denote the number of solar-light installations in which the utility bill is reduced by at least one third out of 5 solar-light selected at random from a lot.

Here, X is a binomial random variable having binomial distribution with parameters $n = 5$ and $p = 0.60$. Hence, the probability distribution of X is given by

$$p(x) = \binom{5}{x}\left(\frac{6}{10}\right)^x\left(\frac{4}{10}\right)^{5-x},\ x = 0, 1, 2, 3, 4, 5$$

(1) The probability of the utility bill is reduced by at least one third in four installations by putting $x = 4$ in $p(x)$.

$$\therefore \quad p(4) = \binom{5}{4}\left(\frac{6}{10}\right)^4\left(\frac{4}{10}\right)^{5-4}$$

$$= 5(0.6)^4\ (0.4)$$

$$= 0.2592 \qquad \text{(i)}$$

(2) The probability that utility bill is reduced by at least one third in at four installations is $p(4) + p(5)$. Now,

$$p(5) = \binom{5}{5}\left(\frac{6}{10}\right)^5\left(\frac{4}{10}\right)^{5-5}$$

$$= (0.6)^5$$

$$= 0.07776$$

Hence, required probability $= p(4) + p(5)$

$$= 0.2592 + 0.07776 \qquad \text{(by (i))}$$

$$= 0.337$$

Example 20 : The mean and variance of a binomial distribution are 3 and 2 respectively. Find the probability that the variate takes values less than or equal to 2.

Solution : If n and p are the parameters of the binomial distribution, then we know that

Mean $= np = 3$ (i)

and Variance $= npq = 2$ (ii)

Dividing (ii) by (i) we get, $\frac{npq}{np} = \frac{2}{3}$

$\therefore \quad q = \frac{2}{3}$. So, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

Substituting in (i) we get, $n \cdot \frac{1}{3} = 3$. So, $n = 9$

$\therefore$ The probability distribution of binomial random variable X is given by

$$p(x) = \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x},\ x = 0, 1, 2,..., 9$$

The probability that the variable takes the value less than or equal to 2 is given by $P(X \leq 2)$.

$$P(x \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= p(0) + p(1) + p(2)$$

$$= \binom{9}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 + \binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7$$

$$= \left(\frac{2}{3}\right)^7\left[\left(\frac{2}{3}\right)^2 + 9 \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{9 \cdot 8}{2}\left(\frac{1}{3}\right)^2\right]$$

$$= \left(\frac{2}{3}\right)^7\left[\frac{4}{9} + 2 + 4\right] = \left(\frac{2}{3}\right)^7 \frac{58}{9} = \left(\frac{2^7}{3^9}\right)58$$

Exercise 7.5

1. An educationist claims that 80 percent of the students passing a higher secondary examination take admission to colleges for university education. What is the probability that out of 10 students (1) 5 students (2) 8 or more students take admission to a college ?
2. It has been found from an experiment that 40 percent of rats get stimulated on administering a particular drug. If 5 rats are given this drug, what is the probability that (1) exactly three and (2) all rats get stimulated ?
3. In a city of some western country, 70 percent of the married persons take divorce. What is the probability that at least three among four persons will take divorce ?
4. Harit participates in a shooting competition. The probability of his shooting a target is 0.2. What is the probability of shooting the target exactly three times out of five trials ?
5. The mean and standard deviation of a binomial random variable X are 8 and 2 respectively. Find the parameters of the probabilty distribution of X and obtain the value of $P(X = 0)$ and $P(1 \leq X \leq 3)$.
6. In a book of 500 pages, there are 50 printing errors. Find the probability of at most two printing errors in 4 pages selected at random from the book.
7. If 4 of 12 scooterists do not carry driving licence, what is the probability that a traffic inspector who randomly selects 4 scooterists will catch (1) 1 for not carrying driving licence. (2) at least 2 for not carrying driving licence.
8. In a shooting competition, the probability of a man hitting a target is $\frac{2}{5}$. If he fires 5 times, what is the probability of hitting the target (1) at least twice (2) at most twice.
9. A quality control engineer inspects a random sample of 3 calculators from a lot of 20 calculators. If such a lot contains 4 slightly defective calculators, what is the probability that the inspector's sample will contain (1) no slightly defective calculators, (2) one slightly defective calculators, (3) at least two slightly defective calculators.
10. If the probability of selecting a defective bolt is 0.1, find (1) the mean (2) the variance for the distribution of defective bolts in a total of 400.

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Miscellaneous Examples :

Example 21 : Suppose E and F be two independent events such that $P(E) < P(F)$. If $P(E \cap F) = \frac{1}{12}$ and $P(E' \cap F') = \frac{1}{2}$, then find P(E) and P(F).

Solution : We are given $P(E \cap F) = \frac{1}{12}$ and $P(E' \cap F') = \frac{1}{2}$.

As E and F are independent events, E' and F' are also independent events.

$P(E \cap F) = \frac{1}{12} \Rightarrow P(E)\,P(F) = \frac{1}{12}$ and

$P(E' \cap F') = \frac{1}{2} \Rightarrow P(E')\,P(F') = \frac{1}{2}$

$\therefore \quad [1 - P(E)]\,[1 - P(F)] = \frac{1}{2}$

$\therefore \quad 1 - P(E) - P(F) + P(E)\,P(F) = \frac{1}{2}$

$\therefore \quad 1 - P(E) - P(F) + \frac{1}{12} = \frac{1}{2}$

$\therefore \quad P(E) + P(F) = 1 + \frac{1}{12} - \frac{1}{2}$

$\therefore \quad P(E) + P(F) = \frac{7}{12}$

We know that the quadratic equation whose roots are a and b is $x^2 - (a + b)x + ab = 0$

$\therefore$ The equation whose roots are P(E) and P(F) is

$x^2 - [P(E) + P(F)]x + P(E)\,P(F) = 0$

$\therefore \quad x^2 - \frac{7}{12}x + \frac{1}{12} = 0$

$\therefore \quad 12x^2 - 7x + 1 = 0$

$\therefore \quad (3x - 1)(4x - 1) = 0$

$\therefore \quad x = \frac{1}{3}, \frac{1}{4}$

Since P(E) < P(F), we have $P(E) = \frac{1}{4}$ and $P(F) = \frac{1}{3}$.

Example 22 : Find the number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.95.

Solution : Let n be the required number of tosses, and X be the number of heads obtained in n tosses. Then X is a binomial random variable having binomial distribution with parameters n and $p = \frac{1}{2}$. Hence, the probability distribution of X is given by

$$p(x) = \binom{n}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x}, \; x = 0, 1, 2, \ldots, n$$

Now, P(at least one head) $= P(X \geq 1)$

$$= 1 - P(X = 0)$$

$$= 1 - p(0)$$

$$= 1 - \binom{n}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{n-0}$$

$$= 1 - \left(\frac{1}{2}\right)^n$$

Given P(at least one head) ≥ 0.95

$\therefore \quad 1 - \left(\frac{1}{2}\right)^n \geq 0.95$

$\therefore \quad \left(\frac{1}{2}\right)^n \leq 0.05$

$\therefore \quad \frac{1}{2^n} \leq \frac{1}{20}$

$\therefore \quad 2^n \geq 20$

$\therefore \quad n \geq 5$

$\therefore$ The least value of n is 5.

Hence, under the given conditions a fair coin must be tossed at least 5 times.

Example 23 : For a random experiment the sample space is U = {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)}. Events A, B, C are defined as follows :

A = {(0, 0, 0), (1, 0, 0)}, B = {(0, 0, 0), (0, 1, 0)}, C = {(0, 0, 0), (0, 0, 1)}

Prove A, B, C are pairwise independent but not mutually independent.

Solution : Here, $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$

$A \cap B = B \cap C = A \cap C = \{(0, 0, 0)\} = A \cap B \cap C$

$\therefore \quad P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4} = P(A \cap B \cap C)$

Now, $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)\ P(B)$

$P(B \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(B)\ P(C)$

$P(A \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)\ P(C)$

$\therefore$ A, B, C are pairwise independent events.

But $P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)\ P(B)\ P(C)$

$\therefore$ A, B, C are not mutually independent.

Note : If we select any vertex of tetrahedron OABC randomly, then sample space

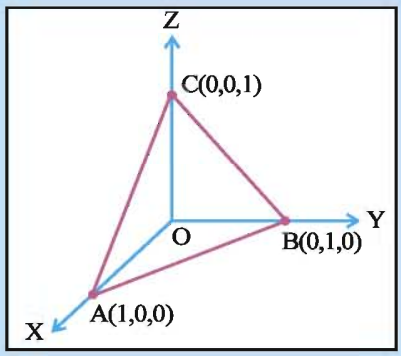
U = {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)}.

Event A : Vertex is on X-axis.

Event B : Vertex is on Y-axis.

Event C : Vertex is on Z-axis.

Events A, B, C are as in Example 23.



Exercise 7

1. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the card drawn is more than 3, what is the probability that it is an even number ?
2. A couple has 2 children. Find the probability that both are boys, if it is known that (1) one of the children is a boy; (2) the older child is a boy.
3. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both the balls drawn are black ?
4. An urn contains 4 red and 7 blue balls. Two balls are drawn at random with replacement. Find the probability of getting (1) both red balls (2) both blue balls (3) one red and one blue ball.

5. A can hit a target 4 times in 5 shots, B can hit it 3 times in 4 shots and C can hit it 2 times in 3 shots. Calculate the probability that,

(1) A, B, C all can hit the target.
(2) B, C can hit and A cannot hit.
(3) Any two of A, B and C will hit the target
(4) None of them will hit the target.

6. A general insurance company insuring vehicles for a period of one year classifies its policy holders into three mutually exclusive group.

Group T_1 : Policy holders with very high risk factor

Group T_2 : Policy holders with high risk factor

Group T_3 : Policy holders with less rick factor

From the past experience of the company, 30 % of its policy holders belong to group T_1, 50 % belong to group T_2 and the rest belong to group T_3. If the probabilities that policy holders belonging to groups T_1, T_2 and T_3 meet with an accident are 0.30, 0.15 and 0.05 respectively, find the proportion of policy holders having a policy for one year will meet with an accident. If a randomly selected policy holder does not meet with an accident, what is the probability that he belongs to group T_2?

7. Rajesh agrees to play a game of tossing a balanced die. If an integer 1 or 2 is obtained on the die, he loses ₹ 2. If an integer 3 or 4 or 5 is obtained, he gets ₹ 5 and if integer 6 is obtained, he gets ₹ 10. If the amount of ₹ X received by Rajesh is treated as a random variable, then obtain probability distribution of X.

8. A random variable X assumes integral values from integers 1 to 100 with equal probability. Find E(X), $E(X^2)$ and σ_X^2.

9. Nine balanced coins are tossed together once. Find probability of getting (1) four heads and (2) at least six heads.

10. The probability function of a binomial distribution is

$$p(x) = \binom{6}{x} p^x q^{6-x}, \; x = 0, 1, 2, \ldots, 6.$$

If $3p(2) = 2p(3)$, then find the value of p.

11. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently.)

12. A restaurant serves two special dishes – A and B to its customers consisting of 60 % men and 40 % women. 80 % of men order dish A and the rest order B. 70 % of woman order B and the rest order A. In what ratio of dishes A to B should the restaurant prepare the two dishes ?

13. In a railway reservation office, two clerks are engaged in checking reservation forms. On an average, the first clerk checks 55 % of the forms, while the second checks the remaining. The first clerk has an error rate of 0.03 and second has an error rate of 0.02. A reservation form is selected at random from the total number of forms checked during a day, and is found to have an error. Find the probability that it was checked by the second clerk.

14. A fair coin tossed two times. Events A, B, C defined as follows :

Event A : First toss shows head

Event B : Second toss shows head

Event C : Same result on both toss

Show that events A, B, C are pairwise independent but not mutually independent.

15. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A : (1 mark)

(1) Two cards are drawn in succession from a standard well shuffled pack of 52 cards. What is the probability that both the cards are aces if the cards are drawn without replacement ? ☐

(a) 0.0045 (b) 0.0385 (c) 0.045 (d) 0.0059

(2) A circular wheel with numbers 1 to 20 on its surface is rolled twice. What is the probability of getting two 13's ? ☐

(a) $\frac{1}{20}$ (b) $\frac{1}{40}$ (c) $\frac{1}{400}$ (d) $\frac{1}{200}$

(3) Let A and B be two events such that $P(A) = 0.4$, $P(A \cup B) = 0.7$ and $P(B) = p$. For what choice of p are A and B independent ? ☐

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$

(4) Two unbiased coins are tossed. If one coin shows head, the probability that the other also shows head is... ☐

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) 1

(5) A problem in mathematics is given to three students A, B, C and their respective probability of solving the problem is $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Probability that the problem is solved is... ☐

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

(6) A die is tossed 5 times. Getting an odd number is considered a success. Then the variance of distribution of success is... ☐

(a) $\frac{8}{3}$ (b) $\frac{3}{8}$ (c) $\frac{4}{5}$ (d) $\frac{5}{4}$

(7) The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on an event is... ☐

(a) $\frac{7}{20}$ (b) $\frac{1}{5}$ (c) $\frac{3}{20}$ (d) $\frac{4}{5}$

(8) If A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P(A \mid B')$ is... ☐

(a) $1 - P(A \mid B')$ (b) $1 - P(A \mid B)$ (c) $\frac{P(A')}{P(B)}$ (d) $1 - P(A' \mid B')$

(9) The probability that a student is not a swimmer is $\frac{4}{5}$. The probability that out of 5 students exactly 4 are swimmers is...

(a) $\left(\frac{1}{5}\right)^3$ (b) $4\left(\frac{1}{5}\right)^4$ (c) ${}_5C_4\left(\frac{4}{5}\right)^4$ (d) $\left(\frac{4}{5}\right)^4$

(10) Let X be a random variable with probability distribution

$X = x$	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Then E(2X + 3) is...

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) 6

Section B : (2 marks)

(11) A study has been done to determine whether or not a certain drug leads to an improvement in symptoms for patients with a particular medical condition. The results are shown in the following table :

	Improvement	No improvement	Total
Drug	270	530	800
No drug	120	280	400
Total	390	810	1200

Based on this table, what is the probability that a patient shows improvement if it is known that the patient was given the drug ?

(a) 0.4375 (b) 0.225 (c) 0.3375 (d) 0.3205

(12) Suppose it is known that the patient shows improvement. Based on the table of example 11, what is the probability that the patient was given the drug ?

(a) 0.225 (b) 0.667 (c) 0.792 (d) 0.692

(13) A box contains four red, two white and three green marbles, all of which are the same size. Two marbles are selected one after the other from the box, without replacement. What is the probability that the marbles are of the same colour ?

(a) 0.67 (b) 0.5 (c) 0.14 (d) 0.28

(14) A company has three plants at which it produces a certain item. 30 % are produced at plant A, 50 % at plant B and remaining at plant C. Suppose that 1 %, 4 % and 3 % of the items produced at plants A, B and C respectively are defective. If an item is selected at random from all of those produced, what is the probability that the item was produced at plant B and is defective ?

(a) 0.5 (b) 0.2 (c) 0.02 (d) 0.04

(15) The mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, then P(X = 1) is...

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{32}$

(16) It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P(A \mid B) = \frac{1}{2}$, $P(B \mid A) = \frac{2}{3}$. Then P(B) is...

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

(17) If two events A and B are such that $P(A') = 0.3$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$, then $P(B \mid A \cup B')$ is...

(a) 0.375 (b) 0.32 (c) 0.31 (d) 0.28

(18) If parameters of a binomial distribution are $n = 5$ and $p = 0.30$, then the mean is ..... and variance is ..... .

(a) 1.5, 1.5 (b) 1.5, 1.05 (c) 1.5, 1.40 (d) 1.5, 1.15

Section C : (3 marks)

(19) A company has three plants at which it produces a certain item. 30 % are produced at plant A, 50 % at plant B and 20 % at plant C. Suppose that 1 %, 4 % and 3 % of the items produced at plants A, B and C respectively are defective. If an item is selected at random from all those produced, what is the probability that the item is defective ?

(a) 0.029 (b) 0.29 (c) 0.025 (d) 0.08

(20) The probability that an event A occurs in a single trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that A occurs at least once is...

(a) 0.936 (b) 0.784 (c) 0.904 (d) 0.874

(21) The variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

$X = x$	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

is...

(a) 6 (b) 36 (c) 4 (d) 8

Section D : (4 marks)

(22) A random variable X has the probability distribution :

X	1	2	3	4	5	6	7	8
$p(x)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, the probability $P(E \cup F)$ is...

(a) 0.35 (b) 0.77 (c) 0.87 (d) 0.50

(23) If a random variable X can take all non-negative integral values and the probability that X takes the value r is proportional to α^r $(0 < \alpha < 1)$, then $P(X = 0)$ is...

(a) $1 - \alpha$ (b) α (c) $\frac{\alpha}{2}$ (d) α^2

(24) The mean and standard deviation of a random veriable X are 10 and 5 respectively. Match the following :

	A		B
(i)	$E(X^2)$	(p)	0
(ii)	$E(X(X+1))$	(q)	135
(iii)	$E\left(\left(\frac{X-10}{5}\right)\right)$	(r)	125
(iv)	$E\left(\left(\frac{X-10}{5}\right)^2\right)$	(s)	1

(a) (i) : (q), (ii) : (r), (iii) : (p), (iv) : (s) (b) (i) : (r), (ii) : (q), (iii) : (s), (iv) : (p)

(c) (i) : (r), (ii) : (q), (iii) : (p), (iv) : (s) (d) (i) : (p), (ii) : (q), (iii) : (r), (iv) : (s)

*

Summary

We studied the following points in this chapter :

1. The conditional probability of an event A, given the occurrence of the event B is given by

   $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$.

2. $0 \leq P(A \mid B) \leq 1$, $P(A' \mid B) = 1 - (A \mid B)$

   $P((A \cup B) \mid C) = P(A \mid C) + P(B \mid C) - P((A \cap B) \mid C)$

3. $P(A \cap B) = P(A) \cdot P(B \mid A)$, $P(A) \neq 0$

   $P(A \cap B) = P(B) \cdot P(A \mid B)$, $P(B) \neq 0$

4. If B_1 and B_2 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, then for any event A of S,

   $P(A) = P(B_1)\, P(A \mid B_1) + P(B_2)\, P(A \mid B_2)$

5. If B_1 and B_2 are mutually exclusive and exhaustive events and A is any event such that $P(A) \neq 0$, then $P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}$, $i = 1, 2, 3$

6. If A and B are independent events then $P(A \cap B) = P(A)\, P(B)$

7. If A and B are independent events then A and B', A' and B and A' and B' are also independent.

8. A random variable is a real valued function whose domain is the sample space of a random experiment.

9. The probability distribution of a random variable X in tabular form is

$X = x$	x_1	x_2	x_3	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_n)$

10. **Mean :** $E(X) = \sum_{i=1}^{n} x_i \, p(x_i)$

Variance : $V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$

$$\sigma_X^2 = \sum_{i=1}^{n} x_i^2 \, p(x_i) - \left[\sum_{i=1}^{n} x_i \, p(x_i)\right]^2$$

Standard deviation : $\sigma_X = \sqrt{V(X)}$

11. $E(aX + b) = aE(X) + b$

12. $V(aX + b) = a^2 \, V(X)$

13. **Bernoulli Trials :**

(1) There is a constant probability of success (S) or failure (F) at each Bernoulli trial.

(2) Bernoulli trials are mutually independent

(3) If the constant probability of getting a success (S) at any Bernoulli trial is p $(0 < p < 1)$, then probability of getting a failure (F) is $q = 1 - p$.

14. **Binomial Distribution :** Suppose X denotes number of successes in a sequence of n Bernoulli trials of a random experiment having a constant probability p of success. The probability distribution of random variable X is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \ldots n$$

where $0 < p < 1$ and $q = 1 - p$ is a binomial distribution with parameters n and p.

15. The mean μ and variance σ_X^2 of binomial distribution with parameters n and p are np and npq respectively.

Ramanujan's notebooks

While still in Madras, Ramanujan recorded the bulk of his results in four notebooks of loose leaf paper. These results were mostly written up without any derivations. This is probably the origin of the misperception that Ramanujan was unable to prove his results and simply thought up the final result directly. Mathematician Bruce C. Berndt, in his review of these notebooks and Ramanujan's work, says that Ramanujan most certainly was able to make the proofs of most of his results, but chose not to.

This style of working may have been for several reasons. Since paper was very expensive, Ramanujan would do most of his work and perhaps his proofs on slate, and then transfer just the results to paper. Using a slate was common for mathematics students in the Madras Presidency at the time. He was also quite likely to have been influenced by the style of G. S. Carr's book studied in his teenage, which stated results without proofs. Finally, it is possible that Ramanujan considered his workings to be for his personal interest alone; and therefore only recorded the results.

The first notebook has 351 pages with 16 somewhat organized chapters and some unorganized material. The second notebook has 256 pages in 21 chapters and 100 unorganised pages, with the third notebook containing 33 unorganised pages. The results in his notebooks inspired numerous papers by later mathematicians trying to prove what he had found. Hardy himself created papers exploring material from Ramanujan's work as did G. N. Watson, B. M. Wilson, and Bruce Bert. A fourth notebook with 87 unorganised pages, the so-called "lost notebook", was rediscovered in 1976 by George Andrews.

LINEAR PROGRAMMING 8

Nature is an infinite sphere of which the centre is everywhere and the circumference is nowhere.

– Blaise Pascal

In order to translate a sentence from English to French, two things are necessary. First we must understand thoroughly the English sentence. Second we must be familiar with the forms of expression peculiar to French language. The situation is very similar when we attempt to express in mathematical symbols a condition proposed in words. First we must understand thoroughly the condition. Second we must be familiar with the forms of mathemaical expression.

– George Polya

8.1 Introduction

Before discussing the basic concepts and applications of linear programming, let us understand the meaning of the words, **'linear'** and **'programming'**. The word linear refers to linear relationship among variables in a model. Thus, a given change in one variable will always result into a proportional change in another variable. For example, doubling the investment on a certain investment will exactly double the return. The word programming refers to the modelling (plan of action) and solving a problem mathematically. Linear Programming was first developed by Leonid Kantorvich, a Russian mathematician, in 1939. During world war II, George B Dentzing while working with the US Air Force, developed linear programming model, primarily for solving military logistics problems.

In earlier classes, we have discussed system of linear equations and their applications in some practical problems. In class XI we have studied linear inequalities and system of linear inequalities in two variables and their solutions by graphical method. In this chapter, we shall apply the system of linear inequalities to some real life problems. The type of problems which seek to maximize (or minimize) profit (or loss) form a general class of problems called **Optimisation problems**. Any optimisation problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimisation problems is **Linear Programming Problems**.

Linear programming problems are of much interest because they are being used extensively in all functional areas of management, airlines, agriculture, military operations, oil refining, education, energy planning, pollution control, transportation planning and scheduling, research and development, health care system etc.

In this chapter, we shall study some linear programming problems and their solutions by graphical method only. There are many other methods also to solve such problems.

8.2 A Linear Programming Problem and its Mathematical Formulation

We begin our discussion with the help of an example which will lead us to a mathematical formulation of the problem in two variables.

A dealer deals in only two items : AC (Air conditioners) and Coolers. He has capital finance ₹ 5,00,000 to invest and has storage space of at most 60 pieces. An AC costs ₹ 25,000 and a cooler costs ₹ 5000. He estimates that from the sale of one AC, he can make profit of ₹ 2500 and from the sale of one cooler

he can make profit of ₹ 750. The dealer wants to know how many AC and coolers he should buy from the available capital so as to maximise his total profit, assuming that he can sell all the items which he buys.

In this example, we observe that,

(1) The dealer can invest his money in buying AC or coolers or a combination thereof. Further he would earn different profits by following different investment strategies.

(2) There are certain **constraints** namely, his investment is limited to a maximum of ₹ 5,00,000 and storage capacity for a maximum of 60 pieces.

Suppose he decides to buy AC only and no collers, so he can buy 5,00,000 ÷ 25,000 = 20 AC. His profit in this case will be ₹ (2500 × 20) = ₹ 50,000.

Suppose he decides to buy coolers only and no AC. With his capital of ₹ 5,00,000 he can buy 100 coolers. But he can store only 60 pieces. Therefore, he has to buy only 60 coolers which will give him a total profit of ₹ (60 × 750) = ₹ 45,000.

There are many other possibilities, for instance, he may buy 10 AC and 50 coolers, as he can store only 60 pieces. Total profit in this case would be ₹ (10 × 2500 + 50 × 750) = ₹ 62,500 and so on. This, dealer can earn different profits by following different investment strategies. So, the problem is : How should the dealer invest his money in order to get maximum profit ? To answer this question, let us try to formulate the problem mathematically.

Mathematical formulation of the problem :

Let x be the number of AC and y be the number of coolers that the dealer buys.

Obviously, $x \geq 0, y \geq 0$ **(non-negative constraints) (i)**

Here, the cost of one AC is ₹ 25,000 and cost of one cooler is ₹ 5000. The dealer can invest at the most ₹ 5,00,000. Mathematically,

$$25{,}000\, x + 5000\, y \leq 5{,}00{,}000$$

$\therefore \quad 5x + y \leq 100$ **(investment constraint) (ii)**

The dealer can store maximum 60 items.

$\therefore \quad x + y \leq 60$ **(storage constraint) (iii)**

The dealer wishes to invest in such a way that he can earn maximum profit, say z.

It is given that the profit earn on selling of an AC is ₹ 2500 and that on a cooler is ₹ 750. So the profit function z is given by

$z = 2500x + 750\, y$ **(called objective function) (iv)**

Mathematically, the given problem now reduces to :

Maximise $z = 2500\, x + 750\, y$

Subject to the constraints :

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0, y \geq 0$$

So, we have to maximise a linear function z subject to certain conditions determined by a set of linear inequalities. The variables are non-negative. There are also some other problems where we have to minimise a linear function (as an example, expenditure) subject to certain conditions determined by a set of linear inequalities with non-negative variables. Such problems are called **Linear Programming Problems.**

Before we proceed further, we now formally define some terms (which have been used above) which we shall be using in the linear programming problems :

The general structure of linear programming model consists of three basic components :

(1) Decision Variables : We need to evaluate various alternatives for arriving at the optimal value of the objective function. The variables in a linear program are a set of quantities that need to be determined in order to solve a problem. i.e., problem is solved when the best values of the variables have been identified. These variables are called decision variables. They are usually denoted by x, y (if there are two variables) or x_1, x_2,..., x_n if there are more variables.

In the example discussed above x, y are decision variables.

(2) The objective function : The objective function of each linear programming problem is expressed in terms of decision variables to optimize the criterion of optimality such as profit, cost, etc. It is expressed as :

Optimize (maximize or minimize)

$z = c_1x + c_2y$ or

$z = c_1x_1 + c_2x_2 + ... + c_nx_n$. In this chapter, we shall find the optimal value of the given objective function by the graphical method.

(3) The constraints : There are always certain limitations on the use of resources, e.g. labour, raw material, space, money, time etc. such limitations are being expressed as linear equalities or inequalities in terms of decision variables. The solution of a linear programming model must satisfy these constraints.

Now on we will denote a linear programming problem as an LP problem.

Thus, the general mathematical model of LP problem is as follows :

Find the values of decision variables x, y so as to optimize (maximize or minimize).

$$z = c_1x + c_2y$$

subject to the linear constraints,

$$a_{11}x + a_{12}y \ (\leq, =, \geq)\ b_1$$
$$a_{21}x + a_{22}y \ (\leq, =, \geq)\ b_2$$
$$a_{31}x + a_{32}y \ (\leq, =, \geq)\ b_3$$
$$x \geq 0,\ y \geq 0$$

In general, we can write as the following :

Find the values of decision variables x_1, x_2,..., x_n, so as to optimize (maximise or minimise)

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

Subject to the linear constrains,

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \quad (\leq, =, \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \quad (\leq, =, \geq)\ b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \quad (\leq, =, \geq)\ b_m$$

and $x_1 \geq 0,\ x_2 \geq 0,\ ....,\ x_n \geq 0$

Here, **a_{ij}'s are coefficients representing the per unit contribution of decision variable x_j, to the value of objective function. a_{ij}'s are called the input-output coefficients and represent the amount of resource. a_{ij}'s can be positive, negative or zero. The b_i's represent the total availability of the ith resource.**

Let us take an example of LP model formulation.

Example 1 : A furniture firm manufactures chairs and tables. Each requires the use of three machines A, B or C. Production of one chair requires 2 hours on machine A, 1 hour on machine B and 1 hour on machine C. Each table requires 1 hour each on machines A and B and 3 hours on machine C. The profit realized by selling one chair is ₹ 300 while that from sale of a table is ₹ 600. The total time available per week on machine A is 70 hours, the time available on machine B is 40 hours and that on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit ? Develop a mathematical formulation.

Solution : Let us represent the given data in a tabular form as following :

Machine	Chair number of hours	Table number of hours	Available time per week (in hours)
A	2	1	70
B	1	1	40
C	1	3	90
Profit per unit	₹ 300	₹ 600	

Let the number of chairs and tables manufactured respectively x and y.

Let z denote the total profit. Then $z = 300x + 600y$ **(i)**

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, the total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours.

This must be less than or equal to total hours available on machine A.

$\therefore \quad 2x + y \leq 70$ **(ii)**

It is given that a chair requires 1 hour on machine B and a table requires 1 hour on machine B. Therefore, total time taken by machine B to produce x chairs and y tables is $(x + y)$ hours. Total time available per week on machine B is 40 hours.

$\therefore \quad x + y \leq 40$ **(iii)**

Similarly, from the consideration of machine C we have the inequality

$x + 3y \leq 90$ **(Why ?) (iv)**

Since the number of chairs and tables cannot be negative.

$\therefore \quad x \geq 0$ and $y \geq 0$ **(v)**

Hence, the mathematical form of the given LPP is as follows :

Maximize $z = 300x + 600y$

Subject to $2x + y \leq 70$

$x + y \leq 40$

$x + 3y \leq 90$

and $x \geq 0, y \geq 0.$

We will now discuss how to find solutions to a linear programming problem. In this chapter we shall study only graphical method.

8.3 Graphical Method of Solving Linear Programming Problems

In this section first we shall discuss some definitions related to the solution of a linear programming problems.

Definition : The set of values of decision variables x_i (i = 1, 2,..., n) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

As an example,

Consider the LP problem.

Maximize $z = 300x + 600y$

subject to $2x + y \leq 70$

$x + y \leq 40$

$x + 3y \leq 90$

and $x \geq 0, y \geq 0$

Here, $x = 1$, $y = 3$; $x = 7$, $y = 6$; $x = 10$, $y = 18$ etc. are solutions of this LP problem as they satisfy the constraints $2x + y \leq 70$, $x + y \leq 40$ and $x + 3y \leq 90$ and $x \geq 0$, $y \geq 0$. Note that $x = 10$, $y = 30$ is not a solution because it does not safisfy $x + 3y \leq 90$.

Feasible Solution : A set of values of the decision variables $x_1, x_2,..., x_n$ is called a feasible solution of an LP problem, if it satisfies both the constraints and non-negativity conditions.

Infeasible Solution : An infeasible solution is a solution for which at least one constraint is violated.

Optimal feasible Solution : A feasible solution of an LP problem is said to be an optimal feasible solution, if it optimizes (maximizes or minimizes) the objective function.

Feasible region (solution region) : When we graph all the constraints, the feasible region is the set of all points which satisfy all the constraints including non-negativity constraints.

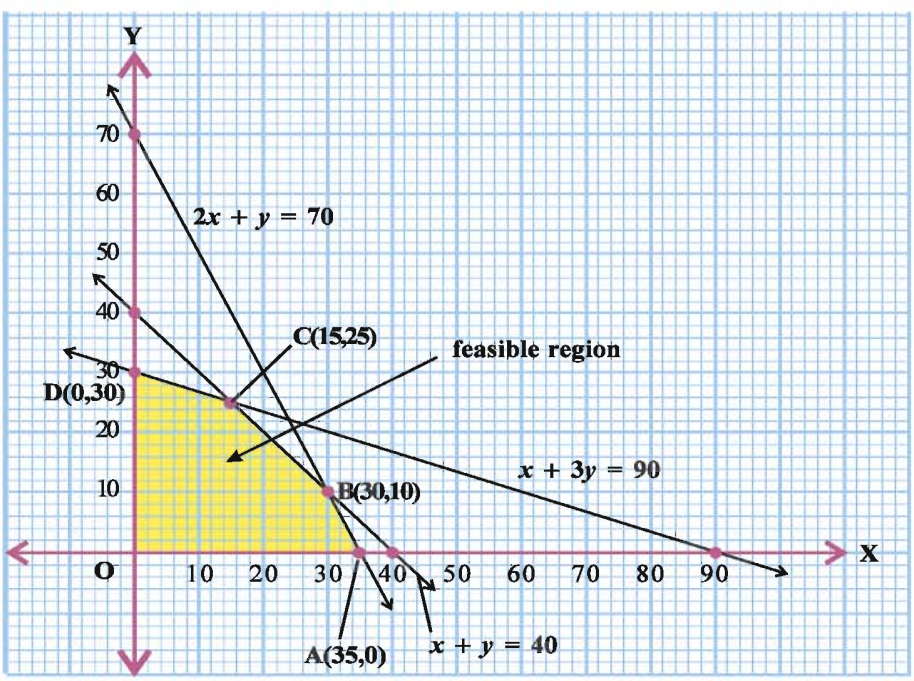


Figure 8.1

In figure 8.1, the region OABCD (yellow coloured) is the feasible region of Example 1.

The region other than the feasible region is called the **infeasible region.**

Note that points within and on the boundary of the feasible region represent feasible solutions of the constraints. In fig. 8.1, every point within or on the boundary of the feasible region OABCD represents feasible solution to the problem.

For example, the point (35, 0), (30, 10), (15, 25), (0, 30), (20, 0), (0, 10), (20, 10) etc. are some of the feasible solutions. The point (30, 20) is an infeasible solution of the problem. We see that every point in the feasible region OABCD satisfies all the constraints of example 1. We also observe that there are infinitely many points in the feasible region. Among them we have to find out one point which gives a maximum value of the objective function $z = 300x + 600y$. To handle this situation, we use the following theorems which are fundamental in solving linear programming problems. We shall not prove these theorems, we just state them.

Theorem 8.1 : Let R be the feasible region (convex polygon) for a linear programming problem and let $z = ax + by$ be the objective function. When z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 8.2 : Let R be the feasible region for a linear programming problem and let $z = ax + by$ be the objective function. If R is bounded, then the objective function z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

In the above example, the corner points (vertices) of the bounded (feasible) region are : O, A, B, C, D and their coordinates are (0, 0), (35, 0), (30, 10), (15, 25) and (0, 30) respectively. Let us now compute the values of z at these points. We have $z = 300x + 600y$.

Vertex of the feasible region	Corresponding value of z (in ₹)	
O(0, 0)	0	
A(35, 0)	10,500	
B(30, 10)	15,000	
C(15, 25)	19,500	←Maximum
D(0, 30)	18,000	

We observe that the maximum profit is earned by the firm by producing 15 chairs and 25 tables.

Note : If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (by theorem 8.1)

This method of solving linear programming problem is known as **Corner Point Method.**

Following steps can be used to solve an LP problem in two variables graphically by using corner-point method.

(1) Formulate the given LP problem in mathematical form, if it is not given in mathematical form.

(2) Find the feasible region of LP problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at the points.

(3) Evaluate the objective function $z = ax + by$ at each corner point. Let M and m respectively denote the largest and the smallest values of z at these points.

(4) When the feasible region is bounded, M and m are the maximum and minimum values of z.

(5) In case, the feasible region is unbounded, we have.

(i) M is the maximum value of z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, z has no maximum value.

(ii) m is the minimum value of z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, z has no minimum value.

We will now illustrate these steps of coner point method in some examples :

Example 2 : Solve the following linear programming problem graphically :

Maximize $z = 20x + 15y$

subject to $180x + 120y \leq 1500$

$x + y \leq 10$

and $x \geq 0, y \geq 0$

Solution : Since $x \geq 0$ and $y \geq 0$, the solution region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$,

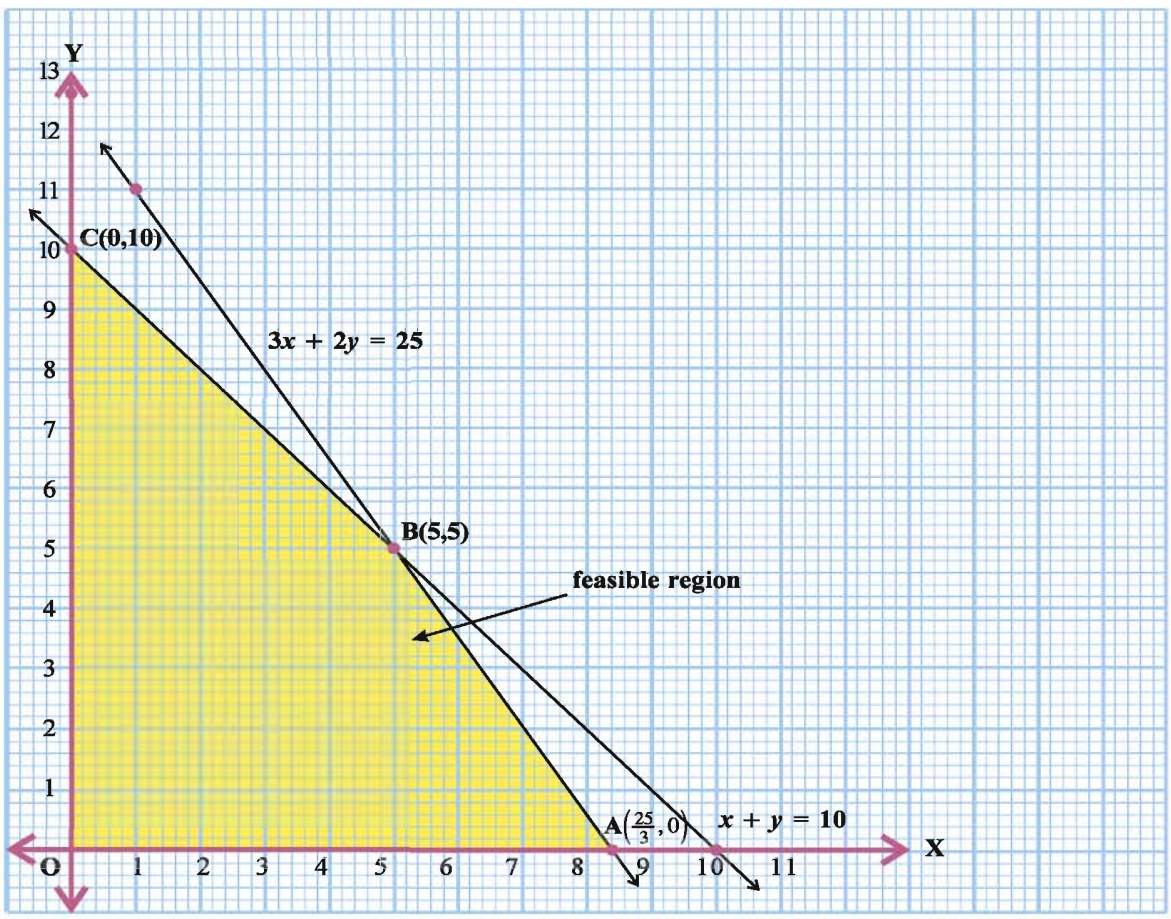


Figure 8.2

(i) $180x + 120y \leq 1500$

$3x + 2y \leq 25$

Draw the line $3x + 2y = 25$

$$y = \frac{25 - 3x}{2}$$

x	0	5	$\frac{25}{3}$	1
y	$\frac{25}{2}$	5	0	11

Determine the region represented by $3x + 2y \leq 25$.

(ii) $x + y \leq 10$

Draw the line $x + y = 10$

$\therefore \quad y = 10 - x$

x	0	10
y	10	0

Determine the region represented $x + y \leq 10$. Colour the intersection of the two regions. Also $x \geq 0$, $y \geq 0$. The yellow coloured region OABC in figure 8.2 is the feasible region. B(5, 5) is the point of intersection of $3x + 2y = 25$ and $x + y = 10$.

The corner points of OABC are O(0, 0), $A\left(\frac{25}{3}, 0\right)$, B(5, 5) and C(0, 10).

Vertex of the feasible region	Corresponding value of $z = 20x + 15y$
O(0, 0)	0
$A\left(\frac{25}{3}, 0\right)$	166.67
B(5, 5)	175
C(0, 10)	150

z is maximum at $x = 5$ and $y = 5$. Maximum value of $z = 175$.

Example 3 : Find the maximum and minimum value of $z = 2x + 5y$,

subject to $3x + 2y \leq 6$, $-2x + 4y \leq 8$, $x + y \geq 1$, $x \geq 0$, $y \geq 0$ using corner point method.

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $3x + 2y \leq 6$

Draw the line $3x + 2y = 6$

$$y = \frac{6 - 3x}{2}$$

x	0	2
y	3	0

Determine the region represented by $3x + 2y \leq 6$.

(2) $-2x + 4y \leq 8$

$\therefore \quad -x + 2y \leq 4$

Draw the line $-x + 2y = 4$.

x	0	2
y	2	3

$\therefore \quad y = \frac{x + 4}{2}$

Determine the region represented by $-x + 2y \leq 4$.

(3) $x + y \geq 1$

Draw the line $x + y = 1$ and determine the region represented by $x + y \geq 1$.

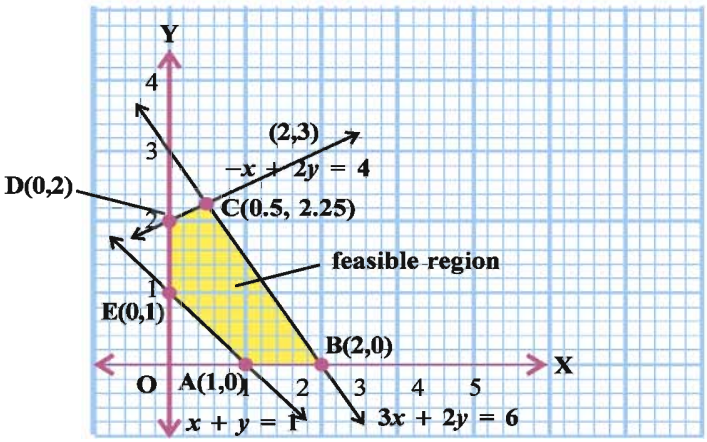


Figure 8.3

Colour the intersection of the three regions.

The yellow coloured region ABCDE in figure 8.3 is the feasible region. The point C(0.5, 2.25) is the point of intersection of $3x + 2y = 6$ and $-2x + 4y = 8$.

The corner points of ABCDE are A(1, 0), B(2, 0), C(0.5, 2.25), D(0, 2), E(0, 1).

Corner point	Value of $z = 2x + 5y$	
A(1, 0)	2	←Minimum
B(2, 0)	4	
C(0.5, 2.25)	12.25	←Maximum
D(0, 2)	10	
E(0, 1)	5	

Hence, $x = 1$, $y = 0$ minimizes $z = 2x + 5y$ and the minimum value is 2.

$x = 0.5$, $y = 2.25$ maximizes $z = 2x + 5y$ and the maximum value is 12.25.

Example 4 : Minimize $2x + 4y$ subject to $x + 2y \geq 10$; $3x + y \geq 10$; $x \geq 0$; $y \geq 0$.

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $x + 2y \geq 10$

Draw the line $x + 2y = 10$

$\therefore \quad y = \frac{10 - x}{2}$

x	0	10
y	5	0

Determine the region represented by $x + 2y \geq 10$.

(2) $3x + y \geq 10$

Draw the line $3x + y = 10$.

$\therefore \quad y = 10 - 3x$

x	0	2
y	10	4

Determine the region represented by $3x + y \geq 10$.

Colour the intersection of the three regions. The feasible region is as shown in the figure 8.4 Observe that the feasible region is unbounded.

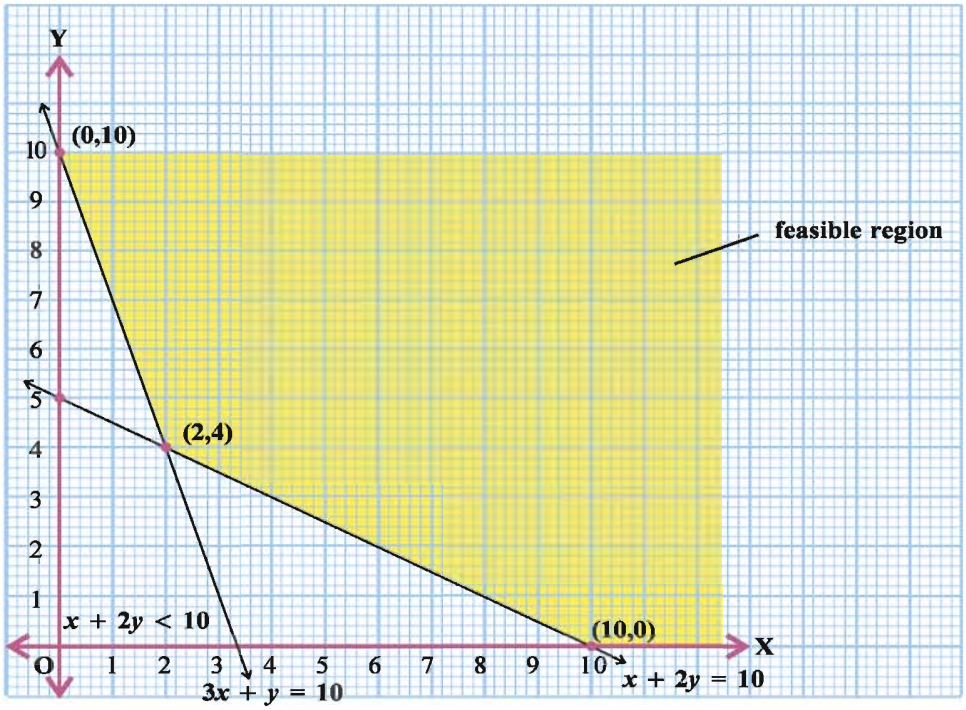


Figure 8.4

The corner points are (0, 10), (2, 4), (10, 0).

Corner point	Value of $z = 2x + 4y$
(0, 10)	40
(2, 4)	20
(10, 0)	20

From the table, we find that 20 may be the smallest value of z at the corner point (2, 4), (10, 0). Since the feasible region is unbounded, 20 may or may not be the minimum value of z. To decide this, we graph the inequality $2x + 4y < 20$ (see step 5(ii) of corner point method).

Now, $2x + 4y < 20$

$\therefore \quad x + 2y < 10$

We have to check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then 20 will not be the minimum value of z. Otherwise, 20 will be the minimum value of z. As shown in the figure 8.4, it has no common point with the feasible region. Hence, 20 is the minimum value of z. In fact, all the points on the line $x + 2y = 10$ give the same minimum value 20. Thus, there is an infinite number of points minimizing $z = 2x + 4y$ subject to the given constraints.

Example 5 : Determine graphically the minimum value of the objective function $z = -50x + 20y$ subject to the constraints.

$2x - y \geq -5$

$3x + y \geq 3$

$2x - 3y \leq 12$

$x \geq 0, y \geq 0,$

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $2x - y \geq -5$

Draw the line $2x - y = -5$

$\therefore \quad y = 2x + 5$

x	0	1
y	5	7

Determine the region represented by $2x - y \geq -5$.

(2) $3x + y \geq 3$

Draw the line $3x + y = 3$

x	0	1
y	3	0

Determine the region represented by $3x + y \geq 3$.

(3) $2x - 3y \leq 12$

Draw the line $2x - 3y = 12$

$\therefore \quad y = \dfrac{2x - 12}{3}$

x	9	6
y	2	0

Determine the region represented by $2x - 3y \leq 12$.

Colour the intersection of the three regions. The feasible region is as shown in the figure 8.5. Observe that the feasible region is unbounded.

The corner points are (0, 5), (0, 3), (1, 0) and (6, 0). We now evaluate z at the corner points.

Corner point	Value of $z = -50x + 20y$	
A(0, 5)	100	
B(0, 3)	60	
C(1, 0)	–50	
D(6, 0)	–300	←Smallest

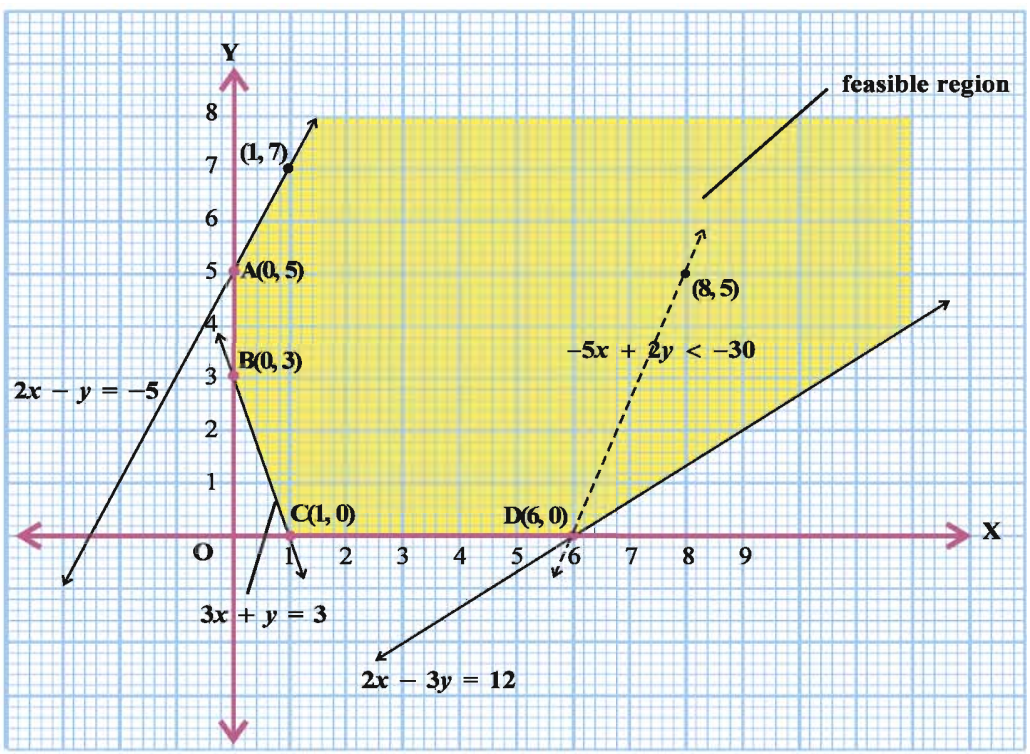


Figure 8.5

From the table, we find that -300 may be the smallest value of z at the corner point (6, 0). Since the feasible region is unbounded, -300 may or may not be the minimum value of z. To decide this, we graph the inequality $-50x + 20y < -300$ i.e. $-5x + 2y < -30$ and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then -300 will not be the minimum value of z. Otherwise -300 will be the minimum value of z. As shown in the figure 8.5, it has common points. Therefore, $z = -50x + 20y$ has no minimum value subject to the given constraints.

[In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0, 5) ?]

Example 6 : Maximize $z = 3x + 4y$, if possible, subject to

$$x - y \leq -1$$
$$-x + y \leq 0$$
$$x \geq 0, y \geq 0$$

Solution : Let us graph the inequalities $x - y \leq -1$, $-x + y \leq 0$, $x \geq 0$ and $y \geq 0$.

From figure 8.6 we can see that there is no point satisfying all the constraints simultaneously. Thus, the problem has no feasible region and hence no feasible solution.

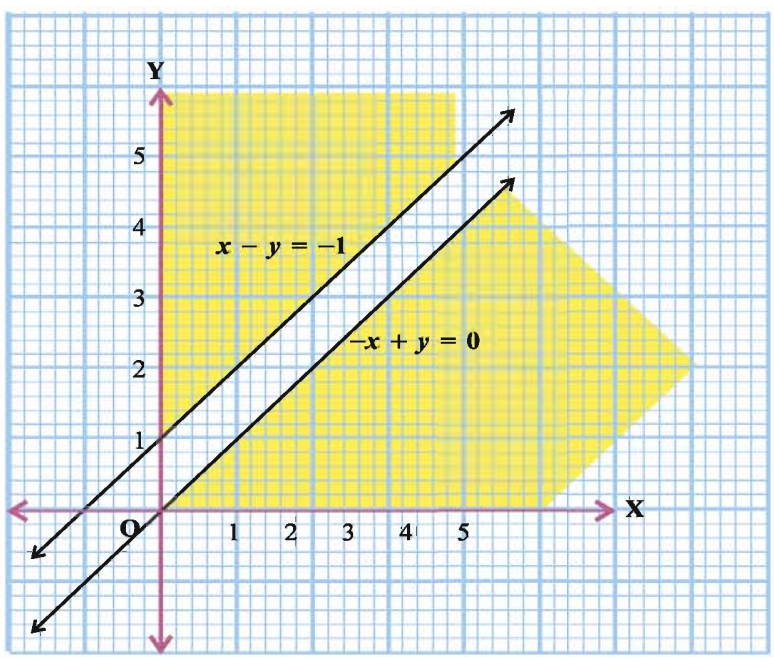


Figure 8.6

From the examples which we have discussed so far, we observed the following :

(1) The feasible region is always a convex region.

(2) The maximum (or minimum) solution of the objective function occurs at the corner of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

Exercise 8.1

1. A company sells two different products A and B, making a profit of ₹ 40 and ₹ 30 per unit on them respectively. The products are produced in a common production process and are sold in two different markets. The production process has a total capacity of 3,000 man-hours. It takes three hours to produce a unit of type A and one hour to produce a unit of type B. The market has been surveyed and company officials feel that the maximum number of units of type A that can be sold is 8,000 and those of type B is 1200. Subject to these constraints, product can be sold in any combination. Formulate this problem as an LP problem mathematically to maximize the profit.
2. Vitamins A and B are found in foods F_1 and F_2. One unit of food F_1 contains three units of vitamin A and four units of Vitamin B. One unit of food F_2 contains six units of vitamin A and three units of vitamin B. One unit of food F_1 and F_2 costs ₹ 4 and ₹ 5 respectively. The minimum daily requirement (for a person) of vitamins A and B is 80 units and 100 units respectively. Assuming that anything in excess of the daily minimum requirement of A and B is not harmful, formulate this problem as an LP problem to find out the optimum mixture of foods F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamins A and B.
3. A pension fund manager is considering investing in two shares A and B. It is estimated that,
   (1) share A will earn a dividend of 12 percent per annum and share B will earn 4 percent dividend per annum.
   (2) growths in the market value in one year of share A respectively are 10 paise per Re 1 invested and 20 paise per Re 1 invested in B.

   He requires to invest the maximum total sum which will give,
   (1) dividend income of at least ₹ 600 per annum; and
   (2) growth in one year of at least ₹ 1000 on the initial investment.

   Formulate this problem as an LP model to compute the minimum sum to be invested to meet the manager's objective.

Solve the following linear programming problems graphically (4 to 12) :

4. Maximize $z = 20x + 10y$

   subject to $x + 2y \leq 40$, $3x + y \geq 30$, $4x + 3y \geq 60$ and $x \geq 0$, $y \geq 0$
5. Maximize $z = 4x + y$

   subject to $x + y \leq 50$, $3x + y \leq 90$ and $x \geq 0$, $y \geq 0$
6. Minimize $z = 200x + 500y$

   subject to $x + 2y \geq 10$, $3x + 4y \leq 24$ and $x \geq 0$, $y \geq 0$
7. Minimize and maximize $z = 3x + 9y$

   subject to $x + 3y \leq 60$, $x + y \geq 10$, $x \geq y$, $x \geq 0$, $y \geq 0$
8. Minimize $z = 3x + 2y$

   subject to $x + y \geq 8$, $3x + 5y \leq 15$, $x \geq 0$, $y \geq 0$
9. Maximize $z = 3x + 4y$

   subject to $x + y \leq 4$, $x \geq 0$, $y \geq 0$
10. Maximize $z = 3x + 4y$

    subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$

11. Maximize $z = -x + 2y$

subject to $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 6$

12. Minimize $z = 5x + 10y$

subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x \geq 0$, $y \geq 0$

*

8.4 Different Types of Linear Programming Problems

Diet Problems : In this type of problems, we have to find the amount of different kinds of constituents / nutrients which should be included in a diet so as to minimize the cost of the desired diet and such that it contains a certain minimum amount of each constituent / nutrient.

Example 7 : A housewife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamim A, at least 12 units of vitamin B and at least 8 units of vitamin C. The vitamin contents of one *kg* of food is given below :

	Vitamin A	Vitamin B	Vitamin C
Food X	1	2	3
Food Y	2	2	1

One kg of food X costs ₹ 60 and one kg of food Y costs ₹ 100 . Find the least cost of the mixture which will produce the diet.

Solution : Let x *kg* of food X and y *kg* of food Y be mixed together to make the required diet.

1 *kg* of food X contains one unit of vitamin A and 1 *kg* of food Y contains 2 units of vitamin A.

Therefore, x *kg* of food X and y *kg* of food Y will contain $x + 2y$ units of vitamin A. It is given that the mixture should contain at least 10 units of vitamin A.

Therefore, $x + 2y \geq 10$ **(i)**

Similarly, x *kg* of food X and y *kg* of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C. The minimum requirements of vitamin B and C are 12 and 8 units respectively.

$\therefore \quad 2x + 2y \geq 12$ **(ii)**

and $3x + y \geq 8$ **(iii)**

Since the quantity of food X and Y cannot be negative.

$\therefore \quad x \geq 0, y \geq 0$ **(iv)**

It is given that one *kg* of food X costs ₹ 60 and one *kg* of food Y costs ₹ 100. So, x *kg* of food X and y kg of food Y will cost ₹ $(60x + 100y)$. Thus, the given linear programming problem is

Minimize $z = 60x + 100y$

Subject to $x + 2y \geq 10$, $2x + 2y \geq 12$, $3x + y \geq 8$ and $x \geq 0$, $y \geq 0$.

Now let us solve this LP problem by graphical method.

To solve this LP problem, we draw the lines $x + 2y = 10$, $2x + 2y = 12$ i.e. $x + y = 6$ and $3x + y = 8$ and obtain the feasible region as shown in the figure 8.7, which is an unbounded one.

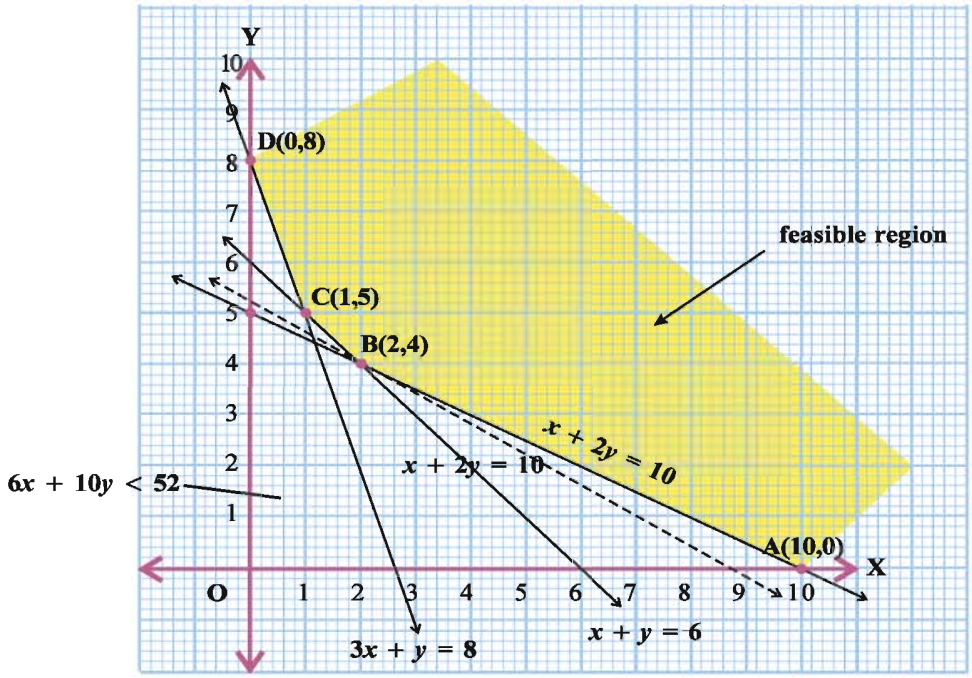


Figure 8.7

The corner points of the coloured region ABCD are A(10, 0), B(2, 4), C(1, 5) and D(0, 8). These points can also be obtained by solving simultaneously the equations of the corresponding intersecting lines,. The values of the objective function at these points are given in the following table :

Corner point	Value of the objective function $z = 60x + 100y$
A(10, 0)	600
B(2, 4)	520 ← Minimum
C(1, 5)	560
D(0, 8)	800

Clearly, z may be minimum at $x = 2$ and $y = 4$. Since the feasible region is unbounded, we have to graph the inequality $60x + 100y < 520$, i.e. $6x + 10y < 52$ and check whether the resulting open half plane has points in common with feasible region or not. We see from the figure 8.7 that it has no point common with the feasible region. So, z has minimum value equal to 520.

The minimum cost of the mixture is ₹ 520.

Manufacturing problems : In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed man-power, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.

Example 8 : A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. The profit on sell of a ring is ₹ 300 and that on sell of a chain is ₹ 190. Find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it an LP problem and solve it graphically.

Solution : Let the number of gold rings to be manufactured be x and that of chains be y. We construct the following table :

Item	Number	Time taken	Profit ₹
Gold ring	x	$1x$ hour	$300x$
Gold chain	y	$\frac{1}{2}y$ hour	$190y$
Total	$x + y$	$\left(x+\frac{1}{2}y\right)$ hour	$300x + 190y$

Our problem is to maximize the profit $z = 300x + 190y$ subject to constraints $x \geq 0$, $y \geq 0$ **(i)**

$x + \frac{1}{2}y \leq 16$

$\therefore \quad 2x + y \leq 32$ **(ii)**

and $x + y \leq 24$ **(iii)**

We draw the lines $2x + y = 32$ and $x + y = 24$ and obtain the feasible region as shown in the figure 8.8

Corner points of the feasible region OABC are O(0, 0), A(16, 0), B(8, 16), C(0, 24).

Let us evaluate z at these corner points.

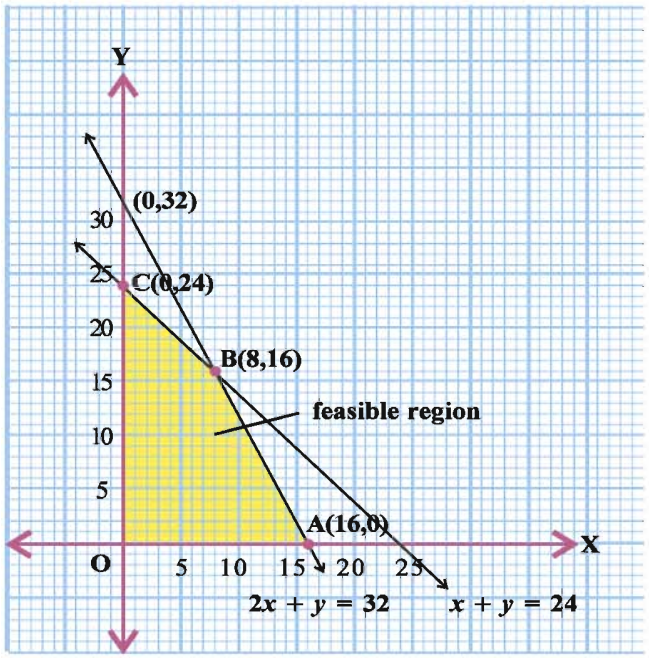


Figure 8.8

Corner point	Value of $z = 300x + 190y$ in ₹
(0, 0)	0
(16, 0)	4800
(8, 16)	5440 ← Maximum
(0, 24)	4560

We observe that profit is maximum when $x = 8$ and $y = 16$ and maximum profit is ₹ 5440.

Thus, to get maximum profit a firm has to produce 8 rings and 16 chains per day.

Transportation problems : In this type of problems, we have to determine transportation schedule for a commodity from different plants or factories situated at different relations to different markets in such a way that the total cost of transportation is minimum.

Example 9 : A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three construction companies P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in ₹ of transporting 1000 bricks to the companies from the depots are given below :

To \ From	P	Q	R
A	80	40	60
B	40	120	80

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum ?

Solution : The given information is as shown in the following figure.

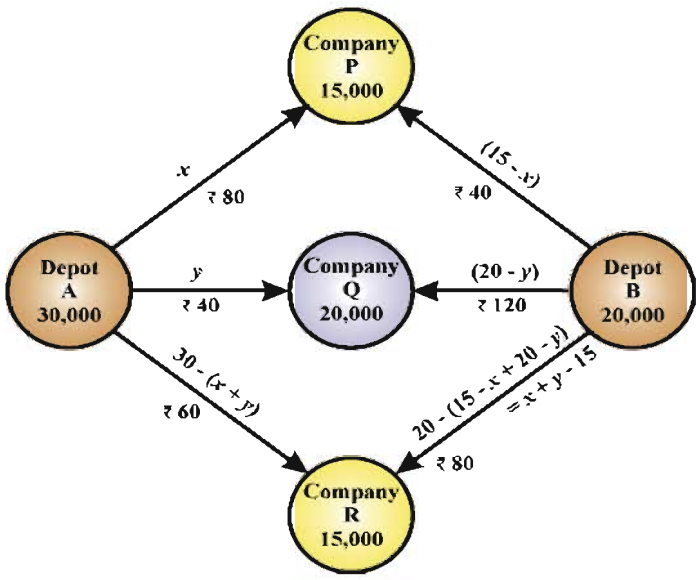


Figure 8.9

Let the depot A transport x thousand bricks to the company P and y thousand bricks to the company Q. Since the depot A has stock of 30,000 bricks, the remaining $30 - (x + y)$ thousand bricks will be transported to the company R. The number of bricks is always non-negative.

We have $x \geq 0$, $y \geq 0$ and $30 - (x + y) \geq 0$ i.e., $x + y \leq 30$ (i)

Now, the requirement of the company P is of 15,000 bricks and x thousand bricks are transported from the depot A, so the remaining $(15 - x)$ thousand bricks are to be transported from the depot B. The requirement of the company Q is of 20,000 bricks and y thousand bricks are transported from depot A. So the remaining $(20 - y)$ thousand bricks are to be transported from depot B. Now, depot B has $20 - (15 - x + 20 - y) = x + y - 15$ thousand bricks which are to be transported to the company R.

Also, $15 - x \geq 0$, $20 - y \geq 0$ and $x + y - 15 \geq 0$

$\therefore \quad x \leq 15$, $y \leq 20$ and $x + y \geq 15$ (ii)

The transportation cost from the depot A to the companies P, Q and R are respectively ₹ $80x$, ₹ $40y$ and ₹ $60(30 - (x + y))$. Similarly, the transportation cost from the depot B to the companies P, Q and R are respectively ₹ $40(15 - x)$, $120(20 - y)$ and $80(x + y - 15)$ respectively. Therefore, the total transportation cost z is given by

$$z = 80x + 40y + 60(30 - x - y) + 40(15 - x) + 120(20 - y) + 80(x + y - 15)$$

$$\therefore \quad z = 60x - 60y + 3600$$

Hence, the above LP problem can be stated mathematically as follows :

Minimize $z = 60x - 60y + 3600$

Subject to $x + y \leq 30$, $x \leq 15$, $y \leq 20$, $x + y \geq 15$ and $x \geq 0$, $y \geq 0$

Here, x and y are in thousands.

Let us solve this problem graphically. We draw the lines $x + y = 30$, $x = 15$, $y = 20$ and $x + y = 15$ and obtain the feasible region as shown in the figure 8.10.

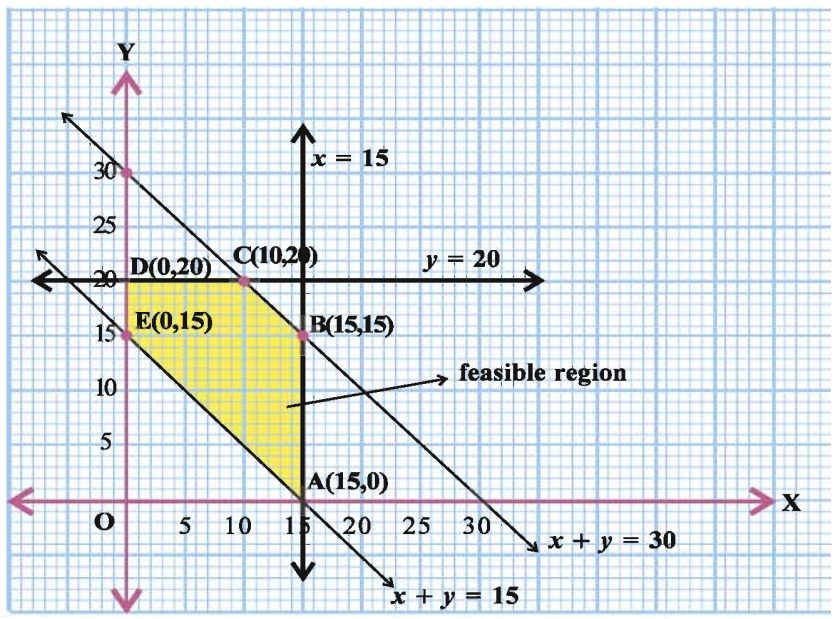


Figure 8.10

Corner points of the feasible region ABCDE are A(15, 0), B(15, 15), C(10, 20), D(0, 20), E(0, 15).

Let us evaluate z at these corner points.

Corner point	Value of $z = 60x - 60y + 3600$
(15, 0)	4500
(15, 15)	3600
(10, 20)	3000
(0, 20)	2400 ← Minimum
(0, 15)	2700

Clearly, z is minimum at $x = 0$, $y = 20$ and the minimum value of z is 2400.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to company P, Q and R from depot A and 15, 0 and 5 thousand bricks to company P, Q, R from depot B respectively.

In this case the minimum transportation cost will be ₹ 2400.

Marketing Problems : Linear programming can be used to determine the right mix of media exposure to use an advertising campaign. Suppose that the available media are radio, television and newspapers. The goal is to determine how many advertisements to place in each medium where the cost of placing an advertisement depends on the medium. Of course, we want to minimize the total cost of the advertising compaign and maximizing the mass where advertisement reaches.

Example 10 : An advertising agency wishes to reach two types of probable customers with annual income greater than one lakh rupees (target audience A) and customers with annual income less than one lakh rupees (target audience B). The total adverising budget is ₹. 2,00,000. One programme of TV advertising costs ₹ 50,000; one programme of radio advertising costs ₹ 20,000. For contract reasons, at least three programmes ought to be aired on TV and the number of radio programmes must be limited to 5. Surveys indicate that a single TV programme reaches 4,50,000 prospective customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 prospective customers in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

Solution : Let us define the following decision variables :

Let x and y be the number of programmes to be aired on TV and radio respectively.

We are given that a single TV programme reaches 4,50,000 in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B.

Hence, we have to maximize.

$$z = (4,50,000 + 50,000)x + (20,000 + 80,000)y$$

$$= 5,00,000x + 1,00,000y \quad \text{(i)}$$

According to budget constraint we have

$$50,000x + 20,000y \leq 2,00,000$$

$$\text{i.e., } 5x + 2y \leq 20 \quad \text{(ii)}$$

Also, there is number of programme constraints as at least 3 TV programmes and at the most 5 radio programme.

$$\therefore \quad x \geq 3 \text{ and } y \leq 5 \quad \text{(iii)}$$

Also, number of programmes is non-negative.

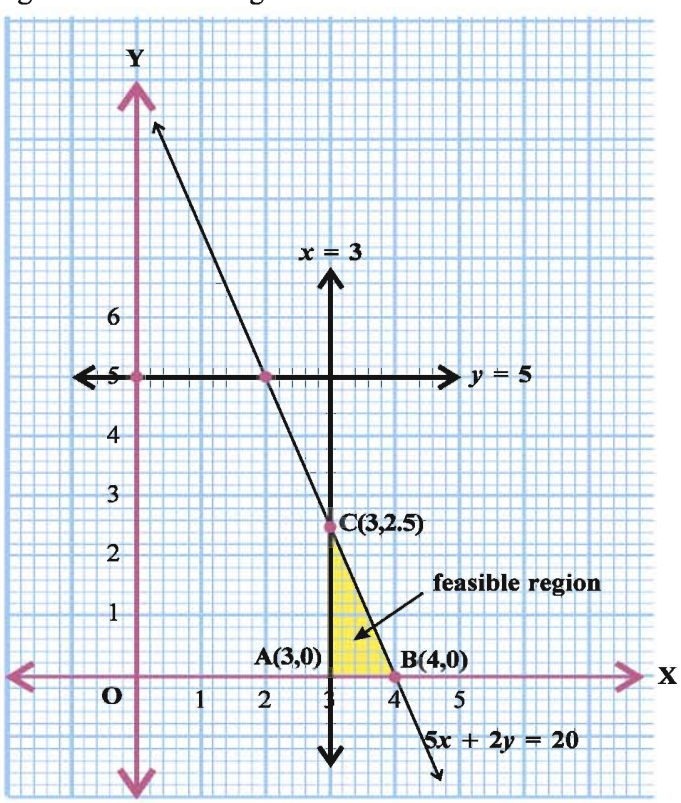


Figure 8.11

$\therefore \quad x \geq 0$ and $y \geq 0$ **(iv)**

Thus, LP problem is maximize $z = 5{,}00{,}000x + 1{,}00{,}000y$

subject to $5x + 2y \leq 20$, $x \geq 3$, $y \leq 5$ and $x \geq 0$, $y \geq 0$

Let us solve this problem graphically. We draw the lines $5x + 2y = 20$, $x = 3$, $y = 5$ and obtain the feasible region as shown in the figure 8.11.

Corner points of the feasible region ABC are A(3, 0), B(4, 0) and $C\left(3, \frac{5}{2}\right)$.

Let us evaluate z at thest corner points.

Corner point	Value of $z = 5{,}00{,}000x + 1{,}00{,}000y$
(3, 0)	15,00,000
(4, 0)	20,00,000 ← Maximum
$\left(3, \frac{5}{2}\right)$	17,50,000

Since the maximum value of $z = 20{,}00{,}000$ occurs at the point B(4, 0), therefore, the agency must release 4 programmes on TV and no programme on radio to achieve the maximum target audiences.

Exercise 8

Use the graphical method to solve the following LP problems : (1 to 6)

1. Maximize $z = 2x + y$

   subject to $x + 2y \leq 10$, $x + y \leq 6$, $x - y \leq 2$, $x - 2y \leq 1$ and $x \geq 0$, $y \geq 0$

2. Minimize $z = -x + 2y$

   subject to $-x + 3y \leq 10$, $x + y \leq 6$, $x - y \leq 2$ and $x \geq 0$, $y \geq 0$

3. Minimize $z = 3x + 2y$

   subject to $5x + y \geq 10$, $x + y \geq 6$, $x + 4y \geq 12$ and $x \geq 0$, $y \geq 0$

4. Maximize $z = 7x + 3y$

   subject to $x + y \geq 3$, $x + y \leq 4$, $0 \leq x \leq \frac{5}{2}$, $0 \leq y \leq \frac{3}{2}$

5. Minimize $z = 20x + 10y$

   subject to $x + 2y \leq 40$, $3x + y \geq 30$, $4x + 3y \geq 60$ and $x \geq 0$, $y \geq 0$

6. Maximize $z = x + y$

   subject to $x + y \leq 1$, $-3x + y \geq 3$ and $x \geq 0$, $y \geq 0$

7. A factory owner purchases two types of machines A and B for his factory. The requirements and limitations for the machines are as folows :

Machine	Area occupied	Labour force	Daily out-put units
A	1000 m^2	12 persons	60
B	1200 m^2	8 persons	40

He has maximum area of 9000 m^2 available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximize the daily output ? Formulate and solve the problem graphically.

8. A diet for a sick person must contain at least 4000 units of vitamin, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of ₹ 5 and ₹ 4 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of minerals and 40 units of calories, while one unit of the food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the foods A and B should be used to have minimum cost, but it must satisfy the requirements of the sick person. Formulate as an LP problem and solve it graphically.

9. A shopkeeper wishes to purchase a number of 5 *l* oil tins and 1 *kg* ghee tins. He has only ₹ 5760 to invest and has a space to store at most 20 items. A 5 *l* oil tin costs him ₹ 360 and a 1 *kg* ghee tin cost him ₹ 240. His expectation is that he can sell an oil tin at a profit of ₹ 22 and a ghee tin at a profit of ₹ 18. Assuming that he can sell all the items he can buy, how should he invest his money in order to maximize the profit ? Formulate this as a linear programming problem and solve it graphically.

10. One kind of cake requires 300 *g* of flour and 15 *g* of fat. Another kind of cake requires 150 *g* of flour and 30 *g* of fat. Find the maximum number of cakes which can be made from 7.5 *kg* of flour and 600 *g* of fat, assuming that there is no shortage of other ingredients used in making the cakes. Formulate it as an LP problem and solve it graphically.

11. An oil company has two depots A and B with capacities of 7000 *l* and 4000 *l* respectively. The company is to supply oil to three petrol pumps, D, E and F, whose requirements are 4500 *l*, 3000 *l* and 3500 *l* respectively. The distances (in *km*) between the depots and the petrol pumps is given in the following table : (Distance in *km*)

From \ To	A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 *l* of oil is ₹ 1 per *km*. How should the delivery be scheduled in order that the transportation cost is minimum ? What is the minimum cost ?

12. An aeroplane can carry a maximum of 200 passangers. A profit of ₹ 1000 occurs on each executive class ticket and a profit of ₹ 600 occurs on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passangers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit ?

13. A manufacturer produces two different models : X and Y, of the same product. Model X generates profit of ₹ 50 per unit and model Y generates profit of ₹ 30 per unit. Raw materials r_1 and r_2 are required for production. At least 18 *kg* of r_1 and 12 *kg* of r_2 must be used daily. Also at most 34 hours of labour are to be utilized. A quantity of 2 *kg* of r_1 is needed for model X and 1 *kg* of r_1 for model Y. For each of X and Y, 1 *kg* of r_2 is required. It takes 3 hours to manufacture model X and 2 hours to manufacture model Y. How many units of each model should be produced to maximize the profit ?

14. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 mark)

(1) Objective function of an LP problems is

(a) a constant (b) a function to be optimized

(c) an inequality (d) a quadratic equation

(2) Let x and y be optimal solution of an LP problem, then

(a) $z = \lambda x + (1 - \lambda)y$, $\lambda \in$ R is also an optimal solution

(b) $z = \lambda x + (1 - \lambda)y$, $0 \leq \lambda \leq 1$ gives an optimal solution.

(c) $z = \lambda x + (1 + \lambda)y$, $0 \leq \lambda \leq 1$ gives an optimal solution.

(d) $z = \lambda x + (1 + \lambda)y$, $\lambda \in$ R gives an optimal solution.

(3) The optimal value of the objective function is attained at the points

(a) given by intersection of lines representing inequations with axes only

(b) given by intersection of lines representing inequations with X-axis only

(c) given by corner points of the feasible region

(d) at the origin

(4) The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of z occurs at both the pooints (15, 15) and (0, 20) is .....

(a) $p = q$ (b) $p = 2q$ (c) $q = 2p$ (d) $q = 3p$

(5) Which of the following statements is correct ?

(a) Every LP problem has at least one optimal solution.

(b) Every LP problem has a unique optimal solution.

(c) If an LP problem has two optimal solutions, then it has infinitely many solutions.

(d) If a feasible region is unbounded then LP problem has no solution.

(6) In solving the LP problem :

"Minimize $z = 6x + 10y$

subject to $x \geq 6$, $y \geq 2$, $2x + y \geq 10$, $x \geq 0$, $y \geq 0$." redundant constraints are

(a) $x \geq 6$, $y \geq 2$ (b) $2x + y \geq 10$, $x \geq 0$, $y \geq 0$

(c) $x \geq 6$ (d) $x \geq 6$, $y \geq 0$

(7) A feasible solution to an LP problem,

(a) must satisfy all of the problem's constraints simultaneously

(b) need not satisfy all of the constraints, only some of them.

(c) must be a corner point of the feasible region.

(d) must optimize the value of the objective function.

Section B (2 marks)

(8) For the LP problem

"Maximize $z = x + 4y$

subject to $3x + 6y \leq 6$, $4x + 8y \geq 16$ and $x \geq 0$, $y \geq 0$."

(a) 4 (b) 8

(c) feasible region is unbounded (d) has no feasible region

(9) For the LP problem

Maximize $z = 2x + 3y$

the coordinates of the corner points of the bounded feasible region are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12). The maximum value of z is .....

(a) 72 (b) 80 (c) 82 (d) 70

(10) For the LP problem

Minimize $z = 2x + 3y$

the coordinates of the corner points of the bounded feasible region are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12). The minimum value of z is .....

(a) 49 (b) 15 (c) 10 (d) 05

Section C (3 marks)

(11) Solution of the following LP problem

Maximize $z = 2x + 6y$

subject to $-x + y \leq 1$, $2x + y \leq 2$ and $x \geq 0$, $y \geq 0$" is

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$ (c) $\frac{26}{3}$ (d) no feasible region

(12) Solution of the following LP problem

Minimize $z = -3x + 2y$

subject to $0 \leq x \leq 4$, $1 \leq y \leq 6$, $x + y \leq 5$ is

(a) -10 (b) 0 (c) 2 (d) 10

Section D (4 marks)

(13) The following graph represents a feasible region. Minimum value of $z = 5x + 4y$ is .....

(a) 150 (b) 145 (c) 160 (d) 250

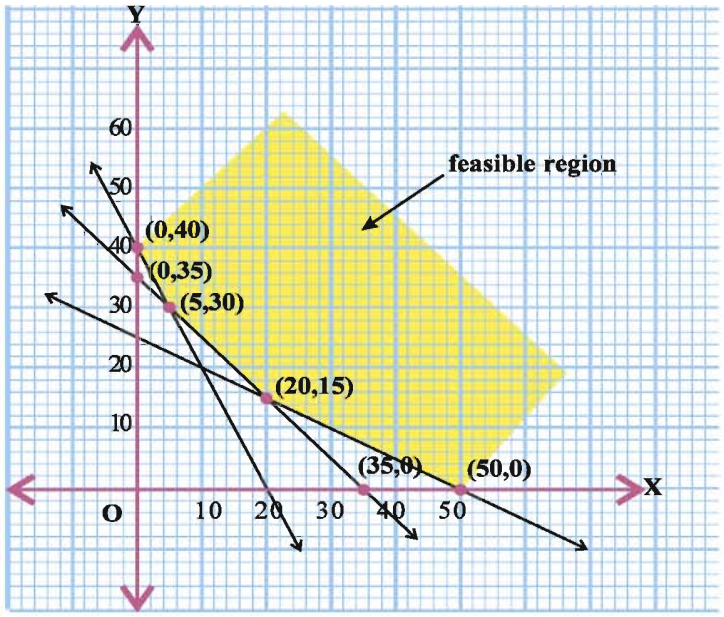


Figure 8.12

(14) Corner points of the bounded feasible region for an LP problem are (0, 4), (6, 0), (12, 0), (12, 16) and (0, 10). Let $z = 8x + 12y$ be the objective function. Match the following :

(i) Minimum value of z occurs at ..... (ii) Maximum value of z occurs at .....

(iii) Maximum of z is ..... (iv) Minimum of z is .....

(a) (i) (6, 0) (ii) (12, 0) (iii) 288 (iv) 48

(b) (i) (0, 4) (ii) (12, 16) (iii) 288 (iv) 48

(c) (i) (0, 4) (ii) (12, 16) (iii) 288 (iv) 96

(c) (i) (6, 0) (ii) (12, 0) (iii) 288 (iv) 96

*

Summary

We have studied the following points in this chapter :

1. Mathematical formulation of linear programming problems.
2. Meaning of the terms : Decision variables, the objective function, the constraints.
3. Graphical method of solving linear programming problems
4. Meaning of the terms : feasible solution, infeasible solution, optimal feasible solution, feasible region, infeasible region.

Fields of Indian mathematics

Some of the areas of mathematics studied in ancient and medieval India include the following :

Arithmetic : Decimal system, Negative numbers (Brahmagupta), Zero (Hindu numeral system), Binary numeral system, the modern positional notation numeral system, Floating point numbers (Kerala school of astronomy and mathematics), Number theory, Infinity (Yajur Veda), Transfinite numbers

Geometry : Square roots (Bakhshali approximation), Cube roots (Mahavira), Pythagorean triples (Sulba Sutras; Baudhayana and Apastamba) statement of the Pythagorean theorem without proof), Transformation (Panini), Pascal's triangle (Pingala)

Algebra : Quadratic equations (Sulba Sutras, Aryabhata, and Brahmagupta), Cubic equations and Quartic equations (biquadratic equations) (Mahavira and Bhaskara II)

Mathematical logic : Formal grammars, formal language theory, the Panini–Backus form (Panini), Recursion (Panini)

General mathematics : Fibonacci numbers (Pingala), Earliest forms of Morse code (Pingala), infinite series, Logarithms, indices (Jain mathematics), Algorithms, Algorism (Aryabhata and Brahmagupta)

Trigonometry : Trigonometric functions (Surya Siddhanta and Aryabhata), Trigonometric series (Madhava and Kerala school)

ANSWERS

(Answers to questions involving some calculations only are given.)

Exercise 1.1

1. (1) Not Reflexive, not symmetric, not transitive (2) Reflexive, not symmetric, transitive
   (3) Reflexive, not symmetric, transitive (4) Reflexive, symmetric, transitive
   (5) Not reflexive, not symmetric, not transitive
2. Equivalence classes : $A_1 = \{...., 1, 7, 13, 19,....\}$
   $A_2 = \{...., 2, 8, 14, 20,....\}$
   $A_3 = \{...., 3, 9, 15, 21,....\}$
   $A_4 = \{...., 4, 10, 16, 22,....\}$
   $A_5 = \{...., 5, 11, 17, 23,....\}$
   $A_6 = \{...., 6, 12, 18, 24,....\}$
3. Reflexive, antisymetric, transitive 4. (1) $\{1\}, \{2\}, \{3\},...$, (2) $\{0\}, \{1, -1\}, \{2, -2\},...$
5. $\{(1, 2)\}$ 6. X-axis and Y-axis and lines parallel to them.

Exercise 1.2

1. f is one-one and onto
2. f is one-one and onto
3. f is not one-one and not onto
4. f is not one-one, not onto
5. f is not one-one, onto
6. f is one-one and onto
7. f is not one-one, but onto
8. f is one-one and onto
9. f is one-one, not onto
10. f is one-one, not onto
11. f is not one-one, not onto
12. $n!$ one-one functions
13. Number of onto functions on $A_1 = 1$
    Number of onto functions on $A_2 = 2$
    Number of onto functions on $A_3 = 6$, in general number of onto functions on $A_n = n!$

Exercise 1.3

2. (1) $(gof)(x) = x^2$, $(fog)(x) = x^2$ (2) $(gof)(x) = x$, $(fog)(x) = x$
3. $(fof)(x) = x$ 4. $(fof)(x) = x^4 - 2x^3 - 4x^2 + 5x + 4$
5. $(fof)(x) = x$ 6. $(fog)(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \in [0, 1) \\ -1, & x < 0 \end{cases}$ $(gof)(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0, gof = f \\ -1, & x < 0 \end{cases}$
7. $(fog)(n) = \begin{cases} 2n + 2, & n \text{ even} \\ \frac{n+3}{2}, & n \text{ odd}, n = 4k + 1, k \in Z \\ n - 2, & n \text{ odd}, n = 4k + 3, k \in Z \end{cases}$ $(gof)(n) = \begin{cases} 2n + 4, & n \text{ even} \\ n - 1, & n \text{ odd} \end{cases}$

Exercise 1.4

1. $f^{-1}(x) = \frac{x-3}{2}$ 2. $f^{-1}(x) = x + 7$ 3. $f^{-1}(x) = x^{\frac{1}{3}}$ 4. $f^{-1}(n) = \frac{n}{2}$

5. $f^{-1}((n, 0)) = 2n$, $f^{-1}((n, 1)) = 2n + 1$ **6.** f^{-1} does not exist

9. **(1)** f^{-1} does not exist **(2)** f^{-1} does not exist **(3)** f^{-1} does not exist
(4) f^{-1} does not exist **(5)** $f^{-1}(z) = \bar{z}$ **(6)** f^{-1} does not exist
(7) $f^{-1}((n, m)) = (m, n)$, $f^{-1} = f$

Exercise 1

4. S is not reflexive, not symmetric, not transitive

8. **(1)** not one-one, not onto **(2)** not one-one, onto
(3) not one-one, not onto **(4)** not one-one, onto
(5) not one-one, onto **(6)** one-one, not onto
(7) one-one, onto **(8)** one-one, onto
(9) not one-one, not onto **(10)** one-one, onto
(11) not one-one, not onto

10. $(gof)(n) = n$ if $5 \mid n$, $(fog)(n) = n$ otherwise $(fog)(n) = 0$

14. fog does not exist

16. f is one-one, onto, $f^{-1}(x) = \frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ **17.** f^{-1} does not exist. f is not onto.

18. If $a * b = a + b + ab$, then $*$ is commutative and associative. If $a * b = a - b + ab$, then $*$ is not commutative and not associative.

19. **(1)** not associative, not commutative **(2)** commutative, associative
(3) not commutative, not associative **(4)** not commutative, not associative
(5) commutative and associative **(6)** not commutative, not associative
(7) commutative, not associative **(8)** not commutative, not associative
(9) commutative, associative **(10)** not commutative, not associative

20. **(1)** $e = 0$, $a^{-1} = -\frac{a}{1+a}$ **(2)** $e = 2$, $a^{-1} = \frac{4}{a}$ **(3)** $e = 2$, $a^{-1} = 4 - a$ **(4)** $e = 0$, $a^{-1} = \frac{a}{a-1}$
(5) e does not exist **(6)** e does not exist **(7)** e does not exist **(8)** e does not exist
(9) X is the identity, $X^{-1} = X$ **(10)** ϕ is the identity, $\phi^{-1} = \phi$

21. **Section A :** **(1)** d **(2)** b **(3)** b **(4)** a **(5)** a **(6)** c **(7)** b **(8)** a **(9)** b **(10)** a
(11) b **(12)** c **(13)** c **(14)** a **(15)** a
Section B : **(16)** a **(17)** b **(18)** a **(19)** a **(20)** b **(21)** b **(22)** d **(23)** a **(24)** b
(25) a **(26)** b **(27)** a
Section C : **(28)** c **(29)** b **(30)** b **(31)** c **(32)** b **(33)** a **(34)** a **(35)** a **(36)** d
(37) d **(38)** c **(39)** d

Exercise 2.1

1. **(1)** $\frac{\pi}{6}$ **(2)** $-\frac{\pi}{6}$ **(3)** $\frac{2\pi}{3}$ **(4)** $-\frac{\pi}{3}$ **(5)** $\frac{\pi}{6}$ **(6)** $-\frac{\pi}{4}$ **2.** **(1)** $\frac{5\pi}{14}$ **(2)** $\frac{3\pi}{10}$ **(3)** $\frac{\pi}{4}$ **(4)** $\frac{3\pi}{8}$

3. **(1)** $\frac{20}{29}$ **(2)** $\frac{1}{5}$ **(3)** $\frac{24}{25}$ **(4)** $\frac{7-3\sqrt{5}}{2}$ **(5)** $\frac{23}{27}$ **4.** $\frac{\pi}{4} - x$

Exercise 2.2

1. **(1)** 0 **(2)** $\frac{7\pi}{6}$ **(3)** $\frac{5\pi}{12}$ **(4)** 7π **(5)** 2 **(6)** 1 **(7)** $\frac{5\pi}{6}$

Exercise 2

3. (1) $\left\{\pm\frac{1}{\sqrt{2}}\right\}$ (2) $\left\{\frac{1}{6}\right\}$ (3) $\left\{\frac{\pi}{4}\right\}$ (4) $\left\{\frac{1}{2}\right\}$ (5) $\{13\}$ (6) $\left\{\frac{1}{4}\right\}$ (7) $\{4\}$

4. **Section A :** (1) a (2) b (3) a (4) b (5) d (6) d (7) b (8) b (9) a (10) c (11) a (12) c (13) d (14) b (15) d

Section B : (16) c (17) b (18) d (19) c (20) b (21) d (22) d (23) b (24) a (25) a (26) c (27) d (28) a (29) d (30) b

Section C : (31) b (32) a (33) a (34) d (35) d (36) d (37) b (38) b (39) a (40) c (41) c (42) c

Section D : (43) d (44) c (45) b (46) b (47) b (48) b (49) b (50) b (51) b

Exercise 3.1

1. (1) 43 (2) 1 (3) 3 **2.** (1) 2 (2) 6, −2 **3.** (1) 0 (2) 131

Exercise 3.2

3. 4, $\frac{-23}{21}$ **5.** $\frac{7\pi}{24}, \frac{11\pi}{24}$ **7.** 4

Exercise 3.3

1. (1) $\{(0, 0), (7, 7)\}$ (2) $\left\{\left(\frac{-39}{7}, \frac{-79}{7}\right)\right\}$ (3) $\left\{\left(1, \frac{1}{2}\right)\right\}$ **2.** −28 **3.** −37

4. (1) 25 (2) 4 **5.** $k = 3; 7$ **6.** $a \in R$

7. (1) $3x + 2y - 5 = 0$ (2) $x = 5$ (3) $x - 4y - 13 = 0$ **8.** 1

Exercise 3

1. $x = \frac{-5}{3}$ **2.** $x = -1, -2$ **3.** $x = 2$ **4.** $x = -7$

10. (1) b (2) c (3) d (4) b (5) d (6) d (7) c (8) b (9) b (10) c (11) b (12) a (13) d (14) d (15) d (16) b

Exercise 4.1

1. $A + B = \begin{bmatrix} -1 & -3 \\ 3 & 7 \\ 3 & -1 \end{bmatrix}$, $A - B = \begin{bmatrix} 5 & -5 \\ 3 & -3 \\ -5 & 3 \end{bmatrix}$, $2A + B = \begin{bmatrix} 1 & -7 \\ 6 & 9 \\ 2 & 0 \end{bmatrix}$, $A - 2B = \begin{bmatrix} 8 & -6 \\ 3 & -8 \\ -9 & 5 \end{bmatrix}$

2. $A + A^T = \begin{bmatrix} 2\sin\theta & 0 \\ 0 & 2\sin\theta \end{bmatrix}$, $A - A^T = \begin{bmatrix} 0 & -2\cos\theta \\ 2\cos\theta & 0 \end{bmatrix}$

3. $B - A = diag[2 \quad 3 \quad -1]$, $2A + 3B = diag[11 \quad 4 \quad 7]$ **4.** $x = 1$ or 7; $y = -2$ or 6

5. $\begin{bmatrix} \frac{1}{3} & 3 \\ 0 & \frac{4}{3} \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & -8 & -1 \\ 1 & -1 & 1 \\ -7 & -2 & -4 \end{bmatrix}$ **7.** $x = 2,\ y = 4;\quad x = 4,\ y = 2$

8. $a = 4,\ b = 1,\ c = 2,\ d = -2$ **9.** $A = \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix},\ B = \begin{bmatrix} -2 & 1 \\ 5 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} 12 & \frac{4}{3} \\ 4 & -\frac{14}{3} \\ \frac{25}{3} & \frac{28}{3} \end{bmatrix}$

11. $\begin{bmatrix} 17 & -1 & 3 \\ -24 & -1 & -16 \\ -7 & 1 & 0 \end{bmatrix}$ **12.** $a = 2,\ b = -8.$

Exercise 4.2

2. $a = 2,\ b = 4,\ c = 1,\ d = 3$ **4.** $AB = [1],\ BA = \begin{bmatrix} 2 & -2 & 4 \\ 3 & -3 & 6 \\ 1 & -1 & 2 \end{bmatrix}$ **5.** $\begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$

6. $\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$ **8.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **9.** $X = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 5 & \frac{13}{2} \\ 5 & \frac{13}{2} & 8 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & \frac{5}{2} \\ -2 & \frac{-5}{2} & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ **11.** $x = \pm\frac{1}{\sqrt{2}} = y$ **12.** $-2,\ -14$

Exercise 4.3

1. **(1)** $\begin{bmatrix} -3 & 2 \\ -1 & 5 \end{bmatrix}$ **(2)** $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ **(3)** $\begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$ **(4)** $\begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$

2. $\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ **7.** $x = 3$

8. **(1)** $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$ **(2)** $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ **(3)** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ **(4)** $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

9. **(1)** $\{(1, -2)\}$ **(2)** $\left\{\left(\frac{11}{24}, \frac{1}{24}\right)\right\}$ **10.** **(1)** $\{(1, 2, 3)\}$ **(2)** $\left\{\left(\frac{9}{5}, \frac{2}{5}, \frac{7}{5}\right)\right\}$

Exercise 4

1. I **3.** $\begin{bmatrix} \frac{-61}{2} & \frac{47}{2} \\ \frac{87}{2} & \frac{-67}{2} \end{bmatrix}$ **4.** $\begin{bmatrix} 52 & -26 & -21 \\ -42 & 21 & 17 \\ 83 & -41 & -34 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$ **6.** $A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$ **8.** $\frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$ **10.** **(1)** $\{(2, 1)\}$ **(2)** $\{(-1, 2)\}$ **11.** **(1)** $\{(1, 1, -1)\}$ **(2)** $\left\{\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)\right\}$

12. $\begin{bmatrix} a^2 & ab \\ ac & bc+1 \end{bmatrix}$ **13.** $(x, y) = \left\{\left(\frac{c_1 - c_2}{m_2 - m_1}, \frac{m_2c_1 - m_1c_2}{m_2 - m_1}\right)\right\}$ **14.** $x = 6$ **15.** $x = \pm 4\sqrt{3}$

24. **(1)** d **(2)** c **(3)** c **(4)** d **(5)** b **(6)** a **(7)** d **(8)** b **(9)** b **(10)** a **(11)** b **(12)** c **(13)** c **(14)** b **(15)** c **(16)** b **(17)** a

Exercise 5.1

4. Discontinuous for $x = 2$ **5.** Continuous **6.** Continuous **7.** Discontinuous for $x = 0$

8. Continuous **9.** Discontinuous for $x = 0$

10. Discontinuous for $x = 0$ **11.** Continuous **12.** Continuous **13.** $k = 3$ **14.** $k = 5$

15. $k = 1$ **16.** $k = 0$ **17.** $a = 4,\ b = -1$ **26.** Discontinuous **27.** $k = \sqrt{2}$ **28.** $n = 5$

Exercise 5.2

4. **(1)** $2sinx\ cosx$ **(2)** $2tanx\ sec^2x$ **(3)** $4x^3$ **(4)** $-4cos^3x\ sinx$

Exercise 5.3

1. $6sin^2(2x + 3) \cdot cos(2x + 3)$ **2.** $3tan^2x \cdot sec^2x$ **3.** $sin^2x \cdot cos^4x\ (3cos^2x - 5sin^2x)$

4. $-2sin(sin(sec(2x + 3))) \cdot cos(sec(2x + 3)) \cdot sec(2x + 3) \cdot tan(2x + 3)$

5. $-(3x^2 - 1) \cdot sec(cot(x^3 - x + 2)) \cdot tan(cot(x^3 - x + 2)) \cdot cosec^2(x^3 - x + 2)$

7. $(2x + 3)^{m-1} \cdot (3x + 2)^{n-1} \cdot [6(m + n)x + 4m + 9n]$

8. $n(sin^{n-1}x \cdot cosx + cos^{n-1} \cdot sinx)$ **9.** $3sin^2x \cdot cos^2x \cdot cos2x = \frac{3}{4}\ sin^2 2x\ cos\ 2x$

10. $6sin^2(4x - 1) \cdot cos^2(2x + 3)\ [2cos(2x + 3)\ cos(4x - 1) - sin(4x - 1)\ sin(2x + 3)]$

Exercise 5.4

1. $-\frac{x}{y}$ **2.** $\frac{1 + cosx}{cosy}$ **3.** $tan^2\ \frac{x + y}{2}$ **4.** $-\frac{4x + 3y}{3x + 2y}$ **5.** $\frac{ysec^2xy - cosx}{cosy - xsec^2xy}$ **6.** $\frac{9x}{4y}$ **7.** $\frac{5}{y}$

8. $\frac{-25x}{16y}$ **9.** $\frac{x - 2}{3 - y}$ **10.** $\frac{cosx}{cosy}$ **11.** $\frac{3}{\sqrt{1 - x^2}}$ **12.** $\frac{2}{1 + x^2}$

13. $f'(x) = \begin{cases} \frac{2}{1 + x^2}, & x > 0 \\ -\frac{2}{1 + x^2}, & x < 0 \end{cases}$

at $x = 0$, $f'(x)$ does not exist.

14. $f'(x) = \begin{cases} \frac{2}{1 + x^2}, & |x| < 1 \\ -\frac{2}{1 + x^2}, & |x| > 1 \end{cases}$

at $x = \pm 1$, $f'(x)$ does not exist.

15. $\frac{3}{1 + x^2}$ **16.** $\frac{-2}{\sqrt{1 - x^2}}$

Exercise 5.5

1. $\frac{b}{a}cosec\theta$ **2.** $\frac{cos\ \theta - 2cos2\theta}{2sin2\theta - sin\ \theta}$ **3.** $cot\ \frac{\theta}{2}$ **4.** $tant$ **5.** $tan\theta$ **6.** $-\frac{bt^3}{2a}$

Exercise 5.6

1. $\left(x + \frac{1}{x}\right)^x \left(\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right) + \left(x + \frac{1}{x}\right)^{\frac{1}{x}} \left(\frac{x^2 - 1}{x^2(x^2 + 1)} - \frac{1}{x^2}\log\left(x + \frac{1}{x}\right)\right)$

2. $x^x \cdot (1 + \log x)(-\sin x^x + \cos x^x)$

3. $\frac{y}{3}\left[\frac{6}{2x+1} + \frac{20}{4x+3} - \frac{42}{7x-1}\right]$ **4.** $(\log x)^{\cos x}\left(-\sin x \log(\log x) + \frac{\cos x}{x \log x}\right)$

5. $y\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$ **6.** $(\log x)^x\left(\frac{1}{\log x} + \log(\log x)\right) + 1 + \log x$

7. $x^{\sin x}(\sin x \log x + x\cos x \log x + \sin x) + (\sin x)^x(\log(\sin x) + x\cot x)$

8. $x^{\left(x+\frac{1}{x}\right)} \cdot \left(1 + \frac{1}{x^2} + \log x - \frac{1}{x^2}\log x\right)$

9. $(\sin x)^x(\log(\sin x) + x\cot x) + \left(\frac{1}{x}\right)^{\cos x}\left(-\frac{\cos x}{x} + \sin x \cdot \log x\right)$

10. $3^{\sin x} \cdot \cos x \log 3 - 4^{\cos x} \cdot \sin x \log 4$ **11.** $\frac{y^2 - xy\log y}{x^2 - xy\log x}$ **12.** $\frac{xy - y}{xy + x}$ **13.** $-\frac{(x\log y + y)\,y}{(y\log x + x)\,x}$

14. $y\left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right)$ or $\frac{15x^{16} - 16x^{15} + 1}{(x-1)^2}$ **15.** $4x^3 - 15x^2 + 48x - 39$

Exercise 5.7

1. $c = 1$ **2.** $c = 2 + \frac{1}{\sqrt{3}}$ **3.** $c = 0$ **4.** $c = \sqrt{ab}$ **5.** $c = \frac{\pi}{4}$ **6.** $c = \pi$ **7.** $c = \frac{\pi}{2}$

8. $c = \frac{\pi}{4}$ **9.** $c = \pm\frac{\pi}{2}$ **10.** $c = \log_2 e$ **12.** **(1)** $c = \sqrt{3}$ **(2)** $c = \sqrt{\frac{4}{\pi} - 1}$ **14.** $\left(\frac{9}{2}, \frac{1}{4}\right)$

Exercise 5

1. Discontinuous at $x = 3$ **2.** Discontinuous at $x = 1$ **3.** Discontinuous at $x = -1$

4. Discontinuous at $x = 2$ **5.** $k = 5$ **6.** $k = 2$ **7.** $k = 7$ **8.** $k = \pm 2$ **9.** $a = 1,\ b = -1$

10. $a = 5,\ b = 0$ **11.** $\frac{2x}{(x^2+1)\log 10}$ **12.** $\frac{2}{1+x^2}$ **13.** $-\tan x \cdot \cos(\log(\cos x))$ **14.** $-\sqrt{\frac{1-y^2}{1-x^2}}$

15. $(\sin x)^{\sin x} \cdot \cos x \cdot (\log \sin x + 1)$ **16.** $(\sin x - \cos x)^{\sin x - \cos x} \cdot (\cos x + \sin x)(1 + \log(\sin x - \cos x))$

17. $x^x(1 + \log x) + \left(x + \frac{1}{x}\right)^x\left(\log\left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{1 + x^2}\right)$ **18.** $x^{x + \frac{1}{x}} \cdot \left(1 + \frac{1}{x^2} + \log x - \frac{\log x}{x^2}\right)$

19. $-\sin x^x \cdot x^x(1 + \log x) + (\tan x)^x\left(\log \tan x + x \sec x \operatorname{cosec} x\right)$

20. $\frac{dy}{dx} = \begin{cases} 0, & 0 < x < 1 \\ \frac{-2}{\sqrt{1-x^2}}, & -1 < x < 0, \text{ not differentiable for } x = 0. \end{cases}$ **21.** 0

22. $\frac{(\sin t)^t(\log \sin t + t\cot t)}{(\cos t)^t(-t\tan t + \log(\cos t))}$ **24.** $\frac{1}{2(1+x^2)}$ **25.** $\frac{1}{2\sqrt{1-x^2}}$ **26.** $\frac{1}{2}$ **37.** $-\left(\frac{a^2+b^2}{y^3}\right)$

40. $\frac{1}{2}$ **41.** 1 **42.** $\frac{-1}{2\sqrt{3}}$ **43.** $\frac{2}{1+x^2}$ **44.** $\frac{7}{1+49x^2} - \frac{3}{1+9x^2}$ **45.** $\frac{1}{1+x^2}$

46. $\frac{-x}{\sqrt{1-x^4}}$ **47.** $-\frac{1}{2}$

48. **Section A :** **(1)** c **(2)** d **(3)** a **(4)** b **(5)** b **(6)** b **(7)** c **(8)** b **(9)** b **(10)** c **(11)** c **(12)** c **(13)** c **(14)** b **(15)** c

Section B : (16) a (17) d (18) b (19) c (20) a (21) c (22) b (23) d (24) a (25) b

Section C : (26) c (27) b (28) d (29) d (30) a (31) a (32) c

Section D : (33) a (34) b (35) b (36) a (37) c (38) d (39) d (40) d

Exercise 6.1

1. $x^3 + \frac{5}{2}x^2 - 4x + 7\log|x| + 4\sqrt{x} + c$ 2. $\frac{10}{7}x^{\frac{7}{2}} + \frac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{1}{2}} + c$

3. $\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6\sqrt{x} + \frac{2}{\sqrt{x}} + c$ 4. $\frac{2a}{7}x^{\frac{7}{2}} + \frac{2b}{5}x^{\frac{5}{2}} + \frac{2c}{3}x^{\frac{3}{2}} + c'$

5. $\frac{x^{e+1}}{e+1} + e^x + e^e x + c$ 6. $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + c$ 7. $\frac{x^2}{2} + 2x + 4\log|x| + c$

8. $\frac{2^x}{\log_e 2} + \log|x + \sqrt{x^2 - 9}| + c$ 9. $x^2 - \frac{1}{3}tan^{-1}\frac{x}{3} + c$ 10. $\frac{2}{3}x^3 + \frac{3}{2}x^2 + c$

11. $\frac{x^3}{3} - \frac{x^2}{2} + x + c$ 12. $\frac{x^5}{5} - \frac{x^3}{3} + x + tan^{-1}x + c$ 13. $\frac{x^3}{3} - x + 2tan^{-1}x + c$

14. $-3cosx + 5sinx + 8tanx + 4cotx - x + c$ 15. $-2cotx - 3cosecx + c$

16. $4tanx - 9cotx - 25x + c$ 17. $-\frac{1}{4}(cotx + tanx) + c$ 18. $cosecx + cotx + x + c$

19. $-cotx + cosecx + c$ 20. $tanx - cotx - 3x + c$ 21. $-cosecx - cotx - x + c$

22. $secx - tanx + x + c$ 23. $a^2tanx - b^2cotx - (a - b)^2x + c$ 24. $x + \frac{\sqrt{3}}{2}\log\left|\frac{x - \sqrt{3}}{x + \sqrt{3}}\right| + c$

25. $2x^4 - x^2 - 20$

Exercise 6.2

1. $\frac{1}{5}\log|5x - 3| + c$ 2. $\frac{1}{7}e^{7x+4} + \frac{(5x-3)^9}{45} + c$ 3. $\frac{7^{2x+3}}{2\log_e 7} - \frac{cot2x}{2} - x + c$

4. $\frac{5^{4x+3}}{4\log_e 5} + \frac{3}{2}cos(2x + 3) + c$ 5. $\frac{1}{\sqrt{5}}\log|\sqrt{5}x + \sqrt{5x^2 - 4}| + c$

6. $\frac{1}{3}sin^{-1}\left(\frac{3x}{4}\right) + c$ 7. $\frac{1}{\sqrt{5}}\log|\sqrt{5}x + \sqrt{5x^2 + 3}| + \frac{1}{12}\log\left|\frac{2x+3}{2x-3}\right| + c$

8. $\frac{1}{\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2 + 3}| + \frac{1}{\sqrt{21}}tan^{-1}\left(\frac{\sqrt{7}x}{\sqrt{3}}\right) + c$

9. $2x^2 + 12x + 25\log|x - 2| + c$ 10. $\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x + \log|x + 1| + c$

11. $-\frac{2}{3}(5 - 3x)^{\frac{1}{2}} + c$ 12. $\frac{3^{5x-2}}{5\log_e 3} + \frac{1}{4(2x+1)^2} + c$ 13. $-\frac{1}{5}cot(3 + 5x) - x + c$

14. $\frac{x}{2} - \frac{1}{12}sin(6x + 10) + c$ 15. $\frac{1}{3}(cosec3x - cot3x) + c$ 16. $2\sqrt{2}sin\frac{x}{2} + c$

17. $\frac{2}{27}[(3x + 4)^{\frac{3}{2}} + (3x + 1)^{\frac{3}{2}}] + c$ 18. $-\frac{1}{6}(5 - 2x)^{\frac{3}{2}} + \frac{1}{6}(3 - 2x)^{\frac{3}{2}} + c$

19. $\log|x + 1| - \frac{1}{x+1} + c$ 20. $x - 2\log|x + 1| - \frac{2}{x+1} + c$

21. $\frac{x^3}{3} + 2x^2 + 6x + 7\log|x - 1| + c$ 22. $\frac{2}{5}(x + 3)^{\frac{5}{2}} - 2(x + 3)^{\frac{3}{2}} + c$

23. $\frac{2}{3}(x + 1)^{\frac{3}{2}} - 2(x + 1)^{\frac{1}{2}} + c$ 24. $\frac{1}{6}(2x + 1)^{\frac{3}{2}} + \frac{1}{2}(2x + 1)^{\frac{1}{2}} + c$

25. $\frac{1}{3}(4x+7)^{\frac{3}{2}} - \frac{1}{2}(4x+7)^{\frac{1}{2}} + c$ **26.** $\frac{3}{8}x + \frac{1}{4}sin2x + \frac{1}{32}sin4x + c$

27. $\frac{-3}{64}cos2x + \frac{1}{192}cos6x + c$ **28.** $\frac{cos^3(2x-1)}{6} - \frac{cos(2x-1)}{2} + c$

29. $\frac{1}{12}sin6x + \frac{1}{4}sin2x + c$ **30.** $\frac{2}{3}sin\,3x + 2sinx + c$

31. $\frac{1}{48}sin12x + \frac{1}{16}sin4x + \frac{1}{32}sin8x + \frac{x}{4} + c$ **32.** $\sqrt{2}\log|cosec\,\frac{x}{2} - cot\,\frac{x}{2}| + c$

33. $2\log\left|sin\,\frac{x}{2}\right| + c$ **34.** $\frac{1}{2}\left[\frac{sin\,(m-n)x}{m-n} - \frac{sin\,(m+n)x}{m+n}\right] + c$

35. $xcosa + sina\log|sin(x-a)| + c$ **36.** $\frac{1}{sin\,(b-a)}\log\left|\frac{sin(x-b)}{sin(x-a)}\right| + c$

37. $-\frac{3x}{2} - \frac{13}{4}\log|3-2x| + c$ **38.** $\frac{1}{5}(3x^2-4x+5)^{\frac{5}{2}} + c$

39. $\sqrt{x^2+6x+4} + c$ **40.** $\frac{1}{30}(5x^4+3)^{\frac{3}{2}} + c$ **41.** $\frac{1}{2}\log x - \frac{1}{4}sin(2\log x) + c$

42. $\frac{2}{3}(\log x+1)^{\frac{3}{2}} + c$ **43.** $\frac{1}{2n\,(m+ncos2x)} + c$ **44.** $\log|sinx + cosx| + c$

45. $tan(xe^x) + c$ **46.** $\frac{1}{2}cot(2e^{-x}+3) + c$ **47.** $\frac{1}{e}\log|x^e + e^x| + c$

48. $\frac{-1}{tan^3x + 2tanx + 9} + c$ **49.** $\frac{1}{(b-a)(asin^2x + bcos^2x)}$ **50.** $\frac{1}{2(b^2-a^2)}\log|a^2cos^2x + b^2sin^2x| + c$

51. $\frac{1}{4}(sin^{-1}x^2)^2 + c$ **52.** $\frac{-2}{\sqrt{tan^{-1}x}} + c$ **53.** $\frac{1}{2}[\log\,(sine^x)]^2 + c$ **54.** $-\frac{1}{2}\left\{\log\left(\frac{x+1}{x}\right)\right\}^2 + c$

55. $\frac{1}{2}tan^2x + \log|cosx| + c$ **56.** $\frac{sec^4x}{4} + c$ **57.** $\frac{tan^5x}{5} - \frac{tan^3x}{3} + tanx - x + c$

58. $2(x+1)^{\frac{1}{2}} - 3(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{6}} - 6\log|(x+1)^{\frac{1}{6}} + 1| + c$

59. $\frac{3}{8}(x+2)^{\frac{8}{3}} - \frac{12}{5}(x+2)^{\frac{5}{3}} + 6(x+2)^{\frac{2}{3}} + c$ **60.** $\frac{1}{ab}tan^{-1}\left(\frac{b}{a}tanx\right) + c$

61. $\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{tanx}{\sqrt{3}}\right) + c$ **62.** $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}cotx}{1-\sqrt{3}cotx}\right| + c$ **63.** $\frac{1}{6}tan^{-1}\left(\frac{2}{3}tanx\right) + c$

64. $\frac{1}{\sqrt{2}}tan^{-1}(\sqrt{2}\,tanx) + c$

Exercise 6.3

1. $\frac{-\sqrt{1-x^2}}{x} + c$ **2.** $\frac{-\sqrt{9-x^2}}{x} - sin^{-1}\frac{x}{3} + c$ **3.** $\frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + c$

4. $\frac{a^6}{6}[sin^{-1}\frac{x^3}{a^3} + \frac{x^3}{a^6}\sqrt{a^6-x^6}] + c$ **5.** $2sin^{-1}\sqrt{\frac{x}{2a}} + c$ **6.** $2sin^{-1}\sqrt{\frac{x}{2}} + \sqrt{2x-x^2} + c$

7. $\sqrt{a^2-x^2} - acos^{-1}\frac{x}{a} + c$ **8.** $\frac{1}{3}sin^{-1}\frac{x^3}{a^3} + c$ **9.** $-\frac{1}{x} - \frac{3}{2}tan^{-1}x - \frac{x}{2(1+x^2)} + c$

10. $\frac{1}{9}\frac{1}{\sqrt{16-9x^2}} + c$ **11.** $\log|x + \sqrt{x^2-a^2}| - \frac{x}{\sqrt{x^2-a^2}} + c$

12. $-\frac{a^2}{2}cos^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4-x^4} + c$ **13.** $2\log|\sqrt{x-1} + \sqrt{x-2}| + c$

14. $-\frac{\sqrt{25-x^2}}{x} - sin^{-1}\frac{x}{5} + c$ **15.** $\log|tan\frac{x}{2} + 1| + c$ **16.** $tan^{-1}\left(1 + tan\frac{x}{2}\right) + c$

17. $\frac{2}{3}\ tan^{-1}\left(\frac{tan\frac{x}{2}}{3}\right) + c$ **18.** $\frac{2}{sin\alpha}\ tan^{-1}\left(tan\frac{\alpha}{2}\ tan\frac{x}{2}\right) + c$ **19.** $\frac{2}{\sqrt{3}}\ tan^{-1}\left(\sqrt{3}tan\frac{x}{2}\right) + c$

20. $\frac{1}{\sqrt{2}}\ \log\left|\frac{tan\frac{x}{2}+1+\sqrt{2}}{tan\frac{x}{2}+1-\sqrt{2}}\right| + c$ **21.** $\frac{sin^5x}{5} - \frac{sin^7x}{7} + c$ **22.** $\frac{cos^{13}x}{13} - \frac{cos^{11}x}{11} + c$

23. $\frac{sin^8x}{8} - \frac{sin^{10}x}{10} + c$ **24.** $-\frac{cos^5x}{5} + \frac{2}{7}cos^7x - \frac{cos^9x}{9} + c$

25. $-cosx + \frac{2}{3}cos^3x - \frac{cos^5x}{5} + c$ **26.** $\frac{1}{32}\left(2x - \frac{1}{2}sin2x - \frac{1}{2}sin4x + \frac{1}{6}sin6x\right) + c$

Exercise 6.4

1. $\frac{2}{\sqrt{3}}\ tan^{-1}\left(\frac{2x+3}{\sqrt{3}}\right) + c$ **2.** $\frac{1}{2\sqrt{2}}\ tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$ **3.** $-\frac{1}{6\sqrt{2}}\ \log\left|\frac{3x+1-\sqrt{2}}{3x+1+\sqrt{2}}\right| + c$

4. $\frac{1}{4}\ \log\left|\frac{x+1}{3-x}\right| + c$ **5.** $\log\left|x - \frac{1}{2} + \sqrt{x^2-x+5}\right| + c$

6. $\frac{1}{\sqrt{2}}\ \log\left|\frac{4x+3}{4} + \sqrt{x^2+\frac{3}{2}x-1}\right| + c$ **7.** $\frac{1}{\sqrt{2}}\ sin^{-1}\left(\frac{4x+3}{\sqrt{65}}\right) + c$

8. $\frac{1}{\sqrt{3}}\ \log\left|x+\frac{5}{6}+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right| + c$ **9.** $\log\left|x-\frac{3}{2}+\sqrt{x^2-3x+2}\right| + c$

10. $sin^{-1}\left(\frac{x-4}{5}\right) + c$ **11.** $2\log|x^2 + 3x + 2| - 5\log\left|\frac{x+1}{x+2}\right| + c$

12. $\frac{3}{4}\log|2x^2 + x + 1| + \frac{5}{2\sqrt{7}}\ tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right) + c$

13. $2\sqrt{x^2+4x+5} - \log\left|x + 2 + \sqrt{x^2+4x+5}\right| + c$ **14.** $-3\sqrt{5-2x-x^2} - 2\ sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$

15. $2\log|sin^2x - 4sinx + 5| + 7tan^{-1}(sinx - 2) + c$ **16.** $sin^{-1}\left(\frac{e^x+2}{3}\right) + c$

17. $\frac{1}{3}\log\left|x^3 + 1 + \sqrt{x^6+2x^3+3}\right| + c$ **18.** $sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right) + c$ **19.** $\frac{1}{\sqrt{2}}\ tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c$

20. $\frac{1}{2\sqrt{2}}\ tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + c$ **21.** $\frac{1}{3}\ tan^{-1}\left(\frac{x^2-1}{3x}\right) + c$

22. $\frac{1}{2\sqrt{2}}\ tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\ \log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + c$

23. $\frac{1}{2}\ \log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$ **24.** $\frac{1}{2\sqrt{3}}\ tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + \frac{1}{4}\ \log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

Exercise 6

1. $x - \frac{6}{5}x^{\frac{5}{6}} + \frac{3}{2}x^{\frac{2}{3}} - 2\cdot x^{\frac{1}{2}} + 3\cdot x^{\frac{1}{3}} - 6\cdot x^{\frac{1}{6}} - 6\log|1 + x^{\frac{1}{6}}| + c$

2. $\frac{1}{2}\ [\log|x + \sqrt{1+x^2}|]^2 + c$ **3.** $-\frac{\sqrt{x^2+2x+2}}{x+1} + c$

4. $-2\sqrt{1-x} + cos^{-1}\sqrt{x} + \sqrt{x-x^2} + c$ 5. $\sqrt{x^2+5x+6} + \frac{1}{2}\log|x+\frac{5}{2}+\sqrt{x^2+5x+6}| + c$

6. $x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + c$ 7. $x - \frac{7}{2}\log|x^2+7x+10| + \frac{29}{6}\log\left|\frac{x+2}{x+5}\right| + c$

8. $\frac{1}{sin(a-b)}\log\left|\frac{cos(x-a)}{cos(x-b)}\right| + c$ 9. $xcos(a-b) + sin(a-b)\log|sin(x+b)| + c$

10. $\frac{-1}{n+1}(1-x)^{n+1} + \frac{1}{n+2}(1-x)^{n+2} + c$

11. $\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{tanx-1}{\sqrt{2\,tanx}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{tanx-\sqrt{2tanx}+1}{tanx+\sqrt{2tanx}+1}\right| + c$

12. $\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{tan^2x-1}{\sqrt{2}tanx}\right) + c$ 13. $\frac{2}{1-a^2}tan^{-1}\left[\left(\frac{1+a}{1-a}\right)tan\frac{x}{2}\right] + c$

14. **Section A :** (1) c (2) b (3) c (4) c (5) c (6) c (7) a (8) c (9) b (10) c (11) b (12) c (13) b (14) c (15) a (16) b (17) a (18) c (19) b (20) a

Section B : (21) d (22) d (23) d (24) c (25) c (26) d (27) c (28) b (29) d (30) b (31) c (32) c (33) d

Section C : (34) a (35) c (36) d (37) a (38) c (39) d (40) d (41) a (42) c (43) c

Section D : (44) b (45) b (46) d (47) c (48) d

Exercise 7.1

1. $\frac{3}{7}$ 2. $\frac{4}{7}$ 3. $\frac{1}{5}, \frac{2}{3}$ 4. $\frac{1}{3}$ 5. $\frac{2}{3}$ 6. (1) $\frac{11}{56}$ (2) $\frac{13}{44}$ 7. $\frac{1}{2}$ 8. (1) $\frac{3}{8}$ (2) $\frac{2}{5}$ 9. $\frac{1}{6}$

10. (1) $\frac{1}{2}$ (2) $\frac{3}{7}$ (3) $\frac{6}{7}$

Exercise 7.2

1. Yes 2. $\frac{10}{13}$ 3. (1) $\frac{15}{91}$ (2) $\frac{15}{91}$ (3) $\frac{5}{21}$ 4. 0.21 5. $\frac{1}{3}$ 6. 0.963

7. (1) $\frac{4}{25}$ (2) $\frac{3}{8}$ 8. (1) $\frac{71}{80}$ (2) $\frac{36}{71}$

Exercise 7.3

1. (1) $\frac{1}{10}$ (2) $\frac{1}{385}$ (3) $\frac{1}{40}$ (4) $\frac{64}{21}$ (5) $\frac{1}{16}$

2. (1) Yes 3. (1) $\frac{3}{20}$ (2) $\frac{3}{4}$ (3) $\frac{3}{10}$ (4) $\frac{11}{20}$

4.

X = x	2	3	4	5	6	7	8	9	10	11	12
p(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

5.

X = x	0	1	2
p(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

6.

X = x	0	1	2
p(x)	$\frac{42}{90}$	$\frac{42}{90}$	$\frac{6}{90}$

7. (1) $c = \frac{1}{3}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$

8.

X = x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Exercise 7.4

1.

X = x	1	2	3	4	5	6
$p(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Mean = 2.53, Variance = 1.96, Standard deviation = 1.4

2. ₹ 25, Yes **3.** **(1)** $k = \frac{1}{10}$ **(2)** Mean = 3.6, Variance = 1.64

4. **(1)** $k = \frac{1}{5}$ **(2)** Mean = 1.1, Variance = 1.69, Standard deviation = 1.3

5. $\frac{35}{12}$ **6.** **(1)** 0 **(2)** 1.6 **(3)** 2 **(4)** 14.4 **7.** ₹ 8 **8.** 125, 135, 0, 1

Exercise 7.5

1. **(1)** $\frac{63 \times 4^6}{5^{10}}$ **(2)** $\frac{4^9 \times 14}{5^{10}}$ **2.** **(1)** $\frac{144}{625}$ **(2)** $\frac{32}{3125}$

3. 0.6517 **4.** 0.0512 **5.** $n = 16$, $p = \frac{1}{2}, \frac{1}{2^{16}}, \frac{696}{2^{16}}$ **6.** 0.9963 **7.** **(1)** 0.3950 **(2)** 0.4074

8. **(1)** 0.6630 **(2)** 0.6826 **9.** **(1)** 0.512 **(2)** 0.384 **(3)** 0.104 **10.** **(1)** 40 **(2)** 36

Exercise 7

1. $\frac{4}{7}$ **2.** **(1)** $\frac{1}{3}$ **(2)** $\frac{1}{2}$ **3.** $\frac{3}{7}$ **4.** **(1)** $\frac{16}{121}$ **(2)** $\frac{49}{121}$ **(3)** $\frac{56}{121}$

5. **(1)** $\frac{2}{5}$ **(2)** $\frac{1}{10}$ **(3)** $\frac{13}{30}$ **(4)** $\frac{1}{60}$ **6.** 0.175, $\frac{17}{33}$

7.

X = x	–2	5	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

8. $\frac{101}{2}, \frac{6767}{2}, \frac{3333}{4}$ **9.** **(1)** $\frac{63}{256}$ **(2)** $\frac{65}{256}$ **10.** $\frac{9}{17}$ **11.** $\frac{3}{16}$ **12.** 3 : 2 **13.** 0.35294

15. **(1)** a **(2)** c **(3)** a **(4)** b **(5)** a **(6)** d **(7)** a **(8)** d **(9)** b **(10)** d
(11) c **(12)** d **(13)** d **(14)** c **(15)** d **(16)** c **(17)** a **(18)** b **(19)** a **(20)** b
(21) c **(22)** b **(23)** a **(24)** c

Exercise 8.1

4. 800 **5.** 120 **6.** 2300 **7.** 60, 180 **8.** Feasible region does not exist
9. 16 **10.** 18 **11.** Maximum value does not exist **12.** 400

Exercise 8

1. 10 **2.** −2 **3.** 13 **4.** 22 **5.** 240 **6.** Feasible region does not exist

7. A type machines 6, B type machines 0, maximum output 360

8. A type food 5 units, B type food 30 units, least cost ₹ 145.

9. 5 *l* oil tins 8, 1 *kg* ghee tins 12, maximum profit ₹ 392.

10. 30 **11.** A to D : 500 *l*, A to E = 3000 *l*, A to F : 3500 *l*, B to D : 4000 *l*

12. 40 executive class tickets, 160 economy class tickets, maximum profit ₹ 1,36,000

13. 6, 6

14. **(1)** b **(2)** b **(3)** c **(4)** d **(5)** c **(6)** b **(7)** a **(8)** d **(9)** a **(10)** b **(11)** c **(12)** a **(13)** b **(14)** b

Srinivasa Ramanujan : Life in England

Ramanujan boarded the S.S. Nevasa on 17 March 1914, and at 10 o'clock in the morning, the ship departed from Madras. He arrived in London on 14 April, with E. H. Neville waiting for him with a car. Four days later, Neville took him to his house on Chesterton Road in Cambridge. Ramanujan immediately began his work with Littlewood and Hardy. After six weeks, Ramanujan moved out of Neville's house and took up residence on Whewell's Court, just a five-minute walk from Hardy's room. Hardy and Ramanujan began to take a look at Ramanujan's notebooks. Hardy had already received 120 theorems from Ramanujan in the first two letters, but there were many more results and theorems to be found in the notebooks. Hardy saw that some were wrong, others had already been discovered, while the rest were new breakthroughs. Ramanujan left a deep impression on Hardy and Littlewood. Littlewood commented, "I can believe that he's at least a Jacobi", while Hardy said "he can compare him only with [Leonhard] Euler or Jacobi."

Ramanujan spent nearly five years in Cambridge collaborating with Hardy and Littlewood and published a part of his findings there. Hardy and Ramanujan had highly contrasting personalities. Their collaboration was a clash of different cultures, beliefs and working styles. Hardy was an atheist and an apostle of proof and mathematical rigour, whereas Ramanujan was a deeply religious man and relied very strongly on his intuition. While in England, Hardy tried his best to fill the gaps in Ramanujan's education without interrupting his spell of inspiration.

Ramanujan was awarded a B.A. degree by research (this degree was later renamed PhD) in March 1916 for his work on highly composite numbers, which was published as a paper in the Journal of the London Mathematical Society. The paper was over 50 pages with different properties of such numbers proven. Hardy remarked that this was one of the most unusual papers seen in mathematical research at that time and that Ramanujan showed extraordinary ingenuity in handling it. On 6 December 1917, he was elected to the London Mathematical Society. He became a Fellow of the Royal Society in 1918, becoming the second Indian to do so, following Ardaseer Cursetjee in 1841, and he was one of the youngest Fellows in the history of the Royal Society. He was elected "for his investigation in Elliptic functions and the Theory of Numbers." On 13 October 1918, he became the first Indian to be elected a Fellow of Trinity College, Cambridge.

TERMINOLOGY

(In Gujarati)

Antiderivation	પ્રતિવિકલન
Antiderivative	પ્રતિવિકલિત
Arbitrary Constant	સ્વૈર અચળ
Binary Operation	દ્વિક્રિયા
Binomial Distribution	દ્વિપદી વિતરણ
Chain Rule	સાંકળનો નિયમ
Cofactor	સહઅવયવ
Column	સ્તંભ
Composite Function	સંયોજિત વિધેય
Conditional Probability	શરતી સંભાવના
Consistent	સુસંગત
Constraints	મર્યાદાઓ
Decision Variables	નિર્ણાયક ચલરાશિઓ
Determinant	નિશ્ચાયક
Equivalence Relation	સામ્ય સંબંધ
Event	ઘટના
Feasible Region	શક્ય ઉકેલનો પ્રદેશ
Feasible Solution	શક્ય ઉકેલ
Implicit Function	ગૂઢ વિધેય
Indefinite Integral	અનિયત સંકલિત
Independent Events	નિરપેક્ષ ઘટનાઓ
Infeasible Solution	અશક્ય ઉકેલ
Integrable	પ્રતિવિકલનીય
Integral	સંકલિત
Integrand	સંકલ્ય
Inverse Function	પ્રતિવિધેય
Linear Programming	સુરેખ આયોજન
Many-one Function	અનેક-એક વિધેય
Mathematical Expectation	ગાણિતિક અપેક્ષા
Matrix	શ્રેણિક
Method of Substitution	આદેશની રીત
Minor	ઉપનિશ્ચાયક
Non-singular Matrix	સામાન્ય શ્રેણિક
Objective Function	હેતુલક્ષી વિધેય
One-one Function	એક-એક વિધેય
Onto Function	વ્યાપ્ત વિધેય
Optimal Feasible Solution	ઈષ્ટતમ શક્ય ઉકેલ
Optimum Value	ઈષ્ટતમ મૂલ્ય
Order	કક્ષા
Primitive	પૂર્વગ
Random Variable	યાદચ્છિક ચલ
Reflexive Relation	સ્વવાચક સંબંધ
Row	હાર
Sample Space	નિદર્શાવકાશ
Skew-symmetric Matrix	વિસંમિત શ્રેણિક
Standard Deviation	પ્રમાણિત વિચલન
Symmetric Matrix	સંમિત શ્રેણિક
Symmetric Relation	સંમિત સંબંધ
Transitive Relation	પરંપરિત સંબંધ
Transpose of a Matrix	પરિવર્ત શ્રેણિક
Universal Relation	સાર્વત્રિક સંબંધ
Variance	વિચરણ