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# STATISTICS

## (Part 2)

## Standard 12

### PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

**Price : ₹ 46.00**

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The Board expresses the pleasure to publish the Textbook of **Statistics (Part 2)** for **Std. 12** which is a translated version of Gujarati. The Textbook Board is thankful to all those who have helped in preparing this textbook. However, we welcome suggestions to enhance the quality of the textbook.

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# FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India : *

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(h) to develop scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(k) to provide opportunities for education by the parent, the guardian, to his child, or a ward between the age of 6-14 years as the case may be.

*Constitution of India : Section 51-A

# CONTENTS



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*"Statistically, the probability of any one of us being here is so small that the mere fact of our existence should keep us all in a state of contented dazzlement."*

**– Lewis Thomas**

# 1

# Probability

**Contents :**

## 1.1 Introduction

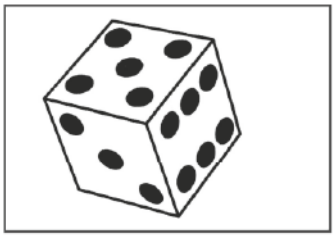
Many events occur in our day-to-day life. We can definitely say for many events that these events will certainly happen. For example, each person taking birth will die, a fruit freely falling from a tree will fall on ground, if the profit per item of a trader is ₹ 10 then he will earn a profit of ₹ 500 by selling 50 items, if a person invests ₹ 1,00,000 in a nationalised bank at an annual interest rate of 7.5 percent then the interest received will be ₹ 7,500, etc. These events are certain but some events are such that we can not be definitely say in advance whether they will happen. For example, getting head on the upper side after tossing a balanced coin, getting number 3 on the upper side of a die when a six faced unbiased die is thrown, the new baby to be born will be a boy, an item produced in a factory is non-defective, what will be the total rainfall in a certain region in the current year, what will be the wheat production in a state in the current year, what will be the result of a cricket match played between teams of two countries, etc. We cannot say with certainty that these events will definitely occur. It is not possible to give precise prediction about the occurrance of such events. We can intuitively get some idea about possibility of happening (or not happening) for these events but there is uncertainty regarding happening (or not happening) of these events. We accept that the occurrence (or non-occurrence) of these events depends upon an unknown element which is called chance. Such events which depend on chance are called random events. Probability is used to numerically express the possibility of these uncertain events. We shall study the theory of probability, the classical definition of probability, its statistical definition and the illustrations showing utility of probability. Now, let us see the explanation of certain terms which are useful to study probability.

## 1.2 Random Experiment and Sample Space

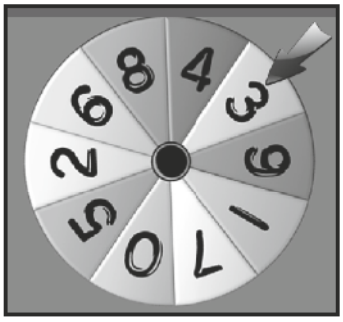
### 1.2.1 Random Experiment

Let us consider the following experiments :

**Experiment 1 :** Toss a balanced coin. Any one outcome is obtained out of two posible outcomes (i) Head-*H* (ii) Tail-*T* for this experiment. (We assume that the coin does not stand on its edge.) Thus, *'H'* and *'T'* are the only possible outcomes for the experiment of tossing a coin. But which of the outcomes will be obtained among these two outcomes cannot be said with certainty before conducting the experiment.



**Experiment 2 :** Throw a balanced die with six faces marked with numbers 1, 2, 3, 4, 5, 6 on it. Note the number appearing on its upper face. Any one outcome among the six possible outcomes 1, 2, 3, 4, 5, 6 will be obtained. There are only six possible outcomes 1, 2, 3, 4, 5, 6 for this experiment of throwing a die but which of the six outcomes will be obtained cannot be said with certainty before conducting the experiment.




**Experiment 3 :** Suppose there is a wheel marked with 10 numbers 0, 1, 2, .... , 9 and a pointer is kept against it. If this wheel is rotated with hand, it will spin and become stable after some time. When the wheel stops, any one of the numbers 0, 1, 2, .... , 9 will appear against the pointer. This number is the winning number. There are total ten possible outcomes 0, 1, 2, .... , 9 for this experiments. But which of the ten numbers will be obtained as a winning number cannot be said with certainty before conducting the experiment.

The experiments 1, 2, 3 shown above are called random experiments. A random experiment is defined as follows. The experiment which can be independently repeated under indentical conditions and all its possible outcomes are known but which of the outcomes will appear cannot be predicted with certainty before conducting the experiment is called a random experiment. The following characteristics of the random experiment can be deduced from its definition :

(1) A random experiment can be independently repeated under almost identical conditions.

(2) All possible outcomes of the random experiment are known but which of the outcomes will appear cannot be predicted before conducting the experiment.

(3) The random experiment results into a certain outcome.

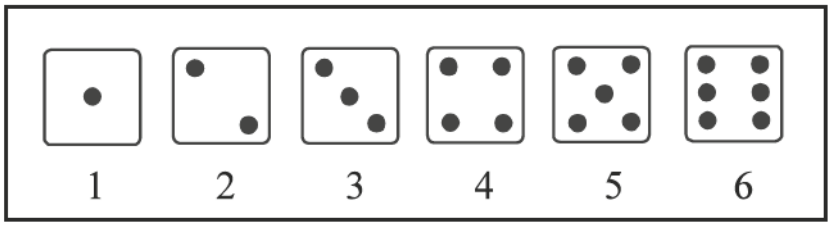
### 1.2.2 Sample Space

The set of all possible outcomes of a random experiment is called a sample space of that random experiment. The sample space is generally denoted by $U$ or $S$. The elements of sample space are called sample points.

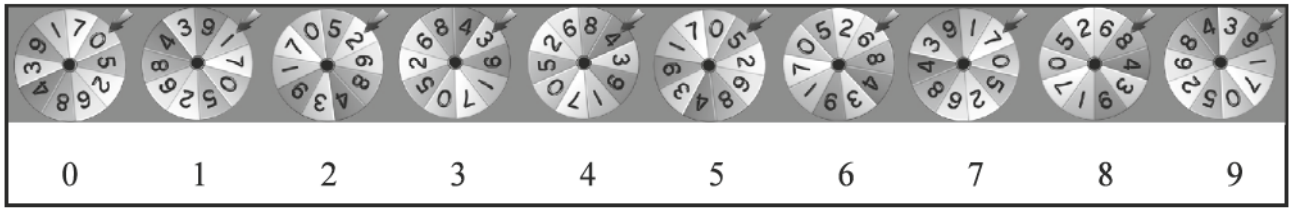
The sample space of the random experiment in the earlier discussion can be obtained as follows :

**Experiment 1 :** Toss a balanced coin. There are total two possible outcomes for this random experiment : $H$ and $T$. Thus, the Sample Space can be written here as $U = \{H, T\}$ or $U = \{T, H\}$.


**Experiment 2 :** Throw a balanced die with six faces with numbers 1, 2, 3, 4, 5, 6 on it. There are total six possible outcomes for this random experiment : 1, 2, 3, 4, 5, 6. Thus, the sample space is $U = \{1, 2, 3, 4, 5, 6\}$.

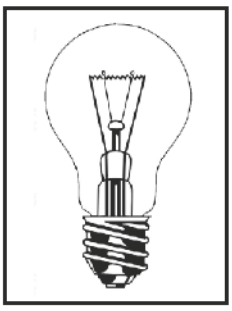



**Experiment 3 :** To decide the winning number by rotating a wheel marked with numbers 0, 1, 2, .... , 9. There are total ten possible outcomes for this random experiment. Thus, the sample space $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.



**Finite Sample Space :** If the total number of possible outcomes in the sample space is finite then it is called a finite sample space. For example, the sample spaces of all the three random experiments given above are finite sample spaces.

**Infinite Sample Space :** If the total number of possible outcomes in the sample space of a random experiment is infinite then it is called an infinite sample space. For example, if the life of electric bulbs ($L$) from a production is recorded in hours then it is a real number. The value of $L$ will be 0 or more. Thus, there will be infinite possible outcomes for an experiment of measuring life of bulbs. The sample space will be $U = \{L \mid L \geq 0,\ L \in R\}$. If the maximum life of electric bulbs is assumed to be 700 hours, the sample space will be $U = \{L \mid 0 \leq L \leq 700;\ L \in R\}$ which is an Infinite sample space.



Now we shall see some more illustrations of sample space of a random experiment.

**Illustration 1 : Two balanced coins are tossed simultaneously. Write the sample space of this random experiment.**

We shall consider any one of the two coins here as the first coin and the other as the second coin. The outcome of this experiment will be as shown in the following diagram.

If we denote the head as $H$ and the tail as $T$, the sample space will be as follows :

$$U = \{HH,\ HT,\ TH,\ TT\}$$

Any one of the outcomes out of $H$ and $T$ can be obtained on the first coin. Thus, this action can be done in two ways and the other coin can also show one of the outcomes $H$ and $T$ which can also be done in two ways. According to the fundamental principle of counting for multiplication, the total number of outcomes will be $2 \times 2 = 2^2 = 4$. It should be noted here that the sample space for the experiment of tossing one balanced coin two times will also be the same as above.

**Illustration 2 : Two balanced dice are thrown where each die has numbers 1 to 6 on the six sides. Write the sample space of this experiment.**

We shall consider any one die as the first die and the other will be called the second die. The number on the first die will be shown as $i$ and the number on the second die will be shown as $j$. The following sample space will be obtained by denoting the pair of numbers on the two dice as $(i,\ j)$ where $i,\ j = 1, 2, 3, 4, 5, 6$.

$$\begin{aligned} U = \{&(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (1,\ 5),\ (1,\ 6),\\ &(2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (2,\ 4),\ (2,\ 5),\ (2,\ 6),\\ &(3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4),\ (3,\ 5),\ (3,\ 6),\\ &(4,\ 1),\ (4,\ 2),\ (4,\ 3),\ (4,\ 4),\ (4,\ 5),\ (4,\ 6),\\ &(5,\ 1),\ (5,\ 2),\ (5,\ 3),\ (5,\ 4),\ (5,\ 5),\ (5,\ 6),\\ &(6,\ 1),\ (6,\ 2),\ (6,\ 3),\ (6,\ 4),\ (6,\ 5),\ (6,\ 6)\} \end{aligned}$$

**OR**

$$U = \{(i,\ j);\ i,\ j = 1, 2, 3, 4, 5, 6\}$$

Any one of the integers 1 to 6 can be shown on the upper side of the first die which can occur in 6 ways and the second die can also show one of the integers among 1 to 6 which will also occur in 6 ways. The total number of outcomes will be $6 \times 6 = 6^2 = 36$ according to fundamental principle of counting for multiplication. Similarly, the sample space for the random experiment of throwing three balanced dice simultaneously will have $6^3 = 216$ total outcomes.

**Illustration 3 : Write the sample space of the random experiment of finding the number of defective items while testing the quality of 1000 items produced in a factory.**

If the defective items are found among 1000 items produced in the factory then the number of defective items in the production can be 0, 1, 2, .... , 1000. Thus, the sample sapce will be as follows :

$$U = \{0, 1, 2, \ldots, 1000\}$$

**Illustration 4 : Write the sample space of random experiment of randomly selecting three numbers from the first four natural numbers.**

If three numbers are selected simultaneously from the first four natural numbers 1, 2, 3, 4 then those three numbers can be $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$ or $(2, 3, 4)$. Thus, the sample space of the random experiment will be as follows :

$$U = \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}$$

3 numbers are to be selected here from the 4 numbers which has ${}^4C_3 = 4$ combinations. Thus, the total number of outcomes for this random experiment is 4.

**Illustration 5 : Write the sample space of a random experiment of randomly selecting any one number from the natural numbers.**

The natural numbers are 1, 2, 3, ..... If one number is randomly selected from these numbers then the sample space will be as follows :

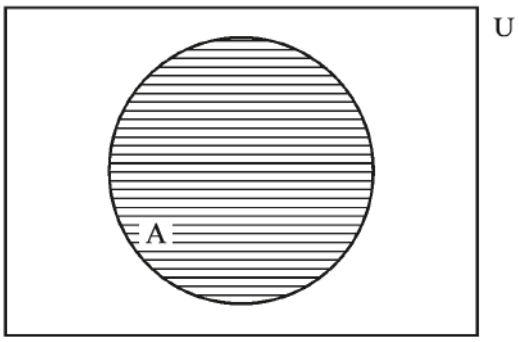
$$U = \{1, 2, 3, 4, \ldots\}$$

It should be noted here that this is an infinite sample space.

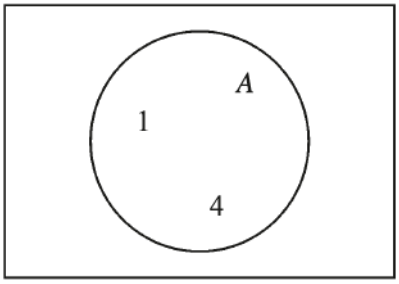
## 1.3 Events : Certain Event, Impossible Event, Special Events

We will study the different types of events by first understanding the meaning of an event.

**(1) Event :** A subset of the sample space of a random experiment is called an event. The events are generally denoted by letters $A$, $B$, $C$, ... or as $A_1$, $A_2$, $A_3$, ... . The set formed by the sample points showing favourable outcomes of an event $A$ will be a subset of the sample space $U$. Thus, any event $A$ associated with the random experiment is the subset of sample space $U$. This is denoted as $A \subset U$.



For example, the sample space of a random experiment of throwing a balanced die is $U = \{1, 2, 3, 4, 5, 6\}$. If the event of obtaining a complete square as a number on the upper side of the die is denoted by $A$ then event $A = \{1, 4\}$.




Now, we shall show that an event is a subset of the sample space by taking a few examples of events in the random experiment of throwing two balanced dice.

- $A_1$ = the sum of numbers on the dice is 6.

  $\therefore\ A_1 = \{(1, 5), (2, 4), (3, 3), (4,2), (5,1)\}$

- $A_2$ = the numbers on the dice are same.

  $\therefore\ A_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

- $A_3$ = the sum of numbers on the dice is more than 9.

  $\therefore\ A_3 = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

All these subsets are called events.

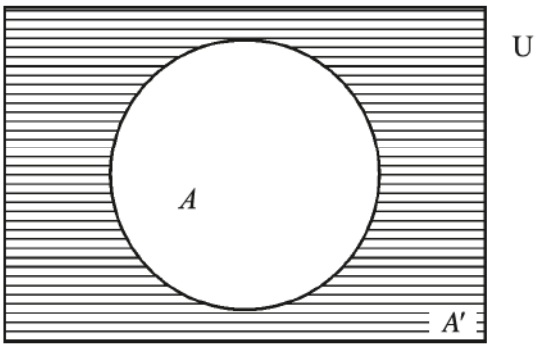
**(2) Impossible Event :** The special subset $\phi$ or $\{\ \}$ of the sample space of a random experiment is called an impossible event. Impossible event is an event which never occurs. It is denoted by $\phi$ or $\{\ \}$.

For example, the event of getting both head ($H$) and tail ($T$) on a balanced coin is an impossible event.

**(3) Certain Event :** The special subset $U$ of the sample space of random experiment is called a certain event. The certain event is an event which always occurs. It is denoted by $U$.

For example, the day next to Saturday is Sunday, the number on the upper side die when a balanced die is thrown is less than 7, etc. are certain events.

**(4) Complementary Event :** Suppose $U$ is a finite sample space and $A$ is one of its events. The set of all the outcomes or elements of $U$ which are not in the event $A$ is called as complementary event of $A$. The complementary event of event $A$ is denoted by $A'$, $\overline{A}$, $A^c$. We will use the notation $A'$ for complementary event of $A$.




$A'$ = Complementary event of event $A$.

= Non-occurrence of event $A$.

= $U - A$.

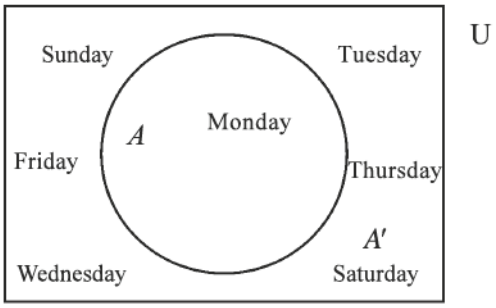
For example, the sample space of the random experiment of finding the day when a cargo ship will reach port $Y$ after leaving from port $X$ will be as follows.

$U$ = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

Suppose $A$ denotes that this ship reaches port $Y$ on Monday. Then the set of days except Monday will be the set of outcomes of event $A'$.

$A$ = {Monday}

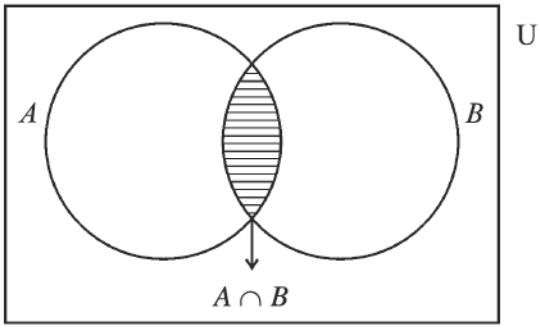
$A'$ = $U - A$ = {Sunday, Tuesday, Wednesday, Thursday, Friday, Saturday}



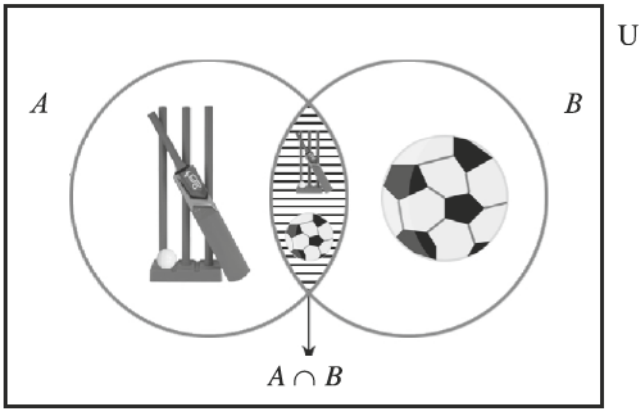

**(5) Intersection of Events :** Suppose $A$ and $B$ are two events of a finite sample space $U$. The event where events $A$ and $B$ occur simultaneously is called the intersection of two events $A$ and $B$. It is denoted by $A \cap B$.

$A \cap B$ = Intersection of two events $A$ and $B$

= Simultaneous occurrence of events $A$ and $B$




For example, some of the students studying in a class of a school are the members of school cricket team and some students are members of school football team. Let us denote the event that a student is a member of cricket team by event $A$ and the event that a student is a member of football team by $B$. If one student is randomly selected from this class then the event that the student is a member of school cricket and football team is called $A \cap B$, the intersection of events $A$ and $B$.

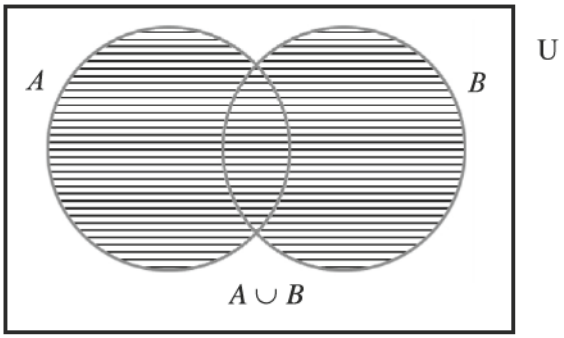



**(6) Union of Events :** Suppose $A$ and $B$ are any two events of a finite sample space $U$. The event where the event $A$ occurs or the event $B$ occurs or both the events $A$ and $B$ occur is called the union of events $A$ and $B$. It is denoted by $A \cup B$.

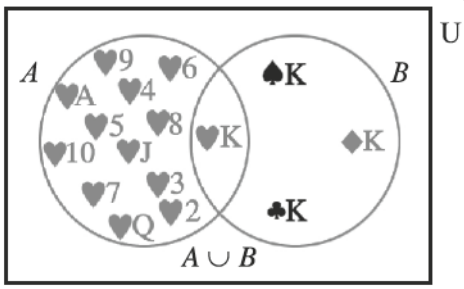
$A \cup B$ = Union of events $A$ and $B$

= Event $A$ occurs or event $B$ occurs or both events $A$ and $B$ occur together

= At least one of the event $A$ and $B$ occurs.



For example, the event $A \cup B$ of getting heart card (say event $A$) or a king (say event $B$) when a card is drawn randomly from a pack of 52 cards will be as follows :




$$A = \{H_A, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_J, H_Q, H_K\}$$

$$B = \{S_K, D_K, C_K, H_K\}$$

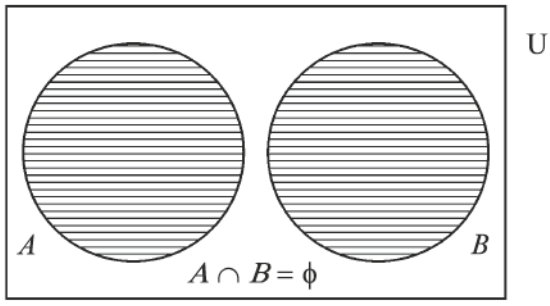
$$A \cup B = \{H_A, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_J, H_Q, H_K, S_K, D_K, C_K\}$$

Thus, the occurrence of $A \cup B$ is selecting any one of these 16 cards.

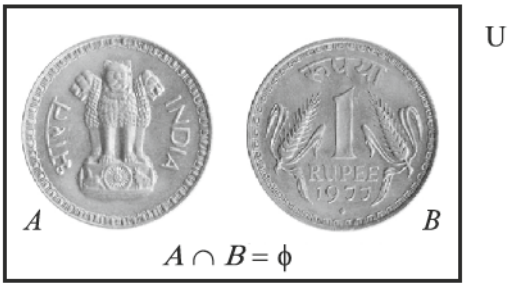
**The suits and types of cards are shown as follows :**

| | | | |
|---|---|---|---|
| Spade - $S$ | Diamond - $D$ | Club - $C$ | Heart - $H$ |
| Ace - $A$ | King - $K$ | Queen - $Q$ | Jack - $J$ |

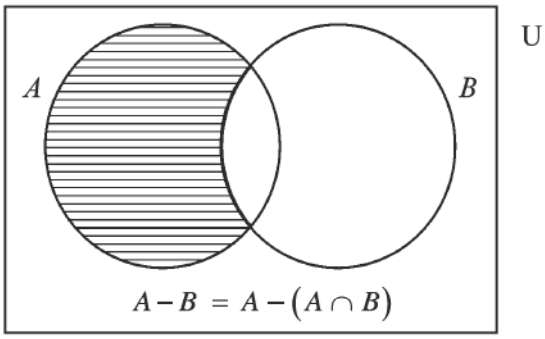
**(7) Mutually Exclusive Events :** Suppose $A$ and $B$ are any two events of a finite sample space $U$. Events $A$ and $B$ do not occur together which means $A \cap B = \phi$ or in other words, event $B$ does not occur when event $A$ occurs and event $A$ does not occur when event $B$ occurs then the events $A$ and $B$ are called mutually exclusive events.




For example, toss a balanced coin. Denote the outcome $H$ on the coin as event $A$ and the outcome $T$ on the coin as $B$. We get $A = \{H\}$ and $B = \{T\}$. It is clear that $A \cap B = \phi$ because when we get $H$ in a trial, it is not possible to get the outcome $T$ in the same trial and vice versa, when we get $T$ in a trial, it is not possible to get the outcome $H$ in the same trial in the random experiment of tossing a balanced coin. Thus, these two events cannot occur simultaneously.




**(8) Difference Event :** Suppose $A$ and $B$ are any two events of a finite sample space $U$. The set of elements or outcomes where event $A$ happens but event $B$ does not happen is called the difference of events $A$ and $B$. It is denoted by $A - B$. It is clear from the venn diagram given here that



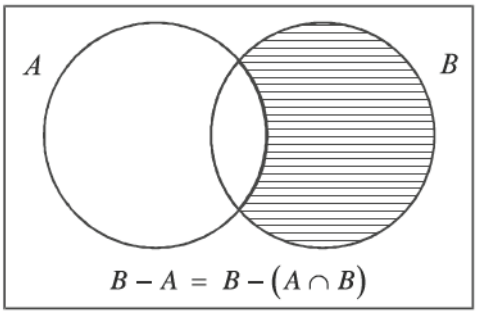

$$A - B = A \cap B' = A - (A \cap B) = (A \cup B) - B$$

$A - B$ = Difference of events $A$ and $B$

= Event $A$ happens but event $B$ does not happen

= Only $A$ happens out of events $A$ and $B$.

Similarly, for two events $A$ and $B$ of a finite sample space $U$, the set of elements or outcomes where $B$ happens but $A$ does not happen is called as the difference of event $B$ and event $A$. It is denoted by $B - A$. It is clear from the venn diagram given here that,



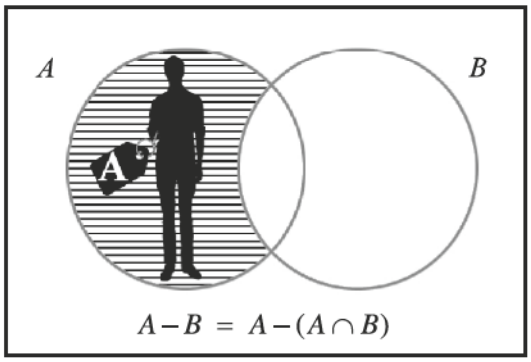

$$B - A = A' \cap B = B - (A \cap B) = (A \cup B) - A$$

$B - A$ = Difference of events $B$ and $A$

= Event $B$ happens but event $A$ does not happen

= Only event $B$ happens out of events $A$ and $B$

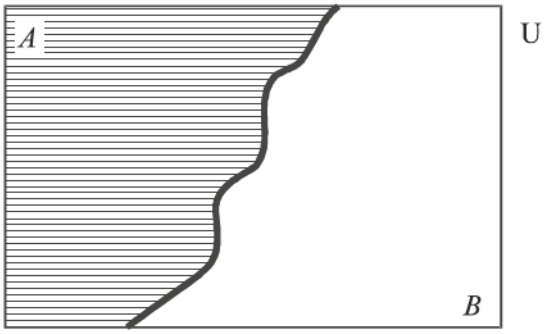
For example, two employees $A$ and $B$ among the employees working in an office are friends. Denote the presence of employee $A$ in the office as event $A$ and the presence of employee $B$ in the office as event $B$. On a ctertain day, if it is said that only the employee $A$ is present in the office out of employees $A$ and $B$ then it is clear that among two employees $A$ and $B$, employee $A$ is present but employee $B$ is not present. Thus, it is called the difference of two events $A - B$ for events $A$ and event B. Here,




$A - B$ = Only employee $A$ is present in the office among the employees $A$ and $B$

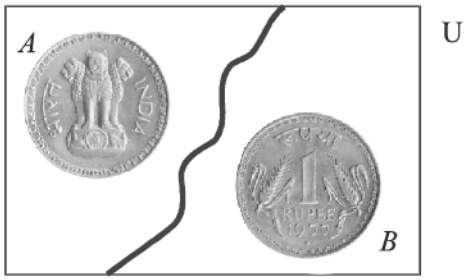
$B - A$ = Only employee $B$ is present in the office among the employees $A$ and B

**(9) Exhaustive Events :** If the group of favourable outcomes of events of random experiment is the sample space then the events are called exhaustive events. Suppose $A$ and $B$ are any two events of a sample space $U$. The events $A$ and $B$ are called the exhaustive events if the union $A \cup B$ of the two events $A$ and $B$ is the sample space $U$, that is $A \cup B = U$.




For example, denote the outcome $H$ as event $A$ and the outcome $T$ as event $B$ when a balanced coin is tossed. It is clear in this case that $A = \{H\}$, $B = \{T\}$ and $A \cup B = \{H, T\} = U$.

$\therefore$ $A$ and $B$ are exhaustive events.



**(10) Mutually Exclusive and Exhaustive Events :** Suppose $A$ and $B$ are two events of a finite sample space $U$. These two events $A$ and $B$ are called the mutually exclusive and exhaustive events if $A \cap B = \phi$ and $A \cup B = U$. It should be noted here that all the mutually exclusive events need not be exhaustive events and similarly, all the exhaustive events need not be the mutually exclusive events.

For example, consider the sample space $U = \{1, 2, 3, 4, 5, 6\}$ of the experiment of throwing a balanced die. Let the event $A$ = getting odd number on the die $= \{1, 3, 5\}$ and event $B$ = getting even number on the die $= \{2, 4, 6\}$. It is clear that $A \cap B = \phi$ and $A \cup B = U$. Thus, the events $A$ and $B$ are mutually exclusive and exhaustive.

**(11) Elementary Events :** The events formed by all the subsets of single elements of the sample space $U$ of a random experiment are called the elementary events. The elementary events are mutually exclusive and exhaustive.

For example, consider the sample space $U = \{H, T\}$ for the random experiment of tossing a balanced coin. The events $A = \{H\}$ and $B = \{T\}$ having single elements are the elementary events. Since $A \cap B = \phi$ and $A \cup B = U$ in this case, it can be said that the elementary events are mutually exclusive and exhaustive.

**Illustration 6 : There are 3 yellow and 2 pink flowers in a basket. One flower is randomly selected from this basket. Denote the selection of yellow flower as an event $A$ and the selection of pink flower as the event $B$. Find the sets representing the following events and answer the given questions.**

**(1)** $U$ **(2)** $A$ **(3)** $B$ **(4)** $A'$ **(5)** $B'$ **(6)** $A \cap B$ **(7)** $A \cup B$ **(8)** $A \cap B'$ **(9)** $A' \cap B$

**(10) State the elementary events of the sample space for this random experiment.**

**(11) Can it be said that the events $A$ and $B$ are mutually exclusive events ? Give reason.**

**(12) Can it be said that the events $A$ and $B$ are exhaustive events ? Give reason.**

We will denote the 3 yellow flowers in the basket as $Y_1, Y_2, Y_3$ and the 2 pink flowers as $P_1, P_2$. The sets representing the required events will be as follows :

(1) $U = \{Y_1, Y_2, Y_3, P_1, P_2\}$

(2) $A = \{Y_1, Y_2, Y_3\}$

(3) $B = \{P_1, P_2\}$

(4) $A' = U - A = \{Y_1, Y_2, Y_3, P_1, P_2\} - \{Y_1, Y_2, Y_3\}$

$= \{P_1, P_2\}$

(5) $B' = U - B = \{Y_1, Y_2, Y_3, P_1, P_2\} - \{P_1, P_2\}$

$= \{Y_1, Y_2, Y_3\}$

(6) $A \cap B = \{Y_1, Y_2, Y_3\} \cap \{P_1, P_2\}$

$= \phi$

(7) $A \cup B = \{Y_1, Y_2, Y_3\} \cup \{P_1, P_2\}$

$= \{Y_1, Y_2, Y_3, P_1, P_2\}$

(8) $A \cap B' = \{Y_1, Y_2, Y_3\} \cap \{Y_1, Y_2, Y_3\}$

$= \{Y_1, Y_2, Y_3\}$

**OR**

$A \cap B' = A - (A \cap B)$

$= \{Y_1, Y_2, Y_3\} - \phi$

$= \{Y_1, Y_2, Y_3\}$

(9) $A' \cap B = \{P_1, P_2\} \cap \{P_1, P_2\}$

$= \{P_1, P_2\}$

**OR**

$A' \cap B = B - (A \cap B)$

$= \{P_1, P_2\} - \phi$

$= \{P_1, P_2\}$

(10) The elementary events are the subsets with one element. If we denote the different elementary events as $E_1, E_2, E_3, \ldots$ then

$E_1 = \{Y_1\}, \; E_2 = \{Y_2\}, \; E_3 = \{Y_3\}, \; E_4 = \{P_1\}, \; E_5 = \{P_2\}$

(11) The events $A$ and $B$ can be called mutually exclusive events because according to the definition of mutually exclusive events, the events $A$ and $B$ are called the mutually exclusive events if $A \cap B = \phi$. It can be seen from the answer to the question 6 that $A \cap B = \phi$.

(12) The events $A$ and $B$ can be called exhaustive events because according to the definition of exhaustive events, the $A$ and $B$ are called the exhaustive events if $A \cup B = U$. It can be seen from the answer to the question 7 that $A \cup B = U$.

**Illustration 7 : The events $A$ and $B$ of a random experiment are as follows :**

$$A = \{1, 2, 3, 4\}, \quad B = \{-1, 0, 1\}$$

**If the sample space $U = A \cup B$ then find the sets showing the following events.**

**(1) $B'$ (2) $A' \cap B$ (3) $A - B$**

Here, $A = \{1, 2, 3, 4\}$

$B = \{-1, 0, 1\}$

$U = A \cup B = \{1, 2, 3, 4\} \cup \{-1, 0, 1\}$

$= \{-1, 0, 1, 2, 3, 4\}$

(1) $B' = U - B$

$= \{-1, 0, 1, 2, 3, 4\} - \{-1, 0, 1\}$

$= \{2, 3, 4\}$

(2) $A' \cap B = B - (A \cap B)$

First we find $A \cap B$,

$A \cap B = \{1, 2, 3, 4\} \cap \{-1, 0, 1\}$
$= \{1\}$

Now, $A' \cap B = B - (A \cap B)$

$= \{-1, 0, 1\} - \{1\}$

$= \{-1, 0\}$

**Alternate Method :**

$A' = U - A = \{-1, 0, 1, 2, 3, 4\} - \{1, 2, 3, 4\} = \{-1, 0\}$

$\therefore A' \cap B = \{-1, 0\} \cap \{-1, 0, 1\} = \{-1, 0\}$

(3) $A - B = \{1, 2, 3, 4\} - \{-1, 0, 1\}$
$= \{2, 3, 4\}$

**Illustration 8 : One number is randomly selected from the first 50 natural numbers. Find the sets showing the following events.**

**(1) The number selected is a multiple of 5 or 7.**

**(2) The number selected is a multiple of both 5 and 7.**

**(3) The number selected is a multiple of 5 but not a multiple of 7.**

**(4) The number selected is only a multiple of 7 out of 5 and 7.**

If one number is selected from the first 50 natural numbers then the group of all possible outcomes of this experiment, which is the sample space $U$, is as follows :

$U = \{1, 2, 3, ....., 50\}$

Event $A$ = Selected number is a multiple of 5

$= \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$

Event $B$ = Selected number is a multiple of 7

$= \{7, 14, 21, 28, 35, 42, 49\}$

Now, the required events are as follows :

(1) The event of selecting a number which is a multiple of 5 or 7 = $A \cup B$

$\therefore A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50\}$

(2) The event of selecting a number which is a multiple of both 5 and 7 = $A \cap B$

$\therefore A \cap B = \{35\}$

(3) The event of selecting a number which is a multiple of 5 but not of 7 $= A \cap B'$

$$\begin{aligned} \therefore\ A \cap B' &= A - (A \cap B) \\ &= \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\} - \{35\} \\ &= \{5, 10, 15, 20, 25, 30, 40, 45, 50\} \end{aligned}$$

(4) The event of selecting a number which is only a multiple of 7 out of 5 and 7 $= A' \cap B$

$$\begin{aligned} \therefore\ A' \cap B &= B - (A \cap B) \\ &= \{7, 14, 21, 28, 35, 42, 49\} - \{35\} \\ &= \{7, 14, 21, 28, 42, 49\} \end{aligned}$$

**Illustration 9 : The events $A_1$ and $A_2$ of a random experiment are defined as follows. Find the sets showing union event $A_1 \cup A_2$ and intersection event $A_1 \cap A_2$.**

$A_1 = \{x \mid x = -1, 0, 1\}$, $\quad A_2 = \{x \mid x = 1, 2, 3\}$

It is given that $A_1 = \{-1, 1, 0\}$ and $A_2 = \{1, 2, 3\}$.

Union of events $A_1 \cup A_2 = \{-1, 0, 1, 2, 3\}$

Intersection of events $A_1 \cap A_2 = \{1\}$

**Illustration 10 : A factory produces screws of different lengths. The length (in cm) of screw is denoted by $x$. The events $A_1$ and $A_2$ are defined as follows in the experiment of finding the length of selected screws. Find the events showing union event $A_1 \cup A_2$ and intersection event $A_1 \cap A_2$.**

$A_1 = \{x \mid 0 < x < 1\}$, $\quad A_2 = \{x \mid \frac{1}{2} \le x < 2\}$

If is given that $A_1 = \{x \mid 0 < x < 1\}$ and $A_2 = \{x \mid \frac{1}{2} \le x < 2\}$.

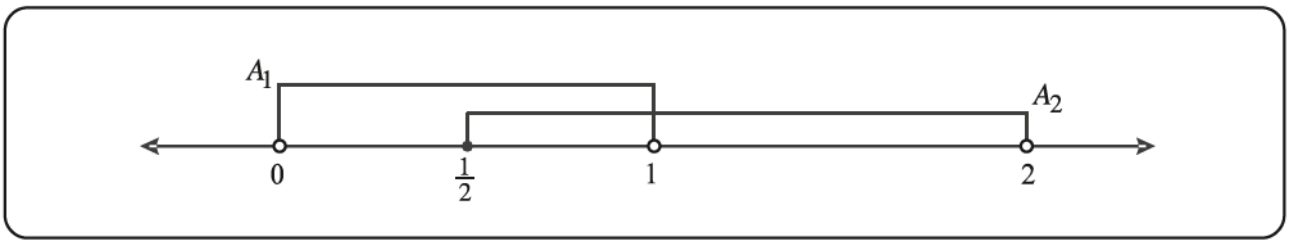
Union of events $A_1 \cup A_2 = \{x \mid 0 < x < 2\}$

$= (0, 2)$ (interval form)

Intersection of events $A_1 \cap A_2 = \{x \mid \frac{1}{2} \le x < 1\}$

$= \left[\frac{1}{2}, 1\right)$ (interval form)

See the following diagram carefully for better explanation of $A_1 \cup A_2$ and $A_1 \cap A_2$.



## Exercise 1.1

**1.** State the sample space for the following random experiments :

(1) A balanced coin is thrown three times.

(2) A balanced die with six sides and a balanced coin are tossed together.

(3) Two persons are to be selected from five persons *a, b, c, d, e*.

**2.** Write the sample space for the marks (in integers) scored by a student appearing for an examination of 100 marks and state the number of sample points in it.

**3.** Write the sample space for randomly selecting one minister and one deputy minister from four persons.

**4.** A balanced coin in thrown in a random experiment till the first head is obtained. The experiment is terminated with a trial of first head. Write the sample space of this experiment and state whether it is finite or infinite.

**5.** Write the sample space for the experiment of randomly selecting three numbers from the first five natural numbers.

**6.** The sample space of a random experiment of selecting a number is $U = \{1, 2, 3, ....., 20\}$. Write the sets showing the following events :

(1) The selected number is odd number

(2) The selected number is divisible by 3

(3) The selected number is divisible by 2 or 3.

**7.** One family is selected from the families having two children. The sex (male or female) of the children from this family is noted. State the sample space of this experiment and write the sets showing the following events :

(1) Event $A_1$ = One child is a female

(2) Event $A_2$ = At least one child is a female.

**8.** Two six faced balanced dice are thrown simultaneously. State the sample space of this random experiment and hence write the sets showing the following events :

(1) Event $A_1$ = The sum of numbers on the dice is 7

(2) Event $A_2$ = The sum of numbers on the dice is less than 4

(3) Event $A_3$ = The sum of numbers on the dice is divisible by 3

(4) Event $A_4$ = The sum of numbers on the dice is more than 12.

**9.** Two numbers are selected at random from the first five natural numbers. The sum of two selected numbers is at least 6 is denoted by event $A$ and the sum of two selected numbers is even is denoted by event $B$. Write the sets showing the following events and answer the given questions :

(1) $U$ (2) $A$ (3) $B$ (4) $A \cup B$ (5) $A \cap B$ (6) $A'$ (7) $A - B$ (8) $A' \cap B$

(9) Can it be said that the events $A$ and $B$ are mutually exclusive ? Give reason.

(10) State the number of sample points in the sample space of this random experiment.

**10.** Three female employees and two male employees are working in an office. One employee is selected from the employees of this office for training. The event that the employee selected for the training is a female is denoted by $A$ and the event that this employee is a male is denoted by $B$. Find the sets showing the following events and answer the given questions :

(1) $U$ (2) $A$ (3) $B$ (4) $A \cup B$ (5) $A \cap B$ (6) $A' \cap B$

(7) Can it be said that the events $A$ and $B$ are mutually exclusive ? Give reason.

(8) Can it be said that the events $A$ and $B$ are exhaustive ? Give reason.

**11.** One card is randomly drawn from a pack of 52 cards. If drawing a spade card is denoted by event $A$ and drawing a card from ace to ten (non-face card) is denoted by $B$ then write the sets showing the following events :

(1) $U$ (2) $A$ (3) $B$ (4) $A \cup B$ (5) $A \cap B$ (6) $B'$

**12.** The events $A_1$ and $A_2$ of a random experiment are as follows. Find the sets showing the union event $A_1 \cup A_2$ and intersection event $A_1 \cap A_2$.

$$A_1 = \{x \mid 0 < x < 5\}, \qquad A_2 = \{x \mid -1 < x < 3, x \text{ is an integer}\}$$

**13.** The events $A_1$ and $A_2$ of a random experiment are as follows. Find the sets showing the union event $A_1 \cup A_2$ and intersection event $A_1 \cap A_2$.

$$A_1 = \{x \mid 2 \le x < 6,\ x \in N\}, \qquad A_2 = \{x \mid 3 < x < 9,\ x \in N\}$$

**14.** The sample space $U$ of a random experiment and its event $A$ are defined as follows. Find the complementary event $A'$ of $A$.

$$U = \{x \mid x = 0, 1, 2, \ldots\ldots, 10\}, \qquad A = \{x \mid x = 2, 4, 6\}$$

**15.** The sample space $U$ of a random experiment and its event $A$ are defined as follows. Find the complementary event $A'$ of $A$.

$$U = \{x \mid 0 < x < 1\}, \qquad A = \{x \mid \tfrac{1}{2} \le x < 1\}$$

*

After getting aquainted with the random experiment, sample space and different events, we shall now study the probability. We shall begin with the definition of probability.

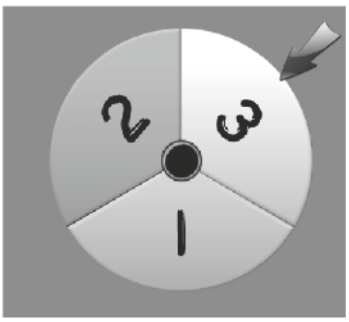
## 1.4 Mathematical Definition of Probability

To understand the mathematical definition of probability, we shall first understand the two important terms namely equiprobable events and favourable outcomes.

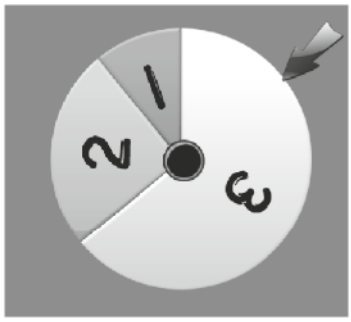
**Equiprobable Events :** If there is no apparent reason to believe that out of one or more events of a random experiment, any one event is more or less likely to occur than the other events then the events are called as equiprobable events.

For example, a manufacturer of a certain item has two machines $M_1$ and $M_2$ in his factory for the production of items. Both the machines produce the same number of items during a day. The lots are made of the produced goods by properly mixing the items produced on both the machines during the day. An item randomly selected from such a lot is made on machine $M_1$ or machine $M_2$ are the elementary events which are equiprobable.

Similarly, the wheels $A$ and $B$ marked with numbers 1, 2, 3 as shown in the following pictures are rotated by hand. The number against the pointer is noted down when the wheels stop rotating after some time. It is clear from the picture that all the three numbers on wheel $A$ will come against the pointer are equiprobable events. But the numbers 1, 2 and 3 coming against the pointer for wheel $B$ are not equiprobable events.




**Wheel *A***




**Wheel *B***

**Favourable Outcomes :** If some outcomes out of all the elementary outcomes in the sample space of random experiment indicate the occurrence of a certain event $A$ then these outcomes are called the favourable outcomes of the event $A$. For example, a card is drawn from a pack of 52 cards. If event $A$ denotes that the card drawn is a face card then the set of favourable outcomes is as follows :

$$A = \left\{S_K, D_K, C_K, H_K, S_Q, D_Q, C_Q, H_Q, S_J, D_J, C_J, H_J\right\}$$

Thus, 12 outcomes are favourable for event $A$.

**Mathematical Definition of probability :** Suppose there are total $n$ outcomes in the finite sample space of a random experiment which are mutually exclusive, exhaustive and equiprobable. If $m$ outcomes among them are favourable for an event $A$ then the probability of the event $A$ is $\frac{m}{n}$. The probability of event $A$ is denoted by $P(A)$.

$$P(A) = \text{Probability of event } A$$

$$= \frac{\text{Favourable outcomes of event } A}{\text{Total number of mutually exclusive, exhaustive and equi-probable outcomes of sample space}}$$

$$= \frac{m}{n}$$

Both the numbers $m(\geq 0)$ and $n(> 0)$ are integers and $m \leq n$. It should be noted here that $n$ can not be zero and infinity. The mathematical definition of probability is also called the classical definition.

The assumptions of the mathematical definition are as follows :

(1) The number of outcomes in the sample space of the random experiment is finite.

(2) The number of outcomes in the sample space of the random experiment is known.

(3) The outcomes in the sample space of the random experiment are equi-probable.

We will accept some of the following important results about probability without proof :

(1) The range for the value of probability $P(A)$ for any event $A$ in the sample space $U$ is 0 to 1. Thus, $0 \leq P(A) \leq 1$.

(2) The probability of an impossible event is zero. Earlier we have denoted an impossible event by $\phi$. Hence, $P(\phi) = 0$.

(3) The probability of certain event is always 1. Earlier we have denoted a certain event by $U$. Hence, $P(U) = 1$.

(4) The probability of complementary event $A'$ of event $A$ in the sample space $U$ is $P(A') = 1 - P(A)$.

(5) If $A \subset B$ for two events $A$ and $B$ in the sample space of a random experiment then

- $P(A) \leq P(B)$
- $P(B - A) = P(B) - P(A)$

(6) For two events $A$ and $B$ in the sample space of a random experiment,

- $P(A \cap B) \leq P(A)$ $\quad [\because A \cap B \subset A]$
- $P(A \cap B) \leq P(B)$ $\quad [\because A \cap B \subset B]$
- $P(A) \leq P(A \cup B)$ $\quad [\because A \subset A \cup B]$
- $P(B) \leq P(A \cup B)$ $\quad [\because B \subset A \cup B]$
- $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$
- $P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$
- $P(A - B) = P(A \cap B') = P(A) - P(A \cap B)$
- $P(B - A) = P(A' \cap B) = P(B) - P(A \cap B)$
- $0 \leq P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

We shall now consider illustrations of finding probability of different events using the mathematical definition.

**Illustration 11 : If two balanced coins are tossed, then find the probability of (1) getting one head and one tail and (2) getting at least one head.**

The sample space for the random experiment of tossing two balanced coins is as follows :

$U = \{HH, HT, TH, TT\}$

$\therefore$ No. of mutually exclusive, exhaustive and equi-probable outcomes $n = 4$.

(1) If $A$ denotes the event of getting one head $H$ and one tail $T$ then $HT$ and $TH$ are two favourable outcomes of event $A$. Thus, $m = 2$.

From the mathematical definition of probability,

$$P(A) = \frac{m}{n}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

Required probability $= \frac{1}{2}$

(2) If $B$ denotes the event of getting at least one head then $HT, TH, HH$ are the favourable outcomes of event $B$. Hence, the number of favourable outcomes $m = 3$ for even $B$.

From the mathematical definition of probability,

$$P(B) = \frac{m}{n}$$

$$= \frac{3}{4}$$

Required probability $= \frac{3}{4}$

**Illustration 12 : Two balanced dice marked with numbers 1 to 6 are thrown simultaneously. Find the probability that (1) sum of numbers on both the dice is 7 (2) sum of numbers on both the dice is more than 10 (3) sum of number on both the dice is at the most 4 (4) both the dice show same numbers (5) sum of numbers on both the dice is 1 (6) sum of numbers on both the dice is 12 or less.**

The sample space for throwing two balanced dice simultaneously is as follows :

$U = \{(i, j) ; i, \ i, j = 1, 2, 3, 4, 5, 6\}$

$\therefore$ Total number of outcomes $n = 36$.

(1) If $A_1$ denotes that sum of the numbers on the dice is 7 then there are total 6 outcomes $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ favourable for this event $A_1$. Thus, the number of favourable outcomes $m = 6$ for event $A_1$. Probability of event $A_1$

$$P(A_1) = \frac{m}{n}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

$\therefore$ Required probability $= \frac{1}{6}$

(2) If $A_2$ denotes the event that the sum of numbers on two dice is more than 10 then $(5, 6), (6, 5), (6, 6)$ are the favourable outcomes of event $A_2$. Thus, the number of favourable outcomes $m = 3$ for even $A_2$. Probability of $A_2$

$$P(A_2) = \frac{m}{n}$$

$$= \frac{3}{36}$$

$$= \frac{1}{12}$$

Required probability $= \frac{1}{12}$

(3) If $A_3$ denotes the event that the sum of numbers on two dice is at the most 4 then total 6 outcomes $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$ are favourable outcomes of event $B$. Thus, the number of favourable outcomes $m = 6$ for event $A_3$. Probability of event $A_3$

$$P(A_3) = \frac{m}{n}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

Required probability $= \frac{1}{6}$

(4) Event $A_4$ = both the dice show the same numbers.

$\therefore$ Total 6 outcomes $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$ are favourable for the event $A_4$.

Thus, the number of favourable outcomes $m = 6$ for event $A_4$. Probability of event $A_4$

$$P(A_4) = \frac{m}{n}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

Required probability $= \frac{1}{6}$

(5) Let $A_5$ be the event that the sum of numbers on two dice is 1. It is obvious that not a single outcome in the sample space is favourable for $A_5$. Hence, the number of favourable outcomes $m = 0$ for event $A_5$. Probability of event $A_5$

$$P(A_5) = \frac{m}{n}$$

$$= \frac{0}{36}$$

$$= 0$$

Required probability = 0

(The probability of impossible event is always 0.)

(6) Let $A_6$ be the event that the sum of numbers on two dice is 12 or less. It is obvious that all the outcomes in the sample space are favourable for event $A_6$. Hence, the number of favourable outcomes $m = 36$ for the event $A_6$. Probability of event $A_6$

$$P(A_6) = \frac{m}{n}$$

$$= \frac{36}{36}$$

$$= 1$$

Required probability = 1

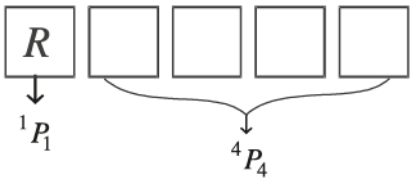
(The probability of certain event is always 1.)

**Illustration 13 : Find the probability of getting *R* in the first place in all possible arrangements of each and every letter of the word *RUTVA*.**

There are 5 letters *R, U, T, V, A* in the word *RUTVA*. These five letters can be arranged in ${}^5P_5 = 5! = 120$ different ways. Thus, total number of outcomes $n = 120$.

Event of getting *R* in the first place of the arrangement = *A*.

The favourable outcomes of event *A* are obtained as follows :




*R* can be arranged in the first place in ${}^1P_1$ ways and the remaining four letters *U, T, V, A* in the rest of the four places can be arranged in ${}^4P_4$ ways. According to the fundamental principle of multiplication, there will be ${}^1P_1 \times {}^4P_4$ arrangements of getting *R* in the first place. Hence, the number of favourable outcomes for event *A* will be

$$m = {}^1P_1 \times {}^4P_4 = 1! \times 4! = 1 \times 24 = 24.$$

Probability of event *A* $P(A) = \frac{m}{n}$

$$= \frac{24}{120}$$

$$= \frac{1}{5}$$

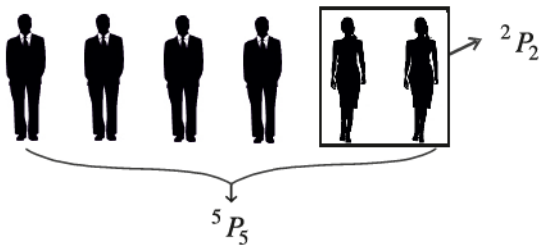
Required probability = $\frac{1}{5}$

**Illustration 14 : Four male employees and two female employees working in a government department are sent one by one in turns to the training centre for training. Find the probability that the two female employees go successively for the training.**

Total 6 persons, 4 males and 2 females can be sent for training at the training centre one by one in ${}^6P_6 = 6! = 720$ ways. Thus, total number of outcomes will be $n = 720$.

Event of two female employees go successively for training = $A$.

The favourable outcomes of event $A$ can be obtained as follows :




Considering the two female employees going successively for the training as one person, total 5 persons can be arranged in ${}^5P_5$ ways and two female employees can be arranged among themselves in ${}^2P_2$ ways in each of these arrangements.

Thus, the number favourable outcomes of event $A$ is $m$ $= {}^5P_5 \times {}^2P_2$

$= 5! \times 2!$

$= 120 \times 2$

$= 240$

Probability of event $A$ $P(A) = \frac{m}{n}$

$= \frac{240}{720}$

$= \frac{1}{3}$

Required probability $= \frac{1}{3}$

**Illustration 15 : Find the probability of having 53 Thursdays in a leap year.**

There are 366 days in a leap year where we have 52 complete weeks (52×7=364 days) and 2 additional days. Each day appears once in each week and thus each day will appear 52 times in 52 weeks. Now, the additional 2 days can be as follows which gives the sample space for this experiment.

$U$ = {Sunday-Monday, Monday-Tuesday, Tuesday-Wednesday, Wednesday-Thursday, Thursday-Friday, Friday-Saturday, Saturday-Sunday}

Thus, total number of outcomes will be $n = 7$.

Event $A$ = leap year has 53 Thursdays.

Wednesday-Thursday and Thursday-Friday are the 2 favourable outcomes of event $A$ from the above 7 outcomes. Thus, $m = 2$.

Probability of event $A$ $\quad P(A) = \frac{m}{n}$

$$= \frac{2}{7}$$

Required probability $= \frac{2}{7}$

**Illustration 16 : There are 2 officers, 3 clerks and 2 peons among the 7 employees working in the cash department of a bank. A committee is formed by randomly selecting two employees from the employees of this department. Find the probability that there are**

**(1) two peons**

**(2) two clerks**

**(3) One officer and one clerk among the two employees selected in the committee.**

There are 7 employees working in the cash department of the bank. If the employees are randomly selected from them then the total number of mutually exclusive, exhaustive and equi-probable outcomes will be $n = {}^7C_2 = \frac{7 \times 6}{2 \times 1} = 21$.

(1) Event of selecting two peons = $A$

Selecting 2 peons from the 2 peons and not selecting any employee from the remaining 5 employees will be the favourable outcomes of event $A$.

The number of such outcomes will be $m = {}^2C_2 \times {}^5C_0 = 1 \times 1 = 1$.

Probability of event $A$ $\quad P(A) = \frac{m}{n}$

$$= \frac{1}{21}$$

Required probability $= \frac{1}{21}$

(2) Event of selecting two clerks = $B$

Selecting 2 clerks from the 3 clerks and not selecting any employee from the remaining four employees will be the favourable outcomes of event $B$.

The number of such outcomes will be $m = {}^3C_2 \times {}^4C_0 = 3 \times 1 = 3$.

Probability of event $B$ $\quad P(B) = \frac{m}{n}$

$$= \frac{3}{21}$$

$$= \frac{1}{7}$$

Required probability $= \frac{1}{7}$

(3) Event of selecting one officer and one clerk = $C$

Selecting 1 officer from 2 officers, one clerk from three clerks and not selecting any peon from two peons will be the favourable outcomes of event $C$.

The number of such outcomes will be $m = {}^2C_1 \times {}^3C_1 \times {}^2C_0 = 2 \times 3 \times 1 = 6$.

Probability of event $C$ $P(C) = \frac{m}{n}$

$= \frac{6}{21}$

$= \frac{2}{7}$

Required probability $= \frac{2}{7}$

**Illustration 17 : A box contains 20 items and 10% of them are defective. Three items are randomly selected from this box. Find the probability that,**

**(1) two items are defective**

**(2) two items are non-defective**

**(3) all three items are non-defective among the three selected items.**

There are 20 items wherein 10% that is 20 × 10 % = 2 items are defective and the rest 18 are non-defective. 3 items are selected from this box of 20 items at random. Hence, the total number of outcomes in the sample space will be $n = {}^{20}C_3 = \frac{20 \times 19 \times 18}{3 \times 2} = 1140$.

(1) Event of getting two defective items among three selected items = $A$

Selecting 2 items from 2 defective items and selecting 1 item from the 18 non-defective items will be the favourable outcomes for the event $A$.

The number of such outcomes $m = {}^2C_2 \times {}^{18}C_1 = 1 \times 18 = 18$.

Probability of event $A$ $P(A) = \frac{m}{n}$

$= \frac{18}{1140}$

$= \frac{3}{190}$

Required probability $= \frac{3}{190}$

(2) Event of getting two non-defective items among three selected items = $B$

Selecting 2 items from 18 non-defective items and selecting one item from 2 defective items will be the favourable outcomes of the event $B$.

The number of such outcomes $m = {}^{18}C_2 \times {}^2C_1 = 153 \times 2 = 306$.

Probability of event $B$ $\quad P(B) = \frac{m}{n}$

$$= \frac{306}{1140}$$

$$= \frac{51}{190}$$

Required probability $= \frac{51}{190}$

(3) Event of getting all three non-defective items $= C$

Selecting 3 items from 18 non-defective items and not selecting any item from the defective items will be the favourable outcomes of event $C$.

The number of such outcomes $m = {}^{18}C_3 \times {}^{2}C_0 = 816 \times 1 = 816$.

Probability of event $C$ $\quad P(C) = \frac{m}{n}$

$$= \frac{816}{1140}$$

$$= \frac{68}{95}$$

Required probability $= \frac{68}{95}$

**Illustration 18 : A box contains 10 chits of which 3 chits are eligible for a prize. A boy named Kathan randomly selects two chits from this box. Find the probability that Kathan gets the prize.**

There are 10 chits of which 3 chits are eligible for a prize and 7 chits are not eligible for prize. If two chits are randomly selected from these 10 chits then the number of mutually exclusive, exhaustive and equiprobable outcomes in the sample space will be $n = {}^{10}C_2 = \frac{10 \times 9}{2} = 45$.

Event of Kathan getting prize $= A$

$\therefore$ Event that Kathan does not get prize $= A'$

The outcomes in which Kathan will draw 2 chits at random from the 7 chits which are not eligible for prize will be the favourable outcomes of the event $A'$.

The number of such outcomes $m = {}^{7}C_2 = 21$.

Probability of $A'$ $\quad P(A') = \frac{m}{n}$

$$= \frac{21}{45}$$

$$= \frac{7}{15}$$

Now $P(A) = 1 - P(A')$

$$= 1 - \frac{7}{15}$$

$$= \frac{8}{15}$$

Thus, probability that Kathan gets prize $= \frac{8}{15}$

**Limitations :** The limitations of the mathematical definition of probability are as follows :

(1) The probability of an event cannot be found by this definition if there are infinite outcomes in the sample space of a random experiment.

(2) The probability of an event cannot be found by this definition if the total number of outcomes in the sample space of a random experiment are not known.

(3) The probability of an event cannot be found by this definition if the elementary outcomes in the sample space of a random experiment are not equi-probable.

(4) The word 'equi-probable' is mentioned in the mathematical definition of probability. Equi-probable events are the events with same probability. Thus, the word probability is used in the definition of probability.

## Exercise 1.2

**1.** A balanced coin is tossed three times. Find the probability of the following events :

(1) Getting all three heads
(2) Not getting a single head
(3) Getting at least one head
(4) Getting more than one head
(5) Getting at the most one head
(6) Getting less than two heads
(7) Getting head and tail alternately
(8) Getting more number of heads than tails

**2.** Two balanced dice are thrown simultaneously. Find the probability of the following events :

(1) The sum of numbers on the dice in 6

(2) The sum of numbers on the dice is not more than 10

(3) The sum of numbers on the dice is a multiple of 3

(4) The product of numbers on the dice is 12

**3.** One family is randomly selected from the families having two children. Find the probability that

(1) One child is a girl and one child is a boy.

(2) At least one child is a girl among the two children of the selected family.

(**Note :** Assume that the chance of the child being a boy or girl is same.)

**4.** One number is selected at random from the first 100 natural numbers. Find the probability that this number is divisible by 7.

**5.** The sample space for a random experiment of selecting numbers is $U = \{1, 2, 3, ....., 120\}$ and all the outcomes in the sample space are equiprobable. Find the probability that the number selected is

(1) a multiple of 3
(2) not a multiple of 3
(3) a multiple of 4
(4) not a multiple of 4
(5) a multiple of both 3 and 4.

6. Find the probability of getting $R$ in the first place and $M$ in the last place when all the letters of the word *RANDOM* are arranged in all possible ways.

7. Find the probability of getting vowels in the first, third and sixth place when all the letters of the word *ORANGE* are arranged in all possible ways.

8. Five members of a family, husband, wife and three children, are randomly arranged in a row for a family photograph. Find the probability that the husband and wife are seated next to each other.

9. Seven speakers $A, B, C, D, E, F, G$ are invited in a programme to deliver speech in random order. Find the probability that speaker $B$ delivers speech immediately after speeker $A$.

10. Find the probability of having 5 Mondays in the month of February of a leap year.

11. Find the probability of having 53 Fridays in a year which is not a leap year.

12. Find the probability of having 5 Tuesdays in the month of August of any year.

13. 4 couples (husband-wife) attend a party. Two persons are randomly selected from these 8 persons. Find the probability that the selected persons are,

    (1) husband and wife (2) one man and one woman

    (3) one man and one woman who are not husband and wife.

14. 8 workers are employed in a factory and 3 of them are excellent in efficiency where as the rest of them are moderate in efficiency. 2 workers are randomly selected from these 8 workers. Find the probability that,

    (1) both the workers have excellent efficiency

    (2) both the workers have moderate efficiency

    (3) one worker is excellent and one worker is moderate in efficiency.

15. Two cards are drawn from a well shuffled pack of 52 cards. Find the probability that,

    (1) both the cards are of different colour (2) both the cards are face cards

    (3) one of the two cards is a king.

16. 3 bulbs are defective in a box of 10 bulbs. 2 bulbs are randomly selected from this box. These bulbs are fixed in two bulb-holders installed in a room. Find the probability that the room will be lighted after starting the electric supply.

17. For two events $A$ and $B$ in the sample space of a random experiment, $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.15$. Find

    (1) $P(A')$ (2) $P(B - A)$ (3) $P(A \cap B')$ (4) $P(A' \cup B')$

18. For two events $A$ and $B$ in the sample space of a random experiment, $P(A') = 2P(B') = 3P(A \cap B) = 0.6$. Find the probability of difference events $A - B$ and $B - A$.

*

## 1.5 Law of Addition of Probability

The rule of obtaining the probability of the occurrence of at least one of the event $A$ and $B$ in the sample space of a random experiment is called the law of addition of probability. We have seen earlier that the occurrence of at least one of the events $A$ and $B$ is denoted by $A \cup B$, the union of events $A$ and $B$. Hence we can say that the law of addition of probability is the rule of obtaining the probability of $A \cup B$, the union of events $A$ and $B$. This rule is stated as follows and we will accept it without proof :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The law of addition of probability can also be used for obtaining the probability of union of more than two events. The law of addition of probability for $A \cup B \cup C$, the union of three events $A$, $B$ and $C$ is as follows :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Some of the important results obtained from this rule are as follows :

(1) If the events $A$ and $B$ in the sample space of a random experiment are mutually exclusive then $A \cap B = \phi$ and $P(A \cap B) = 0$. Hence,

$$P(A \cup B) = P(A) + P(B)$$

(2) If three events $A$, $B$ and $C$ in the sample space of a random experiment are mutually exclusive then,

$A \cap B = \phi$, $A \cap C = \phi$, $B \cap C = \phi$, $A \cap B \cap C = \phi$ and

$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(A \cap B \cap C) = 0$. Hence,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

(3) If two events $A$ and $B$ in the sample space of a random experiment are mutually exclusive and exhaustive then $A \cap B = \phi$ and $A \cup B = U$. As $P(\phi) = 0$ and $P(U) = 1$, $P(A \cap B) = 0$ and $P(A \cup B) = 1$.

$$P(A \cup B) = P(A) + P(B) = 1$$

(4) If three events $A$, $B$ and $C$ in the sample space of a random experiment are mutually exclusive and exhaustive then,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 1$$

**Illustration 19 : A number is randomly selected from the first 50 natural numbers. Find the probability that it is a multiple of 2 or 3.**

If one number is randomly selected from the first 50 natural numbers then the number of mutually exclusive, exhaustive and equi-probable outcomes in the sample space of this random experiment will be $n = {}^{50}C_1 = 50$.

If event $A$ denotes that the number selected is a multiple of 2 and event $B$ denotes that the number selected is a multiple of 3 then the event that the selected number is a multiple of 2 or 3 will be denoted by $A \cup B$. (This event can also be denoted as $B \cup A$. According to set theory, $A \cup B = B \cup A$). To find the probability of $A \cup B$, the union of events $A$ and $B$ by the law of addition of probability, we will first find $P(A), P(B)$ and $P(A \cap B)$.

$A$ = Event that the selected number is a multiple of 2

= {2, 4, 6,..., 50}

Hence, the number of favourable outcomes of event $A$ will be $m = 25$.

Probability of event $A$ $P(A) = \frac{m}{n}$

$= \frac{25}{50}$

$B$ = Event that the selected number is a multiple of 3

= {3, 6, 9,..., 48}

Hence, the number of favourable outcomes of event $B$ will be $m = 16$.

Probability of event $B$ $P(B) = \frac{m}{n}$

$= \frac{16}{50}$

$A \cap B =$ Event that the selected number is a multiple of 2 and 3 that is multiple the LCM of 2 and 3 which is 6.

= {6, 12, 18,..., 48}

Hence, the number of favourable outcomes of event $A \cap B$ will be $m = 8$.

Probability of event $A \cap B$ $P(A \cap B) = \frac{m}{n}$

$= \frac{8}{50}$

From the law of addition of probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{25}{50} + \frac{16}{50} - \frac{8}{50}$$

$$= \frac{25 + 16 - 8}{50}$$

$$= \frac{33}{50}$$

Required probability $= \frac{33}{50}$

**Illustration 20 : One card is randomly selected from a pack of 52 cards. Find the probability that the selected card is**

**(1) club or queen card**

**(2) neither a club nor a queen card.**

If one card is randomly selected from a pack of 52 cards then the number of mutually exclusive, exhaustive and equi-probable outcomes in the sample space of this random experiment $n = {}^{52}C_1 = 52$.

Event that the selected card is a club card = $A$

Event that the selected card is a queen = $B$

(1) Event that the selected card is club or queen card = $A \cup B$

To find the probability of event $A \cup B$ by the law of addition of probability, we will first find $P(A), P(B)$ and $P(A \cap B)$.

$A$ = Event that the selected card is club card.

There are 13 club cards in a pack of 52 cards. Thus, the number of favourable outcomes of event $A$ is $m = 13$.

Probability of event $A$ $\quad P(A) = \frac{m}{n}$

$$= \frac{13}{52}$$

$B =$ Event that the selected card is a queen card.

There are 4 queen cards in a pack of 52 cards. Thus, the number of favourable outcomes of event $B$ is $m = 4$.

Probability of event $B$ $\quad P(B) = \frac{m}{n}$

$$= \frac{4}{52}$$

$A \cap B =$ Event that the selected card is club and queen card that is a club queen.

There is only 1 card in the pack of 52 cards which is club queen. Hence, the number of favourable outcomes of $A \cap B$ is $m = 1$.

Probability of $A \cap B$ $\quad P(A \cap B) = \frac{m}{n}$

$$= \frac{1}{52}$$

From the law of addition of probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= \frac{13 + 4 - 1}{52}$$

$$= \frac{16}{52}$$

$$= \frac{4}{13}$$

Required probability $= \frac{4}{13}$

Event $A \cup B$ can be easily explained by the following diagram :

| Suit | Type of Card | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | A | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | J | Q | K |
| ♠ | ♠A | ♠2 | ♠3 | ♠4 | ♠5 | ♠6 | ♠7 | ♠8 | ♠9 | ♠10 | ♠J | ♠Q | ♠K |
| ♦ | ♦A | ♦2 | ♦3 | ♦4 | ♦5 | ♦6 | ♦7 | ♦8 | ♦9 | ♦10 | ♦J | ♦Q | ♦K |
| ♣ | ♣A | ♣2 | ♣3 | ♣4 | ♣5 | ♣6 | ♣7 | ♣8 | ♣9 | ♣10 | ♣J | ♣Q | ♣K |
| ♥ | ♥A | ♥2 | ♥3 | ♥4 | ♥5 | ♥6 | ♥7 | ♥8 | ♥9 | ♥10 | ♥J | ♥Q | ♥K |

(2) Event that the selected card is not of club = $A'$

Event that the selected card is not queen = $B'$

Hence, the event that the selected card is neither club nor queen is $A' \cap B'$

Thus, the probability of $A' \cap B'$

$$P(A' \cap B') = P(A \cup B)'$$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{4}{13}$$

$$= \frac{9}{13}$$

Required probability $= \frac{9}{13}$

**Illustration 21 : 3 persons from medical profession and 5 persons from engineering profession offer services at a social organization. 2 persons are randomly selected from these persons with the purpose of forming a committee. Find the probability that both the persons selected belong to the same profession.**

There are in all $3 + 5 = 8$ persons. Hence, 2 persons can be selected in ${}^8C_2 = \frac{8 \times 7}{2 \times 1} = 28$ ways.

Thus, the total number of mutually exclusive, exhaustive and equi-probable outcomes in the sample space is $n = 28$.

Event that both the persons selected belong to medical profession = $A$

Even that both the persons selected belong to the engineering profession = $B$

Event that both the persons selected belong to the same profession = $A \cup B$

The two events $A$ and $B$ can not occur together that is $A \cap B = \phi$

Thus, the events $A$ and $B$ are mutually exclusive. Hence, from the law of addition of probability,

$$P(A \cup B) = P(A) + P(B)$$

For which we first find $P(A)$ and $P(B)$.

$A$ = Event that both the persons selected belong to medical profession.

The number of favourable outcomes of $A$ is $m = {}^3C_2 = 3$.

Probability of event $A$ $P(A) = \frac{m}{n}$

$$= \frac{3}{28}$$

$B =$ Event that both the persons selected belong to engineering profession.

The number of favourable outcomes of $B$ is $m = {}^5C_2 = 10$.

Probability of event $B$ $P(B) = \frac{m}{n}$

$$= \frac{10}{28}$$

Now,

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{3}{28} + \frac{10}{28}$$

$$= \frac{3+10}{28}$$

$$= \frac{13}{28}$$

Required probability $= \frac{13}{28}$

**Illustration 22 : The probability that a person from a group reads newspaper $X$ is 0.55, the probability that he read newspaper $Y$ is 0.69 and the probability that he reads both the newspaper $X$ and $Y$ is 0.27. Find the probability that a person selected at random from this group.**

**(1) reads at least one of the newspapers $X$ and $Y$.**

**(2) does not read any of the newspapers $X$ and $Y$.**

**(3) reads only one of the newspapers $X$ and $Y$.**

If the event that a person from the group reads newspaper $X$ is denoted by event $A$ and reads newspaper $Y$ by event $B$ then the given information can be shown as follows :

$P(A) = 0.55,\ P(B) = 0.69,\ P(A \cap B) = 0.27$

(1) Event that the selected person reads at least one of the newspapers $= A \cup B$

From the law of addition of probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.55 + 0.69 - 0.27$$

$$= 0.97$$

Required probability $= 0.97$

(2) Event that the selected person does not read newspaper $A = A'$

Event that the selected person does not read newspaper $B = B'$

Hence, event that the selected person does not read any of the newspaper $X$ and $Y = A' \cap B'$.

Probability of $A' \cap B'$

$$P(A' \cap B') = P(A \cup B)'$$

$$= 1 - P(A \cup B)$$

$$= 1 - 0.97$$

$$= 0.03$$

Required probability $= 0.03$

(3) If the event that the selected person reads only one of the newspapers $X$ and $Y$ is denoted by $C$ then the event $C$ can occur as follows :

The person reads newspaper $X$ (event $A$) and does not read newspaper $Y$ (event $B'$)

**OR**

The person does not read newspaper $X$ (event $A'$) and reads newspaper $Y$ (event $B$)

Thus $C=(A\cap B')\cup(A'\cap B)$

Since the events $A\cap B'$ and $A'\cap B$ are mutually exclusive,

$$\begin{aligned} P(C) &= P(A\cap B')+P(A'\cap B) \\ &= [P(A)-P(A\cap B)]+[P(B)-P(A\cap B)] \\ &= [0.55-0.27]+[0.69-0.27] \\ &= 0.28 + 0.42 \\ &= 0.7 \end{aligned}$$

Required probability = 0.7

**Illustration 23 : For two events $A$ and $B$ in the sample space of a random experiment $P(A)=2P(B)=4P(A\cap B)=0.6$. Find the probability of the following events :**

**(1) $A'\cap B'$ (2) $A'\cup B'$ (3) $A - B$ (4) $B - A$**

It is given that $P(A)=2P(B)=4P(A\cap B)=0.6$. Hence,

| | | |
|---|---|---|
| $P(A)=0.6$ | $2P(B)=0.6$ | $4P(A\cap B)=0.6$ |
| | $\therefore\ P(B)=0.3$ | $\therefore\ P(A\cap B)=0.15$ |

(1) Probability of event $A'\cap B'$

$$\begin{aligned} &= P(A'\cap B') \\ &= P(A\cup B)' \\ &= 1-P(A\cup B) \\ &= 1-[P(A)+P(B)-P(A\cap B)] \\ &= 1-[0.6+0.3-0.15] \\ &= 1-0.75 \\ &= 0.25 \end{aligned}$$

Required probability = 0.25

(2) Probability of event $A'\cup B'$

$$\begin{aligned} &= P(A'\cup B') \\ &= P(A\cap B)' \\ &= 1-P(A\cap B) \\ &= 1-0.15 \\ &= 0.85 \end{aligned}$$

Required probability = 0.85

(3) Probability of event $A-B$ $= P(A-B)$

$= P(A)-P(A\cap B)$

$= 0.6-0.15$

$= 0.45$

Required probability = 0.45

(4) Probability of event $B-A$ $= P(B-A)$

$= P(B)-P(A\cap B)$

$= 0.3-0.15$

$= 0.15$

Required probability = 0.15

**Illustration 24 : For two events $A$ and $B$ in the sample space of a random experiment $P(A')=0.3$, $P(B)=0.6$ and $P(A\cup B)=0.83$. Find $P(A\cap B')$ and $P(A'\cap B)$.**

Here, $P(A')=0.3 \therefore P(A)=1-P(A')=1-0.3=0.7$

$P(B)=0.6$ and $P(A\cup B)=0.83$

First we will find $P(A\cap B)$.

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\therefore\ 0.83=0.7+0.6-P(A\cap B)$

$\therefore\ P(A\cap B)=0.7+0.6-0.83$

$\therefore\ P(A\cap B)=0.47$

Now,

$P(A\cap B')=P(A)-P(A\cap B)$

$= 0.7 - 0.47$

$= 0.23$

Required probability = 0.23

$P(A'\cap B)=P(B)-P(A\cap B)$

$= 0.6 - 0.47$

$= 0.13$

Required probability = 0.13

**Illustration 25 : Two events $A$ and $B$ in the sample space of a random experiment are mutually exclusive. If $3P(A)=4P(B)=1$ then find $P(A\cup B)$.**

Since $3P(A)=4P(B)=1$

| | |
|---|---|
| $3P(A)=1$ | $4P(B) = 1$ |
| $\therefore\ P(A)= \frac{1}{3}$ | $\therefore\ P(B)= \frac{1}{4}$ |

As the events $A$ and $B$ are mutually exclusive $(A\cap B=\phi)$,

$$P(A\cup B)=P(A)+P(B)$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

Required probability $= \frac{7}{12}$

**Illustration 26 : For three mutually exclusive and exhaustive events $A$, $B$ and $C$ in the sample space of a random experiment $2P(A)=3P(B)=4P(C)$. Find $P(A\cup B)$ and $P(B\cup C)$.**

Taking $2P(A)=3P(B)=4P(C)=x$,

| | | |
|---|---|---|
| $2P(A)=x$ | $3P(B)=x$ | $4P(C)=x$ |
| $\therefore\ P(A) = \frac{x}{2}$ | $\therefore\ P(B) = \frac{x}{3}$ | $\therefore\ P(C)= \frac{x}{4}$ |

Since $A$, $B$ and $C$ are mutually exclusive and exhaustive events,

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)=1$$

$$\therefore\ \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 1$$

$$\therefore\ \frac{6x+4x+3x}{12} = 1$$

$$\therefore\ 13x=12$$

$$\therefore\ x=\frac{12}{13}$$

Thus,

$$P(A)=\frac{x}{2} = \frac{\frac{12}{13}}{2} = \frac{6}{13}$$

$$P(B)=\frac{x}{3} = \frac{\frac{12}{13}}{3} = \frac{4}{13}$$

$$P(C)=\frac{x}{4} = \frac{\frac{12}{13}}{4} = \frac{3}{13}$$

Now, the probability of required events,

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{13} + \frac{4}{13}$$

$$= \frac{10}{13}$$

Required probability $= \frac{10}{13}$

$$P(B \cup C) = P(B) + P(C)$$

$$= \frac{4}{13} + \frac{3}{13}$$

$$= \frac{7}{13}$$

Required probability $= \frac{7}{13}$

## Exercise 1.3

**1.** 2 cards are drawn from a pack of 52 cards. Find the probability that both the cards drawn are

(1) of the same suit

(2) of the same colour.

**2.** 3 books of Statistics and 4 of Mathematics are arranged on a shelf. Two books are randomly selected from these books. Find the probability that both the books selected are of the same subject.

**3.** One card is randomly drawn from a pack of 52 cards. Find the probability that it is

(1) Spade card or ace (2) Neighter spade nor ace.

**4.** A number is selected from the natural number 1 to 100. Find the probability of the event that the selected number is a multiple of 3 or 5.

**5.** Two balanced dice are thrown simultaneously. Find the probability that the sum of numbers on two dice is a multiple of 2 or 3.

**6.** The probability that the price of potato rises in the vegetable market during festive days in 0.8. The probability that the price of onion rises is 0.7. The probability of rise in price of both potato and onion is 0.6. Find the probability of rise in price of at least one of the two, potato and onion.

**7.** Two aircrafts drop bomb to destroy a bridge. The probability that a bomb dropped from the first aircraft hits the target is 0.9 and the probability that a bomb from the second aircraft hits the target is 0.7. The probability of one bomb dropped from both the aircrafts hitting the target is 0.63. The bridge is destroyed even if one bomb drops on it. Find the probability that the bridge is destroyed.

8. The probability that a teenager coming to a restaurant for dinner orders pizza is 0.63. The probability of ordering cold-drink is 0.54. The probability that the teenager orders at least one out of pizza and cold-drink is 0.88. Find the probability that the teenager coming for dinner on a certain day orders only one of the two items from pizza and cold-drink.

9. If $A$ and $B$ are mutually exclusive and exhaustive events in a sample space $U$ and $P(A)=2P(B)$ then find $P(A)$.

10. Three events $A$, $B$ and $C$ in a sample space are mutually exclusive and exhaustive. If $4P(A)=5P(B)=3P(C)$ then find $P(A\cup C)$ and $P(B\cup C)$.

11. Find $P(A\cup B\cup C)$ using the following information about three events $A$, $B$ and $C$ in a sample space.

$P(A)=0.65,\ P(B)=0.45,\ P(C)=0.25,\ P(A\cap B)=0.25,\ P(A\cap C)=0.15,\ P(B\cap C)=0.2,$

$P(A\cap B\cap C)=0.05$

12. Three events $A$, $B$ and $C$ in a sample space are mutually exclusive and exhaustive. If $P(C')=0.8$ and $3P(B)=2P(A')$ then find $P(A)$ and $P(B)$.

*

## 1.6 Conditional Probability and Law of Multiplication of Probability

### 1.6.1 Conditional Probability

Suppose $U$ is a finite sample space and $A$ and $B$ are any two events in it. The probability of occurrence of event $B$ under the condition that $A$ occurs is called the conditional probability. If the occurrence of event $B$ under the condition that event $A$ occurs is denoted by $B/A$ then the probability $P(B/A)$ of the conditional event $B/A$ is called the conditional probability. This probability is obtained using the following formula :

$$P(B/A) = \frac{P(A\cap B)}{P(A)}; \quad P(A)\neq 0$$

Similarly, if the occurrence of event $A$ under the condition that event $B$ occurs is denoted by $A/B$ then the probability $P(A/B)$ of the conditional event $A/B$ is obtained using the following formula :

$$P(A/B) = \frac{P(A\cap B)}{P(B)}; \quad P(B)\neq 0$$

Suppose a company produces a certain type of item in its two different factories $A_1$ and $A_2$. One item is randomly selected from a store selling the items produced by this company. Let us denote the event that the selected item is defective as $D$.

- If the selected item is produced at factory $A_1$ then the event that it is defective is denoted by $D/A_1$.
- If the selected item is produced at factory $A_2$ then the event that it is defective is denoted by $D/A_2$.

Thus,

$P(D/A_1)$ = Probability of occurrence of $D$ under the condition that event $A_1$ has occurred and

$P(D/A_2)$ = Probability of occurrence of $D$ under the condition that event $A_2$ has occurred

### 1.6.2 Independent Events

Suppose $A$ and $B$ are any two events in a finite sample space $U$. If the probability of occurrence of event $A$ does not change due to occurrence (or non-occurrence) of event $B$ then the events $A$ and $B$ are called the independent events.

Thus, if $P(A) = P(A/B) = P(A/B')$ and $P(B) = P(B/A) = P(B/A')$ the events $A$ and $B$ are called independent events.

For example, Event $A$ = First throw of a balanced die shows number 1.

Event $B$ = Second throw of a balanced die shows an even number.

It can be said here that the probability of getting an even number in the second throw of the die does not change because the first throw had shown the number 1. This fact can be easily understood by the following calculation :

The total number of outcomes by throwing the dice two times is $n = 6 \times 6 = 36$.

$A$ = Event that the first throw of a balanced die shows the number 1.

The number of favourable outcomes of $A$ is $m = 6$.

Probability of event $A$ $P(A) = \frac{m}{n}$

$$= \frac{6}{36}$$

$B$ = Event that the second throw of balanced die shows an even number.

The number of favourable outcomes of $B$ is $m = 18$

Probability of event $B$ $P(B) = \frac{m}{n}$

$$= \frac{18}{36}$$

$$= \frac{1}{2}$$

$A \cap B$ = Event that the first throw of a balanced die shows the number 1 and the second throw shows even number.

The number of favourable outcomes of $A \cap B$ is $m = 3$.

Probability of event $A \cap B$ $P(A \cap B) = \frac{m}{n}$

$$= \frac{3}{36}$$

Now, if the first throw of the die shows number 1 then the probability $P(B/A)$ for the event $B/A$ of getting an even number in the second throw can be obtained as follows :

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{3}{36}}{\frac{6}{36}}$$

$$= \frac{1}{2}$$

Since we get $P(B) = P(B/A)$, we say that the events $A$ and $B$ are independent.

### 1.6.3 Law of Multiplication of Probability

If $A$ and $B$ are the two events in a sample space $U$ then the rule of obtaining the probability of simultaneous occurrence of events $A$ and $B$ is called the law of multiplication of probability.

For example, Event $A$ = Getting head when a coin is tossed for the first time.

Event $B$ = Getting head when a coin is tossed for the second time.

If the coin is tossed two times then the probability of getting head both the times that is event $A \cap B$ can be obtained by the law of multiplication of probability. The law of multiplication of probability is as follows :

$P(A \cap B) = P(A) \times P(B/A);\ P(A) \neq 0$

$P(A \cap B) = P(B) \times P(A/B);\ P(B) \neq 0$

Some of the important results deduced from this rule are as follows which will be accepted without proof.

(1) If $A$ and $B$ are independent events then $P(A \cap B) = P(A) \times P(B)$

(2) If $A$ and $B$ are independent events then

(i) The events $A'$ and $B'$ are also independent. Hence, $P(A' \cap B') = P(A') \times P(B')$

(ii) The events $A$ and $B'$ are also independent. Hence, $P(A \cap B') = P(A) \times P(B')$

(iii) The events $A'$ and $B$ are also independent. Hence, $P(A' \cap B) = P(A') \times P(B)$

### 1.6.4 Selection with Replacement and without Replacement

When the units are to be randomly selected one by one from the population, the selection can be done in two ways :

(1) **Selection with replacement :** If the selection of a unit from the population in any trial is done by replacing the unit selected in the previous trial back to the population then that selection is called the selection with replacement.

(2) **Selection without Replacement :** If the selection of a unit from the population in any trial is done by not replacing the unit selected in the previous trial back to the population then that selection is called the selection with replacement.

**Illustration 27 : A balanced coin is tossed twice. If the first toss of the coin shows head then find the probability of getting head in both the tosses.**

The sample space of the random experiment of tossing a balanced coin twice is $U = \{HH, HT, TH, TT\}$, where the first symbol shows the outcome of the first toss of the coin and the second symbol shows the outcome of the second toss of the coin. The total number of outcomes in this sample space is $n = 4$.

If $A$ denote the event of getting head in the first toss of the coin and $B$ denotes the event that both the tosses result in head then we have to find $P(B/A)$, probability of $B/A$.

Event $A$ = First toss shows head

$= \{HH, HT\}$

Hence, the number of favourable outcomes of $A$ is $m = 2$.

Probability of event $A$ $P(A) = \frac{m}{n}$

$= \frac{2}{4}$

Event $B$ = Head is shown in both the tosses

$= \{HH\}$

Hence, the number of favourable outcomes of $B$ is $m = 1$.

Probability of event $B$ $P(B) = \frac{m}{n}$

$= \frac{1}{4}$

Event $A \cap B$ = Getting head in the first toss and getting head in both the tosses of the coin (we have $B \subset A$.)

$= \{HH\}$

Hence, the number of favourable outcomes of $A \cap B$ is $m = 1$.

Probability of $A \cap B$ $P(A \cap B) = \frac{m}{n}$

$= \frac{1}{4}$

Now,

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{1}{4}}{\frac{2}{4}}$$

$$= \frac{1}{2}$$

Required probability $= \frac{1}{2}$

**Illustration 28 : A factory has received an order to prepare 50,000 units of an item in a certain time period. The probability of completing this work in the given time is 0.75 and the probability that the workers will not declare strike during that time period is 0.8. The probability that this work will be completed during the given period and the workers will not declare strike is 0.7. Find the probability that**

**(1) The work will be completed as per schedule under the condition that the workers have not declared strike.**

**(2) Find the probability that the workers do not declare strike in the given period knowing that the work is completed as per schedule.**

If we denote event $A$ that the work will be completed as per schedule and event $B$ that the workers will not declare strike then the given information can be written as follows :

$P(A) = 0.75,\ P(B) = 0.8,\ P(A \cap B) = 0.7$

(1) Event that the work will be completed in the given period under the condition that the workers do not declare strike $= A/B$

Probability of $A/B$ from the definition of condition probability,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.7}{0.8}$$

$$= \frac{7}{8}$$

Required probability $= \frac{7}{8}$

(2) If it is given that the work is completed as per schedule then the event that the workers do not declare strike $= B/A$

Probability of $B/A$ from the definition of conditional probability,

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{0.7}{0.75}$$

$$= \frac{14}{15}$$

Required probability $= \frac{14}{15}$

**Illustration 29 : If $P(A') = \frac{7}{25}$, $P(B/A) = \frac{5}{12}$ and $P(A/B) = \frac{1}{2}$ for two events $A$ and $B$ in the sample space of a random experiment then find $P(A \cap B)$ and $P(B)$.**

It is given that $P(A') = \frac{7}{25}$, $P(B/A) = \frac{5}{12}$ and $P(A/B) = \frac{1}{2}$.

$$P(A) = 1 - P(A')$$

$$= 1 - \frac{7}{25}$$

$$= \frac{18}{25}$$

We will find $P(A \cap B)$ from the formula of $P(B/A)$.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\therefore \quad \frac{5}{12} = \frac{P(A \cap B)}{\frac{18}{25}}$$

$$\therefore \quad \frac{5}{12} \times \frac{18}{25} = P(A \cap B)$$

$$\therefore \quad P(A \cap B) = \frac{3}{10}$$

Required probability $= \frac{3}{10}$

Now, we will find $P(B)$ by substituting $P(A/B)$ and $P(A \cap B)$ in the formula of $P(A/B)$.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore \quad \frac{1}{2} = \frac{\frac{3}{10}}{P(B)}$$

$$\therefore \quad P(B) = \frac{\frac{3}{10}}{\frac{1}{2}}$$

$$= \frac{3 \times 2}{10 \times 1}$$

$$= \frac{3}{5}$$

Required probability $= \frac{3}{5}$

**Illustration 30 : A medicine is tested on a group of rabbits and mice to know its effect. It was observed that 7 rabbits show the effect of medicine in a group of 10 rabbits who were given the medicine and 5 mice show the effect of medicine in a group of 9 mice who were given the medicine. One animal is selected at random from each group. Find the probability that (1) both the selected animals show the effect of medicine and (2) one of the two selected animals shows the effect of medicine and the other animal does not show the effect of medicine.**

**The given information will be shown as follows :**

| Animals affected by Medicine | Animals not affected by Medicine |
|---|---|
| Rabbits 7 | Rabbits 3 |
| Mice 5 | Mice 4 |
| **Total 12** | **Total 7** |

(1) Event that a rabbit shows effect of medicine $= A$

Event that a mouse shows effect of medicine $= B$

$\therefore$ Event that both the animals show the effect of medicine $= A \cap B$

The events $A$ and $B$ are independent. Whether the mice show the effect of medicine is not affected by the effect of medicine on rabbits. Hence,

$$P(A \cap B) = P(A) \times P(B)$$

From the total number of outcomes $n = 10$, $m = 7$ outcomes are favourable for event $A$.

$\therefore$ Probability of event $A$ $P(A) = \frac{m}{n}$

$$= \frac{7}{10}$$

From the total number of outcomes $n = 9$, $m = 5$ outcomes are favourable for event $B$.

$\therefore$ Probability of event $B$ $P(B) = \frac{m}{n}$

$$= \frac{5}{9}$$

$$\therefore P(A \cap B) = \frac{7}{10} \times \frac{5}{9}$$

$$= \frac{7}{18}$$

Required probability $= \frac{7}{18}$

(2) Let $C$ denote the event that one animal is affected by the medicine and the other animal is not affected by the medicine. The event $C$ can occur as follows :

Rabbit is affected by the medicine (event $A$) and mouse is not affected by the medicine (event $B'$)

**OR**

Rabbit is not affected by the medicine (event $A'$) and mouse is affected by the medicine (event $B$)

Thus, event $C = (A \cap B') \cup (A' \cap B)$

Since the events $A \cap B'$ and $A' \cap B$ are mutually exclusive,

$$P(C) = P(A \cap B') + P(A' \cap B)$$

$$= [P(A) \times P(B')] + [P(A') \times P(B)] \quad (\because A \text{ and } B \text{ are independent events})$$

Here, $P(A') = 1 - P(A)$

$$= 1 - \frac{7}{10}$$

$$= \frac{3}{10}$$

$P(B') = 1 - P(B)$

$$= 1 - \frac{5}{9}$$

$$= \frac{4}{9}$$

$$\therefore P(C) = [\frac{7}{10} \times \frac{4}{9}] + [\frac{3}{10} \times \frac{5}{9}]$$

$$= \frac{28}{90} + \frac{15}{90}$$

$$= \frac{43}{90}$$

Required probability $= \frac{43}{90}$

**Illustration 31 : A company produces a certain type of item in its two different factories $A_1$ and $A_2$ in the proportion 60% and 40% respectively. The proportions of defectives in the production of these factories are 2% and 3% respectively. One item is randomly selected after mixing the items produced in the two factories. Find the probability that this item is defective.**

Event that the selected item is produced in factory $A_1$ = $A_1$

$$\therefore \quad P(A_1) = \frac{60}{100}$$

$$= \frac{3}{5}$$

Event that the selected item is produced in factory $A_2$ = $A_2$

$$\therefore \quad P(A_2) = \frac{40}{100}$$

$$= \frac{2}{5}$$

Let $D$ denote the event that the item selected from the total production is defective.

Event that the selected item is defective when it is produced in factory $A_1$ = $D/A_1$

$$\therefore \quad P(D/A_1) = \frac{2}{100}$$

$$= \frac{1}{50}$$

Event that the selected item is defective when it is produced in factory $A_2$ = $D/A_2$

$$\therefore \quad P(D/A_2) = \frac{3}{100}$$

Event $D$ can occur as follows.

The selected item is produced in factory $A_1$ and it is defective.

**OR**

The selected item is produced in factory $A_2$ and it is defective.

Thus event $D = (A_1 \cap D) \cup (A_2 \cap D)$

Since the events $A_1 \cap D$ and $A_2 \cap D$ are mutually exclusive,

$$P(D) = P(A_1 \cap D) + P(A_2 \cap D)$$

$$= \left[P(A_1) \times P(D/A_1)\right] + \left[P(A_2) \times P(D/A_2)\right]$$

$$= \left[\frac{3}{5} \times \frac{1}{50}\right] + \left[\frac{2}{5} \times \frac{3}{100}\right]$$

$= \frac{3}{250} + \frac{6}{500}$

$= \frac{12}{500}$

$= \frac{3}{125}$

Required probability $= \frac{3}{125}$

**Illustration 32 : There are 12 screws in a box of which 4 screws are defective. Two screws are randomly selected one by one without replacement from this box. Find the probability that both the screws selected are defective.**

4 screws are defective in the box having 12 screws. Hence, the number of non-defective screws will be 8.

Total number of mutually exclusive, exhaustive and equiprobable outcomes for selecting the first screw are $n = {}^{12}C_1 = 12$.

If $A$ denotes the event that the first screw selected is defective then the number of favourable outcomes of $A$ is $m = {}^{4}C_1 = 4$.

Probability of event $A$ $P(A) = \frac{m}{n} = \frac{4}{12}$

The screws are selected without replacement which means that the first screw is not kept back into the box. Hence, the total number of mutually exclusive, exhaustive and equi-probable outcomes for selecting the second screw is $n = {}^{11}C_1 = 11$.

Let $B$ denote the event that the second screw selected is defective.

The event $B$ occurs under the condition that the event $A$ has occurred. This is the occurrence of event $B/A$.

Since event $A$ has occurred earlier, there are 3 defective screws in the box.

Hence, the number of favourable outcomes for event $B/A$ is $m = {}^{3}C_1 = 3$.

Probability of $B/A$ $P(B/A) = \frac{m}{n}$

$= \frac{3}{11}$

Now, $A \cap B$ = Event that both the screws are defective

From the law of multiplication of probability,

$P(A \cap B) = P(A) \times P(B/A)$

$= \frac{4}{12} \times \frac{3}{11}$

$= \frac{1}{11}$

Required probability $= \frac{1}{11}$

**Illustration 33 : There are 3 boys and 2 girls in a friend-circle. Two persons are randomly selected from this friend-circle one by one with replacement to sing a song. Find the probability that the first person is a boy and the second person is a girl in the two persons selected to sing a song.**

The friend-circle consists of 3 boys and 2 girls that is total 5 persons. Two persons are selected one by one with replacement. This means that the person selected first is sent back to the group before selecting the second person. Hence, the events of selecting two persons one by one are independent events. The total number of mutually exclusive, exhaustive and equi-probable outcomes for selecting the first person is $n = {}^5C_1 = 5$.

Event that the first person selected to sing a song is a boy $= A$

The number of favourable outcomes for event $A$ is $m = {}^3C_1 = 3$

Probability of event $A$ $P(A) = \frac{m}{n}$

$= \frac{3}{5}$

The selection is with replacement here. This means that the total number of mutually exclusive, exhaustive and equi-probable outcomes for selecting the second person is $n = {}^5C_1$.

Event that the second person selected to sing a song is a girl $= B$

The number of favourable outcomes of $B$ is $m = {}^2C_1 = 2$

Probability of event $B$ $P(B) = \frac{m}{n}$

$= \frac{2}{5}$

Now, $A \cap B$ = Event that the first boy and the second girl are the two person selected to sing a song. Since the events $A$ and $B$ are independent,

$P(A \cap B) = P(A) \times P(B)$

$= \frac{3}{5} \times \frac{2}{5}$

$= \frac{6}{25}$

Required probability $= \frac{6}{25}$

**Illustration 34 : Two balanced dice are thrown simultaneously. Find the probability that at least one of the two dice shows the number 3.**

Event that the first die shows number 3 $= A$

Event that the second die shows number 3 $= B$

Event that at least one die shows number 3 $= A \cup B$

The number of favourable outcome for event $A$ is $m=1$

Probability of event $A$ $P(A)=\frac{m}{n}$

$$=\frac{1}{6}$$

The number of favourable outcomes for event $B$ is $m=1$

Probability of event $B$ $P(B)=\frac{m}{n}$

$$=\frac{1}{6}$$

Since the events $A$ and $B$ are independent, the events $A'$ and $B'$ are also independent. Moreover, $P(A')=1-P(A)=1-\frac{1}{6}=\frac{5}{6}$ and $P(B')=1-P(B)=1-\frac{1}{6}=\frac{5}{6}$.

Probability of the event that at least one die shows number 3 $=P(A\cup B)$

$$=1-P(A'\cap B')$$

$$=1-\left[P(A')\times P(B')\right]$$

$$=1-\left[\frac{5}{6}\times\frac{5}{6}\right]$$

$$=1-\frac{25}{36}$$

$$=\frac{11}{36}$$

Required probability $=\frac{11}{36}$

**Illustration 35 : Two cities $A$ and $B$ of different states have rains on 60% and 75% days respectively during the monsoon. For the cities $A$ and $B$, find the probability that on a certain monsoon day,**

**(1) both the cities have rains**

**(2) at least one city has rains**

**(3) only one city has rains.**

**Note : The events of rains on a day in these two cities are independent.**

Let event $A$ denote that it rains in city $A$ and event $B$ denote that it rains in city $B$. The given information can be stated as follows :

$P(A)=\frac{60}{100}=\frac{3}{5}$ $\quad\therefore P(A')=1-P(A)=1-\frac{3}{5}=\frac{2}{5}$

$P(B)=\frac{75}{100}=\frac{3}{4}$ $\quad\therefore P(B')=1-P(B)=1-\frac{3}{4}=\frac{1}{4}$

(1) Event that both the cities $A$ and $B$ have rains $A\cap B$

Since the events $A$ and $B$ are independent,

Probability of event $A\cap B$ $\quad P(A\cap B)=P(A)\times P(B)$

$$=\frac{3}{5}\times\frac{3}{4}$$

$$=\frac{9}{20}$$

Required probability $=\frac{9}{20}$

(2) Event that at least one of the cities $A$ and $B$ has rains $= A \cup B$

Probability of $A \cup B$ $\quad P(A \cup B) = 1 - P(A' \cap B')$

$$= 1 - \left[P(A') \times P(B')\right]$$

$$= 1 - \left[\frac{2}{5} \times \frac{1}{4}\right]$$

$$= 1 - \frac{1}{10}$$

$$= \frac{9}{10}$$

Required probability $= \frac{9}{10}$

(3) Event that only one of cities $A$ and $B$ has rains $= (A \cap B') \cup (A' \cap B)$

If the events $A$ and $B$ are independent then events $A$ and $B'$ as well as $A'$ and $B$ are also independent.

Probability of $(A \cap B') \cup (A' \cap B)$ $\quad = P(A \cap B') + P(A' \cap B)$

$$= \left[P(A) \times P(B')\right] + \left[P(A') \times P(B)\right]$$

$$= \left[\frac{3}{5} \times \frac{1}{4}\right] + \left[\frac{2}{5} \times \frac{3}{4}\right]$$

$$= \frac{3}{20} + \frac{6}{20}$$

$$= \frac{9}{20}$$

Required probability $= \frac{9}{20}$

**Exercise 1.4**

**1.** There are two children in a family. If the first child is a girl then find the probability that both the children in the family are girls.

**2.** Two six-faced balanced dice are thrown simultaneously. If the sum of numbers on both the dice is more than 7 then find the probability that both the dice show same numbers.

**3.** Among the various vehicle-owners visiting a petrol pump, 80% vehicle-owners visit to fill petrol in their vehicle and 60% vehicle-owners visit to fill air in their vehicles. 50% vehicle-owners visit to fill air and petrol in their vehicle. Find the probability for the following events :

(1) If a vehicle-owner has come to fill petrol in his vehicle then that vehicle-owner will fill air in his vehicle.

(2) If a vehicle-owner has come to fill air in his vehicle then that vehicle-owner will fill petrol in his vehicle.

4. 80% customers hold saving account and 50% customers hold current account of a nationalised bank. 90% of the customers hold at least one of the saving account and the current account. If one of the account holders randomly selected from this bank holds a current account, find the probability that he holds a saving account.

5. If $P(A)=\frac{2}{3}$, $P(B)=\frac{3}{5}$ and $P(B/A) = \frac{3}{4}$ for two events in the sample space of a random experiment then find $P(A/B)$.

6. If $P(M)=P(F)=\frac{1}{2}$, $P(A/M)= \frac{1}{10}$ and $P(A/F)=\frac{1}{2}$ for events $A$, $M$ and $F$ then find $P(A\cap M)$ and $P(A\cap F)$.

7. There are 2 gold-coins and 4 silver-coins in a box. The other box contains 3 gold and 5 silver coins. One coin is selected from each box. Find the probability that one of the selected coins is a gold coin and the other is a silver coin.

8. One joint family has 3 sons and 2 daughters whereas the other joint family has 2 sons and 4 daughters. One joint family is selected from two joint families and a child is randomly selected from that family. Find the probability that the selected child is a girl.

9. There are 10 icecream cones in a box of which 3 cones weigh less than the specification and the rest of the 7 cones have the specified weight. Two cones are randomly selected one by one with replacement. Find the probability that both the cones selected weigh less than the specified weight.

10. There are 10 CDs in a CD rack in which 6 are action film CDs and 4 are drama film CDs. Two CDs are randomly selected one by one without replacement from this box. Find the probability that the first selected CD is of action film and the second CD is of drama film.

11. If two balanced dice are thrown then find the probability that
    (1) at least one die shows number 5
    (2) the first die shows the number 5 or 6 and the other die shows an even number.

12. A problem in Mathematics is given to Tania, Kathan and Kirti to solve. The probabilities of them solving the problem correctly are $\frac{2}{3}, \frac{3}{4}$ and $\frac{1}{2}$ repectively. Find the probability that the problem is solved correctly.

13. Person $A$ can hit the target in 3 out of 5 attempts whereas person $B$ can hit the target in 5 out of 6 attempts. If both of them attempt simultaneously, find the probability that the target is hit.

14. Person $A$ speaks truth in 90% cases whereas person $B$ speaks truth in 80% cases. Find the probability that persons $A$ and $B$ differ in stating the same fact.

15. If three events $A$, $B$ and $C$ of a random experiment are independent events and $P(A)=0.2, P(B)=0.3$ and $P(C)=0.5$ then find $P(A\cup B\cup C)$.

*

## 1.7 Statistical Definition of Probability

We have seen the mathematical definition of probability earlier. This definition can help to find the probability only in the cases where the outcomes of the sample space of a random experiment are equi-probable and their number is known. But we find several cases in practice where the outcomes of the sample space are infinite and unknown. For example, there are many fish of different types in a huge lake. We have to find the probability of catching a certain type of fish when a fisherman throws net in the lake to catch fish. The mathematical definition of probability can not be used here as the total number of fish in the lake is unknown. Moreover, we come across many cases in practice where the outcomes of the random experiment are not equi-probable. For example, a trader transports certain goods from his godown to his sales centre. The event that these goods safely reach the sales centre and event that it does not safely reach the sales centre are not equi-probable events. It is not possible to evaluate probability using the mathematical definition of probability in such cases. Let us consider another definition of probability, called the statistical definition of probability, which is generally more useful in such situations.

Let us start with an illustration. We have to find the probability that a customer will purchase while visiting a showroom selling ready made garments for a long time. To know this, we should obtain the data about the customers purchasing from this show-room. These data can be obtained by sample inquiry. As the size of the sample increases, we can say that the information from the sample inquiry is more close to the true (population) information. Suppose it is found that 79 customers purchase out of 100 customers in the sample inquiry. When the number of customers in the sample inquiry was 500 then it was found that 403 customers purchased. The data obtained by increasing the sample size ($n$) are as follows :

| **Size of the sample (No. of customers visiting the show-room)** | **No. of customers purchasing $r$ (Frequency)** | **Proportion of customers purchasing $\frac{r}{n}$ (Expected Frequency)** |
|---|---|---|
| 100 | 79 | 0.79 |
| 500 | 403 | 0.806 |
| 1000 | 799 | 0.799 |
| 5000 | 3991 | 0.7982 |
| 10,000 | 8014 | 0.8014 |

It can be seen from the above data that as the size of the sample $n$ increases, the proportion or expected frequency of customers purchasing the ready-made garments takes values close to 0.8. We accept this value as the probability of the event that the customer visiting the show-room will purchase. Thus, the probability is obtained in the form of relative frequency. The definition of porbability based on the relative frequency is called the statistical definition of probability. It is also called the

empirical definition. The definition is as stated below :

Suppose a random experiment is repeated $n$ times under identical conditions. If an event $A$ occurs in $m$ trials out of $n$ trials then the relative frequency $\frac{m}{n}$ of event $A$ gives the estimate of the probability of event $A$, $P(A)$. When the larger and larger value of $n$ is taken, that is when $n$ tends to infinity, the limiting value of $\frac{m}{n}$ is called the probability of event $A$.

In notation,

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

The limiting value of the ratio $\frac{m}{n}$ when $n$ tends to infinite value is denoted by $\lim_{n \to \infty} \frac{m}{n}$. In practice, the relative frequency $\frac{m}{n}$ itself is taken as the probability of event $A$. Now we shall consider the examples showing the use of the statistical definition of probability.

**Illustration 36 : The sample data obtained about marks scored by a large group of candidates appearing for a public examination of 100 marks are given in the following table.**

| **Marks** | 20 or less | 21–40 | 41–60 | 61–80 | 81–100 |
|---|---|---|---|---|---|
| **No. of Candidates** | 83 | 162 | 496 | 326 | 124 |

**One candidate is randomly selected from those appearing for the public examination. Find the probability that this candidate has scored :**

**(1) less than 41 marks**

**(2) More than 60 marks**

**(3) Marks from 21 to 80.**

The number of candidates selected in the sample is $n = 83 + 162 + 496 + 326 + 124 = 1191$.

(1) Event $A$ = The selected candidate scores less than 41 marks.

$P(A)$ = Relative frequency for the candidates scoring less than 41 marks.

$$= \frac{\text{No. of candidates scoring less than 41 marks}}{\text{Total number of candidates in the sample}} = \frac{m}{n}$$

$m$ = No. of candidates scoring less than 41 marks

$= 83 + 162$

$= 245$

Now, $P(A) = \frac{m}{n}$

$= \frac{245}{1191}$

Required probability $= \frac{245}{1191}$

(2) Event $B$ = The selected candidate scores more than 60 marks

$P(B)$ = relative frequency for candidates scoring more than 60 marks.

$$= \frac{\text{No. of candidates scoring more than 60 marks}}{\text{Total number of candidates in the sample}} = \frac{m}{n}$$

$m$ = No. of candidates scoring more than 60 marks

$= 326 + 124$

$= 450$

Now, $P(B) = \frac{m}{n}$

$= \frac{450}{1191}$

$= \frac{150}{397}$

Required probability $= \frac{150}{397}$

(3) Event $C$ = The selected candidate scores from 21 to 80 marks

$P(C)$ = relative frequency for candidates scoring from 21 to 80 marks.

$$= \frac{m}{n} = \frac{\text{No. of candidates scoring from 21 to 80 marks}}{\text{Total number of candidates in the sample}}$$

$m$ = No. of candidates scoring from 21 to 80 marks

$= 162 + 496 + 326$

$= 984$

Now, $P(C) = \frac{m}{n}$

$= \frac{984}{1191}$

$= \frac{328}{397}$

Required probability $= \frac{328}{397}$

**Illustration 37 : A factory runs in two shifts. The sample data about the quality of items produced in these shifts are shown in the following table :**

| Quality | Shift | | Total |
|---|---|---|---|
| | I | II | |
| **Defective items** | 24 | 46 | 70 |
| **Non-defective items** | 2176 | 2754 | 4930 |
| **Total** | **2200** | **2800** | **5000** |

**One item is randomly selected from the production of the factory.**

**(1) If the item is taken from the production of the first shift then find the probability that it is defective.**

**(2) If the item is defective then find the probability that it is taken from the production of the first shift.**

The total number of units in the sample = 5000

We shall define the events as follows :

Event $A$ = The selected item is from the production of first shift

$$P(A) = \frac{\text{No. of items produced in the first shift}}{\text{Total number of items in the sample}} = \frac{m}{n}$$

$m$ = No. of items produced in the first shift

= 2200

Now, $P(A) = \frac{m}{n}$

$= \frac{2200}{5000}$

Event $D$ = The selected item is defective

$P(D)$ = relative frequency for defective items

$$= \frac{\text{No. of defective items}}{\text{Total number of items in the sample}} = \frac{m}{n}$$

$m$ = No. of defective items

= 70

Now, $P(D) = \frac{m}{n}$

$= \frac{70}{5000}$

Event $A \cap D$ = The selected item is produced in the first shift and it is defective

$P(A \cap D)$ = relative frequency for event $A \cap D$

$$= \frac{\text{No. of items in event } A \cap D}{\text{Total number of items in the sample}} = \frac{m}{n}$$

$m$ = No. of items in event $A \cap D$

= 24

Now, $P(A \cap D) = \frac{m}{n} = \frac{24}{5000}$

(1) The event that the item is defective when it is taken from the first shift = $D/A$

Probability of $D/A$ using the formula of conditional event

$$P(D/A) = \frac{P(A \cap D)}{P(A)}$$

$$= \frac{\frac{24}{5000}}{\frac{2200}{5000}}$$

$$= \frac{24}{2200}$$

$$= \frac{3}{275}$$

Required probability $= \frac{3}{275}$

(This probability can be directly obtained as relative frequency $\frac{24}{2200}$ of the event $D/A$.)

(2) The event that the item is taken from the first shift when it is defective = $A/D$

Probability of $A/D$ using the formula of condition probability

$$P(A/D) = \frac{P(A \cap D)}{P(D)}$$

$$= \frac{\frac{24}{5000}}{\frac{70}{5000}}$$

$$= \frac{24}{70}$$

$$= \frac{12}{35}$$

Required probability $= \frac{12}{35}$

(This probability can be directly obtained as relative frequency $\frac{24}{70}$ of the event $A/D$.)

**Limitations :** The limitations of the statistical definition of probability are as follows :

(1) The value of probability can be obtained by the statistical definition of probability only if $n \to \infty$ that is if $n$ tends to infinity. But the infinite value of $n$ can not be taken in practice.

(2) The probability obtained by this definition is an estimated value. The exact value of probability cannot be known using this definition.

## Exercise 1.5

**1.** The sample data about monthly travel expense (in ₹) of a large group of travellers of local bus in a megacity are given in the following table :

| **Monthly travel expense (₹)** | 501–600 | 601–700 | 701–800 | 801–900 | 901 or more |
|---|---|---|---|---|---|
| **No. of travellers** | 318 | 432 | 639 | 579 | 174 |

One person from this megacity travelling by local bus is randomly selected. Find the probability that the monthly travel expense of this person will be (1) more than ₹ 900 (2) at the most ₹ 700 (3) ₹ 601 or more but ₹ 900 or less.

**2.** The details of a sample inquiry of 4979 voters of constituency are as follows :

| **Details** | **Males** | **Females** |
|---|---|---|
| **Supporters of party *A*** | 1319 | 1118 |
| **Supporters of party *B*** | 1217 | 1325 |

One voter is randomly selected from this constituency.

(1) If this voter is a male, find the probability that he is a supporter of Party $A$.

(2) If this voter is a supporter of Party $A$, find the probability that he is a male.

*

## Summary

- The events based on chance are called random events.
- The experiment which can be independently repeated under identical conditions and all its posible outcomes are known but which of the outcomes will appear can not be predicted with certainty before conducting the experiment is called a random experiment.
- The set of all possible outcomes of a random experiment is called a sample space of that experiment.
- A subset of the sample space of random experiment is called an event of that random experiment.
- $U$ is a finite sample space and $A$ and $B$ are two events in it. If events $A$ and $B$ can never occur together that is if $A \cap B = \phi$ then the events $A$ and $B$ are called mutually exclusive events.
- If the group of favourable outcomes of events of a random experiment is the sample space then the events are called exhaustive events.
- The elementary events are mutually exclusive and exhaustive.
- If there is no apparant reason to believe that out of one or more events of a random experiment, any one event is more or less likely to occur than the other events then the events are called equi-probable events.
- The number of mutually exclusive, exhaustive and equi-probable outcomes in the sample space $U$ of a random experiment is $n$. If $m$ outcomes among them are favourable for the event $A$ then probability of event $A$ is $\frac{m}{n}$.
- The range of values of $P(A)$, the probability of any event $A$ of a sample space $U$, is 0 to 1. That is $0 \le P(A) \le 1$.
- $A$ and $B$ are any two events in a finite sample space $U$. If the probability of occurrence of event $A$ does not change due to occurrence (or non-occurrence) of event $B$ then $A$ and $B$ are independent events.
- Suppose a random experiment is repeated $n$ times under identical conditions. If an event $A$ occurs in $m$ trials out of $n$ trials then the relative frequency $\frac{m}{n}$ of event $A$ gives the estimate of the probability of event $A$, $P(A)$. When the larger and larger value of $n$ is taken that is when $n$ tends to infinity, the limiting value of $\frac{m}{n}$ is called the probability of event $A$.

## List of Formulae

(1) Complementary event of $A$ $A' = U - A$

(2) Difference event of $A$ and $B$ $A - B = A \cap B' = A - (A \cap B)$ (only event $A$ occurrs.)

(3) Difference event of $B$ and $A$ $B - A = A' \cap B = B - (A \cap B)$ (only event $B$ occurrs.)

(4) The probability of an event $A$ of the sample space of a random experiment is $P(A) = \frac{m}{n}$.

(5) Law of addition of probability

For two events $A$ and $B$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For any three events $A$, $B$ and $C$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

If two events $A$ and $B$ are mutually exclusive,

$$P(A \cup B) = P(A) + P(B)$$

If three events $A$, $B$ and $C$ are mutually exclusive,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

If two events $A$ and $B$ are mutually exclusive and exhaustive,

$$P(A \cup B) = P(A) + P(B) = 1$$

If three events $A$, $B$ and $C$ are mutually exclusive and exhaustive,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 1$$

(6) Conditional probability

Event $B$ occurs under the condition that event $A$ occurs

$$P(B/A) = \frac{P(A \cap B)}{P(A)}; \quad P(A) \neq 0$$

Event $A$ occurs under the condition that event $B$ occurs

$$P(A/B) = \frac{P(A \cap B)}{P(B)}; \quad P(B) \neq 0$$

(7) Law of multiplication of probability

For any two events $A$ and $B$,

$P(A \cap B) = P(A) \times P(B/A);\ P(A) \neq 0$

$P(A \cap B) = P(B) \times P(A/B);\ P(B) \neq 0$

- For independent events $A$ and $B$,

$P(A \cap B) = P(A) \times P(B)$

$P(A' \cap B') = P(A') \times P(B')$

$P(A' \cap B) = P(A') \times P(B)$

$P(A \cap B') = P(A) \times P(B')$

(8) According to statistical definition of probability,

$P(A) = \lim_{n \to \infty} \frac{m}{n}$

## Exercise 1

### Section A

**Find the correct option for the following multiple choice questions :**

**1.** Which event is given by a special subset $\phi$ of the sample space $U$ ?

(a) Certain event (b) Complementary event of $\phi$

(C) Union of events U and $\phi$ (d) Impossible event

**2.** What is the value of $P(A \cap A')$ for events $A$ and $A'$ ?

(a) 1 (b) 0 (c) 0.5 (d) between 0 and 1

**3.** Which of the following options is true for any event of the sample space ?

(a) $P(A) < 0$ (b) $0 \leq P(A) \geq 1$ (c) $0 \leq P(A) \leq 1$ (d) $P(A) > 1$

**4.** Which of the following options is not true for any two events $A$ and $B$ in the sample space $U$ where $A \subset B$ ?

(a) $P(A \cap B) = P(B)$ (b) $P(A \cap B) = P(A)$

(c) $P(A \cup B) \geq P(A)$ (d) $P(B - A) = P(B) - P(A)$

5. What is the other name of the classical definition of probability ?

(a) Mathematical definition (b) Axiomatic definition

(c) Statistical definition (d) Geometric definition

6. Which of the following statement for probability of elementary events $H$ and $T$ of random experiment of tossing a balanced coin is not true ?

(a) $P(T)=0.5$ (b) $P(H)+P(T)=1$ (c) $P(H\cap T)=0.5$ (d) $P(H)=0.5$

7. Which random experiment from the following random experiments has an infinite sample space ?

(a) Throwing two dice (b) Selecting two employees from an office

(c) To measure the life of electric bulb (d) Select a card from 52 cards

8. If $A\cup A'=U$ then what type of events are $A$ and $A'$ ?

(a) Independent events (b) Complementary events

(c) Certain events (d) Impossible events

9. If $P(A/B)=P(A)$ and $P(B/A)=P(B)$ then what type of events are $A$ and $B$ ?

(a) Independent events (b) Complementary events

(c) Certain events (d) Impossible events

10. Two events $A$ and $B$ of a sample space are mutually exclusive. Which of the following will be equal to $P(B-A)$ ?

(a) $P(A)$ (b) $P(B)$ (c) $P(A\cap B)$ (d) $P(A\cup B)$

11. What is the total number of sample points in the sample space formed by throwing three six-faced balanced dice simultaneously ?

(a) $6^2$ (b) $3^6$ (c) $6\times 3$ (d) $6^3$

12. If one number is randomly selected between 1 and 20, what is the probability that the number is a multiple of 5 ?

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{5}$ (d) $\frac{1}{3}$

13. If events $A$ and $B$ are independent, which of the following options is true ?

(a) $P(A\cap B)=P(A)\times P(B)$ (b) $P(A\cup B)=P(A)+P(B)$

(c) $P(A\cup B)=P(A)\times P(B)$ (d) $P(A\cap B)=P(A)+P(B)$

14. What is the probability of having 5 Thursdays in the month of February in a year which is not a leap year ?

(a) 0 (b) $\frac{1}{7}$ (c) $\frac{2}{7}$ (d) $\frac{3}{7}$

**15.** If $P(A)=0.4$ and $P(B')=0.3$ for two independent events $A$ and $B$ of a sample space then state the value of $P(A\cap B)$.

(a) 0.12 (b) 0.42 (c) 0.28 (d) 0.18

**16.** For two events $A$ and $B$ of a samples space, state the event $(A\cap B)\cup(A\cap B')$.

(a) $\phi$ (b) $B$ (c) $A$ (d) $U$

**17.** According to the mathematical definition of probability, what is the probability of each outcome among the $n$ outcomes of a random experiment ?

(a) 0 (b) $\frac{1}{n}$ (c) 1 (d) can not say

## Section B

**Answer the following questions in one sentence :**

**1.** Give two examples of random experiment.

**2.** Draw the Venn diagram for $A$-$B$, the difference event of $A$ and $B$.

**3.** Define an event.

**4.** Write the sample space of a random experiment of throwing one balanced die and a balanced coin simultaneously.

**5.** Define conditional probability.

**6.** State the formula for the probability of occurrence of at least one event out of three events $A$, $B$ and $C$.

**7.** Define independent events.

**8.** Write the law of multiplication of probability for two independent events $A$ and $B$ in a sample space.

**9.** Interpret $P(A/B)$ and $P(B/A)$.

**10.** When can we say that three events $A$, $B$ and $C$ in a sample space are exhaustive ?

**11.** Arrange $P(A\cup B), P(A), P(A\cap B), 0, P(A)+P(B)$ in the ascending order.

**12.** Define :

(1) Random Experiment (2) Sample Space

(3) Equi-probable Events (4) Favourable Outcomes

(5) Probability (Mathematical definition) (6) Probability (Statistical definition)

(7) Impossible Event (8) Certain Event

**13.** For two events $A$ and $B$ in a sample space, $A \cap B = \phi$ and $A \cup B = U$. State the values of $P(A \cap B)$ and $P(A \cup B)$.

**14.** If two events $A$ and $B$ in a sample space are independent then state the formula for $P(A \cup B)$.

**15.** If $A = \{x \mid 0 < x < 1\}$ and $B = \{x \mid \frac{1}{4} \le x \le 3\}$ then find $A \cap B$.

**16.** For two independent events $A$ and $B$, $P(A) = 0.5$ and $P(B) = 0.7$. Find $P(A' \cap B')$.

**17.** If $P(A) = 0.8$ and $P(A \cap B) = 0.25$, find $P(A - B)$.

**18.** If $P(A) = 0.3$ and $P(A \cap B) = 0.03$, find $P(B/A)$.

**19.** If $P(A) = P(B) = K$ for two mutually exclusive events $A$ and $B$, find $P(A \cup B)$.

**20.** If $P(A' \cap B) = 0.45$ and $A \cap B = \phi$, find $P(B)$.

**21.** Two events $A$ and $B$ in a sample space are mutually exclusive and exhaustive. If $P(A) = \frac{1}{3}$, find $P(B)$.

**22.** 2% items in a lot are defective. What is the probability that an item randomly selected from this lot is non-defective ?

**23.** State the number of sample points in the random experiment of tossing five balanced coins.

**24.** State the number of sample points in the random experiment of tossing one balanced coin and two balanced dice simultaneously.

**25.** Is it possible that $P(A) = 0.7$ and $P(A \cup B) = 0.45$ for two events $A$ and $B$ in a sample space ?

**26.** Two cards are selected one by one with replacement from 52 cards. State the number of elements in the sample space of this random experiment.

**27.** For two independent events $A$ and $B$, $P(B/A) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{5}$. Find $P(A)$.

**28.** 1998 tickets out of 2000 tickets do not have a prize. If a person randomly selects one ticket from 2000 tickets then what is the probability that the ticket selected is eligible for prize ?

## Section C

**Answer the following questions :**

**1.** Define the following events and draw their venn diagram :

(1) Mutually exclusive events (2) Union of events

(3) Intersection of events (4) Difference event

(5) Exhaustive events (6) Complementary event

**2.** Give the illustrations of finite and infinite sample space.

**3.** Give the illustrations of impossible and certain event.

**4.** State the characteristics of random experiment.

**5.** State the assumptions of mathematical definition of probability.

**6.** State the limitations of mathematical definition of probability.

**7.** State the limitations of statistical definition of probability.

**8.** Explain the equiprobable events with illustration.

**9.** State the law of addition of probability for two events $A$ and $B$. Write the law of addition of probability if these two events are mutually exclusive.

**10.** State the law of multiplication of probability for two events $A$ and $B$. Write the law of multiplication of probability if these two events are independent.

**11.** State the following results for two independent events $A$ and $B$ :

(1) $P(A \cap B)$ (2) $P(A' \cap B')$

(3) $P(A \cap B')$ (4) $P(A' \cap B)$

**12.** If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{6}$ then find $P(A' \cap B')$.

**13.** If $P(B) = 2P(A/B) = 0.4$ then find $P(A \cap B)$.

**14.** If the events $A$ and $B$ are independent and $3P(A) = 2P(B) = 0.12$ then find $P(A \cap B)$.

**15.** If $5P(A) = 3P(B) = 2\,P(A \cup B) = \frac{3}{2}$ for two events $A$ and $B$ then find $P(A' \cup B')$.

**16.** If $P(A \cap B) = 0.12$ and $P(B) = 0.3$ for two independent events $A$ and $B$ then find $P(A \cup B)$.

**17.** If $A = \{x \mid 1 < x < 3\}$ and $B = \{x \mid \frac{1}{2} \leq x < 2\}$ then find $A \cup B$ and $A \cap B$.

**18.** The probability of occurrence of at least one of the two events $A$ and $B$ is $\frac{1}{4}$. The probability that event $A$ occurs but event $B$ does not occur is $\frac{1}{5}$. Find the probability of event $B$.

**19.** If $P(B)=\frac{3}{5}$ and $P(A'\cap B)=\frac{1}{2}$, for two events $A$ and $B$, find $P(A/B)$.

**20.** 6 persons have a passport in a group of 10 persons. If 3 persons are randomly selected from this group, find the probability that

(1) all the three persons have a passport

(2) two persons among them do not have a passport.

**21.** The probability that the tax-limit for income of males increases in the budget of a year is 0.66 and the probability that the tax-limit increases for income of females is 0.72. The probability that the tax-limit increases for income of both the males and females is 0.47. Find the probability that

(1) the tax-limit increases for income of only one of the two, males and females.

(2) the tax-limit does not increase for income of males as well as females in the budget of that year.

**22.** The price of petrol rises in 80% of the cases and the price of diesel rises in 77% of the cases after the rise in price of crude oil. The price of petrol and diesel rises in 68% cases. Find the probability that the price of diesel rises under the condition that there is a rise in the price of petrol.

**23.** As per the prediction of weather bureau, the probabilities for rains on three days, Thursday, Friday and Saturday in the next week are 0.8, 0.7 and 0.6 respectively. Find the probability that it rains on at least one of the three days in the next week.

(**Note :** The events of rains on three days, Thursday, Friday and Saturday of a week are independent.)

## Section D

**Answer the following questions :**

**1.** 6 *LED* televisions and 4 *LCD* televisions are displayed in digital store $A$ whereas 5 *LED* televisions and 3 *LCD* televisions are displayed in digital store $B$. One of the two stores is randomly selected and one television is selected from that store. Find the probability that it is an *LCD* television.

**2.** One number is randomly selected from the natural number 1 to 100. Find the probability that the number selected is either a single digit number or a perfect square.

**3.** A balanced coin is tossed thrice. If the first two tosses have resulted in tail, find the probability that tail appears on the coin in all the three trials.

4. If events $A$, $B$ and $C$ are independent events and $P(A)=P(B)=P(C)=p$ then find the value of $P(A\cup B\cup C)$ in terms of $p$.

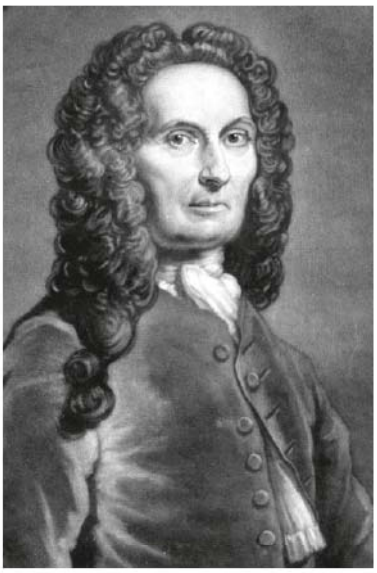
5. The genderwise data of a sample of 6000 employees selected from class 3 and class 4 employees in the government jobs of a state are shown in the following table :

| Class of Employees | Gender | | Total |
|---|---|---|---|
| | Males | Females | |
| **Class 3** | 3600 | 900 | 4500 |
| **Class 4** | 400 | 1100 | 1500 |
| **Total** | 4000 | 2000 | 6000 |

One employee is randomly selected from all the class 3 and class 4 employees in government jobs of this state.

(1) If the selected employee is a male, find the probability that he belongs to class 3.

(2) If it is given that the selected employee belongs to class 3, find the probability that he is a male.



**Abraham de Moivre**
**(1667 -1754)**

Abraham de Moivre was a French mathematician known for de Moivre's formula, one of those that link complex numbers and trigonometry, and for his work on the normal distribution and probability theory. De Moivre wrote a book on probability theory, The Doctrine of Chances. De Moivre first discovered Binet's formula, the closed-form expression for Fibonacci numbers linking the $n^{th}$ power of the golden ratio $\phi$ to the $n^{th}$ Fibonacci number. He also was the first to postulate the Central Limit Theorem, a cornerstone of probability theory. In the later editions of his book, de Moivre included his unpublished result of 1733, which is the first statement of an approximation to the binomial distribution in terms of what we now call the normal or Gaussian function.

De Moivre continued studying the fields of probability and mathematics until his death and several additional papers were published after his death.

# 2

# Random Variable and Discrete Probability Distribution

**Contents :**

## 2.1 Random Variable

We have studied about random experiment, sample space and probability in the chapter of probability. In this chapter, we shall study random variable and discrete probability distribution.

First of all, we shall define random variable and then we shall understand it by illustration.

**Random Variable :** Let $U$ be a sample space of a random experiment. Every element of $U$ need not always be a number. However, we wish to assign a specific number to each outcome.

A function associating a real number with each outcome of $U$ is called a random variable. It is denoted by $X$. That is, a random variable based on a sample space $U$ is denoted by $X : U \to R$.

For example,

(i) The number of heads $(H)$ in tossing an unbiased coin three times

(ii) The number of accidents during a week in a city

(iii) The weight of a person (in kilogram)

(iv) The maximum temperature of a day at a particular place (in Celsius)

Now, let us understand the concept of random variable by some illustrations.

(1) A balanced die is tossed once. If the number observed on the die is denoted by *'u'* then the elements of the sample space $U$ of this experiment can be shown in the notation of a set as follows :

$$U = \{u \mid u = 1, 2, 3, 4, 5, 6\}$$

That is $U = \{1, 2, 3, 4, 5, 6\}$

If we associate a real number $X$ with element $u$ of sample space by

$X(u)$ = the number obtained on the die then we can write

$$X(u) = u, u = 1, 2, 3, 4, 5, 6$$

Thus, variable $X$ will be a random variable assuming values 1, 2, 3, 4, 5 and 6.

In the above illustration, the element of $U$ are numeric. Now, we consider an illustration in which the elements of $U$ are non-numeric.

(2) Suppose a box contains four balls : one red, one blue, one yellow and one white ball. We donote the red ball by $R$, the blue ball by $B$, the yellow ball by $Y$ and the white ball by $W$. A person draws three balls at a time at random from the box. The sample space associated with this experiment is

$$U = \{RBY, RBW, BYW, WYR\}$$

Suppose for the element $u$ of $U$,

$X(u) =$ the number of white ball in $u$ then $X(RBY)=0,\ X(RBW)=1,\ X(BYW)=1,\ X(WYR)=1$.

Thus, random variable $X$ assumes the values in the set $\{0, 1\}$. The outcomes of this sample space are not in numbers but we associate them with real numbers by a random variable.

(3) Suppose the heights of students in a class lie between 120 cm and 180 cm. If we measure the height of a student of this class then it will assume any value between 120 cm and 180 cm.

Here, the sample space is $U = \{u \mid 120 \le u \le 180\}$.

If we denote the height of a selected student by $X$ then $X(u) = u =$ the height (in cm) of a selected student. Thus, $X$ becomes a random variable which will be denoted as $X = x,\ 120 \le x \le 180$.

In the above example (1) and example (2), random variable $X$ assumes particular countable values whereas in example (3), random variable $X$ can assume any value in the interval [120, 180]. This random variable differs from the random variables in earlier two examples.

Now, we shall understand the difference between these random variables in the following section.

### 2.1.1 Discrete Random Variable

A random variable $X$ which can assume a finite or countable infinite number of values in the set $R$ of real numbers is called a discrete random variable.

For example (i) Birth year of a randomly selected student.

(ii) Number of broken eggs in a box of 6 eggs.

Now, we shall understand about the discrete random variable by some specific examples.

(1) Suppose there is one black and two white balls in a box. Suppose the black ball is denoted by $B$ and two white balls by $W_1$ and $W_2$. A person can play the following game by paying ₹ 15.

The person playing a game is asked to select two balls randomly with replacement from the box. He is paid an amount according to the colour of the balls selected by him as per the following conditions :

If a white ball is selected then ₹ 5 are paid for each selected white ball and if a black ball is selected then ₹ 15 are paid per black ball.

If we denote the net amount earned (amount received – amount paid for the game) by the player corresponding to each outcome of the experiment by $X$ then $X$ becomes a discrete random variable. The values assumed by the variable $X$ are denoted in the following table :

| Outcome of the experiment (Event) | The amount received by the person by playing the game | The amount paid to play the game | The value of $X$ (in ₹) |
|---|---|---|---|
| $W_1W_1$ | $5 + 5 = 10$ | 15 | $X(W_1W_1) = 10 - 15 = -5$ |
| $W_1W_2$ | $5 + 5 = 10$ | 15 | $X(W_1W_2) = 10 - 15 = -5$ |
| $W_1B_1$ | $5 + 15 = 20$ | 15 | $X(W_1B_1) = 20 - 15 = 5$ |
| $W_2W_1$ | $5 + 5 = 10$ | 15 | $X(W_2W_1) = 10 - 15 = -5$ |
| $W_2W_2$ | $5 + 5 = 10$ | 15 | $X(W_2W_2) = 10 - 15 = -5$ |
| $W_2B_1$ | $5 + 15 = 20$ | 15 | $X(W_2B_1) = 20 - 15 = 5$ |
| $B_1W_1$ | $15 + 5 = 20$ | 15 | $X(B_1W_1) = 20 - 15 = 5$ |
| $B_1W_2$ | $15 + 5 = 20$ | 15 | $X(B_1W_2) = 20 - 15 = 5$ |
| $B_1B_1$ | $15 + 15 = 30$ | 15 | $X(B_1B_1) = 30 - 15 = 15$ |

Thus, the random variable $X$ assumes the values $-5$, 5 and 15 only. That is the total number of values of $X$ is finite.

(2) Suppose a coin is tossed until either a tail $(T)$ or four heads $(H)$ occur. Let $X$ denote the number of tosses required.

The sample space associated with this random experiment is

$U = \{T, HT, HHT, HHHT, HHHH\}$

The random variable $X$ denotes the number of tosses required for the coin associated with the experiment and it assumes any one value out of 1, 2, 3 and 4 for the sample points of the sample space.

$X(T) = 1,\ X(HT) = 2,\ X(HHT) = 3$

$X(HHHT) = 4,\ X(HHHH) = 4$

The discrete random variable $X$ assumes the finite number of values.

(3) Consider the random variable $X$ denoting the number of tails before getting the first head in the experiment of tossing a coin till the first head is obtained.

In this experiment, head will appear either in the first trial or in the second trial or in the third trial and so on... Similarly, the first head may be obtained after tossing a coin infinite times. Hence, the sample space associated with random experiment becomes

$U = \{H, TH, TTH, TTTH, TTTTH, .....\}$

Thus, the number of tails before getting the first head will be 0, 1, 2, 3, 4....

Thus, the random variable $X$ assumes any one value from the countable infinite number of values 0, 1, 2, 3, 4....

### 2.1.2 Continuous Random Variable

A random variable $X$ which can assume any value in $R$, the set of real numbers or in any interval of $R$ is called a continuous random variable.

For example (i) The actual amount of coffee in a coffee mug having a capacity of 250 millilitre.

(ii) Waiting time for a lift on any one floor of a high-rise office building.

Now, we shall understand more about the continuous random variable by the following examples.

(1) Denote the time taken by a student to finish a test of 3 hours duration by random variable $X$. The sample space here is

$$U = \{u | 0 \le u \le 3\}.$$

Since the time taken by any student for the exam takes any real value from 0 to 3 and the random variable $X$, the actual time taken by a student to complete the exam, will also be any real value from 0 to 3.

Thus,

$$X(u) = u,\ 0 \le u \le 3.$$

That means $X = x, 0 \le x \le 3$

The random variable $X$ assumes any real value from 0 to 3, which is a subset of $R$ and hence $X$ is a continuous random variable.

(2) Suppose there are two stations $A$ and $B$ on an express highway. The distance of station $B$ from station $A$ is 200 km. Let us consider an experiment to know the place of accident between two stations $A$ and $B$. For the sake of simplicity, let us fix the position of station $A$ at 0 km and of station $B$ at 200 km. The sample space of this experiment is any real value between 0 to 200. So, we can write the sample space for this experiment as

$$U = \{u \mid 0 \le u \le 200\}$$

Suppose the random variable $X$ denotes the distance (in kilometer) of the place of the accident between two stations $A$ and $B$ from the station $A$. Then the random variable $X$ is defined as below :

$X(u)$ = distance of the place of accident from the station $A$.

In short, we can define the random variable $X$ as $X = x, 0 \le x \le 200$.

The random variable $X$ assumes any real value from 0 to 200, which is subset of $R$, the set of real numbers. So, $X$ is a continuous random variable.

## 2.2 Discrete Probability Distribution

Suppose $X : U \to R$ is a random variable which assumes all the values of the subset $\{x_1, x_2, ...., x_n\}$ of $R$. Further, suppose $X$ assumes a value $x_i$ with probability $P(X = x_i) = p(x_i)$. If $p(x_i) > 0$, $i = 1, 2, ...., n$ and $\Sigma\, p(x_i) = 1$ then the set of real values $\{x_1, x_2, ...., x_n\}$ and $\{p(x_1), p(x_2), ....., p(x_n)\}$ is called the discrete probability distribution of a random variable $X$. The discrete probability distribution of a random variable $X$ is expressed in a tabular form as follow :

| $X = x$ | $x_1$ | $x_2$ | .... | $x_i$ | ... | $x_n$ |
|---|---|---|---|---|---|---|
| $p(x)$ | $p(x_1)$ | $p(x_2)$ | .... | $p(x_i)$ | ... | $p(x_n)$ |

Here, $0 < p(x_i) < 1, i = 1, 2, ...., n$ and $\Sigma\, p(x_i) = 1$

### 2.2.1 Illustrations for Probability Distribution of Discrete Variable

**Illustration 1 : Determine whether the values given below are appropriate as the values of a probability distribution of a discrete random variable $X$, which assumes the values 1, 2, 3 and 4 only.**

**(i)** $p(1) = 0.25, p(2) = 0.75, p(3) = 0.25, p(4) = -0.25$

**(ii)** $p(1) = 0.15, p(2) = 0.27, p(3) = 0.29, p(4) = 0.29$

**(iii)** $p(1) = \frac{1}{19}$, $p(2) = \frac{9}{19}$, $p(3) = \frac{3}{19}$, $p(4) = \frac{4}{19}$

(i) The value of $P(4)$ is –0.25, which is negative. It does not satisfy the condition $p(x_i) > 0, i = 1, 2, 3, 4$ of discrete probability distribution. So, given values are not suitable for the probability distribution of a discrete variable. Thus, the given distribution cannot be called a probability distribution of a discrete variable.

(ii) For every value 1, 2, 3 and 4 of $X$, $p(x) > 0$, and $p(1) + p(2) + p(3) + p(4) = 1$. Thus, both the conditions of probability distribution of discrete variable are satisfied. So, the given values are appropriate and the given distribution is probability distribution of a discrete variable.

(iii) Here $p(x_i) > 0, i = 1, 2, 3, 4$ but, sum of probabilities

i.e. $p(1) + p(2) + p(3) + p(4) = \frac{17}{19}$, is not 1. So, the given values are not appropriate for the probability distribution. So, the given distribution cannot be called a probability distribution of discrete variable.

**Illustration 2 : Determine when the following distribution is a probability distribution of discrete variable. Hence obtain the probability for $x = 2$ :**

$$p(x) = c\left(\frac{1}{4}\right)^x, \quad x = 1, 2, 3, 4$$

Here, $p(1) = c\left(\frac{1}{4}\right)$, $p(2) = c\left(\frac{1}{4}\right)^2 = c\left(\frac{1}{16}\right)$, $p(3) = c\left(\frac{1}{4}\right)^3 = c\left(\frac{1}{64}\right)$, $p(4) = c\left(\frac{1}{4}\right)^4 = c\left(\frac{1}{256}\right)$

Now, total probability should be 1 for a discrete probability distribution.

$$p(1)+p(2)+p(3)+p(4)=1$$

$$\therefore\ c\left(\frac{1}{4}\right)+c\left(\frac{1}{16}\right)+c\left(\frac{1}{64}\right)+c\left(\frac{1}{256}\right)=1$$

$$\therefore\ c\left[\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}\right]=1$$

$$\therefore\ c\left[\frac{85}{256}\right]=1$$

$$\therefore\ c=\frac{256}{85}$$

Thus, when $c=\frac{256}{85}$, the given distribution becomes probability distribution of a discrete variable.

Now, $P(X=2)=c\left(\frac{1}{4}\right)^2$

$$=\frac{256}{85}\times\frac{1}{16}$$

$$=\frac{16}{85}$$

$\therefore$ The probability of $X=2$ is $\frac{16}{85}$.

**Illustration 3 : A random variable $X$ denotes the number of accidents per year in a factory and the probability distribution of $X$ is given below :**

| $X=x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | $4K$ | $15K$ | $25K$ | $5K$ | $K$ |

**(i) Find the constant $K$ and rewrite the probability distribution.**

**(ii) Find the probability of the event that one or two accidents will occur in this factory during the year.**

**(iii) Find the probability that no accidents will take place during the year in the factory.**

(i) By the definition of discrete probability distribution, we must have

$$p(0)+p(1)+p(2)+p(3)+p(4)=1$$

That is $4K+15K+25K+5K+K=1$

$$\therefore 50K=1$$

$$\therefore K=\frac{1}{50}$$

$$=0.02$$

Thus, when $K=0.02$, the given distribution becomes a probability distribution of a discrete variable, which is given below :

| $X=x$ | 0 | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.08 | 0.30 | 0.50 | 0.10 | 0.02 | 1 |

(ii) Probability of occurrence of one or two accidents

$= P(X=1)+P(X=2)$

$= 0.30 + 0.50$

$= 0.80$

(iii) Probability that accidents do not occur :

$= P(X=0)$

$= 0.08$

**Illustration 4 : In a factory, packets of produced blades are prepared having 50 blades in each packet. A quality control engineer randomly selects a packet from these packets and examines all the blades of the selected packet. If 4 or more defective blades are observed in the selected packet then the packet is rejected. The probability distribution of the defective blades in the packet is given below :**

| **Number of defective blades in the packet** | 0 | 1 | 2 | 3 | 4 | 5 | 6 or more |
|---|---|---|---|---|---|---|---|
| **Probability** | $9K$ | $3K$ | $3K$ | $2K$ | $2K$ | $K-0.02$ | 0.02 |

**From the given probability distribution,**

**(i) Find constant *K*.**

**(ii) Find the probability that the randomly selected packet is accepted by the quality control engineer.**

(i) Let $X$ = number of defective blades found during the inspection of the packet.

By definition of discrete probability distribution

$p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6 \text{ or more})=1.$

$\therefore\ 9K + 3K + 3K + 2K + 2K + K - 0.02 + 0.02 = 1$

$\therefore\ 20K = 1$

$\therefore\ K = \frac{1}{20} = 0.05$

(ii) The randomly selected packet is accepted by the quality control engineer only when 3 or less defective blades are found in the packet.

$\therefore\ P(X \le 3)$

$= p(0)+p(1)+p(2)+p(3)$

$= 9K + 3K + 3K + 2K$

$= 17K$

$= 17(0.05)$

$= 0.85 \qquad (\because K = 0.05)$

**Illustration 5 : There are 4 red and 2 white balls in a box. 2 balls are drawn at random from the box without replacement. Obtain probability distribution of number of white balls in the selected balls.**

Suppose $X$ denotes the number of white balls in the selected two balls. $X$ may assume the values 0, 1 and 2.

$X = 0$ means there will not be any white balls in the selected two balls that means both the selected balls are red.

$$\therefore P(X=0) = P(2 \text{ red balls}) = \frac{{}^4C_2}{{}^6C_2} = \frac{6}{15}$$

Now, $x = 1$ means there will be one white ball and one red ball in the two selected balls.

$$\therefore P(X=1) = P(1 \text{ White ball, } 1 \text{ Red ball})$$

$$= \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2}$$

$$= \frac{2 \times 4}{15} = \frac{8}{15}$$

And $X = 2$ means both the selected ball will be white.

$$\therefore P(X=2) = P(2 \text{ White balls})$$

$$= \frac{{}^2C_2}{{}^6C_2}$$

$$= \frac{1}{15}$$

Thus, probability distribution of random variable $X$ can be written as follows :

| $X = x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$ | $\frac{6}{15}$ | $\frac{8}{15}$ | $\frac{1}{15}$ |

$p(x) > 0$ and $\Sigma p(x) = 1$

### 2.2.2 Mean and Variance

Now, we will discuss two important results based on the probability distribution of discrete random variable. One of them is expected value (mean) of the random variable and the other is variance of the random variable.

Let $X$ be a discrete random variable which assumes one of the values $x_1, x_2, \ldots, x_n$ only and its probability distribution is as follows :

| $X = x$ | $x_1$ | $x_2$ | .... | $x_i$ | ... | $x_n$ |
|---|---|---|---|---|---|---|
| $p(x)$ | $p(x_1)$ | $p(x_2)$ | .... | $p(x_i)$ | ... | $p(x_n)$ |

Where $0 < p(x_i) < 1, i = 1, 2, \ldots, n$ and $\Sigma\, p(x_i) = 1$

The mean of discrete random variable is denoted by $\mu$ or $E(X)$. It is defined as follows :

$$\mu = E(X) = \Sigma\, x_i\, p(x_i)$$

This value is also called expected value of discrete variable $X$.

The variance of discrete random variable $X$ is denoted by $\sigma^2$ or $V(X)$, which is defined as follows :

$$\sigma^2 = V(X) = E(X-\mu)^2$$

$$= E(X^2) - (\mu)^2$$

$$= E(X^2) - (E(X))^2$$

Where $E(X^2) = \Sigma\, x_i^2\, p(x_i)$

**Note :** (i) We will use the following notations for the sake of simplicity.

$\Sigma\, x\, p(x)$ instead of $\Sigma\, x_i\, p(x_i)$

and

$\Sigma\, x^2 p(x)$ instead of $\Sigma\, x_i^2 p(x_i)$

(ii) The mean and variance of variable $X$ are also called mean and variance of the distribution of $X$ respectively.

(iii) The value of the variance of variable $X$ is always positive.

We consider the following examples to find mean and variance of the discrete probability distribution.

**Illustration 6 : Find constant C for the following discrete probability distribution. Hence obtain mean and variance of this distribution.**

$$\boldsymbol{p(x) = C \cdot {}^4P_x,\ x = 0, 1, 2, 3, 4}$$

From the property of discrete probability distribution we must have

$$p(0) + p(1) + p(2) + p(3) + p(4) = 1$$

$$\therefore\ C \cdot {}^4P_0 + C \cdot {}^4P_1 + C \cdot {}^4P_2 + C \cdot {}^4P_3 + C \cdot {}^4P_4 = 1$$

$$\therefore C\left[\frac{4!}{4!} + \frac{4!}{3!} + \frac{4!}{2!} + \frac{4!}{1!} + \frac{4!}{0!}\right] = 1$$

$$\therefore C[1 + 4 + 12 + 24 + 24] = 1$$

$$\therefore C[65] = 1$$

$$\therefore C = \frac{1}{65}$$

Thus, the probability distribution can be written in the tabular form as follow :

| $X = x$ | 0 | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{1}{65}$ | $\frac{4}{65}$ | $\frac{12}{65}$ | $\frac{24}{65}$ | $\frac{24}{65}$ | 1 |

Now, mean of the distribution $= \mu = \Sigma x p(x)$

$$= 0\left(\frac{1}{65}\right) + 1\left(\frac{4}{65}\right) + 2\left(\frac{12}{65}\right) + 3\left(\frac{24}{65}\right) + 4\left(\frac{24}{65}\right)$$

$$= \frac{0 + 4 + 24 + 72 + 96}{65}$$

$$= \frac{196}{65}$$

Now, we obtain $E\left(X^2\right)$.

$$E\left(X^2\right) = \Sigma\, x^2 p(x)$$

$$= (0)^2\left(\frac{1}{65}\right) + (1)^2\left(\frac{4}{65}\right) + (2)^2\left(\frac{12}{65}\right) + (3)^2\left(\frac{24}{65}\right) + (4)^2\left(\frac{24}{65}\right)$$

$$= 0 + \frac{4}{65} + \frac{48}{65} + \frac{216}{65} + \frac{384}{65}$$

$$= \frac{652}{65}$$

Hence, variance of the distribution $= V(X)$

$$= E\left(X^2\right) - \left(E(X)\right)^2$$

$$= \frac{652}{65} - \left(\frac{196}{65}\right)^2$$

$$= \frac{42380 - 38416}{4225} = \frac{3964}{4225}$$

**Illustration 7 : There are two red and one green balls in a box. Two balls are drawn at random with replacement from the box. Obtain probability distribution of number of red balls in the two balls drawn and find its mean and variance.**

Let us denote the number of red balls in the selected two balls by $X$. Then we obtain the probability distribution of $X$ as follow.

Let us denote the two red balls of the box by $R_1$ and $R_2$ and green ball by G.

The number of red balls in the selected balls and its probability can be obtained as in the following table.

| **Selected two balls (Event)** | **Probability of the event** | $X = x$ |
|---|---|---|
| $R_1R_1$ | $\frac{1}{9}$ | 2 |
| $R_1R_2$ | $\frac{1}{9}$ | 2 |
| $R_1G$ | $\frac{1}{9}$ | 1 |
| $R_2R_1$ | $\frac{1}{9}$ | 2 |
| $R_2R_2$ | $\frac{1}{9}$ | 2 |
| $R_2G$ | $\frac{1}{9}$ | 1 |
| $GR_1$ | $\frac{1}{9}$ | 1 |
| $GR_2$ | $\frac{1}{9}$ | 1 |
| $GG$ | $\frac{1}{9}$ | 0 |

From the above table we can say that :

(i) Probability of getting 0 red ball

$= P(X = 0)$

$= \frac{1}{9}$

(ii) Probability of getting 1 red ball

$= P(X = 1)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{4}{9}$

(iii) Probability of getting 2 red balls

$= P(X = 2)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{4}{9}$

Thus, the probability distribution of $X$ can be written in the tabular from as follow :

| $X = x$ | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| $p(x)$ | $\frac{1}{9}$ | $\frac{4}{9}$ | $\frac{4}{9}$ | 1 |

Now, mean of the distribution $= \mu = E(X)$

$$= \Sigma\, x\, p(x)$$

$$= 0\left(\tfrac{1}{9}\right) + 1\left(\tfrac{4}{9}\right) + 2\left(\tfrac{4}{9}\right)$$

$$= \frac{0 + 4 + 8}{9}$$

$$= \frac{12}{9}$$

Now, we first find $E\left(X^2\right)$ to obtain variance of the distribution.

$$E\left(X^2\right) = \Sigma\, x^2 p(x)$$

$$= 0^2\left(\tfrac{1}{9}\right) + 1^2\left(\tfrac{4}{9}\right) + 2^2\left(\tfrac{4}{9}\right)$$

$$= \frac{0+4+16}{9}$$

$$= \frac{20}{9}$$

So, using the formula $V(X) = E\left(X^2\right) - \left(E(X)\right)^2$,

$$V(X) = \frac{20}{9} - \left(\frac{12}{9}\right)^2$$

$$= \frac{20}{9} - \frac{144}{81}$$

$$= \frac{180 - 144}{81}$$

$$= \frac{36}{81}$$

**Illustration 8 : There are 2 black and 2 white balls in a box. Two balls are drawn without replacement from it. Obtain probability distribution of the number of white balls in the selected balls. Hence find its mean and variance.**

Suppose $X$ = number of white balls in the selected two balls then by the formula of probability

(i) Probability of $X = 0$

$$= P(X = 0) = P \quad (0 \text{ white balls}) = \frac{{}^2C_0}{{}^4C_2} = \frac{1}{6}$$

(ii) Probability of $X = 1$

$= P(X = 1) = P$ (1 white ball and 1 black ball)

$$= \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2}$$

$$= \frac{2 \times 2}{6}$$

$$= \frac{4}{6}$$

(iii) Probability of $X = 2$

$= P(X = 2) = P$ (2 white balls)

$$= \frac{{}^2C_2}{{}^4C_2}$$

$$= \frac{1}{6}$$

Thus, the probability distribution of random variable $X$ can be written in the tabular form as,

| $X = x$ | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| $p(x)$ | $\frac{1}{6}$ | $\frac{4}{6}$ | $\frac{1}{6}$ | 1 |

Now, mean of the probability distribution $= E(X)$

$$= \Sigma\, x\, p(x)$$

$$= 0\left(\frac{1}{6}\right) + 1\left(\frac{4}{6}\right) + 2\left(\frac{1}{6}\right)$$

$$= \frac{0+4+2}{6}$$

$$= 1$$

Now, to obtain variance of the probability distribution, we first find $E(X^2)$.

$$E(X^2) = \Sigma\, x^2 p(x)$$

$$= 0^2\left(\frac{1}{6}\right) + 1^2\left(\frac{4}{6}\right) + 2^2\left(\frac{1}{6}\right)$$

$$= \frac{0+4+4}{6}$$

$$= \frac{8}{6}$$

$$\therefore V(X) = E(X^2) - (E(X))^2$$

$$= \frac{8}{6} - (1)^2 \qquad (\because E(X) = 1)$$

$$= \frac{8-6}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

**Illustration 9 : Let $X$ denote the maximum integer among the outcomes of tossing two dice simultaneously. Obtain the probability distribution of variable $X$ and find its mean and variance.**

By tossing two dice simultaneously, we have 36 events in the sample space $U$ and the maximum integer of the outcomes will be one of the numbers 1, 2, 3, 4, 5 or 6. The following table gives the possible outcomes for variable X and the corresponding probability :

| **Event $u$ of $U$** | **Maximum integer** $X(u)=x$ | $P(X=x)$ |
|---|---|---|
| (1, 1) | 1 | $\frac{1}{36}$ |
| (1, 2), (2, 1), (2, 2) | 2 | $\frac{3}{36}$ |
| (1, 3), (2, 3), (3, 3) (3, 2), (3, 1) | 3 | $\frac{5}{36}$ |
| (1, 4), (2, 4), (3, 4), (4, 4) (4, 3), (4, 2), (4, 1) | 4 | $\frac{7}{36}$ |
| (1, 5), (2, 5), (3, 5), (4, 5), (5, 5) (5, 4), (5, 3), (5, 2), (5, 1) | 5 | $\frac{9}{36}$ |
| (1, 6), (2, 6), (3, 6), (4, 6), (5, 6) (6, 6), (6, 5), (6, 4), (6, 3), (6, 2) (6, 1) | 6 | $\frac{11}{36}$ |
| | | **Total 1** |

Now, mean of $X = E(X)$

$$= \Sigma x\, p(x)$$

$$= 1\left(\frac{1}{36}\right)+2\left(\frac{3}{36}\right)+3\left(\frac{5}{36}\right)+4\left(\frac{7}{36}\right)+5\left(\frac{9}{36}\right)+6\left(\frac{11}{36}\right)$$

$$= \frac{161}{36}$$

Now, $E\left(X^2\right)=\Sigma x^2 p(x)$

$$=1^2\left(\frac{1}{36}\right)+2^2\left(\frac{3}{36}\right)+3^2\left(\frac{5}{36}\right)+4^2\left(\frac{7}{36}\right)+5^2\left(\frac{9}{36}\right)+6^2\left(\frac{11}{36}\right)$$

$$=\frac{791}{36}$$

Variance of $X = V(X)$

$$= E\left(X^2\right)-\left(E(X)\right)^2$$

$$= \frac{791}{36} - \left(\frac{161}{36}\right)^2$$

$$= \frac{791}{36} - \frac{25921}{1296}$$

$$= \frac{791 \times 36 - 25921}{1296}$$

$$= \frac{28476 - 25921}{1296}$$

$$= \frac{2555}{1296}$$

**Illustration 10 : It is observed from the life table that the probability that a 40 years old man will live one more year is 0.95. Life insurance company wishes to sell one year life insurance policy of Rs. 10,000 to such a man. What should be the minimum premium of the policy so that expected gain of the company would be positive ?**

Let $X$ be the company's gain and yearly premium of the policy be ₹ K, $K > 0$. Then gain of the company is $X = K$ if 40 year old man will live for one year and gain of the company is $X = K - 10{,}000$ if 40 year old man will die within a year.

Thus, the probability distribution of the gain of the company is as follow :

| $X = x$ | $K$ | $K - 10000$ |
|---|---|---|
| $p(x)$ | 0.95 | 0.05 |

Hence, expected gain of the company

$= E(X)$

$= \Sigma x\, p(x)$

$= K(0.95) + (K - 10000)(0.05)$

$= K(0.95) + K(0.05) - 500$

$= K(0.95 + 0.05) - 500$

$= K - 500$

Now, for positive expected gain, we must have

$K - 500 > 0$

$\therefore K > 500$

So, the company should fix the premium more than ₹ 500 so that the expected gain of the company be will positive.

## EXERCISE 2.1

1. Examine whether the following distribution is a probability distribution of a discrete random variable $X$ :

$$p(x) = \frac{x+2}{25}, \quad x = 1, 2, 3, 4, 5$$

2. If the following distribution is a probability distribution of variable $X$ then find constant $K$.

$$p(x) = \frac{6-|x-7|}{K}, \quad x = 4, 5, 6, 7, 8, 9, 10$$

3. The probability distribution of a random variable $X$ is defined as follows :

$$p(x) = \frac{K}{(x+1)!}, \quad x = 1, 2, 3; K = \text{constant}$$

Hence find (i) constant $K$ (ii) $P(1 < X < 4)$

4. The probability distribution of a random variable $X$ is as follows :

| $X = x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $p(x)$ | $\frac{K}{3}$ | $\frac{K}{3}$ | $\frac{K}{3}$ | $2K$ | $4K$ |

Then (i) determine acceptable value of constant $K$. (ii) Find the Mean of the distribution.

5. The probability distribution of a random variable $X$ is $P(x)$. Variable $X$ can assume the values $x_1 = -2, x_2 = -1, x_3 = 1$ and $x_4 = 2$ and if $4P(x_1) = 2P(x_2) = 3P(x_3) = 4P(x_4)$ then obtain mean and variance of this probability distribution.

6. A die is randomly tossed two times. Determine the probability distribution of the sum of the numbers appearing both the times on the die and obtain expected value of the sum.

7. A box contains 4 red and 2 blue balls. Three balls are simultaneously drawn at random. If $X$ denotes the number of red balls in the selected balls, find the probability distribution of $X$ and find the expected number of red balls in the selected balls.

8. A coin is tossed till either a head or 5 tails are obtained. If a random variable $X$ denotes the necessary number of trials of tossing the coin then obtain probability distribution of the random variable $X$ and calculate its mean and variance.

9. A shopkeeper has 6 tickets in a box. 2 tickets among them are worth a prize of ₹ 10 and the remaining tickets are worth a prize of ₹ 5. If a ticket is drawn at random from the box, find the expected value of the prize.

*

## 2.3 Binomial Probability Distribution

In the earlier sections we considered continuous and discrete random variable and probability distribution of a discrete random variable. Now, we shall study an important probability distribution of a discrete random variable.

In some random experiments, there are only two outcomes. We call such outcomes as success and failure. These outcomes are mutually exclusive. We call such experiments as dichotomous experiments. The illustrations of some of these situations are given in the table below :

| Experiment | | Possible outcomes | |
|---|---|---|---|
| | | **Success** | **Failure** |
| (i) | To know the effect of advertisement given to increase the sale of produced units | sale increased | sale did not increase |
| (ii) | To find the error in a letter typed by a type-writer | Error observed | Error not observed |
| (iii) | To know the effect of a drug on blood pressure given to the patients of high blood pressure | Blood pressure decreased | Blood pressure did not decrease |
| (iv) | To inspect whether produced item is defective | Item is defective | Item is not defective |

If we denote the success by $S$ and failure by $F$ for such types of dichotomous experiment and the probabilities of such outcomes by $p$ and $q$ respectively then

$P(S) = p$ and $P(F) = q,\ 0 < p < 1,\ 0 < q < 1,\ p + q = 1$

Since there are only two outcomes of such an experiment and both are mutually exclusive, we have $p + q = 1$ and hence $q = 1 - p$.

If it is possible to repeat such a dichotomous random experiment $n$ times and each repetition is done under identical conditions then the probability of success $p$ remains constant in each trial. We call such experiments as Bernoulli Trials. Its actual definition can be given as follows :

**Bernoulli Trials :** Suppose dichotomous random experiment has two outcomes, success $(S)$ and failure $(F)$. If this experiment is repeated $n$ times under identical conditions and the probability $p(0 < p < 1)$ of getting a success at each trial is constant then such trials are called Bernoulli Trials.

**Properties of Bernoulli Trials**

(1) The probability of getting a success at each Bernoulli trial remains constant.

(2) Bernoulli trials are mutually independent. That means getting success or failure at any trial does not depend on getting success or failure at the previous trial.

(3) Success and failure are mutually exclusive and exhustive events. Therefore $q = 1 - p$.

**Binomial Probability Distribution**

Suppose $X$ denotes the number of successes in a sequence of success *(S)* and failure $(F)$ obtained in $n$ Bernoulli trials, then $X$ is called a binomial random variables and $X$ assumes any value in the finite set $\{0, 1, 2, ...., n\}$. The probability distribution of the binomial random variable $X$ is defined by the following formula :

$$P(X=x)=p(x)={}^{n}C_{x}p^{x}q^{n-x},\ x=0,\ 1, 2, ....., n, 0<p<1, q=1-p$$

This probability distribution is called binomial Probability Distribution. We shall call such a distribution in short as binomial distribution.

If positive integer $n$ and probability of success $p$ are known here, the whole probability distribution that means the probability of each possible value of $X$ can be determined. Hence, $n$ and $p$ are called parameters of the binomial distribution. We denote binomial distribution having parameters $n$ and $p$ as $b(n, p)$.

**Note :** If we repeat an experiment having such Bernoulli trials $N$ times and $p(x)$ is the probability of getting $x$ successes in the experiment then expected frequency of number of successes in $N$ repetitions $= N \cdot p(x)$

### 2.3.1 Properties of Binomial Distribution

(1) Binomial distribution is a discrete distribution.

(2) Its parameters are $n$ and $p$.

(3) The mean of the distribution is $np$ which denotes average (expected) number of successes in $n$ Bernoulli trials.

(4) The variance of the distribution is $npq$ and its standard deviation is $\sqrt{npq}$.

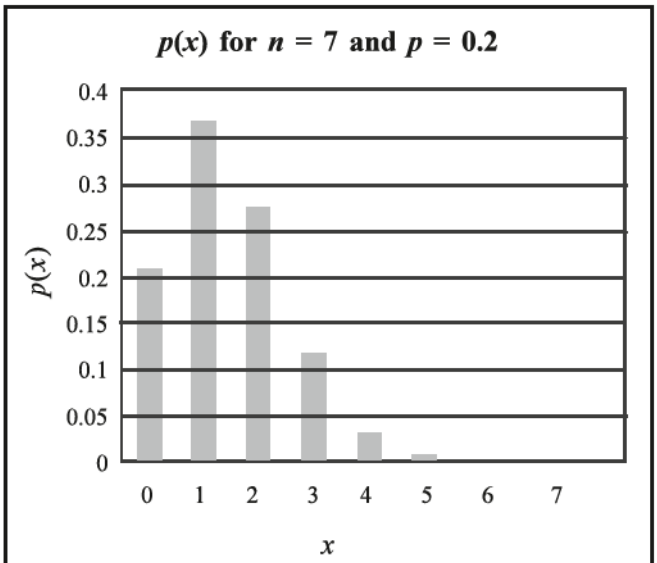
(5) For binomial distribution, mean is always greater than the variance and $\frac{\text{Variance}}{\text{Mean}} = q =$ probability of failure.

(6) If $p < \frac{1}{2}$ then the skewness of the distribution is positive for any value of $n$.

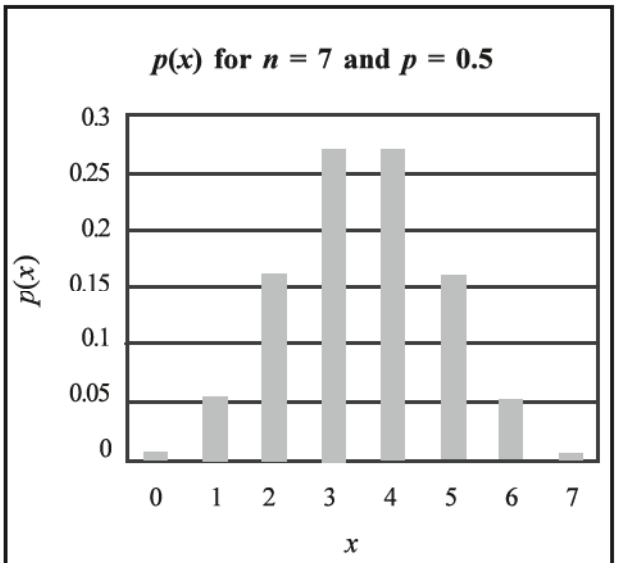
(7) If $p = \frac{1}{2}$ then the distribution becomes symmetric that means the skewness of the distribution is zero for any value of $n$.

(8) If $p > \frac{1}{2}$ then the skewness of the distribution is negative for any value of $n$.

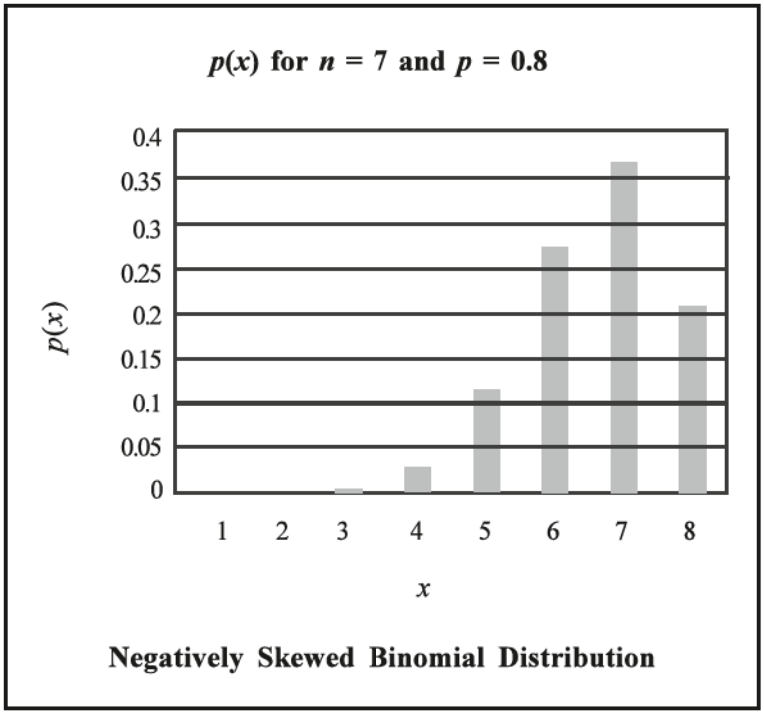
The properties (6), (7) and (8) can be clearly seen from the following graphs :




**Positively Skewed Binomial Distribution**




**Symmetric Binomial Distribution**




## 2.3.2 Illustrations of Binomial Distribution

**Illustration 11 : There are 3 % defective items in the items produced by a factory. 4 items are selected at random from the items produced. What is the probability that there will not be any detective item ?**

If the event that the selected items is defective is considered as success then the probability of success $p = 0.03$ and $n = 4$. None of the selected items is defective means $X = 0$.

Now,

$$p(x) = {}^{n}C_{x} p^{x} q^{n-x}, \; x = 0, 1, 2, \ldots, n$$

Putting the values of $n$, $p$, $q = 1 - p$ and $x$ in the formula,

$$P(X = 0) = {}^{4}C_{0} (0.03)^{0} (0.97)^{4-0}$$

$$= (0.97)^{4}$$

$$= 0.8853$$

Thus, the probability of getting no defective item in the selected 4 items is 0.8853.

**Illustration 12 : The probability that a person living in a city is a non-vegetarian is 0.20. Find the probability of at the most two persons out of 6 persons randomly selected from the city is non-vegetarian.**

If we consider the event that a person is non-vegetarian as success then we are given the probability of success $p = 0.20$ and $n = 6$.

If we take $X =$ number of non-vegetarians among the selected persons then the probability of $X \leq 2$

is obtained by putting the values of $n$, $p$ and $x$ in the formula of binomial probability distribution

$$p(x) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$$

$$p(X \le 2) = p(X = 0 \text{ or } X = 1 \text{ or } X = 2)$$

$$= p(0) + p(1) + p(2)$$

$$= {}^6C_0 (0.20)^0 (0.80)^6 + {}^6C_1 (0.20)^1 (0.80)^{6-1} + {}^6C_2 (0.20)^2 (0.80)^{6-2}$$

$$= 0.2621 + 6(0.20)(0.3277) + 15(0.04)(0.4096)$$

$$= 0.2621 + 0.3932 + 0.2458$$

$$= 0.9011$$

**Illustration 13 : The mean and variance of a binomial distribution are 3.9 and 2.73 respectively. Find the number of Bernoulli trials conducted in this distribution and write $p(x)$.**

Here, variance $= npq = 2.73$ and mean $= np = 3.9$.

$$\therefore q = \frac{\text{Variance}}{\text{Mean}} = \frac{2.73}{3.9} = 0.7 \text{ and } p = 1 - q = 0.3$$

$$\text{Now } n = \frac{np}{p} = \frac{\text{Mean}}{p} = \frac{3.9}{0.3} = 13$$

Thus, the number of Bernoulli trials conducted in this distribution is 13. Since $n = 13$, $p = 0.3$ and $q = 0.7$ in the distribution, its $p(x)$ can be written as follow :

$$p(x) = {}^{13}C_x (0.3)^x (0.7)^{13-x},\ x = 0, 1, 2, \ldots, 13.$$

**Illustration 14 : During a war, on an average one ship out of 9 got sunk in a certain voyage. Find the probability that exactly 5 out of a convoy of 6 ships would arrive safely.**

Suppose $X$ = the number of ships that arrive safely out of a convoy of 6 ships during a war.

$n =$ total number of ships in a convoy $= 6$

$p =$ probability that a ship arrives safely in a certain voyage $= \frac{8}{9}$

$\therefore$ The probability that exactly 5 out of a convoy of 6 ships would arrive safely can be obtained by putting corresponding values in the formula

$$p(x) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, \ldots, n,$$

$$p(5) = {}^6C_5 \left(\frac{8}{9}\right)^5 \left(\frac{1}{9}\right)^1$$

$$= 6\left(\frac{32{,}768}{59{,}049}\right)\left(\frac{1}{9}\right)$$

$$= \frac{196608}{531441}$$

$$= 0.3700$$

**Illustration 15 : Assume that on an average one line out of 4 telephone lines remains busy between 2 pm and 3 pm on week days. Find the probability that out of 6 randomly selected telephone lines (i) not more than 3 (ii) at least three of them will be busy.**

Suppose $p =$ the probability of the event that the selected telephone line remains busy between 2 pm to 3 pm $= \frac{1}{4}$

and $X =$ the number of busy telephone lines out of 6 telephone lines between 2 pm to 3 pm. It is given here that $n = 6$.

(i) The event that not more than 3 lines out of 6 randomlly selected telephone lines will be busy is the event that 3 or less telephone lines will be busy.

That is $X \leq 3$.

$\therefore$ To find probability of this event we use the formula of binomial probability distribution

$p(x) = {}^{n}c_{x} p^{x} q^{n-x}, x = 0, 1, 2, \ldots, n$

$\therefore \quad P(X \leq 3)$

$= P(X = 0 \text{ or } 1 \text{ or } 2 \text{ or } 3)$

$= 1 - P(X = 4 \text{ or } 5 \text{ or } 6)$

$= 1 - [p(4) + p(5) + p(6)]$

$= 1 - \left[{}^{6}C_{4} \left(\frac{1}{4}\right)^{4} \left(\frac{3}{4}\right)^{2} + {}^{6}C_{5} \left(\frac{1}{4}\right)^{5} \left(\frac{3}{4}\right)^{1} + {}^{6}C_{6} \left(\frac{1}{4}\right)^{6} \left(\frac{3}{4}\right)^{0}\right]$

$= 1 - \left[15\left(\frac{1}{256}\right)\left(\frac{9}{16}\right) + 6\left(\frac{1}{1024}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{4096}\right)\right]$

$= 1 - \left[\frac{135}{4096} + \frac{18}{4096} + \frac{1}{4096}\right]$

$= 1 - \frac{154}{4096} = \frac{3942}{4096} = 0.9624$

(ii) The probability that at least 3 telephone lines will be busy

$= P(X \geq 3)$

$= P(X = 3 \text{ or } 4 \text{ or } 5 \text{ or } 6)$

$= p(3) + p(4) + p(5) + p(6)$

Now, from the above calculations we will get the values of $p(4)$, $p(5)$ and $p(6)$. So, we first find the value of $p(3)$.

$$p(3) = {}^6C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^3 = 20 \left(\frac{1}{64}\right)\left(\frac{27}{64}\right)$$

$$= \frac{540}{4096}$$

Now, from the values of $p(4)$, $p(5)$ and $p(6)$ in question (i) and the value of $p(3)$ we obtained,

$$P(X \geq 3) = \frac{540}{4096} + \frac{135}{4096} + \frac{18}{4096} + \frac{1}{4096}$$

$$= \frac{694}{4096} = 0.1694$$

**Illustration 16 : The parameters of binomial distribution of a random variable $X$ are $n = 4$ and $p = \frac{1}{3}$. State the probability distribution of $X$ in a tabular form and hence find the value of $P(X \leq 2)$.**

Here, parameters are $n = 4$ and $p = \frac{1}{3}$ $\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Substituting the values of the parameters in the formula of binomial distribution,

$p(x) = {}^nC_x \, p^x q^{n-x}$, $x = 0, 1, 2, ..., n$ we have $p(x) = {}^4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$, $x = 0, 1, 2, 3, 4$.

Now, we calculate the values of $p(x)$ by putting the different values of $x$ as 0, 1, 2, 3 and 4.

$$p(0) = {}^4C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{4-0} = \frac{16}{81}$$

$$p(1) = {}^4C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{4-1} = 4\left(\frac{1}{3}\right)\left(\frac{8}{27}\right) = \frac{32}{81}$$

$$p(2) = {}^4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{4-2} = 6\left(\frac{1}{9}\right)\left(\frac{4}{9}\right) = \frac{24}{81}$$

$$p(3) = {}^4C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{4-3} = 4\left(\frac{1}{27}\right)\left(\frac{2}{3}\right) = \frac{8}{81}$$

$$p(4) = {}^4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{4-4} = 1\left(\frac{1}{81}\right)1 = \frac{1}{81}$$

These can be put in the tabular form as follows :

| $X = x$ | 0 | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{16}{81}$ | $\frac{32}{81}$ | $\frac{24}{81}$ | $\frac{8}{81}$ | $\frac{1}{81}$ | 1 |

Now, $P(X \le 2)$

$= p(X=0)+p(X=1)+p(X=2)$

$= \frac{16}{81}+\frac{32}{81}+\frac{24}{81}$

$= \frac{72}{81}$

$= \frac{8}{9}$

**Illustration 17 : In a binomial distribution, for $P(X=x)=p(x)$, $n=8$ and $2p(4)=5p(3)$. Find the probability of getting success in all the trials for this distribution.**

Here, we have $2p(4)=5p(3)$ and $n=8$

$\therefore$ Putting $n=8$ in the formula of binomial distribution, we get,

$p(x)={}^8C_x p^x q^{8-x}$, $x = 0, 1, 2, \ldots, 8$

Putting the values of $p(4)$ and $p(3)$ from this formula in the given condition

$2p(4)=3p(3)$

$2\times {}^8C_4\, p^4\, q^{8-4} = 5\times {}^8C_3 p^3 q^{8-3}$

$\therefore\ 2\times(70)\,p^4\,q^4 = 5\times(56)\,p^3q^5$

$\therefore\ 140\,p^4\,q^4 = 280\,p^3q^5$

$\therefore\ p=2q$

$\therefore\ p=2(1-p)$

$\therefore\ p=2-2p$

$\therefore 3p=2$

$\therefore\ p=\frac{2}{3}$ and $q=1-p=1-\frac{2}{3}=\frac{1}{3}$.

Now, getting success in all the trials means the event of getting 8 successes since we have total 8 trials..

The probability of this events is $p(8)$.

$\therefore\ p(8)={}^8C_8\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^{8-8}$

$=1\times\left(\frac{2}{3}\right)^8\times 1$

$=\frac{256}{6561}$

Thus, the probability of getting success in all the trials is $\frac{256}{6561}$.

**Illustration 18 : For a binomial distribution, mean = 18 and variance = 4.5. Determine whether the skewness of this distribution is positive or negative.**

Here, mean $= np = 18$ and variance $= npq = 4.5$

$\therefore q = \frac{\text{Variance}}{\text{Mean}} = \frac{4.5}{18} = 0.25 = \frac{1}{4}$

$\therefore p = 1 - \frac{1}{4} = \frac{3}{4}$

Since the value of $p$ is greater than $\frac{1}{2}$, the skewness of binomial distribution will be negative.

**Illustration 19 : A balacned die is tossed 7 times. If the event of getting a number 5 or more is called success and $X$ denotes the number of success in 7 trials then (i) Write the probability distribution of $X$. (ii) Find the probability of getting 4 successes. (iii) Find the probability of getting at the most 6 successes.**

The sample space associated with tossing of a balanced die once is $U = \{1, 2, 3, 4, 5, 6\}$ and probability of getting each number is $\frac{1}{6}$.

If the event of getting a number 5 or more is called success then probability of success

$p =$ probability of getting 5 or 6 on the die

$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Here, total number of trials is 7. $\therefore n = 7$

(i) Using the probability distribution of $X$ $p(x) = {}^nC_x p^x q^{n-x}$, $x = 0, 1, 2, ..., n$,

$$p(x) = {}^7C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{7-x}, \quad x = 0, 1, 2, 3, 4, 5, 6, 7$$

(ii) Probability of getting 4 successes

$$p(4) = {}^7C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{7-4}$$

$$= 35\left(\frac{1}{81}\right)\left(\frac{8}{27}\right)$$

$$= \frac{280}{2187}$$

(iii) Probability of getting at the most 6 successes

$= p(X \le 6)$

$= 1 - p(X > 6)$

$= 1 - p(X = 7) \quad \because x = 0, 1, 2, ...., 7$

$= 1 - {}^7C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^{7-7}$

$= 1 - \frac{1}{2187} = \frac{2186}{2187}$

**Illustration 20 : A social worker claims that 10 % of the young children in a city have vision problem. A sample survey agency takes a random sample of 10 young children from the city to test the claim. If at the most one young child is affected by the vision problem, the claim of the social worker is rejected. Find (i) the probability that the claim of the social worker is rejected (ii) the expected number of young children having vision problem in the randomly selected 10 young children.**

Suppose $p$ = probability that a young child has eye problem

$= 0.10$ (by accepting the claim of social worker)

And $X$ = the number of young childrens having eye problem in the randomly selected 10 young children

Here, putting $n = 10$ and $p = 0.10$ in the formula

$p(x) = {}^nC_x \; p^x q^{n-x}, \quad x = 0, 1, 2, ..., n$

of binomial distribution,

$p(x) = {}^{10}C_x (0.10)^x \; (0.90)^{10-x}, \; x = 0, 1, 2, ..., 10$

(i) Probability that at the most one young child has eye problem

$= p(0) + p(1)$

$= {}^{10}C_0 \; (0.10)^0 \; (0.90)^{10-0} + {}^{10}C_1 \; (0.10)^1 \; (0.90)^{10-1}$

$= 0.3487 + 10 \; (0.10) \; (0.3874)$

$= 0.3487 + 0.3874$

$= 0.7361$

Now, sample survey agency rejects the claim of the social worker if at the most one young child has eye problem.

$\therefore$ Probability of rejecting the claim of social worker by the sample survey agency $= 0.7361$.

(ii) The expected number of young children having eye problem in the randomly selected 10 young child

$= E(X) = np$

$= 10 \times$ probability that the selected young child has eye problem

$= 10 \times 0.10$

$= 1$

**Illustration 21 : An experiment is conducted to toss five balanced coins simultaneously. If we consider occurrence of head $(H)$ on the coin as success then obtain probability distribution of the number of successes. If such an experiment is repeated 3200 times then obtain expected frequency distribution of the number of successes. For this distribution, obtain expected value of the number successes and also obtain its standard deviation.**

Since the coins are balanced, probability of getting head will be $\frac{1}{2}$.

$p =$ probability of success

$=$ probability of getting head $= \frac{1}{2}$.

$\therefore q = 1 - p = \frac{1}{2}$

Here, $n =$ number of coins $= 5$, $x =$ the number of successes in tossing of five coins.

Putting the values of $n$, $p$ and $q$ in the formula

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x},\ x = 0, 1, 2, \ldots, n$$

of the binomial distribution

$$p(x) = {}^{5}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{5-x}, \quad x = 0, 1, 2, \ldots, 5$$

$$= {}^{5}C_{x} \left(\frac{1}{2}\right)^{x+5-x}$$

$$= {}^{5}C_{x} \left(\frac{1}{2}\right)^{5}$$

$$= \frac{{}^{5}C_{x}}{32}, \quad x = 0, 1, 2, \ldots, 5$$

Now, using the above formula, we calculate the probability for each $x$ and the frequency for the number of successes in 3200 repetition of the experiment $= 3200 \times p(x)$, $x = 0, 1, 2, ....,5$.

We present the calculations in the following table :

| $x$ | $p(x)$ | **Expected Frequency** $= \mathbf{N} \times p(x)$ |
|---|---|---|
| 0 | $\frac{{}^5C_0}{32} = \frac{1}{32}$ | $3200 \times \frac{1}{32} = 100$ |
| 1 | $\frac{{}^5C_1}{32} = \frac{5}{32}$ | $3200 \times \frac{5}{32} = 500$ |
| 2 | $\frac{{}^5C_2}{32} = \frac{10}{32}$ | $3200 \times \frac{10}{32} = 1000$ |
| 3 | $\frac{{}^5C_3}{32} = \frac{10}{32}$ | $3200 \times \frac{10}{32} = 1000$ |
| 4 | $\frac{{}^5C_4}{32} = \frac{5}{32}$ | $3200 \times \frac{5}{32} = 500$ |
| 5 | $\frac{{}^5C_5}{32} = \frac{1}{32}$ | $3200 \times \frac{1}{32} = 100$ |

(ii) Expected value of the number of successes

$= np$

$= 5\left(\frac{1}{2}\right) = 2.5$

(iii) Standard deviation of the number of successes

$= \sqrt{npq}$

$= \sqrt{5 \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)} = \sqrt{\frac{5}{4}} = \sqrt{1.25}$

$= 1.118$

**Illustration 22 : An advertisement company claims that 4 out of 5 house wives do not identify the difference between two different brands of butter. To check the claim, 5000 house wives are divided in groups, each group of 5 house wives. If the claim is true, in how many groups among these groups (i) at the most one house wife (ii) only two house wives can identify the difference between two different brands of butter ?**

As per the claim made by an advertising company, 4 out of 5 house wives do not identify the difference between two different brands of butter.

$\therefore$ Its probability is $\frac{4}{5}$

That means the probability of identifying the difference between two different brands of butter by a house wife $= \frac{1}{5}$.

Let $p$ = probability that the selected house wife can identify the difference between two different brands of butter $= \frac{1}{5}$.

To test the claim, selected 5000 housewives are divided in groups randomly, with each group having 5 house wives. So, there will be 1000 such groups.

If we take $X =$ the number of house wives who identify the difference between two different brands of butter in a group, $x = 0, 1, ...., 5$.

Thus, we have $n = 5$, $p = \frac{1}{5}$, $q = \frac{4}{5}$.

Putting the above values in the formula of binomial distribution

$$p(x) = {}^{n}C_{x}\ p^{x} q^{n-x},\ x = 0, 1, 2, ..., n$$

We get the following $p(x)$

$$p(x) = {}^{5}C_{x} \left(\frac{1}{5}\right)^{x} \left(\frac{4}{5}\right)^{5-x},\quad x = 0, 1, 2, ..., 5$$

Using this formula, we calculate the probabilities for different values of $x$ and multiplying such probabilities by 1000 we get the number of groups out of 1000 groups in which 0, 1, 2, 3, 4 or 5 house wives can identify the difference between two different brands of the butter..

(i) The number of groups out of 1000 groups in which at the most one house wife can identify the difference between two different brands of butter

$$= 1000 \times [p(0) + p(1)]$$

$$= 1000 \times \left[ {}^{5}C_{0} \left(\frac{1}{5}\right)^{0} \left(\frac{4}{5}\right)^{5-0} + {}^{5}C_{1} \left(\frac{1}{5}\right)^{1} \left(\frac{4}{5}\right)^{5-1} \right]$$

$$= 1000 \times \left[ \frac{1024}{3125} + 5 \times \left(\frac{1}{5}\right) \times \left(\frac{256}{625}\right) \right]$$

$$= 1000 \times \left[ \frac{1024}{3125} + \frac{256}{625} \right]$$

$$=1000\times[0.32768+0.4096]$$

$$=1000\times[0.73728]$$

$$=737.28$$

$$\simeq 737 \quad \text{groups}$$

(ii) The number of groups out of 1000 groups in which only two house wives can identify the difference between the two brands of butter.

$$=1000\times p(2)$$

$$=1000\times {}^5C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{5-2}$$

$$=1000\times 10\times\left(\frac{1}{25}\right)\times\left(\frac{64}{125}\right)$$

$$=1000\times\frac{640}{3125}$$

$$=1000\times 0.2048$$

$$=204.8$$

$$\simeq 205 \quad \text{groups}$$

## EXERCISE 2.2

**1.** For a symmetrical binomial distribution with $n=8$, find $p(X\le 1)$.

**2.** Mean of a binomial distribution is 5 and its variance is equal to the probability of success. Find the parameters of this distribution and hence find the probability of the event of getting none of the failures for this distribution.

**3.** A person has kept 4 cars to run on rent. The probability that any car is rented during the day is 0.6. Find the probability that more than one but less than 4 cars are rented during a day.

**4.** There are 200 farms in a Taluka. Among the bore wells made in these 200 farms of the Taluka, salted water is found in 20 farms. Find the probability of the event of not getting salted water in 3 out of 5 randomly selected farms from the Taluka.

**5.** An example is given to 6 students to solve. The probability of getting correct solution of the problem by any student is 0.6. Students are trying to solve the problem independently. Find the probability of getting the correct solution by only 2 out of the 6 students.

*

## Summary

- **Random Variable :** A function associating a real number with each outcome of the sample space of a random experiment is called random variable.
- **Discrete Random Variable :** A random variable $X$ which can assume a finite or countable infinite number of values in the set $R$ of real numbers is called a discrete random variable.
- **Continuous Random Variable :** A random variable $X$ which can assume any value in $R$, the set of real numbers or in any interval of $R$ is called continuous random variable.
- **Discrete Probability Distribution :** Suppose $X : U \to R$ is a random variable which assumes all values of a finite set $\{x_1, x_2, ...., x_n\}$ of $R$. Also suppose $X$ assumes a value $x_i$ with probability $p(x_i)$. If $p(x_i) > 0$ for $i = 1, 2, ..., n$ and $\Sigma\, p(x_i) = 1$ then the set of real values $\{x_1, x_2, ...., x_n\}$ and $\{p(x_1), p(x_2), ...., p(x_n)\}$ is called the discrete probability distribution of a random variable $X$ which is expressed in a tabular form as follows :

| $X = x$ | $x_1$ | $x_2$ | .... | $x_i$ | ... | $x_n$ | Total |
|---|---|---|---|---|---|---|---|
| $p(x)$ | $p(x_1)$ | $p(x_2)$ | .... | $p(x_i)$ | ... | $p(x_n)$ | 1 |

Here $0 < p(x_i) < 1, i = 1, 2, ..., n$

- **Bernoulli Trials :** Suppose dichotomous random experiment has two outcomes success ($S$) and failure ($F$). If this experiment is repeated under identical conditions and the probability $p(0 < p < 1)$ of getting success at each trial is constant then such trials are called Bernoulli Trials.
- **Binomial Random Variable :** Suppose $X$ denotes the number of successes in the sequence of success ($S$) and failure ($F$) obtained in $n$ Bernoulli trials then $X$ is called a binomial random variable.
- **Binomial Probability Distribution :** The probability distribution of a binomial random variable $X$ is called binomial probability distribution.

**List of Formulae**

(1) Mean of discrete probability distribution $= \mu$

$= E(X)$

$= \Sigma\, x\, p(x)$

(2) Variance of discrete probability distribution $= \sigma^2$

$= V(X)$

$= E(X^2) - (E(X))^2$

where $E(X^2) = \Sigma\, x^2\, p(x)$

(3) Binomial Probability Distribution

$P(X = x) = p(x) = {}^nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n.$

$0 < p < 1,\ q = 1 - p$

(4) Mean of binomial probability distribution $= np$

(5) Variance of binomial probability distribution $= npq$

(6) Standard deviation of binomial probability distribution $= \sqrt{npq}$

(7) If an experiment having Bernoulli trials repeats $N$ times and $p(x)$ is the probability of getting $x$ successes in the experiment then expected frequency of the number of successes in $N$ repetitions $= N \cdot p(x)$

## EXERCISE 2

### Section A

**Find the correct option for the following multiple choice questions :**

**1.** Which variable of the following will be an illustration of discrete variable ?

(a) Height of a student (b) Weight of a student

(c) Blood Pressure of a student (d) Birth year of a student

**2.** Which variable of the following will be an illustration of continuous variable ?

(a) Number of accidents occuring at any place

(b) Number of rainy days during a year

(c) Maximum temperature during a day

(d) Number of children in a family

3. A random variable $X$ assume the values $-1$, 0 and 1 with respective probability $\frac{1}{5}$, $K$ and $\frac{1}{3}$, where $0 < K < 1$ and $X$ does not assume any value other than these values. What will be the value of $E(X)$ ?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{2}{15}$ (d) $\frac{3}{15}$

4. A random variable $X$ assumes the values $-2$, 0 and 2 only with respective probabilities $\frac{1}{5}, \frac{3}{5}$ and $K$. If $0 < K < 1$, what will be the value of $K$ ?

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

5. Mean and variance of a discrete probability distribution are 3 and 7 respectively. What will be $E\left(X^2\right)$ for this distribution ?

(a) 10 (b) 4 (c) 40 (d) 16

6. For the probability distribution of a discrete random variable, $E(X)=5$ and $E\left(X^2\right)=35$. What will be the variance of this distribution ?

(a) 40 (b) 30 (c) 20 (d) 10

7. For a positively skewed binomial distribution with $n = 10$, which of the following values might be the value of mean ?

(a) 5 (b) 3 (c) 9 (d) 7

8. For which value of $x$, the value of $p(x)$ of binomial distribution with parameters $n = 4$ and $p = \frac{1}{2}$ becomes maximum ?

(a) 0 (b) 2 (c) 3 (d) 4

9. The binomial distribution has mean 5 and variance $\frac{10}{7}$. What will be the type of this distribution ?

(a) Positively skewed (b) Negatively skewed

(c) Symmetric (d) Nothing can be said about the distribution

10. Which of the following is the formula of probability of an event of not getting a success in the binomial distribution with parameters $n$ and $p$ ?

(a) ${}^nC_0p^nq^0$ (b) ${}^nC_0p^0q^n$ (c) ${}^nC_0pq^n$ (d) ${}^nC_0p^nq$

## Section B

**Answer the following questions in one sentence :**

1. Define discrete random variable.
2. Define continuous random variable.
3. Define discrete probability distribution.
4. State the formula to find mean of discrete variable.
5. State the formula to find variance of discrete variable.
6. Mean of a symmetrical binomial distribution is 7. Find the value of its parameter $n$.
7. The parameters of a binomial distribution are 10 and $\frac{2}{5}$. Calculate its variance.
8. State the relation between the probability of success and failure in Bernoulli trials.
9. State the relation between mean and variance of binomial distribution.
10. The probability of failure in a binomial distribution is 0.6 and the number of trials in it is 5. Find the probability of success.

## Section C

**Answer the following questions :**

1. The probability distribution of a random variable $X$ is as follows :

| $X$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $p(x)$ | 0.2 | 0.3 | $4C$ | $C$ |

Determine the value of constant $C$.

2. Calculate mean of the discrete probability distribution $p(x) = \begin{cases} \frac{x-1}{6}; & x = 2, 3 \\ \frac{1}{2}; & x = 4 \end{cases}$

3. The probability distribution of a random variable is as follows :

$p(x) = \frac{x+3}{10}, \quad x = -2, 1, 2$

Hence calculate $E(X^2)$.

4. If $n=4$ for a symmetrical binomial distribution then find $p(4)$.

5. Define Bernoulli trials.

6. For a binomial distribution, if probability of success is double the probability of failure and $n=4$ then find variance of the distribution.

7. Find the standard deviation of the binomial distribution having $n=8$ and probability of failure $\frac{2}{3}$.

8. Find parameters of the binomial distribution where mean = 4 and variance = 2.

9. For a binomial distribution with $n=10$ and $q-p=0.6$, find mean of this distribution.

10. For a binomial distribution, standard deviation is 0.8 and probability of failure is $\frac{2}{3}$, find the mean of this distribution.

## Section D

**Answer the following questions :**

1. The probability distribution of a random variable $X$ is as follows :

$$p(x)=\begin{cases} K(x-1); & x=2,3 \\ K; & x=4 \\ K(6-x)\ ; & x=5 \end{cases}$$

Find the value of constant $K$ and the probability of the event that variable $X$ assumes even numbers.

2. The probability distribution of a random variable $X$ is as follows :

$$p(x)=C\left(x^2+x\right),\ x=-2,1,2$$

Find the value of $C$ and show that $p(2)=3p(-2)$.

3. The distribution of a random variable $X$ is $p(x)=K\cdot{}^5P_x,\ x=0,1,2,3,4,5$

Find constant $K$ and mean of this distribution.

4. What is discrete probability distribution ? State its properties.

5. State properties of binomial distribution.

6. In a game of hitting a target, the probability that Ramesh will fail in hitting the target is $\frac{2}{5}$. If he is given 3 trials to hit the target, find the probability of the event he hits the target successfully in 2 trials. State mean of this distribution.

7. A person is asked to select a number from positive integers 1 to 7. If the number selected by him is odd then he is entitled to get the prize. If he is asked to take 5 trials then find the probability of the event that he will be entitled to get a prize in only one trial.

8. The mean and variance of the binomial distribution are 2 and $\frac{6}{5}$ respectively. Find $p(1)$ and $p(2)$ for this binomial distribution.

9. 10 % apples are rotten in a box of apples. Find the probability that half of the 6 apples selected from the box with replacement will be rotten and find the variance of the number of rotten apples.

## Section E

**Solve the following :**

1. The probability distribution of the monthly demand of laptop in a store is as follows :

| **Demand of laptop** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Probability** | 0.10 | 0.15 | 0.20 | 0.25 | 0.18 | 0.12 |

Determine the expected monthly demand of laptop and find variance of the demand.

2. Two dice are thrown simultaneously once. Obtain the discrete probability distribution of the number of dice for which the number '6' comes up.

3. If the probability that any 50 year old person will die within a year is 0.01, find the probability that out of a group of 5 such persons

(i) none of them will die within a year

(ii) at least one of them will die within a year.

4. The probability that a student studying in 12th standard of science stream will get admission to engineering branch is 0.3. 5 students are selected from the students who studied in this stream. Find the probability of the event that the number of students admitted to engineering branch is more than the number of students who did not get admission to the engineering branch.

5. The probability that a bomb dropped from a plane over a bridge will hit the bridge is $\frac{1}{5}$. Two bombs are enough to destroy the bridge. If 6 bombs are dropped on the bridge, find the probability that the bridge will be destroyed.

6. Normally, 40 % students fail in one examination. Find the probability that at least 4 students in a group of 6 students pass in this examination.

7. There are 3 red and 4 white balls in a box. Four balls are selected at random with replacement from the box. Find the probability of the event of getting (i) 2 red balls and 2 white balls (ii) all four white balls among the selected balls using binomial distribution.

## Section F

**Solve the following :**

1. There are one dozen mangoes in a box of which 3 mangoes are rotten. 3 mangoes are randomly selected from the box without replacement. If $X$ denotes the number of rotten mangoes in the selected mangoes, obtain the probability distribution of $X$ and hence find expected value and variance of the rotten mangoes in the selected mangoes.

2. It is known that 50 % of the students studying in the 10th standard have a habit of eating chocolate. To examine the information, 1024 investigators are appointed. Every investigator randomly selects 10 students from the population of such students and examines them for the habit of eating chocolate. Find the number of investigators who inform that less than 30 percent of the students have a habit of eating chocolate.

**James Bernoulli**
**(1654 –1705)**

James (Jacob) Bernoulli was born in Basel, Switzerland. He was one of the many prominent mathematicians in the Bernoulli family. Following his father's wish, he studied theology (divinity) and entered the ministry. But contrary to the desires of his parents, he also studied mathematics and astronomy. He travelled throughout Europe from 1676 to 1682; learning about the latest discoveries in mathematics and the sciences under leading figures of the time. He was an early proponent of Leibnizian calculus and had sided with Leibniz during the Leibniz-Newton calculus controversy. He is known for his numerous contributions to calculus, and along with his brother Johann, was one of the founders of the calculus of variations. However, his most important contribution was in the field of probability, where he derived the first version of the law of large numbers. He was appointed as professor of mathematics at the University of Basel in 1687, remained in this position for the rest of his life.

*"Normal Distribution is father of all probability distributions. For larger sample size almost all theoritical distributions follow normal distribution".*

**– Unknown**

# 3

# Normal Distribution

**Contents :**

## 3.1 Normal Distribution : Introduction, Probability Density Function

In the previous chapter, we have studied the probability distribution for a discrete random variable. Now, we shall study the probability distribution for a continuous random variable. We know that, if a random variable $X$ can assume any value of real set $R$ or within any interval of real set $R$ then it is called continuous random variable. If a random variable can assume any value between the definite interval $a$ to $b$ then it is denoted by $a < x < b$. A function for obtaining probability that a continuous random variable assumes value between specified interval is called probability density function of that variable and it satisfies the following two conditions:

(1) The probability that the value of random variable lies within the specified interval is non negative.

(2) The total probability that the random variable assumes any value within the specified interval is one.

Thus, probability density function is used to determine probability that the value of random variable $X$ lies within the specified interval $a$ to $b$ and it is denoted as $P(a < x < b)$. It is necessary to note here that probability for the definite value of continuous random variable $X$ obtained by the probability density function is always zero (0). i.e. $P(x = a) = 0$. Thus, the probabilities $P(a < x < b)$ and $P(a \le x \le b)$ obtained by using probability density function are always equal i.e. $P(a < x < b) = P(a \le x \le b)$.

Normal distribution is very important probability distribution among probability distributions for continuous random variable and is very useful distribution for higher statistical study. It can be defined as under:

If $X$ is a random variable with mean $\mu$ and standard deviation $\sigma$ and if its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\infty < x < \infty$$

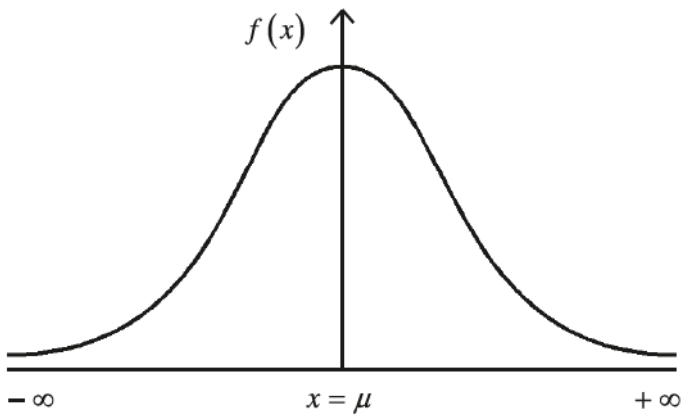
$$-\infty < \mu < \infty$$

$$0 < \sigma < \infty$$

where $\pi = 3.1416$ and $e = 2.7183$ are the constants

then $X$ is called normal random variable and $f(x)$ is called probability density function of normal random variable. The distribution of this normal random variable $X$ is called normal distribution and is denoted by $N(\mu, \sigma^2)$

A curve drawn by considering different values of normal random variable $X$ and its respective values of probability density function $f(x)$ is called normal curve and is shown as under:




As shown in the above diagram, normal curve is completely bell shaped which shows that it is symmetric distribution.

## 3.2 Standard Normal Variable and Standard Normal Distribution

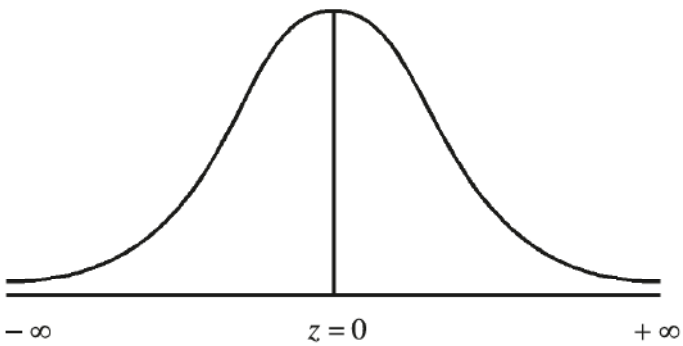
If X is a random normal variable with mean $\mu$ and standard deviation $\sigma$ then random variable $Z = \frac{X-\mu}{\sigma}$ is called standard normal random variable and its probability density function is given below.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \; ; -\infty < z < \infty$$

It can be seen here that probability density function of standard normal variable is a normal density function with mean zero (0) and standard deviation 1.

**Note :** During the further study of this chapter, we shall call normal variable $X$ instead of normal random variable $X$ and standard normal variable $Z$ instead of standard normal random variable $Z$.

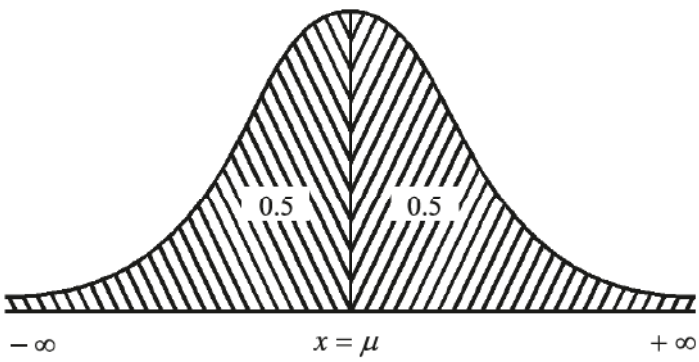
By plotting different values of standard normal variable $Z$ and its respective values of $f(z)$ on graph paper, a completely bell shaped curve is obtained as under :




This curve is called standard normal curve and it is symmetrical to both the sides of $Z = 0$.

## 3.3 Method for Finding the Probability (area) from the Tables of Standard Normal Curve

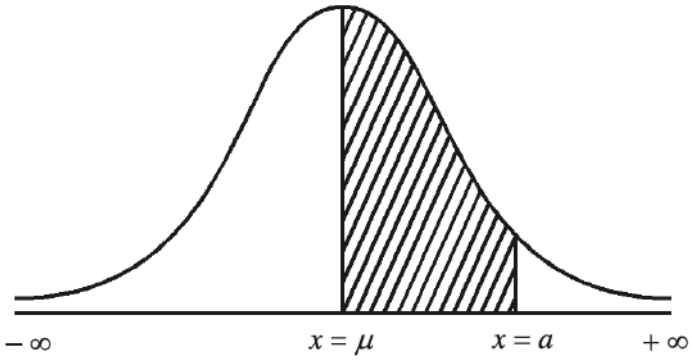
We know that a normal curve is a curve of normal density function and it can be seen as under :




The area (probability) of the shaded region between the curve and $X$- axis is equal to 1. The normal curve of normal variable $X$ is symmetrical about mean $\mu$ on both the sides and hence the perpendicular line at the point $X = \mu$ on X-axis divides the area (probability) of normal curve in two equal parts. The

area (probability) to the right side of $X=\mu$ is 0.5 and is denoted by $P(X \geq \mu)=0.5$, whereas the area (probability) to the left side of $X=\mu$ is 0.5 and is denoted by $P(X \leq \mu)=0.5$.

In normal curve, probability that value of normal variable $X$ lies between mean $\mu$ and its any specific value $a$ $(a>\mu)$ can be shown by the area of shaded region between the $x$-axis and the perpendicular lines at $X=\mu$ and $X=a$. It can be shown as under :




In notation, this can be shown as $P(\mu \leq X \leq a)$.

For obtaining area under normal curve, first of all the normal variable $X$ is changed into standard normal variable $Z$. By considering different positive values of standard normal variable $Z$, a table is prepared for obtaining area under normal curve for 0 to $Z$ and by using this table the area can be obtained.

**Note :** A table for different values of standard normal variable is given on the last page of the book.

Suppose probability that a normal variable $X$ assumes the value between mean $\mu$ and constant $a$ $(a>\mu)$ is to be obtained then it is denoted as $P(\mu \leq X \leq a)$. Now, if standard deviation of normal variable $X$ is $\sigma$,

When $X=\mu$ then $Z=\frac{X-\mu}{\sigma}=\frac{\mu-\mu}{\sigma}=\frac{0}{\sigma}=0$ and

When $X=a$ then $Z=\frac{X-\mu}{\sigma}=\frac{a-\mu}{\sigma}=Z_1$

Thus, $P(\mu \leq X \leq a)=P(0 \leq Z \leq Z_1)$

= area between $Z=0$ to $Z=Z_1$ obtained from tables of standard normal variable.

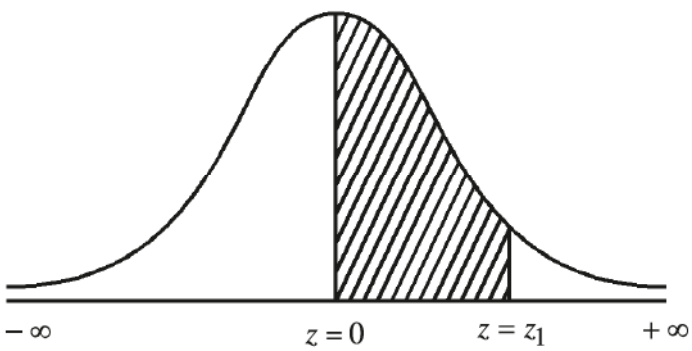



**Illustration 1 : A normal distribution has mean 10 and standard deviation 2. Find the probabilities of (1) Normal variable $X$ will take value between 10 and 12. (2) Normal variable $X$ has the value between 8 and 10.**

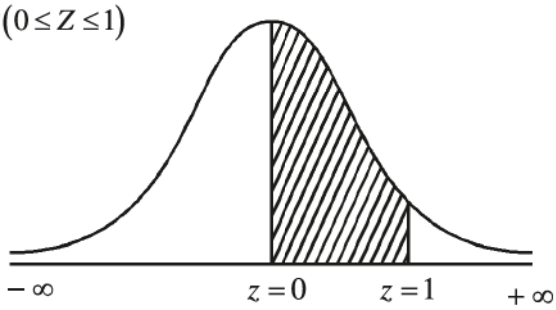
Here, mean $\mu = 10$ and standard deviation $\sigma = 2$.

(1) The probability that a normal variable $X$ will take value between 10 and 12 is to be determined, i.e. to determine $P(10 \le X \le 12)$

$$\therefore P(10 \le X \le 12) = P\left(\frac{10-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{12-\mu}{\sigma}\right)$$

$$= P\left(\frac{10-10}{2} \le Z \le \frac{12-10}{2}\right)$$

$$= P(0 \le Z \le 1)$$



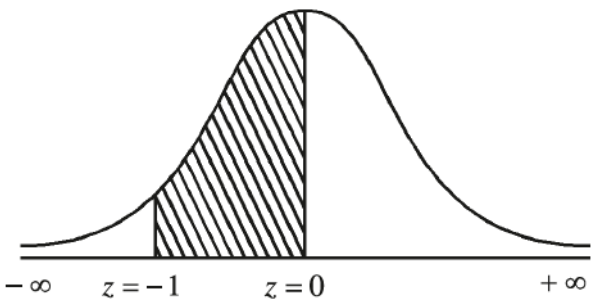

$= 0.3413$ (from the tables of standard normal variable)

(2) The probability that a normal variable $X$ will take value between 8 and 10 is to be determined, i.e. to determine $P(8 \le X \le 10)$

$$\therefore P(8 \le X \le 10) = P\left(\frac{8-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{10-\mu}{\sigma}\right)$$

$$= P\left(\frac{8-10}{2} \le Z \le \frac{10-10}{2}\right)$$

$$= P(-1 \le Z \le 0)$$

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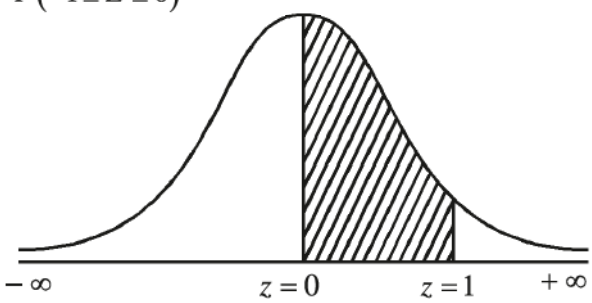



$= P(0 \le Z \le 1)$ ($\because$ Symmetry)

$= 0.3413$ (from the tables of standard normal variable)

**Illustration 2 : A normal distribution has mean 20 and variance 16. Find the probabilities of (1) Normal variable $X$ will take value less than 26 (2) Normal variable $X$ has the value more than 14.**

Here, mean $\mu = 20$ and variance $\sigma^2 = 16$.

$\therefore$ standard deviation $\sigma = 4$.

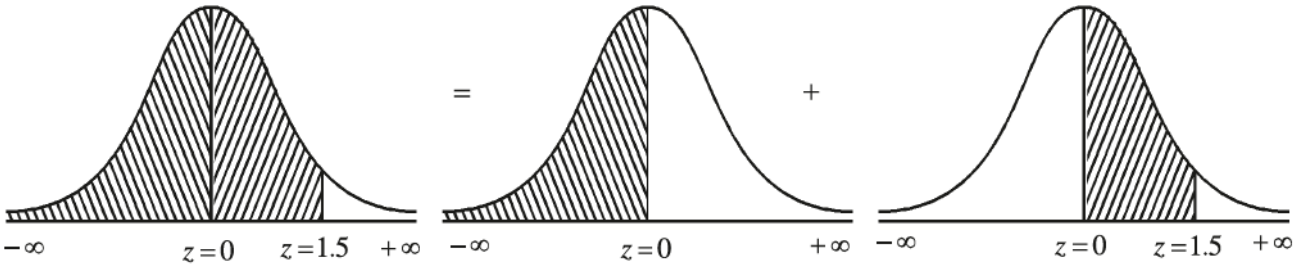
(1) The probability that a normal variable $X$ will take value less than 26

$$= P(X \le 26)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{26-\mu}{4}\right)$$

$$= P\left(Z \le \frac{26-20}{4}\right)$$

$$= P(Z \le 1.5)$$




$= P(-\infty < Z \le 0) + P(0 \le Z \le 1.5)$

$= 0.5 + 0.4332$ (from the tables of standard normal variable)

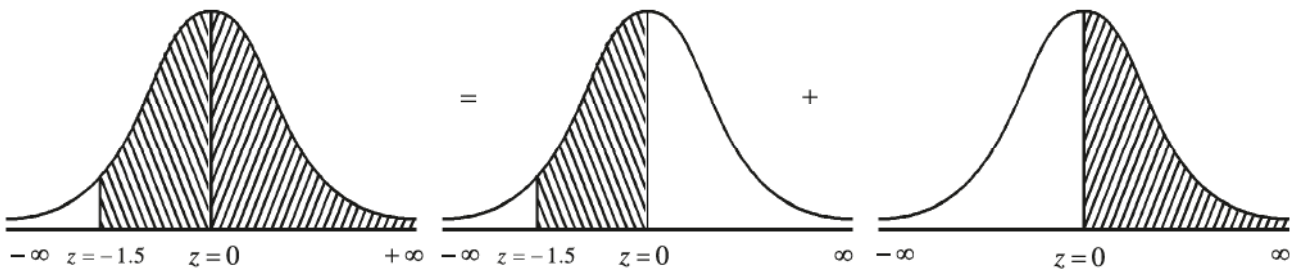
$= 0.9332$

(2) The probability that a normal variable $X$ will take value more than 14

$$= P(X \ge 14) = P\left(\frac{X-\mu}{\sigma} \ge \frac{14-\mu}{\sigma}\right)$$

$$= P\left(Z \ge \frac{14-20}{4}\right)$$

$$= P(Z \ge -1.5)$$




$= P(-1.5 \le Z \le 0) + P(0 \le Z < \infty)$

$= P(0 \le Z \le 1.5) + 0.5$ ($\because$ Symmetry)

$= 0.4332 + 0.5$ (from the tables of standard normal variable)

$= 0.9332$

**Illustration 3 : The number of students in classes of higher secondary schools of a city follows normal distribution. Average number of students in the classes is 50 and standard deviation is 15. If a class is selected at random then find the following probabilities (i) a class consists of more than 68 students (ii) a class consists of less than 32 students.**

It is given here that the number of students in the class follows normal distribution.

Normal variable $X =$ number of students in a class

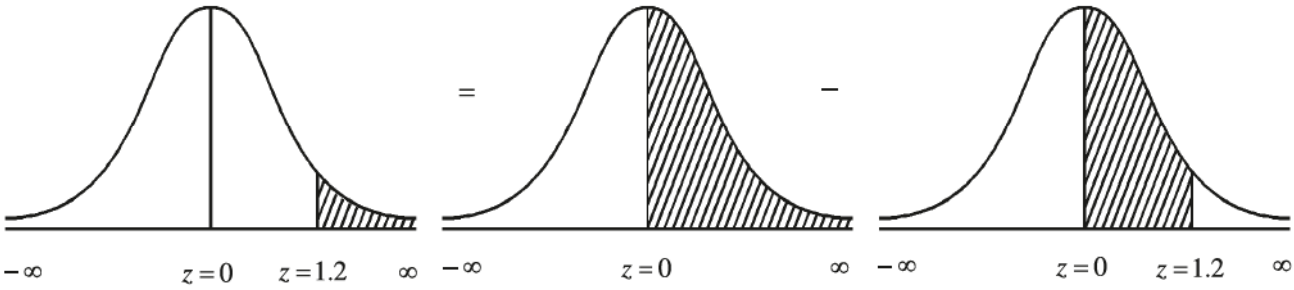
Also, mean $\mu = 50$ students and standard deviation $\sigma = 15$ students

(1) The probability that a randomly selected class consists of more than 68 students

$$= P(X \geq 68) = P\left(\frac{X-\mu}{\sigma} \geq \frac{68-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{68-50}{20}\right)$$

$$= P(Z \geq 1.2)$$




$$= P(0 \leq Z < \infty) - P(0 \leq Z \leq 1.2)$$

$= 0.5 - 0.3849$ (from the tables of standard normal variable)

$= 0.1151$

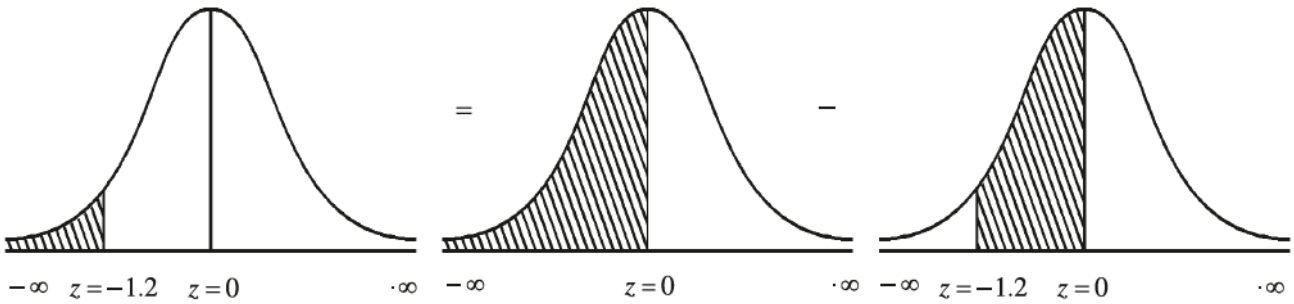
Thus, the probability that a randomly selected class consists of more than 68 students is 0.1151.

(2) The probability that a randomly selected class consists of less than 32 students

$$= P(X \leq 32) = P\left(\frac{X-\mu}{\sigma} \leq \frac{32-\mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{32-50}{20}\right)$$

$$= P(Z \leq -1.2)$$




$$= P(-\infty < Z \leq 0) - P(-1.2 \leq Z \leq 0)$$

$$= 0.5 - P(0 \leq Z \leq 1.2) \; (\because \text{ Symmetry})$$

$= 0.5 - 0.3849$ (from the tables of standard normal variable)

$= 0.1151$

Thus, the probability that a randomly selected class consists of less than 32 students is 0.1151

**Illustration 4 : The average weight of grown up children living in a large society is 50 kg and its standard deviation is 5 kg. If their weight follows normal distribution and a grown up child is selected at random then find**

**(1) the probability that his weight is between 55 kg and 65 kg.**

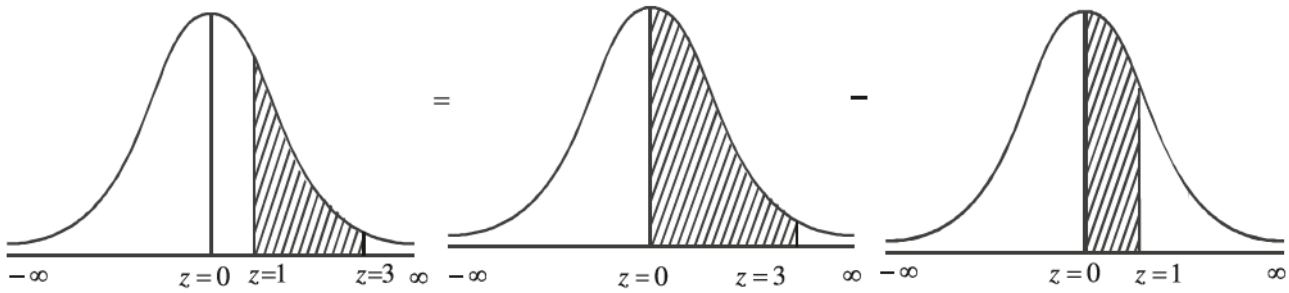
**(2) the probability that his weight is between 35 kg and 45 kg.**

Normal variable $X$ = weight of a grown up child, average weight $\mu = 50$ kg and standard deviation $\sigma = 5$ kg.

(1) Probability that a randomly selected grown up child has weight between 55 kg to 65 kg

$$= P(55 \le X \le 65) = P\left(\frac{55-50}{5} \le \frac{X-\mu}{\sigma} \le \frac{65-50}{5}\right)$$

$$= P(1 \le Z \le 3)$$




$$= P(0 \le Z \le 3) - P(0 \le Z \le 1)$$

$$= 0.4987 - 0.3413$$

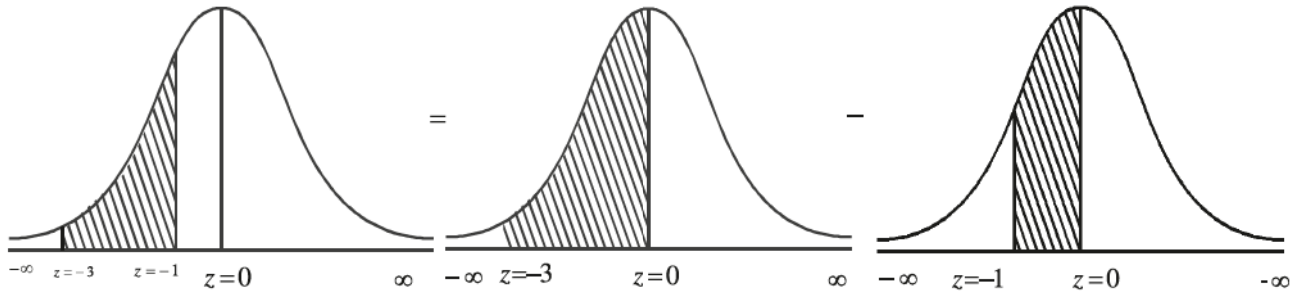
$$= 0.1574$$

Thus, the probability that a randomly selected grown up child has weight between 55 kg to 65 kg is 0.1574.

(2) Probability that a randomly selected grown up child has weight between 35 kg to 45 kg

$$= P(35 \le X \le 45) = P\left(\frac{35-50}{5} \le \frac{X-\mu}{\sigma} \le \frac{45-50}{5}\right)$$

$$= P(-3 \le Z \le -1)$$




$$= P(-3 \le Z \le 0) - P(-1 \le Z \le 0)$$

$$= P(0 \le Z \le 3) - P(0 \le Z \le 1) \quad (\because \text{ symmertry})$$

$$= 0.4987 - 0.3413$$

$$= 0.1574$$

Thus, the probability that a randomly selected grown up child has weight between 35 kg to 45 kg is 0.1574.

**Note :** From the above illustrations it is clear that the area under the normal curve for $Z = 0$ to $Z = a$ is equal to the area between $Z = -a$ to $Z = 0$. This is because the normal distribution is a symmetric distribution.

**Illustration 5 : The monthly income of workers working in a production house follows normal distribution. Their average monthly income is ₹ 15,000 and standard deviation is ₹ 4000.**

**(1) If a worker is selected at random then find the probability that his monthly income is between ₹ 10,000 and ₹ 25,000.**

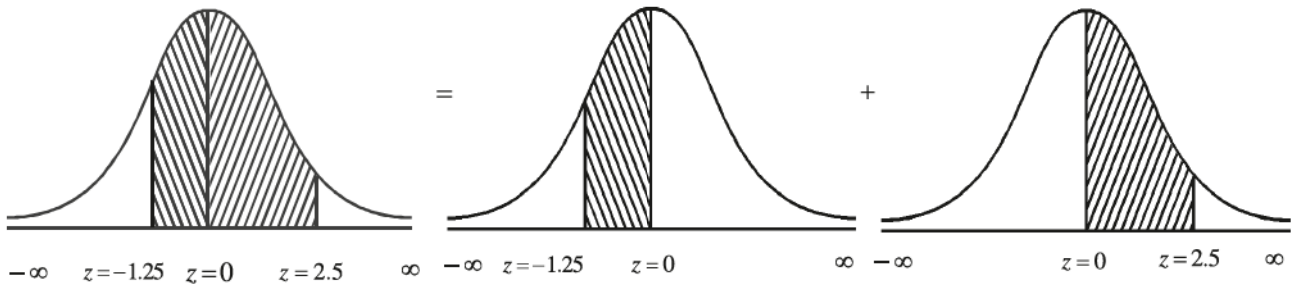
**(2) Find the percentage of workers having monthly income between ₹ 12,000 and 22,000 in the production house.**

Here, normal variable $X =$ monthly income of worker, average income $\mu =$ ₹ 15,000 and standard deviation $\sigma =$ ₹ 4000.

(1) The probability that a randomly selected worker has income between ₹ 10,000 and ₹ 25,000

$$= P(10000 \le X \le 25000) = P\left(\frac{10000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{25000-15000}{4000}\right)$$

$$= P(-1.25 \le Z \le 2.5)$$




$$= P(-1.25 \le Z \le 0) + P(0 \le Z \le 2.5)$$

$$= P(0 \le Z \le 1.25) + P(0 \le Z \le 2.5) \; (\because \text{ symmertry})$$

$$= 0.3944 + 0.4938$$

$$= 0.8882$$

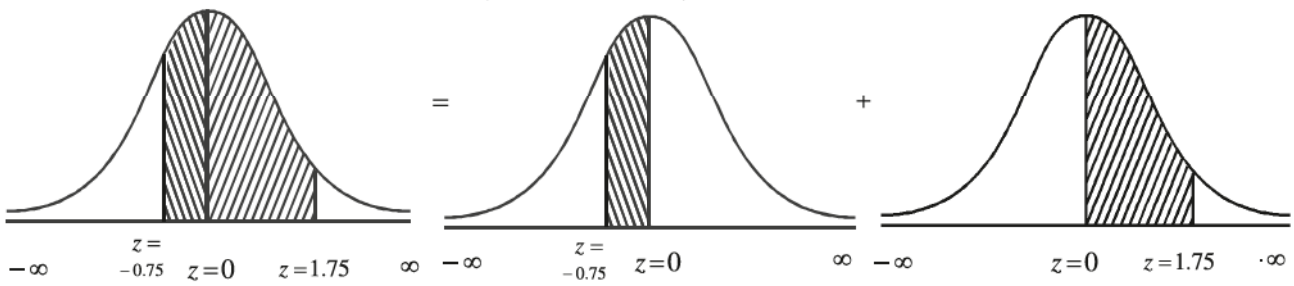
Thus, the probability that a randomly selected worker has monthly income between ₹ 10,000 and ₹ 25,000 is 0.8882.

(2) The probability that a randomly selected worker has income between ₹ 12,000 and ₹ 22,000

$$= P(12000 \le X \le 22000)$$

$$= P\left(\frac{12000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{22000-15000}{4000}\right)$$

$$= P(-0.75 \le Z \le 1.75)$$




$$= P(-0.75 \le Z \le 0) + P(0 \le Z \le 1.75)$$

$$= P(0 \le Z \le 0.75) + P(0 \le Z \le 1.75) \quad (\because \text{ symmertry})$$

$= 0.2734 + 0.4599$

$= 0.7333$

$\therefore$ The percentaged that a randomly selected worker has monthly income between ₹ 12,000 and ₹ 22,000

$= 0.7333 \times 100$

$= 73.33\ \%$

Thus, 73.33 % of the wokers in the production house have monthly income between ₹ 12,000 and ₹ 22,000.

**Note :** In order to express probability in percentage, probability is multiplied by 100.

## 3.4 Properties of Normal Distribution

Some important properties of normal distribution are as under :

(1) It is a distribution of continuous random variable.

(2) The constants μ and $\sigma$ are the parameters of distribution which indicate mean and standard deviation respectively.

(3) The distribution is symmetric about μ and its skewness is zero (0).

(4) For this distribution, the value of mean, median and mode are same. In notation, $\mu = M = M_0$

(5) For this distribution, quartiles are equidistant from median i.e. $Q_3 - M = M - Q_1$ and $M = \frac{Q_3+Q_1}{2}$

(6) The probability curve is completely bell shaped.

(7) Normal curve is asymptotic to *X*-axis. The tails never touch *X*-axis.

(8) The approximate value of quartiles of normal distribution can be obtained from the following formula

$Q_1 = \mu - 0.675\,\sigma$

$Q_3 = \mu + 0.675\,\sigma$

(9) For this distribution, quartile deviation = $\frac{2}{3}\sigma$ (approximately)

(10) For this distribution, mean deviation = $\frac{4}{5}\sigma$ (approximately)

(11) Important areas under normal curve are as below :

(i) Total area under normal curve is 1 and area on both the sides of perpendicular line at $X = \mu$ is 0.5

(ii) The area under the curve between the perpendicular lines at $\mu - \sigma$ and $\mu + \sigma$ is 0.6826 i.e. area under the normal curve between the perpendicular lines at $\mu \pm \sigma$ is 0.6826

(iii) The area under the curve between the perpendicular lines at $\mu - 2\sigma$ and $\mu + 2\sigma$ is 0.9545

(iv) The area under the curve between the perpendicular lines at $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.9973

(v) The area under the curve between the perpendicular lines at $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$ is 0.95.

(vi) The area under the curve between the perpendicular lines at $\mu - 2.575\sigma$ and $\mu + 2.575\sigma$ is 0.99.

## 3.5 Properties of Standard Normal Distribution

Some important properties of standard normal distribution are as under :

(1) It is a distribution of continuous random variable.

(2) For this distribution, mean is zero (0) and its standard deviation is 1.

(3) The distribution is symmetric to $Z = 0$ and its skewness is zero (0).

(4) The probability curve is completely bell shaped and is asymptotic to $X$-axis.

(5) The approximate value of the first quartile of standard normal distribution is –0.675 and that of the third quartile is 0.675.

(6) For this distribution, quartile deviation $= \frac{2}{3}$ (approximately).

(7) For this distribution, mean deviation $= \frac{4}{5}$ (approximately).

(8) Important areas under normal curve are as below :

(i) Total area under normal curve is 1 and area on both the sides of perpendicular line at $Z = 0$ is 0.5

(ii) The area under the curve between the perpendicular lines at $Z = -1$ and $Z = +1$ is 0.6826 i.e. area under the normal curve between the perpendicular lines at $Z = \pm 1$ is 0.6826.

(iii) The area under the curve between the perpendicular lines at $Z = -2$ and $Z = +2$ is 0.9545.

(iv) The area under the curve between the perpendicular lines at $Z = -3$ and $Z = +3$ is 0.9973.

(v) The area under the curve between the perpendicular lines at $Z = -1.96$ and $Z = +1.96$ is 0.95

(vi) The area under the curve between the perpendicular lines at $Z = -2.575$ and $Z = +2.575$ is 0.99

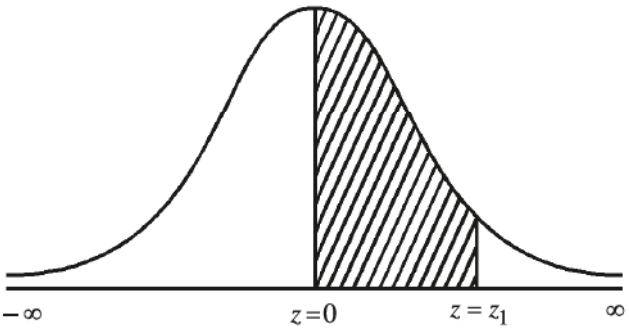
It should be noted here that the probability distribution of standard normal variable $Z$ is a distribution of normal variable with mean zero and variance 1. $Z$ is called standard score or $Z$-score and it is independent of unit of measurement.

We have seen earlier that when a value of normal variable $X$ and values of parameters are known then corresponding value of $Z$-score is obtained and by using the table of standard normal variable, the respective probability can be obtained. Now, if the probability is known then to determine value of $Z$-score we shall study the following illustrations :

**Illustration 6 : If the probability that value of standard normal variable $Z$ lies between 0 and $Z$-score $(z_1)$ is 0.3925 then obtain the possible values of $Z$-score $(z_1)$.**

The probability that value of standard normal variable $Z$ lies between $Z = 0$ and $Z = z_1$ is 0.3925. This probability is equal to the area under the curve between $Z = 0$ and $Z = z_1$. The value of $z_1$ may be positive or negative.

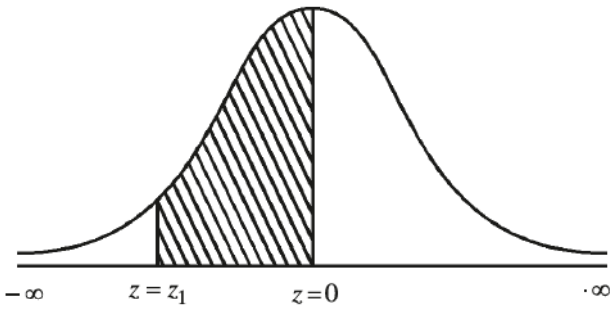
Suppose the value of $z_1$ is positive then $P(0 \le Z \le z_1) = 0.3925$.




For obtaining the value of $z_1$, see the first column of table of standard normal variable ($Z$-table). For $Z = 1.20$, the area is 0.384 which is less than 0.3925. Now, read the values in this row. For the value 0.3925, the corresponding value of $Z$ is 1.24. Therefore, one possible value of $Z$-score is 1.24.

Now, suppose the value of $z_1$ is negative

$\therefore P(z_1 \le Z \le 0) = 0.3925$




Now, since the normal distribution is symmetric, $P(z_1 \le Z \le 0) = P(0 \le Z \le z_1) = 0.3925$. Thus, as above, $z_1 = 1.24$ but it can be seen in the above diagram that the perpendicular line at $z_1$ is to the left of $Z = 0$ hence $Z$-score $z_1 = -1.24$.

Thus, if the probability that the value of standard normal variable lies between $Z = 0$ and $Z = z_1$ is 0.3925 then the possible values of $Z$-score are $\pm 1.24$.

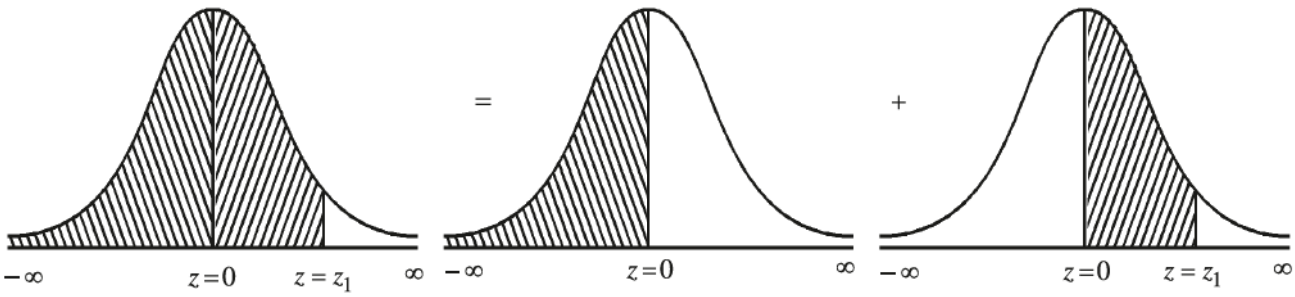
Thus, the sign of $z_1$ is positive if it is on right hand side of $Z = 0$ and negative if it is on left hand side of $Z = 0$.

**Illustration 7 : If the probabilities for standard normal variable $Z$ are as under then obtain the value of $Z$-socre $(z_1)$ :**

**(1) Area to the left of $Z = z_1$ is 0.95**

**(2) Area to the right of $Z = z_1$ is 0.05.**

(1) Area to the left of $Z = z_1$ is 0.95 i.e. $P(Z \le z_1) = 0.95$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from left to right so that the area under the curve is 0.95. The figure is as under :




$$\text{Thus} \quad P(Z \le z_1) = P(-\infty < Z \le 0) + P(0 \le Z \le z_1) = 0.95$$

$$\therefore \; 0.5 + P(0 \le Z \le z_1) = 0.95$$

$$\therefore \; P(0 \le Z \le z_1) = 0.95 - 0.5$$

$$\therefore \; P(0 \le Z \le z_1) = 0.45$$

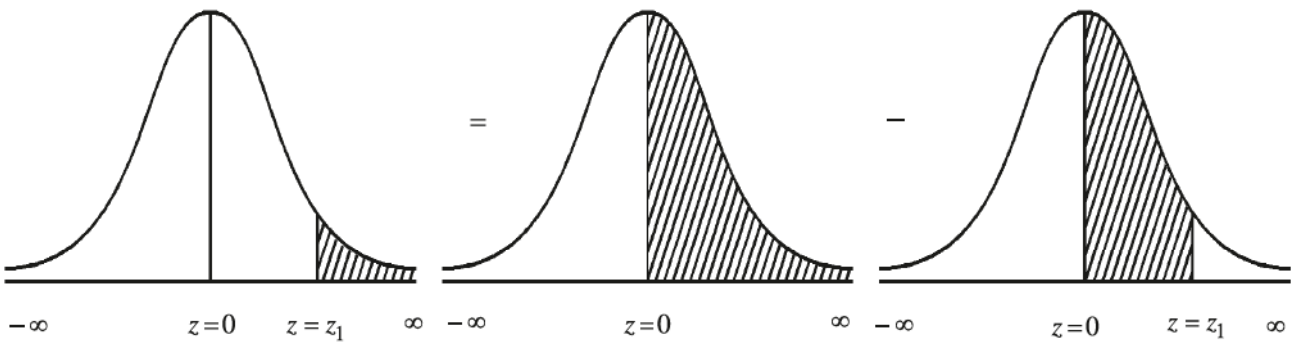
Now, corresponding to probability 0.45, it is not possible to obtain value of $z_1$ directly from the table of standard normal variable. Hence, the approximate value of $z_1$ is determined as follows

| From table | Area | $Z$-score |
|---|---|---|
| Nearest value before 0.45 | 0.4495 | 1.64 |
| Nearest value after 0.45 | 0.4505 | 1.65 |
| Average value | 0.4500 | 1.645 |

From the above table, it can be seen that $z_1 = 1.645$.

Thus, for $z_1 = 1.645$, $P(Z \le z_1) = 0.95$.

(2) Area to the right of $Z = z_1$ is 0.05 i.e. $P(Z \ge z_1) = 0.05$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from right to left so that the area under the curve is 0.05. The figure is as under




$$\therefore P(Z \ge z_1) = P(0 \le Z < \infty) - P(0 \le Z \le z_1) = 0.05$$

$$\therefore 0.5 - P(0 \le Z \le z_1) = 0.05$$

$$\therefore P(0 \le Z \le z_1) = 0.45$$

As calculated earlier, $z_1 = 1.645$.

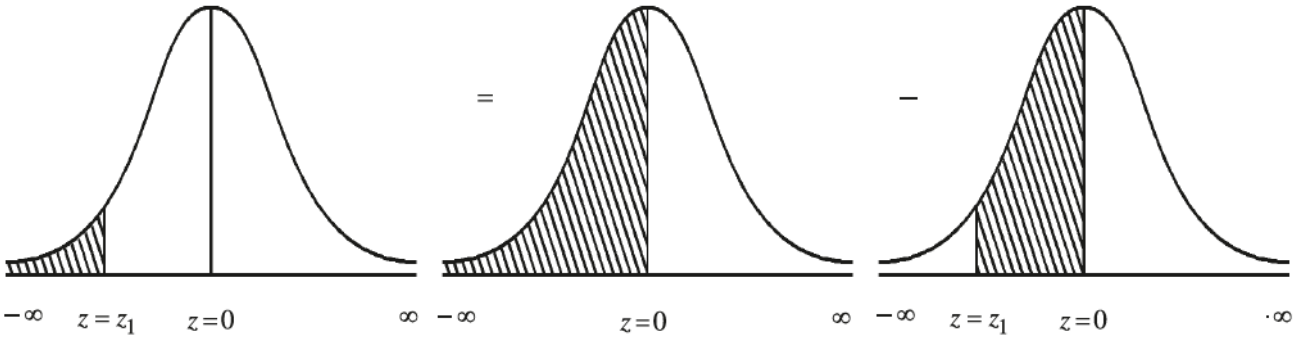
Thus, for $z_1 = 1.645$, $P(Z \ge z_1) = 0.05$.

**Illustration 8 : If the probabilities for standard normal variable $Z$ are as under then obtain the value of $Z$-score $(z_1)$ :**

**(1) Area to the left of $Z = z_1$ is 0.10.**

**(2) Area to the right of $Z = z_1$ is 0.90.**

(1) Area to the left of $Z = z_1$ is 0.10 i.e. $P(Z \le z_1) = 0.10$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from left to right so that the area under the curve is 0.10. The figure is as follows




$$\therefore\ P(Z \le z_1) \ = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0) = 0.10$$

$$\therefore\ 0.50 - P(z_1 \le Z \le 0) \ = 0.10$$

$$\therefore\ P(z_1 \le Z \le 0) \ = 0.5 - 0.10$$

$$\therefore\ P(0 \le Z \le z_1) \ = 0.40 \quad (\because \text{ symmetry})$$

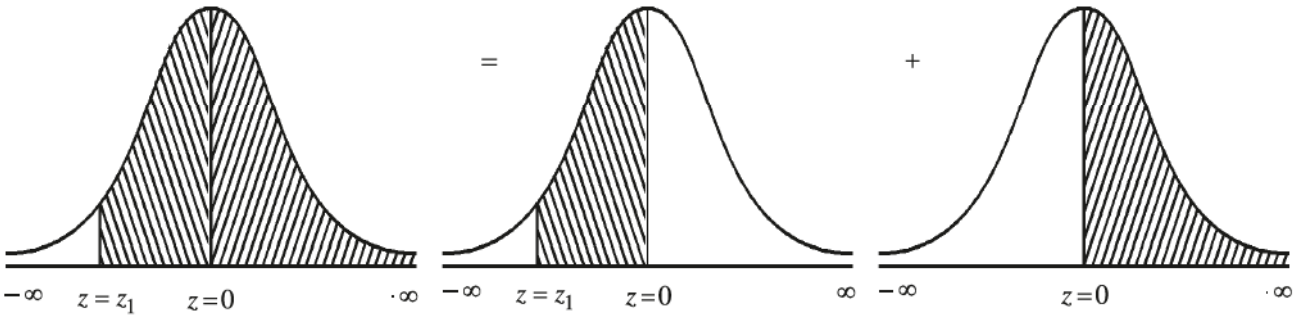
Now, corresponding to probability 0.4, it is not possible to obtain value of $z_1$ directly from the table of standard normal variable. Hence, the approximate value of $z_{1,}$ is determined as under

| **From table** | **Area** | ***Z*-score** |
|---|---|---|
| Nearest value before 0.40 | 0.3997 | 1.28 |
| Nearest value after 0.40 | 0.4015 | 1.29 |
| Average value | 0.4006 | 1.285 |

From the above table, it can be seen that the nearest value to 0.40 is 0.3997 and the respective value of *Z*-score is 1.28. Also, $z_1$ is to the left of $Z = 0$, hence $z_1 = -1.28$.

Thus for $z_1 = -1.28$, $P(Z \le z_1) = 0.10$.

(2) Area to the right of $Z = z_1$ is 0.90 i.e $P(Z \ge z_1) = 0.90$. In the curve of standard normal variable, the perpendicular line at $Z = z_1$ is to be drawn by moving from right to left so that the area under the curve is 0.90. The figure is as under




$$P(Z \ge z_1) \ = P(z_1 \le Z \le 0) + P(0 \le Z < \infty) = 0.90$$

$$\therefore\ P(z_1 \le Z \le 0) + 0.50 \ = 0.90$$

$$\therefore\ P(z_1 \le Z \le 0) = 0.40$$

$$\therefore\ P(0 \le Z \le z_1) = 0.40 \quad (\because \text{ Symmetry})$$

As seen earlier $z_1 = -1.28$

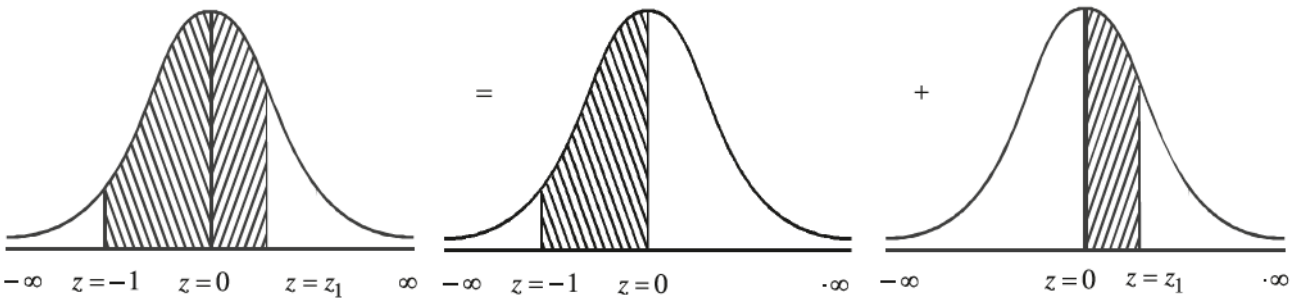
Thus, for $z_1 = -1.28$, $P(Z \ge z_1) = 0.90$

**Illustration 9 : If $Z$ is standard normal variable and $z_1$ is $Z$-score then obtain the values of $z_1$ satisfying the following conditions**

**(1)** $\boldsymbol{P(-1 \le Z \le z_1) = 0.5255}$

**(2)** $\boldsymbol{P(z_1 \le Z \le 2) = 0.7585}$

(1) It is given that $P(-1 \le Z \le z_1) = 0.5255$. A perpendicular line at $Z = -1$ is drawn and then a perpendicular line at $Z = z_1$ is drawn to its right side so that the area between them is 0.5255. The figure is as under




$$P(-1 \le Z \le z_1) = P(-1 \le Z \le 0) + P(0 \le Z \le z_1) = 0.5255$$

$$\therefore\ P(0 \le Z \le 1) + P(0 \le Z \le z_1) = 0.5255 \quad (\because \text{ Symmetry})$$

$$\therefore\ 0.3413 + P(0 \le Z \le z_1) = 0.5255$$

$$\therefore\ P(0 \le Z \le z_1) = 0.5255 - 0.3413$$

$$\therefore\ P(0 \le Z \le z_1) = 0.1842$$

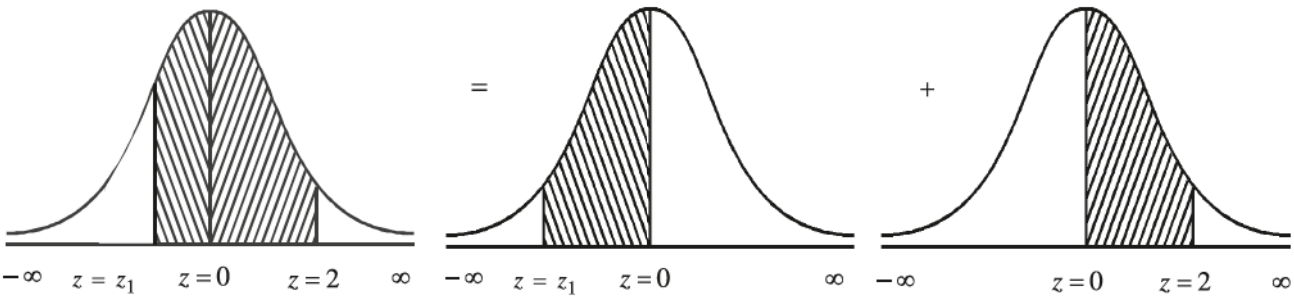
By using the table of standard normal variable the estimated value of $Z$-score, $z_1$ can be determined as under

| **From table** | **Area** | ***Z*-score** |
|---|---|---|
| Nearest value before 0.1842 | 0.1808 | 0.47 |
| Nearest value after 0.1842 | 0.1844 | 0.48 |
| Average value | 0.1826 | **0.475** |

From the above table, it can be seen that the nearest value to 0.1842 is 0.1844 and the respective value of $Z$-score is 0.48. Hence we take $z_1 = 0.48$.

Thus, for $z_1 = 0.48$, $P(-1 \le Z \le z_1) = 0.5255$

(2) It is given that $P(z_1 \le Z \le 2) = 0.7585$. A perpendicular line at $Z = 2$ is drawn and then a perpendicular line at $Z = z_1$ is drawn to its left so that the area between them is 0.7585. The figure is as under




$$\therefore P(z_1 \le Z \le 2) = P(z_1 \le Z \le 0) + P(0 \le Z \le 2) = 0.7585$$

$$\therefore P(z_1 \le Z \le 0) + 0.4772 = 0.7585$$

$$\therefore P(z_1 \le Z \le 0) = 0.7585 - 0.4772$$

$$\therefore P(0 \le Z \le z_1) = 0.2813 \quad (\because \text{ Symmetry})$$

By using the table of standard normal variable the estimated value of $Z$-score, $z_1$ can be determined as under

| From table | Area | $Z$-score |
|---|---|---|
| Nearest value before 0.2813 | 0.2794 | 0.77 |
| Nearest value after 0.2813 | 0.2823 | 0.78 |
| Average value | 0.2809 | 0.775 |

From the above table, it can be seen that the nearest value to 0.2813 is 0.2809 and the respective value of $Z$-score is 0.775. Since $Z$-score is to the left of $Z = 0$, therefore $z_1 = -0.775$.

Thus, for $z_1 = -0.775$, $P(z_1 \le Z \le 2) = 0.7585$

**Activity**

For 30 persons residing around your residence, collect information of their weight (in kg) and obtain its mean (in kg) and standard deviation (in kg). Assuming that the weight of selected person follows normal distribution with the obtained mean and standard deviation, estimate (1) minimum weight of 5% persons having maximum weight (2) maximum weight of 15% of persons having minimum weight.

## 3.6 Illustrations

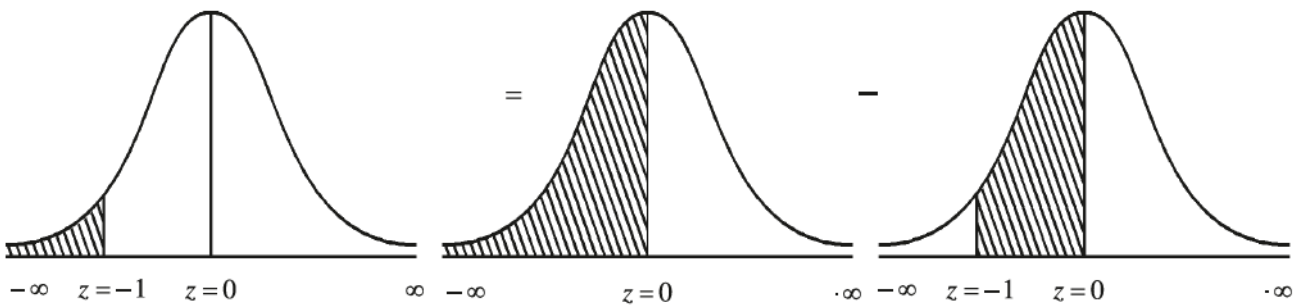
**Illustration 10 : In a city, daily sale of petrol at a petrol pump follows normal distribution and its mean and standard deviation are 33,000 litre and 3000 litre respectively. (1) Obtain the percentage of days of a month during which the daily sales of petrol is less than 30,000 litre. (2) During the month of May, how many days are expected so that the sale of petrol is between 32,000 litre to 35,000 litre?**

Here, $X$ = daily sale of petrol at petrol pump (in litre). Also $\mu = 33{,}000$ litre and $\sigma = 3000$ litre.

(1) Probability that the sale of petrol is less than 30,000 litre

$$= P(X \le 30000) \qquad = P\left(\frac{X-\mu}{\sigma} \le \frac{30000-33000}{3000}\right)$$

$$= P(Z \le -1)$$




$$= P(-\infty < Z \le 0) - P(-1 \le Z \le 0)$$

$$= 0.5 - P(0 \le Z \le 1) \; (\because \text{ Symmetry})$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

$\therefore$ Percentage of days during a certain month where the daily sale of petrol is less than 30,000 litre

$$= 0.1587 \times 100$$

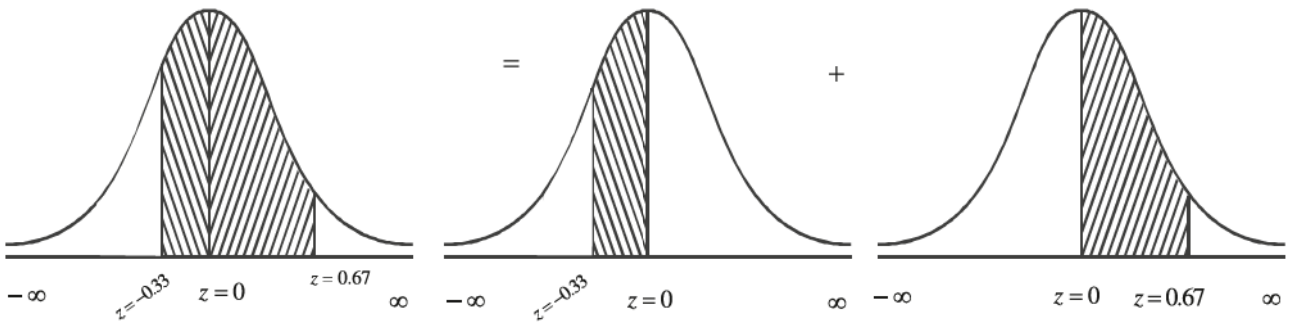
$$= 15.87 \ \%$$

Thus, during 15.87 % of the days of a month, the daily sale of petrol is less than 30,000 litres.

(2) The probability that during the month of May, the daily demand of petrol is between 32,000 and 35,000 litre

$$= P(32000 \le X \le 35000) \quad = P\left(\frac{32000-33000}{3000} \le \frac{X-\mu}{\sigma} \le \frac{35000-33000}{3000}\right)$$

$$= P(-0.33 \le Z \le 0.67)$$



$$= P(-0.33 \le Z \le 0) + P(0 \le Z \le 0.67)$$

$$= P(0 \le Z \le 0.33) + 0.2486 \qquad (\because \text{ Symmetry})$$

$$= 0.1293 + 0.2486$$

$$= 0.3779$$

The number of days in the month of May is $N = 31$. Therefore, the expected number of days in the month of May during which the daily sale of petrol is

between 32,000 litter to 35,000 litre $= 31 \times 0.3779$

$= 11.71$

$\simeq$ 12 Days (approximately)

Thus, in the month of May, approximately during 12 days the demand of petrol is between 32,000 litre and 35,000 litre.

**Illustration 11 : 200 students are selected from all the students of a school and the marks obtained by them in an examination of 100 marks follows normal distribution. The mean marks of the distribution is 60 and its standard deviation is 8.**

**(1) If 70 or more marks are required for the special scholarship then obtain the number of students getting special scholarship.**

**(2) Obtain the minimum marks of 10% of the students getting maximum marks.**

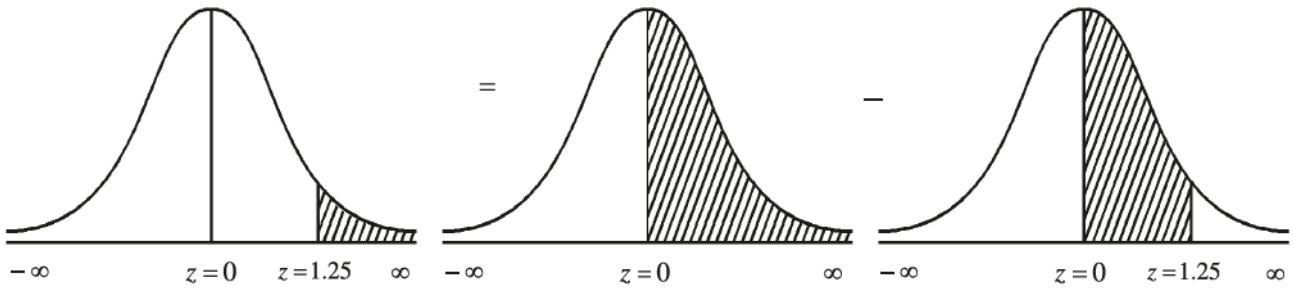
Here, $X$ = marks obtained by a student

Also, $N = 200$, $\mu = 60$ and $\sigma = 8$.

(1) Probability that the marks of the student is 70 or more

$$= P(X \ge 70) \quad = P\left(\frac{X-\mu}{\sigma} \ge \frac{70-60}{8}\right)$$

$$= P(Z \ge 1.25)$$




$$= P(0 \le Z < \infty) - P(0 \le Z \le 1.25)$$

$$= 0.5 - 0.3944$$

$$= 0.1056$$

$\therefore$ the expected number of students getting 70 or more marks

$= 200 \times 0.1056$

$= 21.12$

$\simeq$ 21 (approximately)

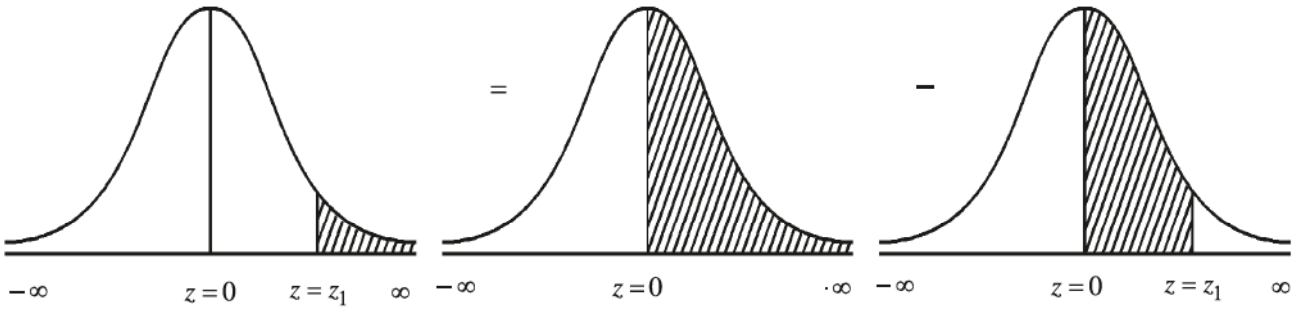
Thus, the approximate number of students getting special scholarship is 21.

(2) Suppose the minimum marks of 10% of the students getting maximum marks is $x_1$. The probability that a student gets $x_1$ or more marks is 0.10.

$\therefore \quad P(X \ge x_1) = 0.10$

$$\therefore \quad P\left(\frac{X-\mu}{\sigma} \geq \frac{x_1-60}{8}\right) = 0.10$$

$$\therefore \quad P(Z \geq z_1) = 0.10, \quad \text{where} \quad z_1 = \frac{x_1-60}{8}$$




$$\therefore 0.10 = P(0 \leq Z < \infty) - P(0 \leq Z \leq z_1)$$

$$\therefore 0.10 = 0.5 - P(0 \leq Z \leq z_1)$$

$$\therefore P(0 \leq Z \leq z_1) = 0.40$$

By using the table of standard normal variable the estimated value of $z_1$ can be determined as under

| From table | Area | $Z$-score |
|---|---|---|
| Nearest value before 0.40 | 0.3997 | 1.28 |
| Nearest value after 0.40 | 0.4015 | 1.29 |
| Average value | 0.4006 | 1.285 |

From the above table it can be seen that the nearest value to 0.40 is 0.3997 and the respective value of $Z$-score is 1.28.

Therefore $z_1 = 1.28$

$$\therefore \quad \frac{x_1-60}{8} = 1.28$$

$$\therefore \quad x_1 - 60 = 10.24$$

$$\therefore \quad x_1 = 70.24$$

Thus, the minimum marks of most intelligent 10% of the students is $70.24 \simeq 70$.

**Illustration 12 : The monthly income of a group of employees follows normal distribution. The mean of the distribution is ₹ 15,000 and its standard deviation is ₹ 4000. From this information, (1) obtain range of monthly income for middle 60% of the employees. (2) if monthly income of 250 employees is between ₹ 15000 and certain fixed income ₹ $x_1$ then find the value of $x_1$.**

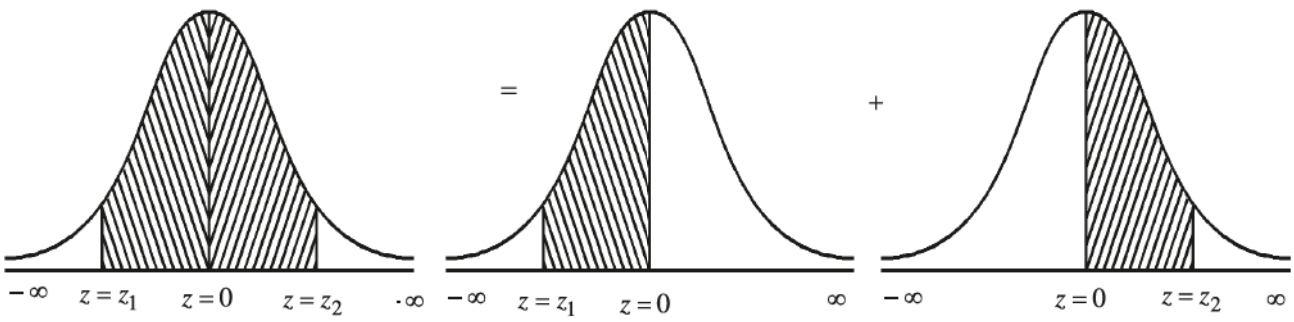
Here, $X$ = monthly income of an employee. Also, $N = 1000$, $\mu = ₹\,15{,}000$ and $\sigma = ₹\,4000$.

(1) Suppose the range of monthly income of exactly middle 60% of employee is ₹ $x_1$ and ₹ $x_2$ where $x_1$ and $x_2$ are at equal distance from mean $\mu$. Now the probability that the monthly income of employee is between $x_1$ and $x_2$ is 0.60.

i.e. $P(x_1 \le X \le x_2) = 0.60.$

$$\therefore\ P\left(\frac{x_1-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{x_2-\mu}{\sigma}\right) = 0.60$$

$$\therefore\ P(z_1 \le Z \le Z_2) = 0.60 \text{ where } z_1 = \frac{x_1-15000}{4000} \text{ and } z_2 = \frac{x_2-15000}{4000}$$




$$0.60 = P(z_1 \le Z \le 0) + P(0 \le Z \le z_2)$$

Now, since $x_1$ and $x_2$ are at equal distance from mean $\mu$, the perpendicular line at $Z = 0$ divides the total area (probability) under the curve between $Z = z_1$ and $Z = z_2$ into two equal parts. So, $z_1 = -z_2$ and also $P(z_1 \le Z \le 0) = 0.30$ and $P(0 \le Z \le z_2) = 0.30$.

By using the table of standard normal variable the estimated value of $z_1$ and $z_2$ can be obtained as under

| From table | Area | $Z$-score |
|---|---|---|
| Nearest value before 0.30 | 0.2995 | 0.84 |
| Nearest value after 0.30 | 0.3023 | 0.85 |
| Average value | 0.3009 | 0.845 |

The nearest value to 0.30 is 0.2995 and the respective value of $Z$-score is 0.84.

$\therefore\ z_1 = -0.84$ and $z_2 = 0.84$

$$\therefore\ \frac{x_1-15000}{4000} = -0.84 \text{ and } \frac{x_2-15000}{4000} = 0.84$$

$\therefore\ x_1 - 15000 = -3360$ and $\therefore\ x_2 - 15000 = 3360$

$\therefore\ x_1 = 11640$ and $\therefore\ x_2 = 18360$

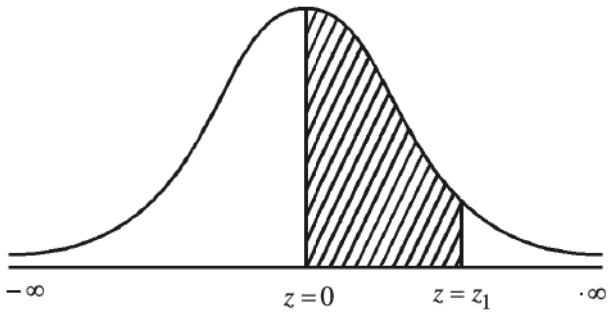
Thus, the range of monthly income for middle 60 % of the employees will be ₹ 11,640 to ₹ 18,360.

(2) Monthly income of 250 employees is between ₹ 15,000 and ₹ $x_1$

Therefore $P(15000 \le X \le x_1) = \frac{250}{1000}$

$$\therefore\ P\left(\frac{15000-15000}{4000} \le \frac{X-\mu}{\sigma} \le \frac{x_1-15000}{4000}\right) = 0.25$$

$$\therefore\ P(0 \le Z \le z_1) = 0.25 \text{ where } z_1 = \frac{x_1-15000}{4000}$$




By using the table of standard normal variable the estimated value of $z_1$ can be obtained as under

| From table | Area | Z-score |
|---|---|---|
| Nearest value before 0.25 | 0.2486 | 0.67 |
| Nearest value after 0.25 | 0.2518 | 0.68 |
| Average value | 0.2502 | 0.675 |

It is clear from the above table that $z_1 = 0.675$

$$\therefore\ \frac{x_1-15000}{4000} = 0.675$$

$$\therefore\ x_1 - 15000 = 2700$$

$$\therefore\ x_1 = 17700$$

Thus, monthly income of 250 employees will be between ₹ 15,000 and ₹ 17,700.

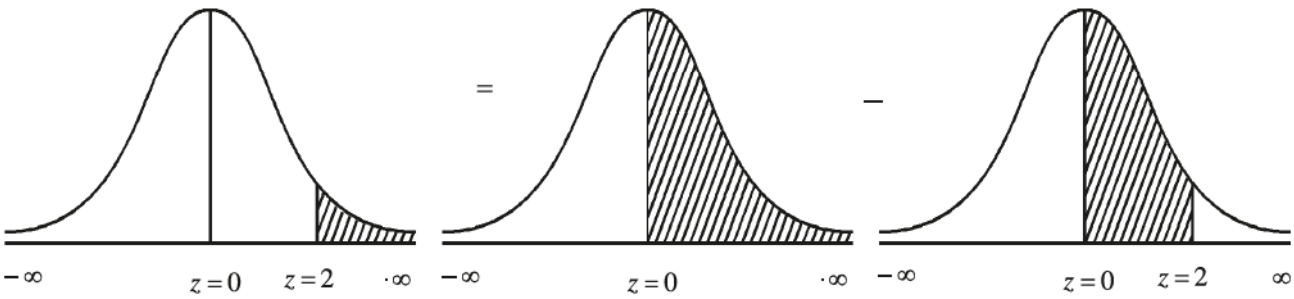
**Illustration 13 : The bill amount of purchase by the customers in departmental store follows normal distribution and its mean is ₹ 800 and standard deviation is ₹ 200. On a day, 57 customers had the bill amount more than ₹ 1200. Estimate the number of customers who visited the store on that day.**

Here $X$ = bill amount of purchase by the customer. $\mu = 800$ and $\sigma = 200$. Suppose $N$ customers visited that departmental store during that day.

The probability that the bill amount of purchase by the customer is more than ₹ 1200

$$= P(X \geq 1200) \quad = P\left(\frac{X-\mu}{\sigma} \geq \frac{1200-800}{200}\right)$$

$$= P(Z \geq 2)$$




$$= P(0 \leq Z < \infty) - P(0 \leq Z \leq 2)$$

$$= 0.5 - 0.4772 \text{ (From } Z\text{-table)}$$

$$= 0.0228$$

Now, the expected number of customer whose bill amount of purchase is more than ₹ 1200 $= N \times P(x \geq 1200)$

$$57 = N \times 0.0228$$

$$\therefore \quad N = \frac{57}{0.0228}$$

$$\therefore \quad N = 2500$$

$\therefore$ 2500 customers visited the store on that day.

**Illustration 14 : For a group of 1000 persons, the average height is 165 cms and variance is 100 (cms)$^2$. The distribution of height of these persons follows normal distribution. From this information, determine the third decile and the 60th percentile and interpret it.**

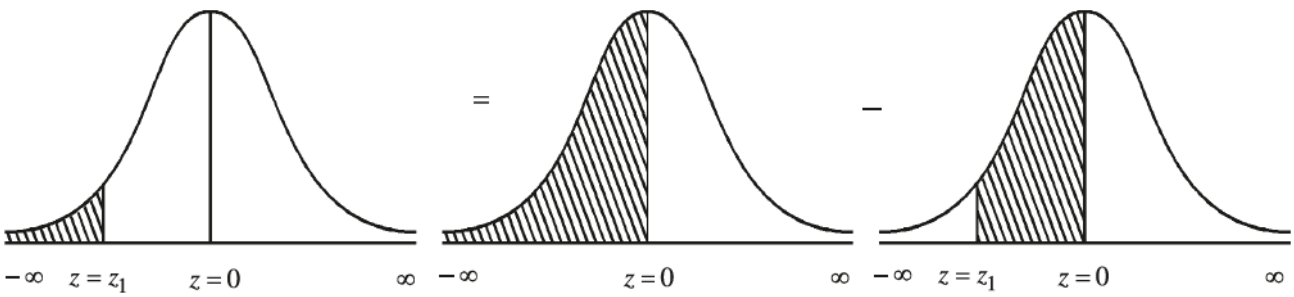
Here, $X$ = height of a person in the group. Also, $\mu = 165$ and $\sigma^2 = 100$ therefore $\sigma = 10$

The third decile $(D_3)$ is to be determined. According to the definition of $D_3$, 30% of the observations in the data have the value less than or equal to $D_3$.

$$\therefore P(X \leq D_3) = \frac{30}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \leq \frac{D_3-165}{10}\right) = 0.30$$

$$\therefore P(Z \leq z_1) = 0.30 \text{ where } z_1 = \frac{D_3-165}{10}$$




$$0.30 = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0)$$

$$0.30 = 0.50 - P(z_1 \le Z \le 0)$$

$$\therefore P(z_1 \le Z \le 0) = 0.50 - 0.30$$

$$\therefore P(0 \le Z \le z_1) = 0.20 \quad (\because \text{ Symmetry})$$

By using the table of standard normal variable the estimated value of $z_1$ can be obtained as under

| From table | Area | *Z*-score |
|---|---|---|
| Nearest value before 0.2 | 0.1985 | 0.52 |
| Nearest value after 0.2 | 0.2019 | 0.53 |
| Average value | 0.2002 | 0.525 |

The nearest value to 0.2 is 0.2002 and the respective value of *Z*-score is 0.525 and it is to the left of $Z = 0$

Therefore $z_1 = -0.525$

$$\therefore \frac{D_3 - 165}{10} = -0.525$$

$$\therefore D_3 - 165 = -5.25$$

$$\therefore D_3 = 159.75$$

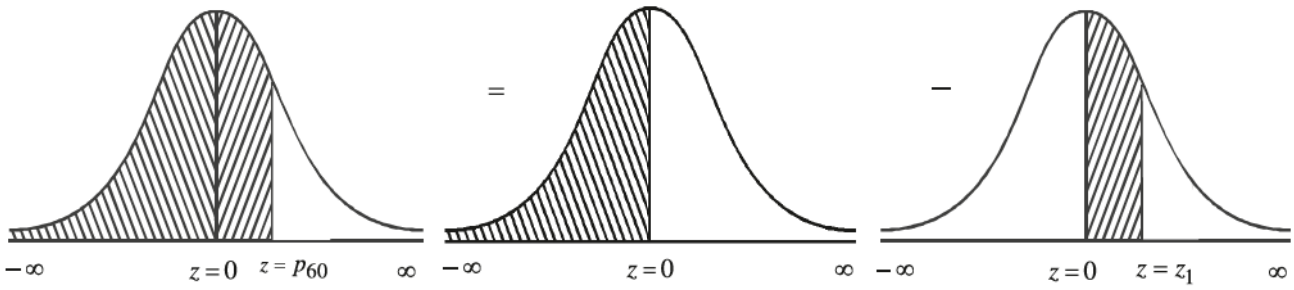
Thus, 30% of the persons in the group have height less than or equal to 159.75 cms.

Now, according to the definition of the 60th percentile ( $P_{60}$ ), 60% of the observations in the given data have the value less than or equal to $P_{60}$.

$$\therefore P(X \le P_{60}) = \frac{60}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \le \frac{P_{60}-165}{10}\right) = 0.60$$

$$\therefore P(Z \le z_1) = 0.60 \text{ where } z_1 = \frac{P_{60}-165}{10}$$




$$0.60 = P(-\infty < Z \le 0) + P(0 \le Z \le z_1)$$

$$0.60 = 0.50 + P(0 \le Z \le z_1)$$

$$\therefore P(0 \le Z \le z_1) = 0.10$$

By using the table of standard normal variable the estimated value of $z_1$ can be obtained as under

| From table | Area | *Z*-score |
|---|---|---|
| Nearest value before 0.10 | 0.0987 | 0.25 |
| Nearest value after 0.10 | 0.1026 | 0.26 |
| Average value | 0.10065 | 0.255 |

The nearest value to 0.10 is 0.10065 and the respective value of *Z*-score is 0.255.

$$\therefore z_1 = 0.255$$

$$\therefore \frac{P_{60} - 165}{10} = 0.255$$

$$\therefore P_{60} - 165 = 2.55$$

$$\therefore P_{60} = 167.55$$

Thus, 60% of the persons in the group have height less than or equal to 167.55 cms.

**Illustration 15 : A manufacturing company produces electric bulb and life of the electric bulb (in hours) follows normal distribution. Its average life is 2040 hours. If 3.36% of bulbs have life more than 2150 hours then find variance of the life of bulbs.**

Here, $X$ = life of electric bulb. Also, $\mu = 2040$. Suppose its variance is $\sigma^2$.

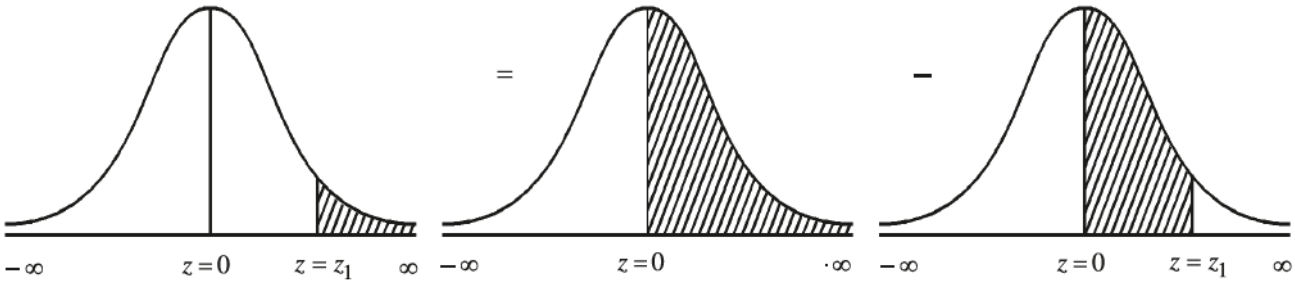
Now, 3.36% of the bulbs have life more than 2150 hours.

$$\therefore P(X \ge 2150) = \frac{3.36}{100}$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \ge \frac{2150-2040}{\sigma}\right) = 0.0336$$

$$\therefore P\left(Z \ge \frac{110}{\sigma}\right) = 0.0336$$

$$\therefore P(Z \ge z_1) = 0.0336 \quad \text{where } z_1 = \frac{110}{\sigma}$$




$$0.0336 = P(0 \le Z < \infty) - P(0 \le Z \le z_1)$$

$$0.0336 = 0.5 - P(0 \le Z \le z_1)$$

$$\therefore P(0 \le Z \le z_1) = 0.5 - 0.0336$$

$$\therefore P(0 \le Z \le z_1) = 0.4664$$

From the table of standard normal variable, for $Z$-score 1.83, $P(0 \le Z \le 1.83) = 0.4664$

$$\therefore z_1 = 1.83$$

$$\therefore \frac{110}{\sigma} = 1.83$$

$$\therefore \sigma = \frac{110}{1.83}$$

$$\therefore \sigma = 60.11$$

$$\therefore \sigma^2 = 3613.21$$

Thus, the variance of the life of electric bulbs produced is 3613.21 (hours)$^2$.

**Illustration 16 : The profit in daily business of a businessman having grocery shop follows normal distribution. Variance of profit is 22500 (₹)$^2$, and the probability that the daily profit is less than Rs. 1000 is 0.0918. Find the average daily profit.**

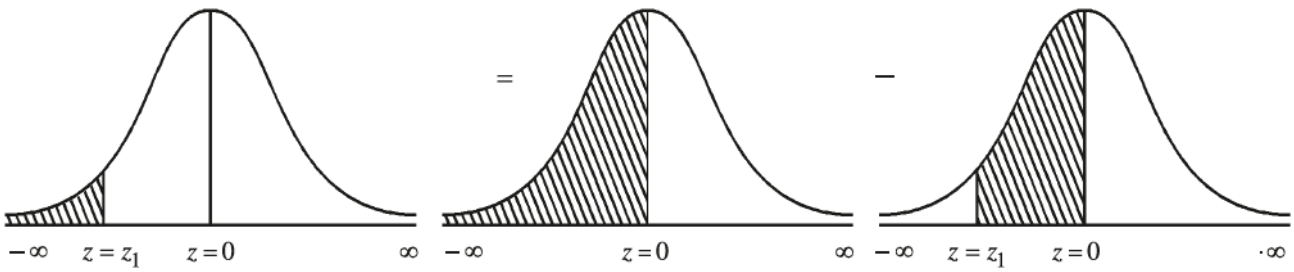
Here, $X$ = daily profit of the businessman in his business. As $\sigma^2 = 22{,}500$, therefore $\sigma = 150$ and suppose the average profit is $\mu$.

Now, the probability that the daily profit is less than ₹ 1000 = 0.0918

$$\therefore P(X \le 1000) = 0.0918$$

$$\therefore P\left(\frac{X-\mu}{\sigma} \le \frac{1000-\mu}{150}\right) = 0.0918$$

$$\therefore P(Z \le z_1) = 0.0918 \text{ where } z_1 = \frac{1000-\mu}{150}$$



$$0.0918 = P(-\infty < Z \le 0) - P(z_1 \le Z \le 0)$$

$$0.0918 = 0.5 - P(z_1 < Z \le 0)$$

$$\therefore\ P(z_1 \le Z \le 0) = 0.5 - 0.0918$$

$$\therefore\ P(0 \le Z \le z_1) = 0.4082 \quad (\because \text{ Symmetry})$$

From the table of standard normal variable, $Z$-score is 1.33.

$$\therefore\ z_1 = -1.33$$

$$\therefore\ \frac{1000-\mu}{150} = -1.33$$

$$\therefore\ \mu = 1199.5$$

Thus, the average daily profit of the businessman in his business is ₹ 1199.5

**Illustration 17 : The maximum temperature of a city during summer follows normal distribution. On a particular day, the probability that the maximum temperature of the city is more than 31° Celsius is 0.3085, whereas the probability that during some other day, the maximum temperature is less than 27° is 0.0668. Find mean and standard deviation of the maximum temperature of the city.**

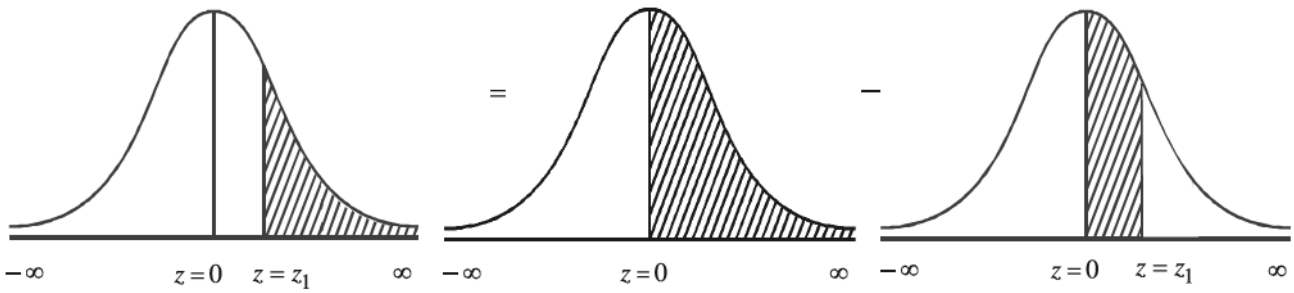
Here, $X$ = maximum temperature (in Celsius) of the city. Suppose $\mu$ and $\sigma$ are the mean and standard deviation.

Now, the probability that the maximum temperature is more than 31° Celsius = 0.3085

$$\therefore\ P(X \ge 31) = 0.3085$$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} \ge \frac{31-\mu}{\sigma}\right) = 0.3085$$

$$\therefore\ P(Z \ge z_1) = 0.3085 \quad \text{where} \quad z_1 = \frac{31-\mu}{\sigma}$$




$$0.3085 = P(0 \le Z < \infty) - P(0 \le Z \le z_1)$$

$$0.3085 = 0.5 - P(0 \le Z \le z_1)$$

$$\therefore\ P(0 \le Z \le z_1) = 0.5 - 0.3085$$

$$\therefore\ P(0 \le Z \le z_1) = 0.1915$$

From the table of standard normal variable, $Z$-score is 0.5.

$$\therefore\ z_1 = 0.5$$

$$\therefore\ \frac{31-\mu}{\sigma} = 0.5$$

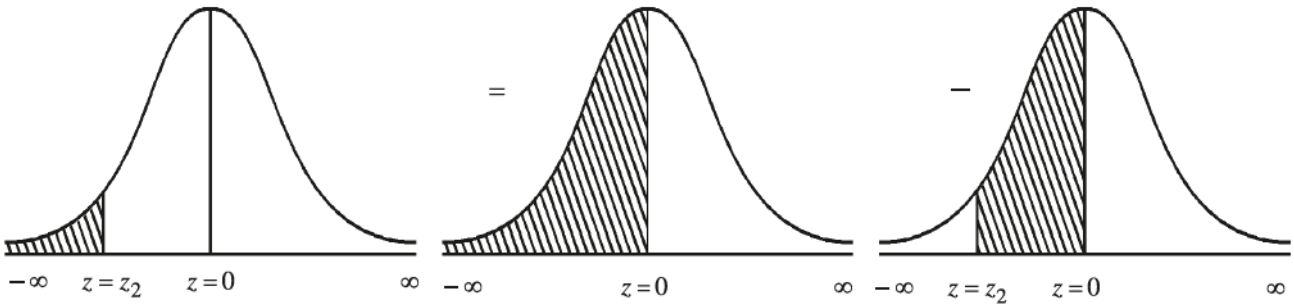
$$\therefore\ 31 - \mu = 0.5\sigma \quad \text{..... (1)}$$

The probability that the maximum temperature is less than 27° Celsius = 0.0668

$$\therefore\ P(X \le 27) = 0.0668$$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} \le \frac{27-\mu}{\sigma}\right) = 0.0668$$

$$\therefore\ P(Z \le z_2) = 0.0668 \text{ where } z_2 = \frac{27-\mu}{\sigma}$$




$$0.0668 = P(-\infty < Z \le 0) - P(z_2 \le Z \le 0)$$

$$\therefore\ 0.0668 = 0.5 - P(z_2 \le Z \le 0)$$

$$\therefore\ P(z_2 \le Z \le 0) = 0.5 - 0.0668$$

$$\therefore\ P(0 \le Z \le z_2) = 0.4332\ (\because \text{ Symmetry})$$

From the table of standard normal variable, $Z$-score is 1.5

$$\therefore\ z_2 = -1.5$$

$$\therefore\ \frac{27-\mu}{\sigma} = -1.5$$

$$\therefore\ 27 - \mu = -1.5\sigma \text{ ..... (2)}$$

Solving equations (1) and (2),

$$31 - \mu = 0.5\sigma$$

$$27 - \mu = -1.5\sigma$$

$$- \quad + \quad +$$

$$4 = 2\sigma$$

$$\therefore\ \sigma = 2$$

By substituting $\sigma = 2$ in equation (1),

$$31 - \mu = 0.5(2)$$

$$\therefore\ 31 - \mu = 1$$

$$\therefore\ \mu = 30$$

Thus, the mean of maximum temperature of a city is 30° Celsius and its standard deviation is 2° Celsius.

**Illustration 18 : The probability density function of a normal variable is as under**

$$f(x) = \frac{1}{4\sqrt{2\pi}} \quad e^{-\frac{1}{32}(x-50)^2} \quad ; \quad -\infty < x < \infty$$

**Obtain parameters of this distribution and find the values of following :**

**(1)** $P(52 \le X \le 58)$ **(2)** $P(|X-45| \le 4)$

By comparing the given probability density function with the probability density function of normal variable $X$

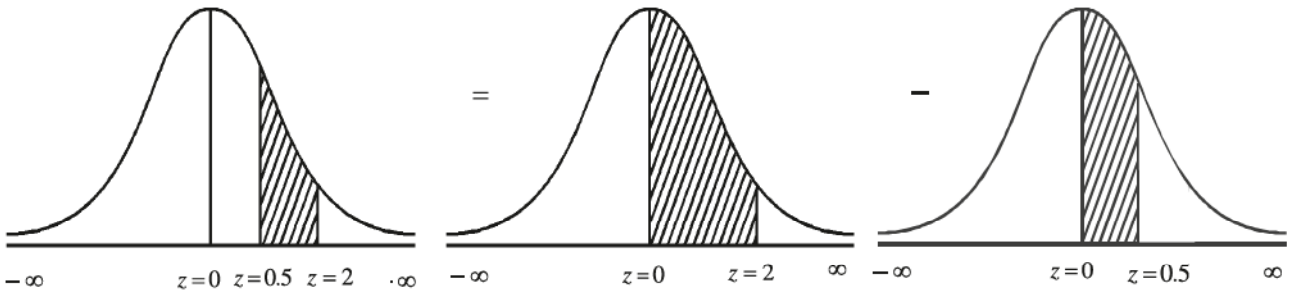
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \quad e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty$$

Here, $\sigma\sqrt{2\pi} = 4\sqrt{2\pi}$ and $\mu = 50$

$\therefore \ \sigma = 4$

(1) $P(52 \le X \le 58) = P\left(\frac{52-50}{4} \le \frac{X-\mu}{\sigma} \le \frac{58-50}{4}\right)$

$= P(0.5 \le Z \le 2)$




$= P(0 \le Z \le 2) - P(0 \le Z \le 0.5)$

$= 0.4772 - 0.1915$

$= 0.2857$

Thus, $P(52 \le X \le 58) = 0.2857$

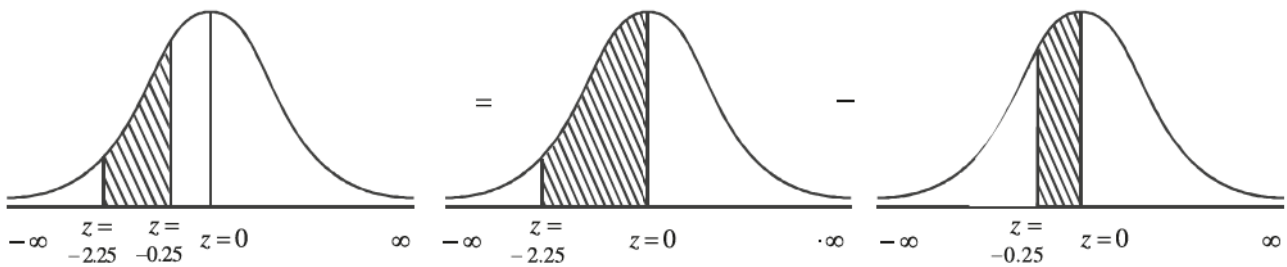
(2) $P(|X-45| \le 4) = P(-4 \le (X-45) \le 4)$ (definition of modulus)

$= P(-4+45 \le (X-45)+45 \le 4+45)$

$= P(41 \le X \le 49)$

$= P\left(\frac{41-50}{4} \le \frac{X-\mu}{\sigma} \le \frac{49-50}{4}\right)$

$= P(-2.25 \le Z \le -0.25)$



$$= P(-2.25 \le Z \le 0) - P(-0.25 \le Z \le 0)$$

$$= P(0 \le Z \le 2.25) - P(0 \le Z \le 0.25) \quad (\because \text{ Symmetry})$$

$$= 0.4878 - 0.0987$$

$$= 0.3891$$

Thus, $P(|X-45| \le 4) = 0.3891$.

**Illustration 19 : The probability density function of a normal variable $X$ is defined as under**

$$f(x) = \text{ constant } \cdot e^{-\frac{1}{2}\left(\frac{x-25}{10}\right)^2} \quad ; \quad -\infty < x < \infty$$

**From this normal distribution estimate the values of the following :**

**(1) Third quartile (2) Quartile deviation (3) Mean deviation**

By comparing the given probability density function with the probability density function of normal variable $X$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \quad e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty$$

Here, $\mu = 25$ and $\sigma = 10$.

(1) Third quartile $Q_3 = \mu + 0.675\,\sigma$

$$= 25 + 0.675\ (10)$$

$$= 25 + 6.75$$

$$= 31.75$$

(2) Quartile deviation $= \frac{2}{3}\ \sigma$

$$= \frac{2}{3}\ (10)$$

$$= \frac{20}{3}$$

(3) Mean deviation $= \frac{4}{5}\ \sigma$

$$= \frac{4}{5}\ (10)$$

$$= 8$$

Thus, for the given normal distribution the estimates of the required values are 31.75, $\frac{20}{3}$ and 8 respectively.

**Illustration 20 : The extreme quartiles for a normal distribution are 20 and 50 respectively. Obtain the limits which include 95% of the observations of the distribution.**

Here, $Q_1 = 20$ and $Q_3 = 50$. For the normal distribution

Mean = Median = Mode = $\frac{Q_3+Q_1}{2}$

$$\therefore \mu = \frac{50+20}{2}$$

$$\therefore \mu = 35$$

Quartile deviation = $\frac{2}{3}\sigma$

$$\therefore \frac{Q_3-Q_1}{2} = \frac{2}{3}\sigma$$

$$\therefore \frac{50-20}{2} = \frac{2}{3}\sigma$$

$$\therefore \frac{50-20}{2} \times \frac{3}{2} = \sigma$$

$$\therefore \sigma = 22.5$$

For the normal distribution the limits including 95% of the observations are $\mu \pm 1.96\sigma$. Hence, the interval (limits) is

$$(\mu - 1.96\sigma,\ \mu + 1.96\sigma)$$

$$\therefore (35 - 1.96\,(22.5),\ 35 + 1.96\,(22.5))$$

$$\therefore (35 - 44.1,\ 35 + 44.1)$$

$$\therefore (-9.1,\ 79.1)$$

Thus, from the given information, the limits including 95% of the observations are –9.1 to 79.1.

**Illustration 21 : For a normal distribution, the first quartile and the mean deviation are 20 and 24 respectively. Obtain an estimate of the value of mode.**

Here, $Q_1$ = 20 and mean deviation = 24.

$$\therefore \frac{4}{5}\sigma = 24$$

$$\therefore \sigma = 24 \times \frac{5}{4}$$

$$\therefore \sigma = 30$$

Now, quartile deviation = $\frac{2}{3}\sigma$

$$\therefore \frac{Q_3-Q_1}{2} = \frac{2}{3}\sigma$$

$$\therefore \frac{Q_3-20}{2} = \frac{2}{3}(30)$$

$$\therefore Q_3 - 20 = 20 \times 2$$

$$\therefore Q_3 = 40 + 20$$

$$\therefore Q_3 = 60$$

Now, for normal distribution, Mean = Median = Mode = $\frac{Q_3+Q_1}{2}$

$$= \frac{60+20}{2}$$

$$= 40$$

Thus, from the given information the estimated value of mode is 40.

**Illustration 22 : The number of vehicles arriving at toll station during busy hours of national highway follows normal distribution. The mean of this distribution is $\mu$ and its standard deviation is $\sigma$. The number of vehicles arriving at two different busy time periods are 88 and 64 and if the respective values of *Z*-score for these values are 0.8 and –0.4 then find mean and standard deviation of number of vehicles arriving at the toll station during busy period.**

Here, $X$ = number of vehicles arriving at toll station during busy period.

The mean of this distribution is $\mu$ and its standard deviation is $\sigma$

$$Z\text{-score} = \frac{X-\mu}{\sigma}$$

When $X = 88$ then $Z = 0.8$ therefore $0.8 = \frac{88-\mu}{\sigma}$

$$\therefore\ 0.8\sigma = 88-\mu \qquad (1)$$

When $X = 64$ then $Z = -0.4$ therefore $-0.4 = \frac{64-\mu}{\sigma}$

$$\therefore -0.4\sigma = 64-\mu \qquad (2)$$

Solving equations (1) and (2),

$$\begin{array}{rl} & 0.8\sigma = 88-\mu \\ - & 0.4\sigma = 64-\mu \\ + & \quad - \quad + \\ \hline & 1.2\sigma = 24 \\ & \therefore\ \sigma = 20 \end{array}$$

By putting $\sigma = 20$ in equation (1), $0.8(20) = 88-\mu$

$$\therefore\ 16 = 88-\mu$$

$$\therefore\ \mu = 72$$

Thus. the mean of given data is $\mu = 72$ vehicles and its standard deviation is $\sigma = 20$ vehicles.

**Activity**

Collect the information of average monthly expenses of 30 families residing around your residence. Assuming that the average monthly expense of these families follows normal distribution with the mean and standard deviation determined by you,

(1) Obtain the limits of average monthly income of middle 60% of the families.

(2) Find the percentage of observations lying between the range $\mu \pm \sigma$ from your data.

## Summary

- A function for obtaining probability that a continuous random variable assumes value between specified interval is called probability density function of that variable.
- The probability for a definite value of continuous random variable $X$ obtained by the probability density function is always zero (0).
- A curve drawn by considering different values of normal variable $X$ and its respective values of probability density function $f(x)$ is called normal curve.
- Normal curve is completely bell shaped and its skewness is zero.
- If $X$ is a random normal variable with mean $\mu$ and standard deviation $\sigma$ then $Z = \frac{X-\mu}{\sigma}$ is called standard normal variable.
- Standard normal probability distribution is a probability distribution of normal variable with mean zero and standard deviation 1.
- The observed value of standard normal variable $Z$ is called standard score or $Z$-score and it is independent of unit of measurement.
- Normal distribution is also defined as $N(\mu, \sigma^2)$ where $\mu$ and $\sigma$ are parameters of the distribution which indicate its mean and standard deviation respectively.
- In order to express probability in percentage, the probability is multiplied by 100.
- In order to obtain expected number of observations, the probability is multiplied by the total number of observations ($N$).

### List of Formulae

If $X$ is a normal variable with mean $\mu$ and standard deviation $\sigma$ then

(1) Standard normal variable $Z = \frac{X-\mu}{\sigma}$

(2) Mean = Median = Mode = $\frac{Q_3+Q_1}{2}$

(3) Approximate value of the first quartile $Q_1 = \mu - 0.675\,\sigma$

(4) Approximate value of the third quartile $Q_3 = \mu + 0.675\,\sigma$

(5) Quartile deviation = $\frac{2}{3}\ \sigma$ (approximately)

(6) Mean deviation = $\frac{4}{5}\ \sigma$ (approximately)

## EXERCISE 3

### Section A

**Find the correct option for the following multiple choice questions :**

**1.** Which of the following is probability density function for normal variable $X$ with mean $\mu$ and standard deviation $\sigma$ ?

(a) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$ ; $-\infty<x<\infty$ (b) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$ ; $-\infty<x<\infty$

(c) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ ; $-\infty<x<\infty$ (d) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ ; $0\le x<\infty$

**2.** For a normal variable X with mean $\mu$ and standard deviation $\sigma$, which of the following is standard normal variable $Z$ for it ?

(a) $Z = \frac{x-\sigma}{\mu}$ (b) $Z = \frac{\sigma-x}{\mu}$ (c) $Z = \frac{\mu-x}{\sigma}$ (d) $Z = \frac{x-\mu}{\sigma}$

**3.** Which of the following is probability density function for standard normal variable?

(a) $f(z)= e^{-\frac{1}{2}z^2}$ ; $-\infty<z<\infty$ (b) $f(z)=\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2}$ $-\infty<z<\infty$

(c) $f(z)=\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2}$ ; $0<z<\infty$ (d) $f(z)=\frac{1}{\sqrt{2\pi}}\ e^{-z^2}$ $-\infty<z<\infty$

**4.** Which of the following are mean and variance of standard normal variable?

(a) Mean = 0, Variance = 1 (b) Mean = 1, Variance = 0

(c) Mean = 0, Variance = 0 (d) Mean = 1, Variance = 1

**5.** What is the total area under normal curve among the following ?

(a) –1 (b) 0 (c) 1 (d) 0.5

**6.** What is the area under the normal curve to the right hand side of perpendicular line at $X = \mu$ ?

(a) 0 (b) 0.5 (c) 1 (d) –0.5

**7.** In normal distribution, usually which limits include 99 % of the observations?

(a) $\mu\pm1.96\sigma$ (b) $\mu\pm2\sigma$ (c) $\mu\pm3\sigma$ (d) $\mu\pm2.575\sigma$

**8.** In normal distribution, usually what percentage of the observations are included in the limits $\mu\pm\sigma$ ?

(a) 34.13 % (b) 95.45 % (c) 68.26 % (d) 50 %

**9.** Which of the following is approximate value of mean deviation for normal variable?

(a) $\frac{4}{5}\sigma$ (b) $\frac{4}{5}\mu$ (c) $\frac{2}{3}\sigma$ (d) $\frac{2}{3}\mu$

**10.** Which of the following is approximate value of quartile deviation for standard normal variable?

(a) $\frac{2}{3}\sigma$ (b) $\frac{2}{3}$ (c) $\frac{4}{5}\sigma$ (d) $\frac{4}{5}$

**11.** Mean and the first quartile for a normal distribution are 11 and 3 respectively. Which of the following is the value of the third quartile?

(a) 8 (b) 14 (c) 19 (d) 10

**12.** For a normal distribution, approximate value of mean deviation is 20. Which of the following is the value of quartile deviation?

(a) $\frac{25}{3}$ (b) $\frac{32}{3}$ (c) 24 (d) $\frac{50}{3}$

**13.** In usual notation of normal distribution, $x = 25$, $\mu = 20$ and $\sigma = 5$ then which of the following is the value of standard normal variable?

(a) 1 (b) –1 (c) 4 (d) $\frac{10}{3}$

**14.** Mean of a normal variable $X$ is 50. If the value of $Z$–score is –2.5 for $x$ = 25 then which of the following is a variance of the distribution ?

(a) 10 (b) 100 (c) 50 (d) 25

**15.** If the distribution of normal variable is shown as $N(20, 4)$ then which of the following intervals includes 99.73% of observations?

(a) (18, 22) (b) (16, 24) (c) (14, 26) (d) (12, 28)

## Section B

**Answer the following questions in one sentence :**

**1.** Give the values of the constants used in probability density function of normal variable.

**2.** What is the probability that a continuous random variable takes definite value?

**3.** What is the shape of normal curve?

**4.** What is the skewness of normal distribution?

**5.** "Standard score is independent of unit of measurement". Is this statement true or false?

**6.** For which value of standard normal variable, the standard normal curve is symmetric on both the sides?

**7.** Which value of normal variable divides the area of normal curve in two equal parts?

**8.** What percentage of area is covered under the normal curve within the range $\mu - 2\sigma$ to $\mu + 2\sigma$ ?

**9.** Mean of a normal distribution is 13.25 and its standard deviation is 10. Estimate the value of its third quartile.

**10.** For a normal distribution having mean 10 and standard deviation 6, estimate the value of quartile deviation.

**11.** The approximate value of mean deviation for a normal distribution is 8. Find its standard deviation.

**12.** For a normal distribution, the estimated value of quartile deviation is 12. Find the value of its standard deviation.

13. For a probability distribution of standard normal variable, state the estimated limits for the middle 50 % observations.

14. The extreme quartiles of normal distribution are 20 and 30. Find its mean.

15. The monthly expense of a group of persons follows normal distribution with mean ₹ 10,000 and standard deviation ₹ 1000. A student has obtained a $Z$- score = ₹ 1 for randomly selected person having monthly expense more than 11,000. Is this calculation of $Z$- score true ? Give reason.

16. The age of a group of persons follows normal distribution with mean 45 years and standard deviation 10 years. Calculate $Z$-score for a randomly selected person having age 60 years.

17. Marks obtained by students of a school in Economics subject follows normal distribution with mean $\mu$ and standard deviation $\sigma$ . The value of standard score that a randomly selected student obtained 60 marks is 1. If the variance of variable is 100 (marks)$^2$ then find average marks.

## Section C

**Answer the following questions :**

1. Define probability density function of continuous random variable.

2. Write the conditions for probability density function for continuous variable.

3. How is the normal curve drawn?

4. Define probability density function for normal variable.

5. What is the shape of standard normal curve? To which value of variable it is symmetric?

6. Define standard normal variable and write its probability density function.

7. A normal variable $X$ has the probability density function as,

$$f(x) = \text{constant} \times e^{-\frac{1}{50}(x-10)^2} \; ; \; -\infty < x < \infty$$

Find the first quartile from this information.

8. The extreme quartiles of a normal variable are 10 and 30. Find its mean deviation.

9. For a normal variable, mean deviation is 48 and its third quartile is 120. Estimate its first quartile.

10. A normal variable $X$ has the probability density function as,

$$f(x) = \frac{1}{10\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} \; ; \; -\infty < x < \infty$$

For this distribution, obtain the limits which include middle 68.26% of the observations.

11. The probability that the value of standard normal variable lies between 0 and $Z$-score ($z_1$) is 0.4950. Find the possible values of $Z$-score.

## Section D

**Answer the following questions :**

1. Define normal distribution and state the characteristics of normal curve.
2. State the properties of normal distribution
3. State the properties of standard normal distribution.
4. A normal distribution has mean 50 and variance 9. Find the probability that

   (1) The value of normal variable $X$ lies between 50 and 53.

   (2) The value of normal variable $X$ lies between 47 and 53.
5. If $X$ is a normal variable with mean 100 and standard deviation 15 then find the percentage of observations

   (1) Having value more than 85.

   (2) Having value less than 130.
6. The weight of randomly selected 500 adult persons from a region of a city follows normal distribution. The average weight of these persons is 55 kg and its standard deviation is 7 kg.

   (1) Estimate the number of persons having weight between 41 kg to 62 kg.

   (2) Estimate the number of persons having weight less than 41 kg.
7. If probabilities for the value of standard normal variable $Z$ are as under then estimate the value of $Z$-score $(z_1)$ :

   (1) Area to the left of $Z = z_1$ is 0.9928.

   (2) Area to the rightt of $Z = z_1$ is 0.0250.
8. If $Z$ is a standard normal variable then estimate the value of $Z$-score $(z_1)$ such that the following conditions are satisfied :

   (1) Area to the left of $Z = z_1$ is 0.15.

   (2) Area to the rightt of $Z = z_1$ is 0.75.
9. If $Z$ is a standard normal variable and $z_1$ represents the $Z$-score then estimate the value of $z_1$ so that the following conditions are satisfied :

   (1) $P(-2 \le Z \le z_1) = 0.2857$ (2) $P(z_1 \le Z \le 1.75) = 0.10$
10. The monthly production of units in a factory is normally distributed with mean $\mu$ and standard deviation $\sigma$. The $Z$-scores corresponding to the production of 2400 units and 1800 units are 1 and –0.5 respectively. Find its mean and standard deviation.

**11.** A normal variable $X$ has the following probability density function

$$f(x) = \frac{1}{6\sqrt{2\pi}} \cdot e^{-\frac{1}{72}(x-100)^2} \quad ; \quad -\infty < x < \infty$$

For this distribution, obtain the estimated limits for the exact middle 50% of the observations.

**12.** For a normal distribution, the first quartile is 35 and its third quartile is 65. Estimate the limits that includes exactly middle 95.45% of the observations.

**13.** For a normal distribution, the third quartile and mean deviations are 36 and 24 respectively. Find the mean of the distribution.

**14.** A normal variable $X$ has mean 200 and variance 100.

(1) Estimate the values of extreme quartiles.

(2) Find the approximate value of quartile deviation.

(3) Find the approximate value of mean deviation.

## Section E

**Solve the following :**

**1.** An amount of purchase of a customer in a mall of a city follows normal distribution with mean ₹ 800 and standard deviation ₹ 200. If a customer is selected at random then find the probabilities for the following events :

(1) Amount of purchase made by him is in between ₹ 850 to ₹ 1200.

(2) Amount of purchase made by him is in between ₹ 600 to ₹ 750.

**2.** The average weight of 500 persons of age between 20 years and 26 years of certain area is 55 kg and its variance is 100 $(kg)^2$. The weight of these persons follows normal distribution. According to the weight of persons they can be categorized as under :

(1) Person having weight more than 70 kg is in the fat persons group

(2) Person having weight between 50 kg to 60 kg is in the healthy persons group.

(3) Person having weight less than 35 kg is in the physically weak person's group

From this information, estimate the number of fat persons, number of healthy persons and number of physically weak persons in that area.

**3.** The average monthly expense of students residing in university hostel is ₹ 2000 and its standard deviation is ₹ 500. If the monthly expense of a student follows normal distribution then

(1) Find percentage of students having expense between ₹ 750 and ₹ 1250.

(2) Find percentage of students having expense more than ₹ 1800.

(3) Find percentage of students having expense less than ₹ 2400.

**4.** The monthly average salary of workers working in a production house is ₹ 10,000 and its standard deviation is ₹ 2000. By assuming that the monthly salary of a worker follows normal

distribution, estimate the maximum salary of 20% of the workers having lowest salary. Also estimate the minimum salary of 10% of the workers having highest salary.

5. A normal distribution has mean 52 and variance 64. Obtain estimated limits which include exactly middle 25% of the observations.

6. In a big showroom of electronic items, on an average 52 electronic units are sold every week and its variance is 9 (unit)$^2$. Sale of electronic items follows normal distribution. The probability that the sale of electronic items during a week out of 52 weeks is from $x_1$ units to 61 units is 0.1574. Estimate the value of $x_1$. Also estimate the number of weeks during which the sale of electronic items is more than 55 units.

7. It is known that on an average a person spends 61 minutes in a painting exhibition. If this time is normally distributed and 20% of the persons spent less than 30 minutes in the exhibition then find variance of the distribution. Also determine the probability that a person spends more than 90 minutes in the exhibition.

8. If the diameter of pipes produced by a company manufacturing pipes is 20 mm to 22 mm then it is accepted by specified group of customers. The standard deviation of produced pipes is 4 mm and it is known that 70% of the pipes produced in the unit have diameter more than 19.05 mm. Find the average diameter of the produced pipes. Also find the percentage of pipes rejected by specified group of customers.

   **Note :** The diameter of the produced pipes follows normal distribution.

9. A normal variable $X$ has mean 400 and variance 900. Find the fourth decile and 90th percentile for this distribution and also interpret the values.

10. A normal variable $X$ has the following density function :

$$f(x) = \frac{1}{50\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-150}{50}\right)^2} \quad ; \quad -\infty < x < \infty$$

For this distribution,

(i) If $P(x_1 \le x \le 250) = 0.4772$ then estimate $x_1$.

(ii) If $P(75 \le x \le x_2) = 0.3539$ then estimate $x_2$.

## Section F

**Solve the following :**

1. An intelligence test is conducted for 500 children and it is found that the average marks are 68 and standard deviations is 22. If the marks obtained by the children is normally distributed then (1) find the number of children getting marks more than 68. (2) Find the percentage of children getting marks between 70 and 90. (3) Find the minimum score of most intelligent 50 children.

2. Age of 500 employees working in a private company follows normal distribution with mean 40 years and standard deviation 6 years. The company wants to reduce its staff by 25% in the following manner :

(i) To retrench 5% of the employees having minimum age

(ii) After retrenching 5% of employees having minimum age, next 10% of the employees are to be transferred to another company.

(iii) To retire 10% of employees having maximum age.

From this information, find the age of employees who are to be retrenched, transferred and retired from the company.

**3.** An entrance test of 200 marks is conducted for higher study. 20,000 students remain presents in the examination and the marks obtained by them follows normal distribution with mean 120 and standard deviation 20. The rules for the result are as under :

(a) Students who acquire less than 40 percent marks are failed.

(b) An additional test is conducted for the students acquiring marks between 40 percent and 48 percent.

(c) Students who acquire mark between 48 percent and 75 percent are called for personal interview.

(d) Students who acquire marks more than 75 percent get direct admission for the higher studies.

Find approximate number of students who : (1) failed in test (2) appeared for additional 100 marks test (3) appeared for personal interview and (4) got direct admission for the higher studies.

**4.** The monthly income of a group of persons follows normal distribution with mean ₹ 20,000 and standard deviation ₹ 5000. If the minimum monthly income of 50 richest person is ₹ 31,625 then how many persons are in the group? Also, what is the maximum income of 50 persons having lowest monthly income?

**5.** Analysis of result of 12th standard students of a school is as under :

| | | |
|---|---|---|
| Pass with distinction | : | 15 % of total students |
| Pass without distinction | : | 75 % of total students |
| Fail | : | 10 % of total students |

For passing the examination, minimum 40 % of the total marks and for distinction minimum 80 % marks are required. If the percentage of result of the students follows normal distribution then find mean and standard deviation and by using it determine the percentage marks for which 75 % of the students have less than that percentage marks.

**6.** The monthly bill amount of regular customers of a provision store follows normal distribution. If 7.78 % customers have monthly bill amount less than ₹ 3590 and 94.52 % customers have bill amount less than ₹ 5100 then determine the parameters of the normal distribution. Also determine the interval for monthly bill amount of exactly middle 95% customers.

**7.** A normal variable $X$ has following probability density function :

$$f(x) = \frac{1}{\sqrt{5000\pi}} e^{-\frac{1}{5000}(x-75)^2} \quad ; \quad -\infty \le x \le \infty$$

From this, answer the following questions :

(i) If $P(60 \le x \le x_2) = 0.5670$ then find $x_2$.

(ii) If $P(x_1 \le x \le 125) = 0.3979$ then find $x_1$.

(iii) Find $P(|x-50| \le 10)$.

**8.** A normal variable $X$ has following probability density function :

$$f(x) = \text{ constant } \cdot e^{-\frac{1}{200}(x-50)^2} \quad ; \quad -\infty < x < \infty$$

From this distribution, answer the following questions :

(1) Find median.

(2) Find estimated values of the extreme quartiles.

(3) Find approximate value of quartile deviation.

(4) Find approximate value of mean deviation.

**Johann Carl Friedrich Gauss**
**(1777 – 1855)**

Carl Friedrich Gauss was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics astronomy, matrix theory, and optics. He was referred to as the Princeps mathematicorum. (Latin, "the foremost of mathematicians") and "greatest mathematician since antiquity". Gauss had an exceptional influence in many fields of mathematics and science and has several theories and results in his name.

In the area of probability and statistics, Gauss introduced what is now known as Gaussian or normal distribution, the Gaussian function and the Gaussian error curve. He showed how probability could be represented by a bell shaped or "normal" curve, which peaks around the mean or expected value and quickly falls off towards plus/minus infinity, which is basic to descriptions of statistically distributed data.

# 4

# Limit

**Contents :**

## 4.1 Introduction

We have studied function in 11th Standard. We studied in the chapter that when we substitute a particular value of a variable in the function, we got the corresponding value of the function. For example, if we substitute $x=2$ in the function $f(x)=2x+3$, we get $f(2)=7$. And if we substitute $x=1$ in the function $f(x)=\frac{3-x}{3x+2}$, we get $f(1)=\frac{2}{5}$. But this is not possible for all functions and all values of $x$. Let us consider a function $f(x)=\frac{x^2-9}{x-3}$ and if we substitute $x=3$ in $f(x)$, we get $f(3)=\frac{0}{0}$ which is an indeterminate value. To find approximate value of $f(3)$ for this function, we need to know the concept of limit of a function. So, limits can be used to approximate the value of a function when the value of the function is indeterminate.

We consider the following illustration to clarify the above concept.

Assume that we are watching a football game through internet. Unfortunately, the connection is choppy and we missed what happened at 14:00 (14 minutes after the commencement of match.)





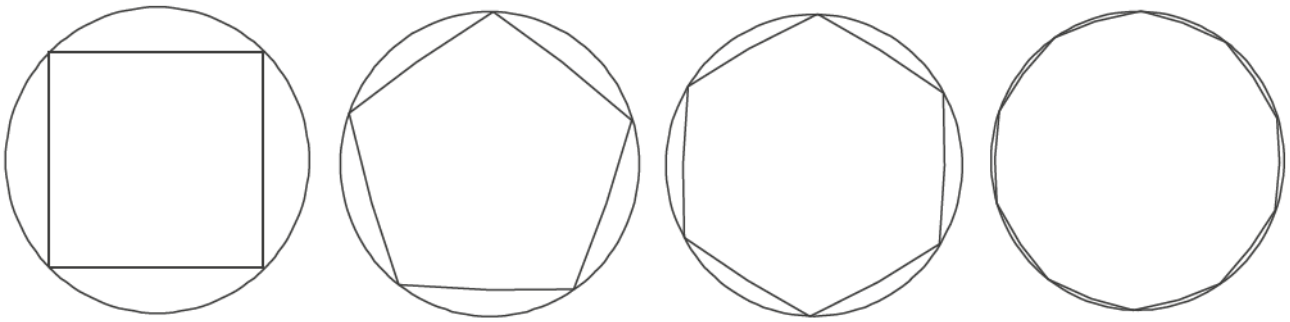
What would be the position of the ball at 14:00 ? We have seen the position of the ball at 13:58 (13 minutes and 58 seconds after the commencement of match), 13:59, 14:01, 14:02.

We will see the neighbouring instants of 14:00, (13:59 and 14:01) and estimate the position of the ball at 14:00. Our estimation is "At 14:00, the ball was somewhere between its position at 13:59 and 14:01." With a slow-motion camera, we might even say "At 14:00, the ball was somewhere between its positions at 13:59.99 and 14:00.01". It means that our estimation improves as we take closer and closer instants to 14:00. The approximate position of the ball thus obtained will be the limiting value of the position of the ball.

Thus, we can say that, "Limit is a method for finding confident approximate value."

We consider one more illustration.

Suppose we want to find the area of a circle. We can estimate the area of circle from the area of polygon drawn inside the circle.



4 side 5 side 6 side 10 side

We can see from the above figures that as the number of sides of polygon increases, area of the polygon approaches nearer the area of circle. The limiting value of the area of polygon is the best approximate value of the area of the circle.

Thus, limit can be used to approximate the unknown values by using its nearby values. Closer the neighbouring values, better is the approximation.

To understand the concept of limit, we shall understand the following basic terms.

## 4.2 Real Line and its Interval

**Real line :** The real line or real number line is a line where its points are the real numbers.

**Interval :** A set of real numbers between any two real numbers is an interval. We shall study different types of intervals.

**Closed Interval :** If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between $a$ and $b$ including $a$ and $b$ is called a closed interval. It is denoted by $[a, b]$.

$$[a, b] = \{x \mid a \le x \le b, \ x \in R\}$$

**Open Interval :** If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between $a$ and $b$ not including $a$ and $b$ is called an open interval. It is denoted by $(a, b)$.

$$(a, b) = \{x \mid a < x < b, \ x \in R\}$$

**Closed-Open Interval :** If $a \in R$, $b \in R$ and $a < b$ then the set of all real numbers between $a$ and $b$ including $a$ but not including $b$ is called a closed open interval. It is denoted by $[a, b)$.

$[a, b) = \{x \mid a \le x < b, \ x \in R\}$

**Open-Closed Interval :** If $a \in R$, $b \in R$ and $a < b$ then set of all real numbers between $a$ and $b$ not including $a$ but including $b$ is called an open closed interval. It is denoted by $(a, b]$.

$(a, b] = \{x \mid a < x \le b, \ x \in R\}$

## 4.3 Modulus

If $x \in R$ then

$|x| = x$ if $x \ge 0$

$= -x$ if $x < 0$

Modulus of any real number is always non-negative.

**e.g.** $|3| = 3$, $|-4| = 4$, $|0| = 0$

**Meaning of $|x-a| < \delta$ (*Delta*)**

Using the definition of modulus

$|x-a| < \delta = (x-a) < \delta$ if $x \ge a$ **or** $x < a+\delta$ if $x \ge a$

$= (a-x) < \delta$ if $x < a$ **or** $x > a-\delta$ if $x < a$

$\therefore \ |x-a| < \delta \Leftrightarrow x \in (a-\delta, a+\delta)$

## 4.4 Neighbourhood

Any open interval containing $a$, $a \in R$ is called a **neighbourhood** of $a$.

**$\delta$ neighbourhood of $a$ :**

If $a \in R$ and $\delta$ is non-negative real number then the open interval $(a-\delta, a+\delta)$ is called $\delta$ neighbourhood of $a$. It is denoted by $N(a, \delta)$.

Here, it can be understood that

$N(a, \delta) = \{x \mid a-\delta < x < a+\delta, \ x \in R\}$

$= \{x \mid |x-a| < \delta, \ x \in R\}$

$\delta$ neighbourhood of $a$ can be expressed in the following different ways.

| Neighbourhood form | Modulus form | Interval form |
|---|---|---|
| $N(a, \delta)$ | $\lvert x-a\rvert < \delta$ | $(a-\delta, a+\delta)$ |

**Illustration 1 : Express $N(5, 0.2)$ in modulus and interval form.**

Comparing $N(5, 0.2)$ with $N(a, \delta)$, we get $a = 5$ and $\delta = 0.2$.

**Modulus form :** $|x-a| < \delta$

Putting $a = 5$ and $\delta = 0.2$,

$N(5, 0.2) = |x-5| < 0.2$

**Interval form :** $(a-\delta,\ a+\delta)$

Putting $a = 5$ and $\delta = 0.2$,

$$N(5, 0.2) = (5-0.2,\ 5+0.2)$$
$$= (4.8,\ 5.2)$$

**Illustration 2 : Express 0.001 neighbourhood of 3 in modulus and interval form.**

Comparing 0.001 neighbourhood of 3 with $\delta$ neighbourhood of $a$, we get $a = 3$ and $\delta = 0.001$.

**Modulus form :** $|x-a| < \delta$

Putting $a = 3$ and $\delta = 0.001$,

0.001 neighbourhood of 3 $= |x-3| < 0.001$

**Interval form :** $(a-\delta,\ a+\delta)$

Putting $a = 3$ and $\delta = 0.001$,

$$\text{0.001 neighbourhood of 3} = (3-0.001,\ 3+0.001)$$
$$= (2.999,\ 3.001)$$

**Illustration 3 : Express $|x+1| < 0.5$ in neighbourhood and interval form.**

Comparing $|x+1| < 0.5$ with $|x-a| < \delta$, we get $a = -1$ and $\delta = 0.5$.

**Neighbourhood form :** $N(a,\ \delta)$

Putting $a = -1$ and $\delta = 0.5$,

$|x+1| < 0.5 = N(-1,\ 0.5)$

**Interval form :** $(a-\delta,\ a+\delta)$

Putting $a = -1$ and $\delta = 0.5$,

$$|x+1| < 0.5 = (-1-0.5,\ -1+0.5)$$
$$= (-1.5,\ -0.5)$$

**Illustration 4 : Express $(0.9, 1.1)$ in neighbourhood and modulus form.**

Comparing $(0.9, 1.1)$ with $(a-\delta,\ a+\delta)$, we get $a-\delta = 0.9$ and $a+\delta = 1.1$.

Adding $a-\delta = 0.9$ and $a+\delta = 1.1$, we get $2a = 2$ $\therefore\ a = 1$.

Putting $a = 1$ in $a+\delta = 1.1$, we get $\delta = 0.1$.

**Neighbourhood form :** $N(a,\ \delta)$

Putting $a = 1$ and $\delta = 0.1$,

$(0.9,\ 1.1) = N(1,\ 0.1)$

**Modulus form :** $|x-a| < \delta$

Putting $a = 1$ and $\delta = 0.1$,

$(0.9,\ 1.1) = |x-1| < 0.1$

**Punctured $\delta$ neighbourhood of $a$ :**

If $a \in R$ and $\delta$ is a non-negative real number then the open interval $(a-\delta,\ a+\delta) - \{a\}$ is called punctured $\delta$ neighbourhood of $a$. It is denoted by $N^*(a,\ \delta)$.

Here, it can be understood that

$$N^*(a,\ \delta) = N(a,\ \delta) - \{a\}$$

$$= \{x \mid a-\delta < x < a+\delta,\ x \neq a,\ x \in R\}$$

$$= \{x \mid |x-a| < \delta,\ x \neq a,\ x \in R\}$$

**e.g.** $N^*(5,\ 2) = N(5,\ 2) - \{5\}$

$$= \{x \mid 3 < x < 7,\ x \neq 5,\ x \in R\}$$

$$= \{x \mid |x-5| < 2,\ x \neq 5,\ x \in R\}$$

## EXERCISE 4.1

**1.** Express the following in modulus and interval form :

(1) 0.4 neighbourhood of 4 (2) 0.02 neighbourhood of 2

(3) 0.05 neighbourhood of 0 (4) 0.001 neighbourhood of –1

**2.** Express the following in interval and neighbourhood form :

(1) $|x-2| < 0.01$ (2) $|x+5| < 0.1$

(3) $|x| < \frac{1}{3}$ (4) $|x+3| < 0.15$

**3.** Express the following in modulus and neighbourhood form :

(1) $3.8 < x < 4.8$ (2) $1.95 < x < 2.05$

(3) $-0.4 < x < 1.4$ (4) $1.998 < x < 2.002$

**4.** Express $N(16,\ 0.5)$ in the interval and modulus form.

**5.** If $N(3,\ b) = (2.95,\ k)$ then find the values of $b$ and $k$.

**6.** If $|x-10| < k_1 = (k_2,\ 10.01)$ then find the values of $k_1$ and $k_2$.

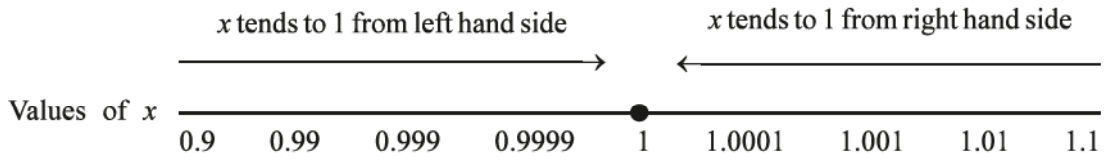
*

**Meaning of $x \to a$ :**

If the value of variable $x$ is brought very close to a number '$a$' by increasing or decreasing its value then it can be said that $x$ tends to $a$. It is denoted by $x \to a$.

It is necessary here to note that $x \to a$ means value of $x$ approaches very close to $a$ but it will not be equal to $a$.

e.g. Let us understand the meaning of $x \to 1$.

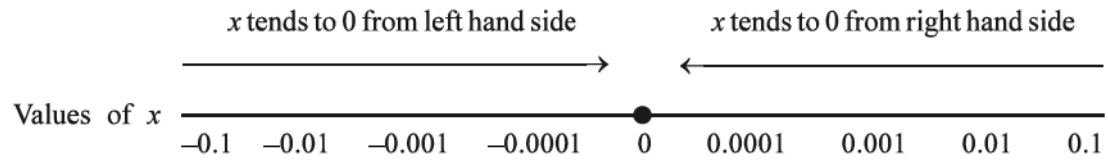



**Meaning of $x \to 0$ :**

If by decreasing the positive value of a variable $x$ or by increasing negative value of the variable $x$, the value of $x$ is brought very close to '0' then it can be said that $x$ tends to 0. It is denoted by $x \to 0$.

It is necessary here to note that $x \to 0$ means, the value of $x$ approaches very close to 0 but it will not be equal to 0.

Let us understand the meaning of $x \to 0$.




## 4.5 Limit of a function

When the value of a variable $x$ is brought closer and closer to a number '$a$', the value of function $f(x)$ reaches closer and closer to a definite number '$l$' then we can say that as $x$ tends to $a$, $f(x)$ tends to $l$. i.e. as $x \to a$, $f(x) \to l$. Symbolically it can be written as $\lim\limits_{x \to a} f(x) = l$. $l$ is called the limiting value of the function.

**Definition :** The function $f(x)$ has a limit $l$ as $x$ tends to '$a$' if for each given predetermined $\varepsilon > 0$, however small, we can find a positive number $\delta$ such that $|f(x) - l| < \varepsilon$ (*Epsilon*) for all $x$ such that $|x - a| < \delta$.

Now, we shall understand how limit of a function is obtained.

Suppose, we want to find the value of the function $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

If we put $x = 1$ in $f(x) = \frac{x^2-1}{x-1}$ we get $f(1) = \frac{0}{0}$ which is indeterminate. So, we cannot find the value of $f(1)$ but assuming value of $x$ very close to 1, we can approximate the value of $f(1)$. Let us see the changes in $f(x)$ as $x$ tends to 1.

| $x$ (towards 1 from LHS of 1) | $f(x)$ | $x$ (towards 1 from RHS of 1) | $f(x)$ |
|---|---|---|---|
| 0.9 | 1.9 | 1.1 | 2.1 |
| 0.99 | 1.99 | 1.01 | 2.01 |
| 0.999 | 1.999 | 1.001 | 2.001 |
| 0.9999 | 1.9999 | 1.0001 | 2.0001 |
| . . . | . . . | . . . | . . . |

We can assume any value of $x$ close to 1. Generally, we start with a value at a distance 0.1 on both sides of $x = 1$. i.e. we start with $x = 0.9$ and 1.1 and bring values of $x$ closer to 1 from both the sides.

It is clear from the table that when the value of $x$ is brought nearer to 1 by increasing or decreasing its values, the value of $f(x)$ approaches to 2.

This can symbollically be expressed as $\lim\limits_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Limit of a function is obtained by putting different values of $x$ in $f(x)$ and tabulating them. So, this method of obtaining the limit of a function is called a **tabular method.**

**Illustration 5 : Find $\lim\limits_{x \to 3} 2x+5$ by tabular method.**

We have $f(x) = 2x+5$. We shall take the values of $x$ very near to 3 and prepare a table in the following way :

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 2.9 | 10.8 | 3.1 | 11.2 |
| 2.99 | 10.98 | 3.01 | 11.02 |
| 2.999 | 10.998 | 3.001 | 11.002 |
| 2.9999 | 10.9998 | 3.0001 | 11.0002 |
| . . . | . . . | . . . | . . . |

It is clear from the table that when the value of $x$ is brought nearer to 3 by increasing or decreasing its values, the value of $f(x)$ approaches to 11. That is, when $x \to 3$, $f(x) \to 11$.

$\therefore \lim\limits_{x \to 3} 2x+5 = 11$

**Illustration 6 : Find $\lim\limits_{x \to -1} \frac{x^2-1}{x+1}$, $x \in R - \{-1\}$ by preparing table.**

We have $f(x) = \frac{x^2-1}{x+1}$. We shall take the values of $x$ very near to –1 and prepare a table in the following way :

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| –1.1 | –2.1 | –0.9 | –1.9 |
| –1.01 | –2.01 | –0.99 | –1.99 |
| –1.001 | –2.001 | –0.999 | –1.999 |
| –1.0001 | –2.0001 | –0.9999 | –1.9999 |
| . . . | . . . | . . . | . . . |

It is clear from the table that when the value of $x$ is brought nearer to –1 by increasing or decreasing its value, the value of $f(x)$ approaches to –2. That is, when $x \to -1$, $f(x) \to -2$.

$$\therefore \lim_{x \to -1} \frac{x^2-1}{x+1} = -2$$

**Illustration 7 : Find $\lim\limits_{x \to 0} \frac{2x^2+3x}{x}$, $x \in R - \{0\}$ using tabular method.**

We have $f(x) = \frac{2x^2+3x}{x}$. We shall take the values of $x$ very near to 0 and prepare a table in the following way :

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| –0.1 | 2.8 | 0.1 | 3.2 |
| –0.01 | 2.98 | 0.01 | 3.02 |
| –0.001 | 2.998 | 0.001 | 3.002 |
| –0.0001 | 2.9998 | 0.0001 | 3.0002 |
| . . . | . . . | . . . | . . . |

It is clear from the table that when the value of $x$ is brought nearer to 0 by increasing or decreasing its value, the value of $f(x)$ approaches to 3. That is, when $x \to 0$, $f(x) \to 3$.

$$\therefore \lim_{x \to 0} \frac{2x^2+3x}{x} = 3$$

**Illustration 8 : Find $\lim\limits_{x\to 1} \frac{1}{x-1}$, $x \in R-\{1\}$ by tabular method.**

We have $f(x) = \frac{1}{x-1}$. We shall take the values of $x$ very near to 1 and prepare a table in the following way :

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 0.9 | –10 | 1.1 | 10 |
| 0.99 | –100 | 1.01 | 100 |
| 0.999 | –1000 | 1.001 | 1000 |
| 0.9999 | –10000 | 1.0001 | 10000 |
| . . . | . . . | . . . | . . . |

It is clear from the table that when the value of $x$ is brought nearer to 1 by increasing or decreasing its value, the value of $f(x)$ does not approach to a particular value. That is, when $x \to 1$, $f(x)$ does not tend to a particular value. Thus, limit of the function does not exist.

$\therefore \lim\limits_{x\to 1} \frac{1}{x-1}$ does not exist.

**Illustration 9 : Find $\lim\limits_{x\to 2} \frac{3x^2-4x-4}{x^2-4}$, $x \in R-\{2\}$ by tabular method.**

We have $f(x) = \frac{3x^2-4x-4}{x^2-4}$. We can obtain the value of limit as calculated in previous illustrations. But for simplification we shall obtain the value of limit of $f(x)$ after eliminating the common factor $(x-2)$ from numerator and denominator.

$$\lim_{x\to 2} \frac{3x^2-4x-4}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(3x+2)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{3x+2}{x+2} \qquad (\because \ x-2 \neq 0)$$

We shall take the values of $x$ very near to 2 and prepare a table in the following way :

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1.9 | 1.9744 | 2.1 | 2.02439 |
| 1.99 | 1.9975 | 2.01 | 2.002494 |
| 1.999 | 1.9997 | 2.001 | 2.0002499 |
| 1.9999 | 1.9999 | 2.0001 | 2.000025 |
| . . . | . . . | . . . | . . . |

It is clear from the table that when the value of $x$ is brought very near to 2 by increasing or decreasing its value, the value of $f(x)$ approaches to 2. That is, when $x \to 2$, $f(x) \to 2$.

$$\therefore \lim_{x\to 2} \frac{3x^2-4x-4}{x^2-4} = 2$$

## EXERCISE 4.2

**1. Find the values of the following using tabular method :**

(1) $\lim\limits_{x \to 1} 2x+1$

(2) $\lim\limits_{x \to 3} \dfrac{x^2-2x-3}{x-3}$

(3) $\lim\limits_{x \to 2} \dfrac{2x^2+3x-14}{x-2}$

(4) $\lim\limits_{x \to -3} \dfrac{2x^2+9x+9}{x+3}$

(5) $\lim\limits_{x \to 2} x$

**2.** Using tabular method, show that $\lim\limits_{x \to 3} \dfrac{2}{x-3}$ does not exist.

**3.** If $y = \dfrac{x^2+x-6}{x-2}$, show that as $x \to 2$ then $y \to 5$ using tabular method.

**4.** If $y = 5-2x$, show that as $x \to -1$ then $y \to 7$ using tabular method.

*

## 4.6 Working rules of limit

The following rules will be accepted without proof :

If $f(x)$ and $g(x)$ are two real valued functions of a real variable $x$ and $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

(1) $\lim\limits_{x \to a} \left[f(x) \pm g(x)\right] = l \pm m$

The limit of the sum (or difference) of two functions is equal to the sum (or difference) of their limits.

(2) $\lim\limits_{x \to a} \left[f(x) \times g(x)\right] = l \times m$

The limit of the product of two functions is equal to the product of their limits.

(3) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{l}{m}, \quad m \neq 0$

The limit of the division of two functions is equal to the division of their limits, provided the limit of the function in denominator is not zero.

(4) $\lim\limits_{x \to a} k\, f(x) = kl$, $k$ is the constant.

The limit of the product of a function with a constant is equal to the product of the limit of the function with the same constant.

## 4.7 Standard forms of limit

(1) **Limit of a polynominl**

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_n x^n$ then using the working rules of limit

$$\lim_{x \to b} f(x) = a_0 + a_1 b + a_2 b^2 + \ldots.. + a_n b^n$$

(2) $\lim_{x \to a} \left[ \frac{x^n - a^n}{x - a} \right] = n a^{n-1}, \quad n \in Q$

We will see some illustrations based on the standard forms and working rules of limit.

**Illustration 10 : Find the value of $\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3}$.**

$$\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 2x + 3} = \frac{(0)^2 + 5(0) + 6}{(0)^2 + 2(0) + 3}$$

$$= \frac{6}{3}$$

$$= 2$$

**Illustration 11 : Find the value of $\lim_{x \to 2} \frac{2x + 3}{x - 1}$.**

$$\lim_{x \to 2} \frac{2x + 3}{x - 1} = \frac{2(2) + 3}{2 - 1}$$

$$= \frac{7}{1}$$

$$= 7$$

**Illustration 12 : Find the value of $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6}$.**

If we put $x = 3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate. Hence, we shall factorize numerator and denominator. Since $x \to 3$, $(x - 3)$ will be a common factor of numerator and denominator.

**Note :** If we put $x = a$ in the given function and we get $\frac{0}{0}$ then $(x - a)$ will be the common factor of numerator and denominator.

**Numerator** $= x^2 - 2x - 3$

$= x^2 - 3x + x - 3$

$= x(x - 3) + 1(x - 3)$

$= (x - 3)\ (x + 1)$

**Denominator** $= x^2 - 5x + 6$

$= x^2 - 3x - 2x + 6$

$= x(x - 3) - 2(x - 3)$

$= (x - 3)\ (x - 2)$

Now, $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{(x - 3)(x + 1)}{(x - 3)(x - 2)}$

$$= \lim_{x \to 3} \frac{(x + 1)}{(x - 2)} \qquad (\because\ x - 3 \neq 0)$$

$$= \frac{3+1}{3-2}$$

$$= \frac{4}{1}$$

$$= 4$$

**Illustration 13 : Find the value of $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x^2 - 1}$.**

If we put $x = 1$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

**Numerator** $= 2x^2 + x - 3$

$= 2x^2 + 3x - 2x - 3$

$= x(2x + 3) - 1(2x + 3)$

$= (2x + 3)\ (x - 1)$

**Denominator** $= x^2 - 1$

$= (x + 1)\ (x - 1)$

Now, $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{(2x + 3)(x - 1)}{(x + 1)(x - 1)}$

$= \lim\limits_{x \to 1} \dfrac{2x + 3}{x + 1} \qquad (\because\ x - 1 \neq 0)$

$= \dfrac{2(1) + 3}{1 + 1}$

$= \dfrac{5}{2}$

**Illustration 14 : Find the value of $\lim\limits_{x \to -3} \dfrac{2x^2 + 7x + 3}{3x^2 + 8x - 3}$.**

If we put $x = -3$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

**Numerator** $= 2x^2 + 7x + 3$

$= 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3)$

$= (x + 3)\ (2x + 1)$

**Denominator** $= 3x^2 + 8x - 3$

$= 3x^2 + 9x - x - 3$

$= 3x(x + 3) - 1(x + 3)$

$= (x + 3)\ (3x - 1)$

Now, $\lim\limits_{x \to -3} \dfrac{2x^2 + 7x + 3}{3x^2 + 8x - 3} = \lim\limits_{x \to -3} \dfrac{(x+3)(2x+1)}{(x+3)(3x-1)}$

$$= \lim_{x \to -3} \frac{2x+1}{3x-1} \qquad (\because \ x + 3 \neq 0)$$

$$= \frac{2(-3)+1}{3(-3)-1}$$

$$= \frac{-6+1}{-9-1}$$

$$= \frac{-5}{-10}$$

$$= \frac{1}{2}$$

**Illustration 15 : Find the value of $\lim\limits_{x \to -\frac{1}{2}} \dfrac{2x^2 - x - 1}{4x^2 + 8x + 3}$.**

If we put $x = -\frac{1}{2}$ in the function $f(x)$, we get the value of the function as $\frac{0}{0}$, which is indeterminate.

**Numerator** $= 2x^2 - x - 1$

$$= 2x^2 - 2x + x - 1$$

$$= 2x(x-1) + 1(x-1)$$

$$= (x-1)(2x+1)$$

**Denominator** $= 4x^2 + 8x + 3$

$$= 4x^2 + 6x + 2x + 3$$

$$= 2x(2x+3) + 1(2x+3)$$

$$= (2x+3)(2x+1)$$

Now, $\lim\limits_{x \to -\frac{1}{2}} \dfrac{2x^2 - x - 1}{4x^2 + 8x + 3} = \lim\limits_{x \to -\frac{1}{2}} \dfrac{(x-1)(2x+1)}{(2x+3)(2x+1)}$

$$= \lim_{x \to -\frac{1}{2}} \frac{x-1}{2x+3} \qquad (\because \ 2x + 1 \neq 0)$$

$$= \frac{-\frac{1}{2} - 1}{2\left(-\frac{1}{2}\right) + 3}$$

$$= \frac{-\frac{3}{2}}{-1+3}$$

$$= \frac{-\frac{3}{2}}{2}$$

$$= -\frac{3}{4}$$

**Illustration 16 : Find the value of $\lim\limits_{x \to 2} \left[ \frac{1}{x-2} - \frac{2}{x^2-2x} \right]$.**

$$\lim_{x \to 2} \left[ \frac{1}{x-2} - \frac{2}{x^2-2x} \right] = \lim_{x \to 2} \left[ \frac{1}{x-2} - \frac{2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \left[ \frac{x-2}{x(x-2)} \right]$$

$$= \lim_{x \to 2} \frac{1}{x} \qquad (\because \ x - 2 \neq 0)$$

$$= \frac{1}{2}$$

**Illustration 17 : Find the value of $\lim\limits_{x \to 0} \frac{1}{x} \left[ \frac{2x+3}{3x-5} + \frac{3}{5} \right]$.**

$$\lim_{x \to 0} \frac{1}{x} \left[ \frac{2x+3}{3x-5} + \frac{3}{5} \right] = \lim_{x \to 0} \frac{1}{x} \left[ \frac{5(2x+3) + 3(3x-5)}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[ \frac{10x + 15 + 9x - 15}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[ \frac{19x}{5(3x-5)} \right]$$

$$= \lim_{x \to 0} \frac{19}{5(3x-5)} \qquad (\because \ x \neq 0)$$

$$= \frac{19}{5[3(0)-5]}$$

$$= \frac{19}{5(-5)}$$

$$= -\frac{19}{25}$$

**Illustration 18 : If $f(x) = x^2 + x$ then find the value of $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x^2 - 4}$.**

Here, $f(x) = x^2 + x$

$$\therefore \ f(2) = (2)^2 + 2$$

$$= 4 + 2$$

$$= 6$$

Now, $\lim_{x \to 2} \frac{f(x) - f(2)}{x^2 - 4} = \lim_{x \to 2} \frac{\left(x^2 + x\right) - 6}{x^2 - 4}$

**Numerator** $= x^2 + x - 6$

$= x^2 + 3x - 2x - 6$

$= x(x + 3) - 2(x + 3)$

$= (x + 3)(x - 2)$

**Denominator** $= x^2 - 4$

$= (x + 2)(x - 2)$

So, $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x + 3)(x - 2)}{(x + 2)(x - 2)}$

$= \lim_{x \to 2} \frac{x + 3}{x + 2} \qquad (\because \ x - 2 \neq 0)$

$= \frac{2 + 3}{2 + 2}$

$= \frac{5}{4}$

**Illustration 19 : If $f(x) = x^3$ then find the value of $\lim_{h \to 0} \frac{f(3 + h) - f(3 - h)}{2h}$.**

Here, $f(x) = x^3$

$\therefore \ f(3 + h) = (3 + h)^3$

$= 27 + 27h + 9h^2 + h^3$

and

$f(3 - h) = (3 - h)^3$

$= 27 - 27h + 9h^2 - h^3$

Now, $\lim_{h \to 0} \frac{f(3 + h) - f(3 - h)}{2h} = \lim_{h \to 0} \frac{\left(27 + 27h + 9h^2 + h^3\right) - \left(27 - 27h + 9h^2 - h^3\right)}{2h}$

$= \lim_{h \to 0} \frac{27 + 27h + 9h^2 + h^3 - 27 + 27h - 9h^2 + h^3}{2h}$

$$= \lim_{h \to 0} \frac{54h + 2h^3}{2h}$$

$$= \lim_{h \to 0} \frac{h\left(54 + 2h^2\right)}{2h}$$

$$= \lim_{h \to 0} \frac{54 + 2h^2}{2} \qquad (\because \ h \neq 0)$$

$$= \frac{54 + 2(0)^2}{2}$$

$$= \frac{54}{2}$$

$$= 27$$

**Illustration 20 : Find the value of $\lim\limits_{x \to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x}$.**

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x}$$

(multiplying numerator and denominator by $\sqrt{3+x} + \sqrt{3}$ )

$$= \lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \times \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{3+x}\right)^2 - \left(\sqrt{3}\right)^2}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{3 + x - 3}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{3+x} + \sqrt{3}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(\sqrt{3+x} + \sqrt{3}\right)} \qquad (\because \ x \neq 0)$$

$$= \frac{1}{\sqrt{3+0} + \sqrt{3}}$$

$$= \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

**Illustration 21 : Find the value of $\lim\limits_{x \to 2} \dfrac{\sqrt{x+7} - 3}{x-2}$ .**

$$\lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x-2}$$

(multiplying numerator and denominator by $\sqrt{x+7} + 3$)

$$= \lim_{x \to 2} \frac{\sqrt{x+7} - 3}{x-2} \times \frac{\sqrt{x+7} + 3}{\sqrt{x+7} + 3}$$

$$= \lim_{x \to 2} \frac{\left(\sqrt{x+7}\right)^2 - (3)^2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x+7-9}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{x-2}{(x-2)\left(\sqrt{x+7} + 3\right)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x+7} + 3} \qquad (\because \ x - 2 \neq 0)$$

$$= \frac{1}{\sqrt{2+7} + 3}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

**Illustration 22 : Find the value of $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ where $f(x) = \sqrt{x}$, $x > 0$.**

$$f(x) = \sqrt{x}$$

$$\therefore \ f(x+h) = \sqrt{x+h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(multiplying numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ )

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad (\because \ h \neq 0)$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

**Illustration 23 : Find the value of $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{\sqrt{x} - \sqrt{2}}$ .**

$$\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x} - \sqrt{2}}$$

(multiplying numerator and denominator by $\sqrt{x} + \sqrt{2}$ )

$$= \lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x} - \sqrt{2}} \times \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}}$$

$$= \lim_{x \to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right)}{\left(\sqrt{x}\right)^2 - \left(\sqrt{2}\right)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right)}{(x-2)}$$

$$= \lim_{x \to 2} \left(x^2 + 2x + 4\right)\left(\sqrt{x} + \sqrt{2}\right) \qquad (\because \ x - 2 \neq 0)$$

$$= \left[(2)^2 + 2(2) + 4\right]\left[\sqrt{2} + \sqrt{2}\right]$$

$$= (4 + 4 + 4)\left(2\sqrt{2}\right)$$

$$= 12\left(2\sqrt{2}\right)$$

$$= 24\sqrt{2}$$

**Illustration 24 : Find the value of $\lim\limits_{x \to 1} \dfrac{\sqrt{x+3}-2}{\sqrt{x+8}-3}$.**

$$\lim_{x \to 1} \frac{\sqrt{x+3}-2}{\sqrt{x+8}-3}$$

(multiplying numerator and denominator by $\sqrt{x+3}+2$ and $\sqrt{x+8}+3$)

$$= \lim_{x \to 1} \frac{\sqrt{x+3}-2}{\sqrt{x+8}-3} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \times \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}$$

$$= \lim_{x \to 1} \frac{\left(\sqrt{x+3}\right)^2-(2)^2}{\left(\sqrt{x+8}\right)^2-(3)^2} \times \frac{\sqrt{x+8}+3}{\sqrt{x+3}+2}$$

$$= \lim_{x \to 1} \frac{(x+3-4) \times \left(\sqrt{x+8}+3\right)}{(x+8-9) \times \left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{(x-1)\left(\sqrt{x+8}+3\right)}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{\sqrt{x+8}+3}{\sqrt{x+3}+2} \qquad (\because \ x-1 \neq 0)$$

$$= \frac{\sqrt{1+8}+3}{\sqrt{1+3}+2}$$

$$= \frac{\sqrt{9}+3}{\sqrt{4}+2}$$

$$= \frac{3+3}{2+2}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

**Illustration 25 : Find the value of $\lim\limits_{x \to 2} \dfrac{x^5-32}{x-2}$.**

$$\lim_{x \to 2} \frac{x^5-32}{x-2} = \lim_{x \to 2} \frac{x^5-2^5}{x-2}$$

$$= 5(2)^{5-1} \qquad \left[\because \ \lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 5(2)^4$$

$$= 5(16)$$

$$= 80$$

**Illustration 26 : Find the value of $\lim\limits_{x \to 3} \dfrac{x^5 - 243}{x^3 - 27}$.**

$$\lim_{x \to 3} \frac{x^5 - 243}{x^3 - 27} = \lim_{x \to 3} \frac{x^5 - 3^5}{x^3 - 3^3}$$

(multiplying numerator and denominator by $(x - 3)$)

$$= \lim_{x \to 3} \frac{x^5 - 3^5}{x - 3} \times \frac{x - 3}{x^3 - 3^3}$$

$$= \lim_{x \to 3} \left[ \frac{x^5 - 3^5}{x - 3} \div \frac{x^3 - 3^3}{x - 3} \right]$$

$$= \frac{5(3)^{5-1}}{3(3)^{3-1}} \quad \left[ \because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= \frac{5(3)^4}{3(3)^2}$$

$$= \frac{5 \times 81}{3 \times 9}$$

$$= 15$$

**Illustration 27 : Find the value of $\lim\limits_{x \to -2} \dfrac{x^7 + 128}{x + 2}$.**

$$\lim_{x \to -2} \frac{x^7 + 128}{x + 2} = \lim_{x \to -2} \frac{x^7 - (-2)^7}{x - (-2)}$$

$$= 7(-2)^{7-1} \quad \left[ \because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= 7(-2)^6$$

$$= 7(64)$$

$$= 448$$

**Illustration 28 : Find the value of $\lim\limits_{h \to 0} \dfrac{(x + h)^5 - x^5}{h}$.**

$$\lim_{h \to 0} \frac{(x + h)^5 - x^5}{h}$$

(Taking $x + h = t$, when $h \to 0$, $t \to x$)

$$= \lim_{t \to x} \frac{t^5 - x^5}{t - x} \quad (\because x + h = t)$$

$$= 5(x)^{5-1} \quad \left[ \because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= 5x^4$$

**Illustration 29 : Find the value of $\lim\limits_{x \to 0} \dfrac{\sqrt[n]{x+1} - 1}{x}$.**

$$\lim_{x \to 0} \frac{\sqrt[n]{x+1} - 1}{x} = \lim_{x \to 0} \frac{(x+1)^{\frac{1}{n}} - 1^{\frac{1}{n}}}{x}$$

(Taking $x + 1 = t$, when $x \to 0$, $t \to 1$)

$$= \lim_{t \to 1} \frac{t^{\frac{1}{n}} - 1^{\frac{1}{n}}}{t - 1} \quad (\because x + 1 = t \quad \therefore x = t - 1)$$

$$= \frac{1}{n}(1)^{\frac{1}{n} - 1} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}\right]$$

$$= \frac{1}{n} \times 1$$

$$= \frac{1}{n}$$

**Summary**

- **Neighbourhood :** Let $a \in R$. Any open interval containing '$a$' is called a neighbourhood of '$a$'.
- **$\delta$ neighbourhood of $a$ :** If $a \in R$ and $\delta$ is a non-negative real number then the open interval $(a - \delta, a + \delta)$ is called $\delta$ neighbourhood of $a$.
- **Meaning of $x \to a$ :** If the value of a variable $x$ is brought very close to a number '$a$' by increasing or decreasing then it can be said that $x$ tends to $a$. It is symbolically denoted by $x \to a$.
- **Meaning of $x \to 0$ :** If by decreasing the positive values of a variable $x$ or by increasing negative values of a variable $x$ is brought very close to '0' then it can be said that $x$ tends to 0. It is symbolically denoted by $x \to 0$.
- **Limit of a function**

  The function $f(x)$ has limit $l$ as $x$ tends to '$a$' if for each given predetermined $\varepsilon > 0$, however small, we can find a positive number $\delta$ such that $|f(x) - l| < \varepsilon$ for all $x$ such that $|x - a| < \delta$.

**List of Formulae :**

- $\lim\limits_{x \to a} \left[ f(x) \pm g(x) \right] = l \pm m$
- $\lim\limits_{x \to a} \left[ f(x) \times g(x) \right] = l \times m$
- $\lim\limits_{x \to a} \left[ \dfrac{f(x)}{g(x)} \right] = \dfrac{l}{m}, \quad m \neq 0$
- $\lim\limits_{x \to a} k\, f(x) = kl, \quad k$ is constant.
- If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_n x^n$ then

  $\lim\limits_{x \to b} f(x) = a_0 + a_1 b + a_2 b^2 + \ldots.. + a_n b^n$
- $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n a^{n-1}, \quad n \in Q$

## EXERCISE 4

### Section A

**Choose the correct option for the following multiple choice questions :**

**1.** What is the modulus form of 0.3 neighbourhood of 3 ?

(a) $|x - 0.3| < 3$ (b) $|x - 3| < 0.3$ (c) $|x + 3| < 0.3$ (d) $|x - 3| > 0.3$

**2.** What is the interval form of 0.02 neighbourhood of –2 ?

(a) $(1.98,\ 2.02)$ (b) $(-1.98,\ 2.02)$ (c) $(-2.02,\ -1.98)$ (d) $(-2.02,\ 1.98)$

**3.** What is the interval form of $|x - 5| < 0.25$ ?

(a) $(4.75,\ 5.25)$ (b) $(-4.75,\ +5.25)$ (c) $(-5.25,\ -4.75)$ (d) $(-5.25,\ 4.75)$

**4.** What is the interval form of $|2x + 1| < \frac{1}{5}$ ?

(a) $\left(-\frac{6}{5},\ -\frac{4}{5}\right)$ (b) $\left(-\frac{6}{10},\ -\frac{4}{10}\right)$ (c) $\left(\frac{4}{10},\ \frac{6}{10}\right)$ (d) $\left(-\frac{6}{10},\ \frac{4}{10}\right)$

**5.** What is the modulus form of $N(5,\ 0.02)$ ?

(a) $|x + 5| < 0.02$ (b) $|x - 0.02| < 5$ (c) $|x - 5| > 0.02$ (d) $|x - 5| < 0.02$

**6.** If modulus form of $N(a,\ 0.07)$ is $|x - 10| < k$ then what will be the value of $k$ ?

(a) $a$ (b) 0.7 (c) 0.07 (d) 9.93

**7.** What is the value of $\lim\limits_{x \to 3} 3x - 1$ ?

(a) 9 (b) 10 (c) $\frac{4}{3}$ (d) 8

8. What is the value of $\lim_{x \to 4} \sqrt{4x+9}$ ?

(a) 5 (b) 25 (c) $\frac{7}{4}$ (d) 7

9. What is the value of $\lim_{x \to -2} 10$ ?

(a) 10 (b) –2 (c) 8 (d) Indeterminate

10. What is the value of $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$ ?

(a) 192 (b) 324 (c) 36 (d) 108

11. What is the value of $\lim_{x \to -1} \frac{x^5 + 1}{x + 1}$ ?

(a) –5 (b) 5 (c) 4 (d) –4

12. If $y = 10 - 3x$ and $x \to -3$ then $y$ tends to which value ?

(a) 1 (b) 9 (c) 19 (d) 7

## Section B

**Answer the following questions in one sentence :**

1. Express 0.09 neighbourhood of 0 in interval form.
2. Express 0.001 neighbourhood of –5 in modulus form.
3. Express $|x - 10| < \frac{1}{10}$ in neighbourhood form.
4. Express $|2x| < \frac{1}{2}$ in interval form.
5. Express $N(50, 0.8)$ in modulus form.
6. If $N(a, 0.2) = |x - 7| < b$ then find the value of $a$.
7. If $|x + 4| < 0.04 = (k, -3.96)$ then find the value of $k$.
8. Find the value of $\lim_{x \to 5} (3x + 5)$.
9. Find the value of $\lim_{x \to -3} \sqrt[3]{2 - 2x}$.
10. Find the value of $\lim_{x \to 0} \left( \frac{3x^2 - 4x + 10}{2x + 5} \right)$.
11. Find the value of $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$.
12. Find the value of $\lim_{x \to -a} \frac{x^m + a^m}{x + a}$ where $m$ is an odd number.

13. If $\lim_{x \to -1} 4x + k = 6$ then find the value of $k$.

14. If $\lim_{x \to 3} \frac{2}{3x + k} = \frac{1}{7}$ then find the value of $k$.

## Section C

**Answer the following questions :**

1. Define an open interval.
2. Define the $\delta$ neighbourhood of $a$.
3. Define the punctured $\delta$ neighbourhood of $a$.
4. Express the interval form $(-0.5, 0.5)$ in modulus form.
5. Express the interval form $(-8.75, -7.25)$ in neighbourhood form.
6. If $N(k_1, 0.5) = (19.5, k_2)$ then find the value of $k_1$ and $k_2$.
7. Express $|3x + 1| < 2$ in neighbourhood and interval form.
8. If $|x - A_1| < 0.09 = (A_2, 4.09)$ then find the value of $A_1$ and $A_2$.
9. Explain the meaning of $x \to a$.
10. Explain the meaning of $x \to 0$.
11. Define limit of a function.
12. State multiplication working rule of limit.
13. State division working rule of limit.
14. State the standard form of limit of a polynomial.

## Section D

**Find the values of the following :**

1. $\lim_{x \to 1} \frac{3x^2 - 4x + 1}{x - 1}$
2. $\lim_{x \to 3} \frac{x - 3}{2x^2 - 3x - 9}$
3. $\lim_{x \to -1} \frac{3x^2 - 2x - 5}{x + 1}$
4. $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1}$
5. $\lim_{x \to \frac{1}{2}} \frac{2x^2 + 5x - 3}{4x^2 - 1}$
6. $\lim_{x \to -3} \frac{2x^2 + 9x + 9}{2x^2 + 7x + 3}$
7. $\lim_{x \to -\frac{1}{2}} \frac{2x^2 + 3x + 1}{2x^2 - x - 1}$
8. $\lim_{x \to -2} \frac{9x^2 + 5x - 26}{5x^2 + 17x + 14}$
9. $\lim_{x \to 0} \frac{1}{x}\left[\frac{5x + 14}{3x + 7} - 2\right]$
10. $\lim_{x \to 2} \left[\frac{2}{x - 2} - \frac{4}{x^2 - 2x}\right]$

11. $\lim_{x \to 0} 1 + \frac{2}{3 + \frac{4}{x}}$

12. $\lim_{x \to -p} \frac{x^4 - p^4}{x^3 + p^3}$

13. $\lim_{x \to 3} \frac{x^6 - 729}{x^4 - 81}$

14. $\lim_{x \to -2} \frac{x^{10} - 1024}{x^5 + 32}$

15. $\lim_{x \to -1} \frac{x^{2017} + 1}{x^{2018} - 1}$

16. $\lim_{x \to 1} \frac{x^{\frac{7}{2}} - 1}{x^{\frac{3}{2}} - 1}$

17. $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

## Section E

**I. Answer the following :**

1. If $y = 5x + 7$ then using tabular method, prove that when $x \to 2$, $y \to 17$.

2. If $y = \frac{3x^2 + 16x + 16}{x + 4}$ then using tabular method, prove that when $x \to -4$, $y \to -8$.

3. Using tabular method, prove that $\lim_{x \to -1} \frac{3}{x + 1}$ does not exist.

**II. Find the values of the following using tabular method :**

1. $\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5}$

2. $\lim_{x \to 1} \frac{2x^2 + 3x - 5}{x - 1}$

3. $\lim_{x \to -1} \frac{4x^2 + 5x + 1}{x + 1}$

4. $\lim_{x \to 0} 3x - 1$

**III. Find the values of the following :**

1. $\lim_{h \to 0} \frac{(x + h)^7 - x^7}{h}$

2. $\lim_{x \to 0} \frac{\sqrt[10]{1 + x} - 1}{x}$

3. $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x}$

4. $\lim_{x \to \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{2x - 1}$ where $f(x) = x^2 + x - 1$

5. $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ where $f(x) = x^3$

6. $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ where $f(x) = x^7$

7. $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ where $f(x) = \sqrt{x + 7}$

8. $\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$ where $f(x) = 2x^2 + 3$

9. $\lim_{x \to 0} \frac{f(2 + x) - f(2 - x)}{2x}$ where $f(x) = x^2$

10. $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ where $f(x) = x^2 + x$

**Srinivasa Ramanujan**
**(1887 - 1920)**

Srinivasa Ramanujan was one of the greatest mathematical geniuses of India. He made substantial contributions to the analytical theory of numbers and worked on elliptic functions, continued fractions and infinite series. Ramanujan independently discovered results of Gauss, Kummer and others on hyper geometric series. Ramanujan initially developed his own mathematical research in isolation; it was quickly recognized by Indian mathematicians. When his skills became obvious and known to the wider mathematical community, centered in Europe at the time, he began a famous partnership with the English mathematician G. H. Hardy, who realized that Ramanujan had rediscovered previously known theorems in addition to producing new ones. On 18th February 1918, Ramanujan was elected as fellow of the Cambridge Philosophical Society. On the 125th anniversary of his birth, India declared the birthday of Ramanujan, December 22nd as 'National Mathematics Day and also declared that the year 2012 would be celebrated as the National Year of Mathematics.

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# 5

# Differentiation

**Contents :**

## 5.1 Introduction

We have studied about functions in Standard 11. Let $f(x)$ be a function of $x$. Differentiation is a technique which is used for analyzing the way in which function $f(x)$ changes and how much does it change with a change in the value of $x$. That is, we can know how rapidly a function is changing at any point using differentiation. In real life, we have functions like production cost, revenue, profit, etc. and it is often important to know how rapidly these functions change with respect to change in produced units or sold units $x$.

Consider $y = f(x) = 2x^2 + 3$, a function of $x$. If the value of independent variable ($x$) is changed there will be a corresponding change in the dependent variable ($y$). If the value of $x$ is 2 then the value of dependent variable $y$ will be 11. Now we shall find the increase in $y$ for a small increase in $x$. For a small increase in value of $x$, i.e. if we take values of $x$ as 2.1, 2.01, 2.001, 2.0001, ..... etc then we get corresponding value of $y$ as 11.82, 11.082, 11.008, 11.0008, ...., etc. We denote the increase in $x$ by $\delta_x$ and increase in $y$ by $\delta_y$. The ratio $\frac{\delta_y}{\delta_x}$ is termed as incrementary ratio. Let us observe this incrementary ratio for the above values of $x$ and corresponding values of y.

| $x$ | $\delta_x$ | $y = f(x)$ | $\delta_y$ | $\frac{\delta_y}{\delta_x}$ |
|---|---|---|---|---|
| 2.1 | 0.1 | 11.82 | 0.82 | 8.2 |
| 2.01 | 0.01 | 11.0802 | 0.0802 | 8.02 |
| 2.001 | 0.001 | 11.0080 | 0.0080 | 8.002 |
| 2.0001 | 0.0001 | 11.0008 | 0.0008 | 8.0002 |
| . . . | . . . | . . . | . . . | . . . |

We make the following observations from the above table :

(i) $\delta_y$ varies when $\delta_x$ varies

(ii) $\delta_y \to 0$ when $\delta_x \to 0$

(iii) The ratio $\frac{\delta_y}{\delta_x}$ tends to 8.

Hence, this example illustrates that $\delta_y \to 0$ when $\delta_x \to 0$ but $\frac{\delta_y}{\delta_x}$ tends to a finite value, not necessarily zero. The limit of $\frac{\delta_y}{\delta_x}$ is represented by $\frac{dy}{dx}$ and is called the derivative of $y$ with respect to $x$.

In the above example $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta_y}{\delta_x} = 8$

In many business problems, we are interested in the rate of change of a function and, in particular, the range of values of independent variable for which the rate of change of a function may be positive or negative.

Differentiation is used in production, replacement, pricing and other management decision problems.

In short, differentiation is used to determine the rate of change in the dependent variable (function of independent variable) with respect to the independent variable.

## 5.2 Definition : Differentiation and Derivative

Let us consider a function $y = f(x)$.

When we take $x = a$, the value of the function will be $f(a)$. Now, when the value of $x$ changes from $a$ to $a + h$, the value of the function will change from $f(a)$ to $f(a + h)$. So, for a change of $(a + h) - a = h$ in the value of $x$, there is a change of $f(a + h) - f(a)$ in the value of $f(x)$. If there is a change of $h$ in value of $a$ then the relative change in the function will be $\frac{f(a+h) - f(a)}{h}$. If $h$ is made very small then the limit of this relative change is called derivative of $f(x)$ with respect to $x$ at $x = a$ and it is denoted by $f'(a)$.

**Definition :** Let $f : A \to R$ and $a \in A$, where $A$ is an open interval of $R$. If $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists, then this limit of a function $f$ is called **derivative** at $x = a$. It is denoted by $f'(a)$.

The process of obtaining derivative of a function is called **differentiation**.

Thus, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

For any value of $x$ of the domain of $f$, we have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. $f'(x)$ is called a derivative of $f(x)$ with respect to $x$.

If $y$ is a function of $x$ then its derivative is denoted by $\frac{dy}{dx}$.

We shall now find derivatives of some functions using this definition of derivative.

**Illustration 1 : Obtain the derivative of $f(x) = x$ with the help of definition.**

Here, $f(x) = x$

$\therefore \; f(x + h) = x + h$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)-x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1 \qquad (\because \ h \neq 0)$$

Hence, if $f(x) = x$ then $f'(x) = 1$.

**Illustration 2 : Obtain derivative of $f(x) = x^3$ with the help of definition.**

Here, $f(x) = x^3$

$$\therefore \ f(x+h) = (x+h)^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 \qquad (\because \ h \neq 0)$$

$$= 3x^2 + 3x(0) + (0)^2$$

$$= 3x^2$$

Hence, if $f(x) = x^3$ then $f'(x) = 3x^2$

**Illustration 3 : Obtain derivative of $f(x) = x^n$ with the help of definition.**

Here, $f(x) = x^n$

$$\therefore \ f(x+h) = (x+h)^n$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

(Taking $x + h = t$, when $h \to 0$ then $t \to x$)

$$= \lim_{t \to x} \frac{t^n - x^n}{t - x} \qquad (\because \ x + h = t)$$

$$= nx^{n-1} \quad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

Hence, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

**Illustration 4 : Obtain derivative of $f(x) = \sqrt{x}$ with the help of definition.**

Here, $f(x) = \sqrt{x}$

$\therefore \ f(x+h) = \sqrt{x+h}$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(Multiplying numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ )

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad (\because \ h \neq 0)$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Hence, if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$

**Illustration 5 : Obtain derivative of $f(x) = \frac{1}{x}$ with the help of definition.**

Here, $f(x) = \frac{1}{x}$

$$\therefore \quad f(x+h) = \frac{1}{x+h}$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{x - x - h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} \qquad (\because \ h \neq 0)$$

$$= \frac{-1}{x(x+0)}$$

$$= \frac{-1}{x^2}$$

Hence, if $f(x) = \frac{1}{x}$ then $f'(x) = \frac{-1}{x^2}$

**Illustration 6 : Obtain derivative of $f(x) = k$ ($k$ is constant) with the help of definition.**

Here, $f(x) = k$

$$\therefore \quad f(x+h) = k$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= 0$$

Hence, if $f(x) = k$ then $f'(x) = 0$

## EXERCISE 5.1

**Obtain the derivatives of the following functions with the help of definition :**

1. $f(x) = 2x + 3$
2. $f(x) = x^2$
3. $f(x) = x^7$
4. $f(x) = \frac{1}{x+1}, \quad x \neq -1$
5. $f(x) = \sqrt[3]{x}$
6. $f(x) = \frac{2}{3x-4}, \quad x \neq \frac{4}{3}$
7. $f(x) = 10$

*

## 5.3 Some Standard Derivatives

We shall use derivatives of following functions.

**1.** If $y = x^n$ (where $n \in R$ and $x \in R^+$)

then $\frac{dy}{dx} = nx^{n-1}$

**2.** If $y = k$ (where $k$ is constant)

then $\frac{dy}{dx} = 0$

## 5.4 Working Rules for Differentiation

We shall accept certain working rules for differentiation without proof.

If $u$ and $v$ are differentiable functions of $x$ then,

**Rule 1 :** If $y = u \pm v$ then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

**Rule 2 :** If $y = u \cdot v$ then

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

**Rule 3 :** If $y = \frac{u}{v}, \quad v \neq 0$ then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

**Rule 4 :** **(Chain Rule)**

If $y$ is a differentiable function of $u$ and $u$ is a differentiable function of $x$ then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

We shall see some illustrations explaining the use of working rules for differentiation mentioned above.

**Illustration 7 : Find $\frac{dy}{dx}$ for $y = x^4 - 3x^2 + 2x - 3$.**

$$y = x^4 - 3x^2 + 2x - 3$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 3x^2 + 2x - 3\right)$$

$$= \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}(2x) - \frac{d}{dx}(3)$$

$$= \frac{d}{dx}\left(x^4\right) - 3\frac{d}{dx}\left(x^2\right) + 2\frac{d}{dx}(x) - \frac{d}{dx}(3)$$

$$= 4x^3 - 3(2x) + 2(1) - (0)$$

$$= 4x^3 - 6x + 2$$

**Illustration 8 : Find $\frac{dy}{dx}$ for $y = x^3 + \sqrt{x} - \frac{4}{x} + \frac{1}{\sqrt[3]{x}} + \frac{1}{4}$.**

$$y = x^3 + \sqrt{x} - \frac{4}{x} + \frac{1}{\sqrt[3]{x}} + \frac{1}{4}$$

$$= x^3 + x^{\frac{1}{2}} - 4x^{-1} + x^{-\frac{1}{3}} + \frac{1}{4}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 4\frac{d}{dx}\left(x^{-1}\right) + \frac{d}{dx}\left(x^{-\frac{1}{3}}\right) + \frac{d}{dx}\left(\frac{1}{4}\right)$$

$$= 3x^2 + \frac{1}{2}x^{\frac{1}{2}-1} - 4\left(-1\,x^{-1-1}\right) + \left(\frac{-1}{3}\right)x^{\frac{-1}{3}-1} + 0$$

$$= 3x^2 + \frac{1}{2}x^{-\frac{1}{2}} + 4x^{-2} - \frac{1}{3}x^{-\frac{4}{3}}$$

$$= 3x^2 + \frac{1}{2x^{\frac{1}{2}}} + \frac{4}{x^2} - \frac{1}{3x^{\frac{4}{3}}}$$

**Illustration 9 : If $y = \left(2x^2 + 3\right)(3x - 2)$ then find derivative of $y$ with respect to $x$.**

$$y = \left(2x^2 + 3\right)(3x - 2)$$

Take, $u = 2x^2 + 3$ and $v = 3x - 2$.

$$\therefore \quad \frac{du}{dx} = 4x \quad \text{and} \quad \frac{dv}{dx} = 3$$

Now, $y = u \cdot v$.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \left(2x^2 + 3\right)(3) + (3x - 2)(4x)$$

$$= 6x^2 + 9 + 12x^2 - 8x$$

$$= 18x^2 - 8x + 9$$

**Note :** Illustration 9 can also be solved using working rule 1 by simplifying $y$ i.e. multiplying two terms of $y$.

**Illustration 10 : Find $\frac{dy}{dx}$, $y = \frac{2x+3}{3x-2}$.**

$$y = \frac{2x+3}{3x-2}$$

Take $u = 2x + 3$ and $v = 3x - 2$.

$$\therefore \quad \frac{du}{dx} = 2 \text{ and } \frac{dv}{dx} = 3$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(3x-2)(2) - (2x+3)(3)}{(3x-2)^2}$$

$$= \frac{(6x-4) - (6x+9)}{(3x-2)^2}$$

$$= \frac{6x - 4 - 6x - 9}{(3x-2)^2}$$

$$= \frac{-13}{(3x-2)^2}$$

**Illustration 11 : If $y = \frac{3}{4x+5}$ then differentiate $y$ with respect to $x$.**

$$y = \frac{3}{4x+5}$$

Take $u = 3$ and $v = 4x + 5$.

$$\therefore \quad \frac{du}{dx} = 0 \text{ and } \frac{dv}{dx} = 4$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(4x+5)(0) - 3(4)}{(4x+5)^2}$$

$$= \frac{0 - 12}{(4x+5)^2}$$

$$= \frac{-12}{(4x+5)^2}$$

**Illustration 12 : If $y = \dfrac{2x^2 + 3x + 4}{x^2 + 5}$ then find $\dfrac{dy}{dx}$.**

$$y = \frac{2x^2 + 3x + 4}{x^2 + 5}$$

Take $u = 2x^2 + 3x + 4$ and $v = x^2 + 5$.

$$\therefore \quad \frac{du}{dx} = 4x + 3 \text{ and } \frac{dv}{dx} = 2x$$

Now, $y = \dfrac{u}{v}$.

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\left(x^2 + 5\right)(4x + 3) - \left(2x^2 + 3x + 4\right)(2x)}{\left(x^2 + 5\right)^2}$$

$$= \frac{\left(4x^3 + 20x + 3x^2 + 15\right) - \left(4x^3 + 6x^2 + 8x\right)}{\left(x^2 + 5\right)^2}$$

$$= \frac{4x^3 + 20x + 3x^2 + 15 - 4x^3 - 6x^2 - 8x}{\left(x^2 + 5\right)^2}$$

$$= \frac{-3x^2 + 12x + 15}{\left(x^2 + 5\right)^2}$$

**Illustration 13 : Differentiate $y = (3x + 7)^8$ with respect to $x$.**

$$y = (3x + 7)^8$$

Taking $u = 3x + 7$, $y = u^8$

$$\therefore \quad \frac{du}{dx} = 3 \text{ and } \frac{dy}{du} = 8u^7$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$= \left(8u^7\right)(3)$$

$$= 24u^7$$

Putting value of $u$,

$$\frac{dy}{dx} = 24(3x + 7)^7$$

**Illustration 14 : Find $\frac{dy}{dx}$, $y = \sqrt{x^2 + 3}$.**

$$y = \sqrt{x^2 + 3}$$

Taking $u = x^2 + 3$, $y = \sqrt{u}$

$$\therefore \quad \frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = \frac{1}{2\sqrt{u}}.$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \left(\frac{1}{2\sqrt{u}}\right)(2x)$$

$$= \frac{x}{\sqrt{u}}$$

Putting value of $u$,

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 3}}$$

**Illustration 15 : Obtain derivative of $y = 1 + \frac{2}{3 + \frac{4}{x}}$ with respect to $x$.**

$$y = 1 + \frac{2}{3 + \frac{4}{x}}$$

$$= 1 + \frac{2x}{3x + 4}$$

$$= \frac{(3x + 4) + 2x}{3x + 4}$$

$$\therefore \quad y = \frac{5x + 4}{3x + 4}$$

Here, take $u = 5x + 4$ and $v = 3x + 4$

$$\therefore \quad \frac{du}{dx} = 5 \text{ and } \frac{dv}{dx} = 3$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(3x + 4)(5) - (5x + 4)(3)}{(3x + 4)^2}$$

$$= \frac{(15x + 20) - (15x + 12)}{(3x + 4)^2}$$

$$= \frac{15x + 20 - 15x - 12}{(3x+4)^2}$$

$$= \frac{8}{(3x+4)^2}$$

**Illustration 16 : If $2xy + 3x + y - 4 = 0$ then find $\frac{dy}{dx}$.**

$$2xy + 3x + y - 4 = 0$$

$$\therefore \quad 2xy + y = 4 - 3x$$

$$\therefore \quad y(2x+1) = 4 - 3x$$

$$\therefore \quad y = \frac{4-3x}{2x+1}$$

Here, take $u = 4 - 3x$ and $v = 2x + 1$.

$$\therefore \quad \frac{du}{dx} = -3 \text{ and } \frac{dv}{dx} = 2$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2x+1)(-3) - (4-3x)(2)}{(2x+1)^2}$$

$$= \frac{-6x - 3 - 8 + 6x}{(2x+1)^2}$$

$$= \frac{-11}{(2x+1)^2}$$

**Illustration 17 : If $y = 2 + 3x + 4x^2 + \frac{5}{6-7x}$ then find $\frac{dy}{dx}$.**

$$y = 2 + 3x + 4x^2 + \frac{5}{6-7x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[2 + 3x + 4x^2 + \frac{5}{6-7x}\right]$$

$$= 0 + 3(1) + 4(2x) + \frac{d}{dx}\left(\frac{5}{6-7x}\right)$$

$$= 3 + 8x + \frac{(6-7x)(0) - 5(-7)}{(6-7x)^2} \qquad [\because \text{ Division rule}]$$

$$= 3 + 8x + \frac{35}{(6-7x)^2}$$

**Illustration 18 : If** $y = \left(x + \frac{6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$ **then find** $\frac{dy}{dx}$.

$$y = \left(x + \frac{6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \left[\frac{x(x+5)+6}{x+5}\right]\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \left(\frac{x^2+5x+6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \frac{3x+2}{x+5}$$

Here, take $u = 3x + 2$ and $v = x + 5$.

$$\therefore \quad \frac{du}{dx} = 3 \text{ and } \frac{dv}{dx} = 1$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x+5)(3) - (3x+2)(1)}{(x+5)^2}$$

$$= \frac{(3x+15) - (3x+2)}{(x+5)^2}$$

$$= \frac{3x+15-3x-2}{(x+5)^2}$$

$$= \frac{13}{(x+5)^2}$$

**Illustration 19 : If** $f(x) = 3x^2 + 2x + 1$ **then find** $f'(x)$ **and hence obtain** $f'(1)$.

Here, $f(x) = 3x^2 + 2x + 1$

$$\therefore \quad f'(x) = 6x + 2$$

$$\therefore \quad f'(1) = 6(1) + 2$$

$$= 8$$

**Illustration 20 : If $f(x) = x^2 - x + 3$ then for which value of $x$, $f'(x) = 0$ ?**

Here, $f(x) = x^2 - x + 3$

$\therefore \; f'(x) = 2x - 1 + 0$

Now, $f'(x) = 0$ is given

$\therefore \; 2x - 1 = 0$

$\therefore \; 2x = 1$

$\therefore \; x = \frac{1}{2}$

## 5.5 Second Order Differentiation

As seen in many of the previous illustrations, the derivative of a function of $x$ is generally also a function of $x$. The derivative of $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$. This derivative is called the first order derivative of the function. The second order derivative of the function means the derivative of the first order derivative of the function. It is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. Second order derivative along with the first order derivative can be useful in maximization or minimization of a function. This can be applied to minimize cost function, maximize revenue function and maximize profit function.

We shall now see the method of obtaining second order derivative with few illustrations.

**Illustration 21 : Obtain $\frac{dy}{dx}$ for $y = 3x^4 - 2x^3 + x^2 - 8x + 7$. Also obtain its value at $x = 1$.**

$$y = 3x^4 - 2x^3 + x^2 - 8x + 7$$

$$\therefore \; \frac{dy}{dx} = 12x^3 - 6x^2 + 2x - 8$$

$$\therefore \; \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[12x^3 - 6x^2 + 2x - 8\right]$$

$$= 36x^2 - 12x + 2$$

**Putting $x = 1$,**

$$\frac{d^2y}{dx^2} = 36(1)^2 - 12(1) + 2$$

$$= 36 - 12 + 2$$

$$= 26$$

**Illustration 22 : If $f(x) = 4x^3 + 2x^2 + 7x + 9$ then for which value of $x$, $f''(x) = 52$ ?**

$$f(x) = 4x^3 + 2x^2 + 7x + 9$$

$$\therefore\ f'(x) = 12x^2 + 4x + 7$$

$$\therefore\ f''(x) = 24x + 4$$

Now, $f''(x) = 52$

$$\therefore\ 24x + 4 = 52$$

$$\therefore\ 24x = 48$$

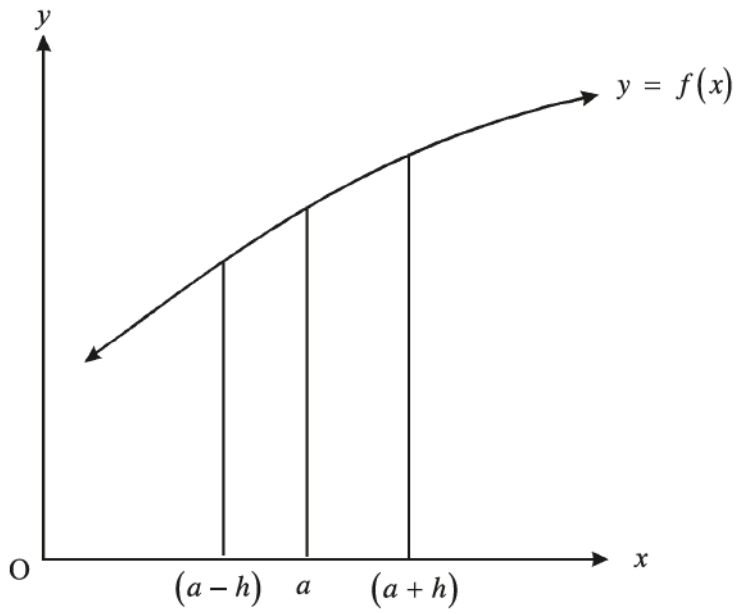
$$\therefore\ x = 2$$

## 5.6 Increasing Function and Decreasing Function

**Increasing function**

In the adjacent figure, the curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If $h$ is a very small positive number and if $f(a+h) > f(a)$ and also $f(a) > f(a-h)$ then $f(x)$ is said to be an increasing function at $x = a$.

**If the function is increasing at $x = a$ then $f'(a) > 0$**




**Decreasing function**

In the adjacent figure, the curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If $h$ is a very small positive number and if $f(a+h) < f(a)$ and also $f(a) < f(a-h)$, then $f(x)$ is said to be a decreasing function of $x = a$.

**If the function is decreasing at $x = a$ then $f'(a) < 0$**

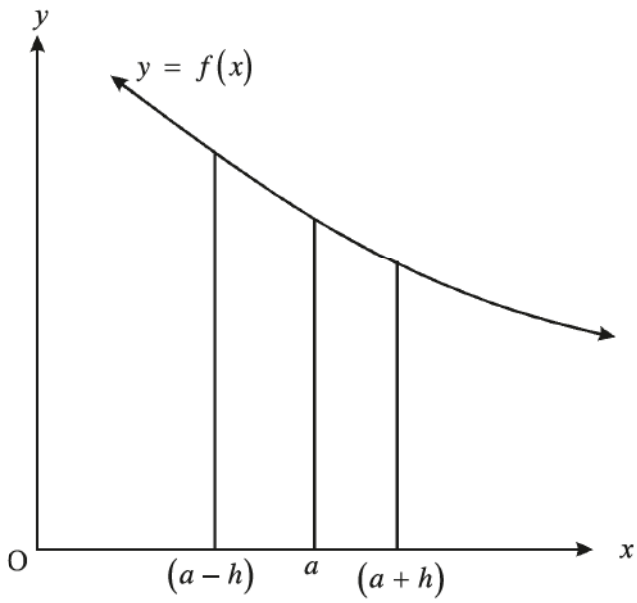


**Illustration 23 : If $f(x) = x^2 - 4x$ then decide whether the function is increasing or decreasing at $x = -1$, $x = 0$ and $x = 3$.**

$$f(x) = x^2 - 4x$$

$$\therefore \quad f'(x) = 2x - 4$$

**At $x = -1$**

$$f'(-1) = 2(-1) - 4$$

$$= -6 < 0$$

$\therefore$ Function is decreasing at $x = -1$.

**At $x = 0$**

$$f'(0) = 2(0) - 4$$

$$= -4 < 0$$

$\therefore$ Function is decreasing at $x = 0$.

**At $x = 3$**

$$f'(3) = 2(3) - 4$$

$$= 2 > 0$$

$\therefore$ Function is increasing at $x = 3$.

**Illustration 24 : Decide whether the function $y = x^3 - 3x^2 + 7$ is increasing or decreasing at $x = 1$ and $x = 3$.**

$$y = x^3 - 3x^2 + 7$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x$$

**At $x = 1$**

$$\frac{dy}{dx} = 3(1)^2 - 6(1)$$

$$= 3 - 6$$

$$= -3 < 0$$

$\therefore$ Function is decreasing at $x = 1$.

**At $x = 3$**

$$\frac{dy}{dx} = 3(3)^2 - 6(3)$$

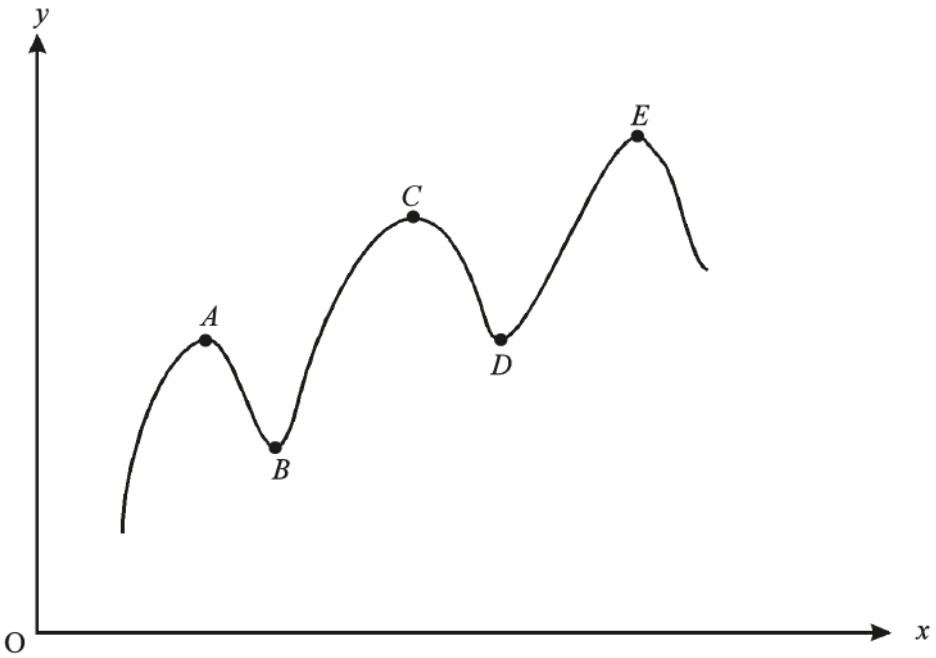
$$= 27 - 18$$

$$= 9 > 0$$

$\therefore$ Function is increasing at $x = 3$.

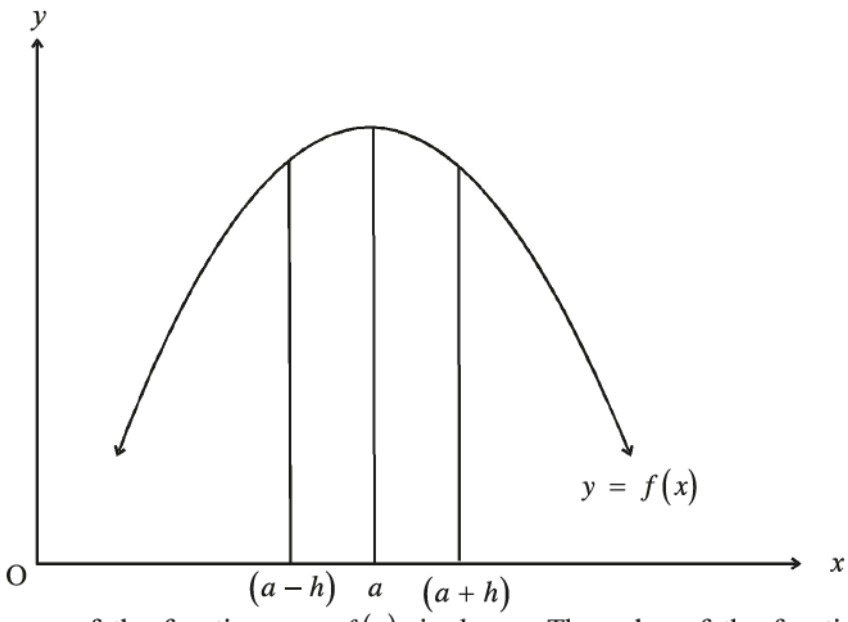
## 5.7 Maximum and Minimum Values of a Function

We discussed about increasing and decreasing function. Now, we shall study the method of obtaining maximum and minimum value of a function. Suppose the graph of a function $y = f(x)$ is obtained as follows.




It can be seen that the curve obtains maximum values at points *A, C* and *E* while its values are minimum at points *B* and *D*. Thus, the function may have more than one maximum or minimum values.

**Maximum Value :**

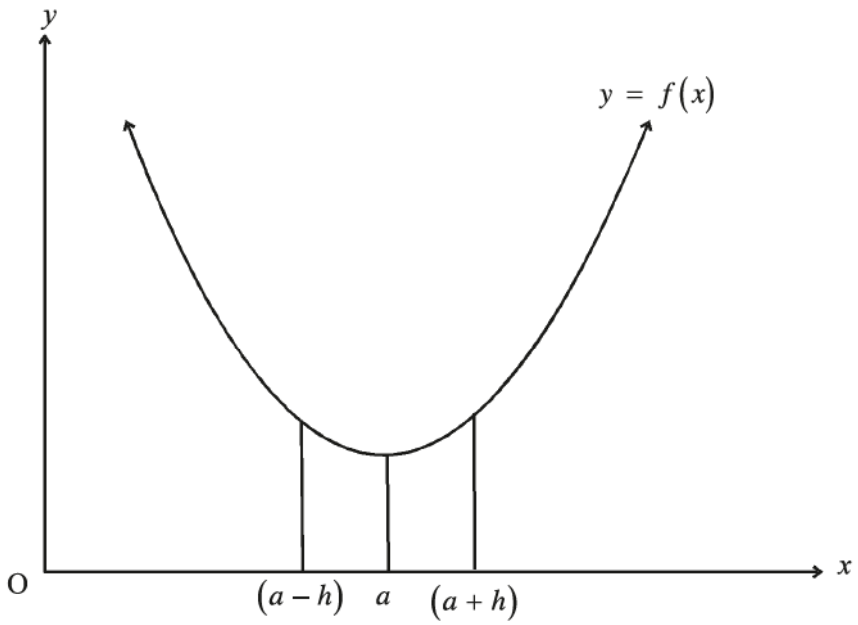



In the figure, curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If $h$ is a small positive number and if $f(a) > f(a+h)$ and also $f(a) > f(a-h)$ then $f(x)$ is said to be maximum at $x = a$.

The necessary and sufficient conditions for a function to be maximum at $x = a$ are as follows :

(i) $f'(a) = 0$ (ii) $f''(a) < 0$

**Minimum Value :**




In the figure, curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If $h$ is a small positive number and if $f(a) < f(a+h)$ and $f(a) < f(a-h)$ then $f(x)$ is said to be minimum at $x = a$.

The necessary and sufficient conditions for a function to be minimum at $x = a$ are as follows :

(i) $f'(a) = 0$ (ii) $f''(a) > 0$

The maximum and minimum values of a function are known as stationary maximum and stationary minimum values of function.

Maximum or minimum values do not mean the largest or the smallest value of a function. The function is maximum of $x = a$ only means that the value of the function is maximum in a small interval around $x = a$. Similarly, the function is minimum at $x = b$ only means that the value of the function is minimum in a small interval around $x = b$. The points where maximum or minimum values occur are known as **stationary points**. The necessary condition to obtain a stationary value is $\frac{dy}{dx} = 0$.

**Method of obtaining maximum and minimum values of a function :**

- Find the first derivative $\frac{dy}{dx} = f'(x)$ of the given function.
- Putting $\frac{dy}{dx} = 0$, solve the equation and obtain the values of $x$. These values of $x$ give the stationary points.
- Find the second order derivative and put these values of $x$ alternatively in the second derivative.
- The value of $x$ at the stationary points for which the second order derivative is negative gives the maximum value of the function while the value of $x$ at the stationary points for which the second order derivative is positive gives the minimum value of the function.
- The maximum and minimum values of a function are obtained by putting these values of $x$ in the given function.

We shall now see the method of obtaining the maximum and minimum values of a function with a few illustrations.

**Illustration 25 : Find the maximum and minimum values of $f(x) = 2x^3 + 3x^2 - 12x - 4$.**

Here, $f(x) = 2x^3 + 3x^2 - 12x - 4$

$\therefore \quad f'(x) = 6x^2 + 6x - 12$

For stationary values, $f'(x) = 0$

$\therefore \quad 6x^2 + 6x - 12 = 0$

$\therefore \quad x^2 + x - 2 = 0$

$\therefore \quad (x+2)(x-1) = 0$

$\therefore \quad x = -2$ or $x = 1$

Now, $f''(x) = 12x + 6$

**At $x = -2$**

$$f''(-2) = 12(-2) + 6$$
$$= -18 < 0$$

$\therefore$ We get the maximum value of the function at $x = -2$.

**At $x = 1$**

$$f''(1) = 12(1) + 6$$
$$= 18 > 0$$

$\therefore$ We get the minimum value of the function at $x = 1$.

**Minimum value of $f(x)$**

Putting $x = 1$ in the function $f(x)$,

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) - 4$$
$$= 2 + 3 - 12 - 4$$
$$= -11$$

**Maximum value of $f(x)$**

Putting $x = -2$ in the function $f(x)$,

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 4$$
$$= -16 + 12 + 24 - 4$$
$$= 16$$

Thus, the maximum value of $f(x)$ is 16 and the minimum value is $-11$.

**Illustration 26 : Find the maximum and minimum values of $y = x^3 - 2x^2 - 4x - 1$.**

Here, $y = x^3 - 2x^2 - 4x - 1$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 4x - 4$$

For stationary values, $\frac{dy}{dx} = 0$

$$\therefore \quad 3x^2 - 4x - 4 = 0$$

$$\therefore \quad 3x^2 - 6x + 2x - 4 = 0$$

$$\therefore \quad 3x(x-2) + 2(x-2) = 0$$

$$\therefore \quad (x-2)(3x+2) = 0$$

$$\therefore \quad x = 2 \text{ or } x = -\frac{2}{3}$$

Now, $\frac{d^2y}{dx^2} = 6x - 4$

**At $x = 2$**

$$\frac{d^2y}{dx^2} = 6(2) - 4$$

$$= 8 > 0$$

$\therefore$ Function is minimum at $x = 2$.

**At $x = -\frac{2}{3}$**

$$\frac{d^2y}{dx^2} = 6\left(\frac{-2}{3}\right) - 4$$

$$= -4 - 4$$

$$= -8 < 0$$

$\therefore$ Function is maximum at $x = -\frac{2}{3}$.

**Minimum value of function $y$**

Putting $x = 2$ in the function $y$,

$$y = (2)^3 - 2(2)^2 - 4(2) - 1$$

$$= 8 - 8 - 8 - 1$$

$$= -9$$

**Maximum value of function $y$**

Putting $x = -\frac{2}{3}$ in the function $y$,

$$y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) - 1$$

$$= \frac{-8}{27} - \frac{8}{9} + \frac{8}{3} - 1$$

$$= \frac{13}{27}$$

Thus, the maximum value of $y$ is $\frac{13}{27}$ and the minimum value is $-9$.

## 5.8 Marginal Income and Marginal Cost

The differentiation is used to obtain solutions of economic and business problems. We have seen that the first and second order derivatives can be used to obtain the maximum and minimum values of a function.

First order derivative of a function can also be used to obtain marginal income and marginal cost.

In study of economics, the relation between price and demand of a commodity can be represented as a function. If we denote the price of a commodity by $p$ and its demand by $x$ then, we get the relation $x = f(p)$, which is called the **demand function**. If the income or revenue obtained by selling $x$ units of a commodity is denoted by $R$ then,

$$R = xp$$

Thus, revenue $R$ is a function of demand $x$.

The change in revenue due to small change in demand is called **marginal revenue**.

Marginal revenue can be obtained by taking the derivative of revenue function with respect to $x$. Thus, when the demand is $x$ then

$$\text{Marginal revenue} = \frac{dR}{dx}$$

If we denote the cost of producing $x$ units by $C$ then $C$ can also be represented as function of $x$.

The change in cost due to small change in production is called **marginal cost**.

Marginal cost can be obtained by taking the derivative of cost function with respect to $x$. Thus, when the production is $x$ then

$$\text{Marginal cost} = \frac{dC}{dx}$$

**Illustration 27 : If the demand function of pizza is $p = 150 - 4x$ then find the marginal revenue when demand is of 3 pizzas and interpret it.**

Here, demand function $p = 150 - 4x$

Now, revenue function $R = p \cdot x$

$$= (150 - 4x)\,x$$

$$\therefore \quad R = 150x - 4x^2$$

Marginal revenue $\frac{dR}{dx} = 150 - 8x$

When demand of pizza is $x = 3$ then

Marginal revenue $\frac{dR}{dx} = 150 - 8(3)$

$$= 126$$

**Interpretation :** Revenue for selling the 4th pizza is approximately ₹ 126.

**Illustration 28 : If the demand function of a commodity is $x = \frac{50 - p}{2}$ then find the marginal revenue when price is ₹ 30.**

Demand function $x = \frac{50 - p}{2}$

$\therefore \quad 2x = 50 - p$

$\therefore \quad p = 50 - 2x$

Now, revenue function $R = p \cdot x$

$= (50 - 2x)x$

$\therefore \quad R = 50x - 2x^2$

Marginal revenue $\frac{dR}{dx} = 50 - 4x$

When price $p = 30$ then

$x = \frac{50 - 30}{2}$

$\therefore \quad x = 10$

When demand $x = 10$ then

Marginal Revenue $= \frac{dR}{dx} = 50 - 4(10)$

$= 10$

**Interpretation :** Revenue for selling the 11th unit is approximately ₹ 10.

**Illustration 29 : The cost function of a commodity is $C = 5x^2 + 6x + 2000$, where $x$ is the number of units produced. Find marginal cost when production is 50 units.**

Cost function $C = 5x^2 + 6x + 2000$

$\therefore$ Marginal Cost $\frac{dC}{dx} = 10x + 6$

When $x = 50$ then

Marginal Cost $\frac{dC}{dx} = 10(50) + 6$

$= 506$

**Interpretation :** The cost of producing the 51st unit is approximately ₹ 506.

## 5.9 Elasticity of Demand

Generally, a change in price of a commodity results in change in its demand in opposite direction. When the price of a commodity increases, its demand decreases and when the price of a commodity decreases, its demand increases. But these changes are not equal for all the commodities. For example, a sudden increase in price of luxury commodities results in a major decrease in its demand. While increase in the price of necessary commodities does not result in a major decrease in its demand. The change in demand for a commodity due to change in its price can be studied using elasticity of demand.

**Definition :** The ratio of percentage change in the demand of a commodity due to percentage change in its price is called elasticity of demand.

i.e.

$$\text{Elasticity of demand} = -\frac{\text{Percentage change in demand}}{\text{Percentage change in price}}$$

The ratio is negative as the changes in price and demand of a commodity is in opposite direction. For convenience, the value of elasticity of demand is taken positive and hence the negative sign is taken in the formula. If we denote the demand as $x$ and price as $p$ and the demand function $x = f(p)$ is given then

$$\text{Elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}.$$

**Illustration 30 : The demand function of a commodity is $x = 50 - 4p$. Find elasticity of demand when price is $p = 5$ and interpret it.**

Demand function $x = 50 - 4p$

$$\therefore \quad \frac{dx}{dp} = 0 - 4(1)$$

$$= -4$$

$$\text{Now, elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-p}{(50 - 4p)} \times (-4)$$

$$= \frac{4p}{50 - 4p}$$

When price $p = 5$ then

$$\text{Elasticity of demand} = \frac{4(5)}{50 - 4(5)}$$

$$= \frac{20}{50 - 20}$$

$$= \frac{20}{30}$$

$$= 0.67$$

**Interpretation :** When the price changes by 1 percent, demand changes by 0.67 percent (in opposite direction) when the price is 5.

**Illustration 31 : The demand function of a commodity is $p = 12 - \sqrt{x}$. Find the elasticity of demand when the price is 9 units and interpret it.**

Demand function $p = 12 - \sqrt{x}$

$$\therefore \quad \frac{dp}{dx} = 0 - \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}$$

$$\therefore \quad \frac{dx}{dp} = -2\sqrt{x} \qquad \left[\because \frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}\right]$$

$$\text{Now, elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-\left(12 - \sqrt{x}\right)}{x} \times \left(-2\sqrt{x}\right)$$

$$= \frac{\left(12 - \sqrt{x}\right)\left(2\sqrt{x}\right)}{x}$$

When demand is 9 units then

$$\text{Elasticity of demand} = \frac{\left(12 - \sqrt{9}\right)\left(2\sqrt{9}\right)}{9}$$

$$= \frac{(12 - 3)(2 \times 3)}{9}$$

$$= \frac{9 \times 6}{9}$$

$$= 6$$

**Interpretation :** When price changes by 1 percent, demand changes by 6 percent (in opposite direction) when demand is 9 units.

## 5.10 Minimization of cost function and maximization of Revenue function and Profit function

In practice, problems of minimizing the production cost of an item, maximizing the revenue by selling produced items and maximizing profits are to be solved. We know that the production cost $C$ or revenue $R$ by selling produced items and profit $P$ can be represented as functions of $x$. Using the derivatives, we can decide when it will be maximum or minimum.

The conditions for minimizing the production cost function $C$ are

$$\frac{dC}{dx} = 0 \text{ and } \frac{d^2C}{dx^2} > 0.$$

Similarly, the conditions for maximizing the revenue function $R$ are

$$\frac{dR}{dx} = 0 \text{ and } \frac{d^2R}{dx^2} < 0.$$

And conditions for maximizing the profit function $P$ are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0.$$

We shall now see the method of obtaining minimum cost, maximum revenue and maximum profit with few illustrations.

**Illustration 32 : The daily cost of production for $x$ tons of a commodity is $10x^2 - 1000x + 50000$. How many units should be produced for the minimum cost ? Also find the minimum cost.**

Production cost function $C = 10x^2 - 1000x + 50000$

$$\therefore \quad \frac{dC}{dx} = 20x - 1000$$

Putting $\frac{dC}{dx} = 0$,

$$20x - 1000 = 0$$

$$\therefore \quad 20x = 1000$$

$$\therefore \quad x = 50$$

Now $\frac{d^2C}{dx^2} = 20$

Here, putting $x = 50$ in $\frac{d^2C}{dx^2}$,

$$\frac{d^2C}{dx^2} = 20 > 0$$

$\therefore$ Production cost is minimum at $x = 50$.

To find minimum cost, put $x = 50$ in the production cost function,

$$\text{Minimum Cost} = 10(50)^2 - 1000(50) + 50000$$

$$= 10(2500) - 50000 + 50000$$

$$= 25000$$

**Illustration 33 : A factory produces $x$ units and its production capacity is 60,000 units per day. Its daily total production cost is $C = 250000 + 0.08x + \frac{200000000}{x}$. How many units should be produced for minimum production cost ?**

Production cost function $C = 250000 + 0.08x + \frac{200000000}{x}$

$$\therefore \quad \frac{dC}{dx} = 0.08 - \frac{200000000}{x^2}$$

Putting $\frac{dC}{dx} = 0$

$$0.08 - \frac{200000000}{x^2} = 0$$

$$\therefore \quad 0.08 = \frac{200000000}{x^2}$$

$$\therefore \quad 0.08\ x^2 = 200000000$$

$$\therefore \quad x^2 = 2500000000$$

$$\therefore \quad x = 50000 \quad \text{or} \quad x = -50000$$

Production cannot be negative, so we will take $x = 50000$.

Now $\frac{d^2C}{dx^2} = \frac{400000000}{x^3}$

Here, putting $x = 50000$ in $\frac{d^2C}{dx^2}$,

$$\frac{d^2C}{dx^2} = \frac{400000000}{(50000)^3} > 0$$

$\therefore$ Production cost is minimum at $x = 50000$.

Thus, 50,000 units should be produced so that the production cost is minimum.

**Illustration 34 : The demand function of a watch is $p = 6000 - 2x$. Find the demand which maximizes the revenue and also find the corresponding price.**

Demand function $p = 6000 - 2x$

Now, revenue function $R = p \cdot x$

$$= (6000 - 2x)x$$

$$\therefore \quad R = 6000x - 2x^2$$

$$\therefore \quad \frac{dR}{dx} = 6000 - 4x$$

Putting $\frac{dR}{dx} = 0$,

$$6000 - 4x = 0$$

$$\therefore \quad 6000 = 4x$$

$$\therefore \quad x = 1500$$

Now $\frac{d^2R}{dx^2} = 0 - 4$

$= -4$

Here, putting $x = 1500$ in $\frac{d^2R}{dx^2}$,

$$\frac{d^2R}{dx^2} = -4 < 0$$

$\therefore$ Revenue is maximum at $x = 1500$.

Now we shall find the corresponding price.

Putting $x = 1500$ in demand function $p = 6000 - 2x$,

Price $p = 6000 - 2(1500)$

$= 6000 - 3000$

$p = 3000$

**Illustration 35 : If the production cost function for a producer is $C = 100 + 0.015x^2$ and revenue function is $R = 3x$ then find the profit function. How many units should be produced by the producer for maximum profit ?**

Production cost function $C = 100 + 0.015x^2$ and revenue function $R = 3x$

Now, profit function $P = R - C$

$= 3x - \left(100 + 0.015x^2\right)$

$\therefore \quad P = 3x - 100 - 0.015x^2$

$\therefore \quad \frac{dP}{dx} = 3 - 0.015(2x)$

$= 3 - 0.03x$

Putting $\frac{dP}{dx} = 0$

$3 - 0.03x = 0$

$\therefore \quad 3 = 0.03x$

$\therefore \quad x = \frac{3}{0.03}$

$x = 100$

Now $\frac{d^2P}{dx^2} = 0 - 0.03(1)$

$= -0.03$

Here putting $x = 100$ in $\frac{d^2P}{dx^2}$,

$$\frac{d^2P}{dx^2} = -0.03 < 0$$

$\therefore$ At $x = 100$, profit is maximum.

## Summary

- Derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- If $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$
- If $y = k$ (constant), $\frac{dy}{dx} = 0$
- If $u$ and $v$ are differentiable functions of $x$ then,

  (1) If $y = u \pm v$ then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

  (2) If $y = u \cdot v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

  (3) If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

  (4) Chain Rule : $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- If the function $f(x)$ is increasing at $x = a$ then $f'(a) > 0$.
- If the function $f(x)$ is decreasing at $x = a$ then $f'(a) < 0$.
- The necessary and sufficient conditions for a function to be maximum at $x = a$ : $f'(a) = 0$ and $f''(a) < 0$.
- The necessary and sufficient conditions for a function to be minimum at $x = a$ : $f'(a) = 0$ and $f''(a) > 0$.
- Marginal Cost $= \frac{dC}{dx}$
- Marginal Revenue $= \frac{dR}{dx}$
- Elasticity of demand $= -\frac{p}{x} \cdot \frac{dx}{dp}$
- The necessary and sufficient conditions for minimizing the production cost function $C$ : $\frac{dC}{dx} = 0$ and $\frac{d^2C}{dx^2} > 0$.
- The necessary and sufficient conditions for maximizing the revenue function $R$ : $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$.
- The necessary and sufficient conditions for maximizing the profit function $P$ : $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$.

## EXERCISE 5

### Section A

**Choose the correct option for the following multiple choice questions :**

**1.** What is the formula for derivative of function $f(x)$ ?

(a) $\lim_{h \to x} \frac{f(x+h) - f(x)}{h}$ (b) $\lim_{h \to 0} \frac{f(x+h) + f(x)}{h}$

(c) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (d) $\lim_{h \to x} \frac{f(x) - f(x+h)}{h}$

**2.** What is $\frac{dy}{dx}$ if $y = ax^n$, $a$ is a constant ?

(a) $nx^{n-1}$ (b) $an\,x^{n-1}$ (c) 0 (d) $an\,x^{n+1}$

**3.** If $y = ax + b$, $a$ and $b$ are constant then what will be $\frac{dy}{dx}$ ?

(a) $a$ (b) $b$ (c) $a + b$ (d) 0

**4.** What is the derivative of $f(x) = \frac{4}{x^2}$ ?

(a) $\frac{4}{2x}$ (b) $-\frac{8}{x^3}$ (c) $\frac{8}{x^3}$ (d) 0

**5.** If $u$ and $v$ are two functions of $x$ then what is the formula of derivative of their product ?

(a) $u\frac{du}{dx} + v\frac{dv}{dx}$ (b) $u\frac{dv}{dx} - v\frac{du}{dx}$ (c) $\frac{du}{dx} \times \frac{dv}{dx}$ (d) $u\frac{dv}{dx} + v\frac{du}{dx}$

**6.** If $u$ and $v$ are functions of $x$ then what is the formula for derivative of $\frac{v}{u}$ ?

(a) $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (b) $\frac{v\frac{du}{dx} + u\frac{dv}{dx}}{v^2}$ (c) $\frac{u\frac{dv}{dx} + v\frac{du}{dx}}{u^2}$ (d) $\frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}$

**7.** If the function $f(x)$ is increasing at $x = a$ then which is the correct option from the following ?

(a) $f'(a) < 0$ (b) $f'(a) > 0$ (c) $f'(a) = 0$ (d) $f''(a) > 0$

**8.** What are the necessary and sufficient conditions for a function to be minimum at $x = a$ ?

(a) $f'(a) = 0,\ f''(a) < 0$ (b) $f'(a) > 0,\ f''(a) > 0$

(c) $f'(a) = 0,\ f''(a) > 0$ (d) $f'(a) < 0,\ f''(a) > 0$

**9.** What is the formula for elasticity of demand ?

(a) $-\frac{p}{x} \cdot \frac{dx}{dp}$ (b) $\frac{p}{x} \cdot \frac{dx}{dp}$ (c) $-\frac{x}{p} \cdot \frac{dp}{dx}$ (d) $-\frac{p}{x} \cdot \frac{dp}{dx}$

**10.** What are the conditions of revenue function $R$ to be maximum ?

(a) $\frac{dR}{dx} = 0, \frac{d^2R}{dx^2} < 0$ (b) $\frac{dR}{dx} = 0, \frac{d^2R}{dx^2} > 0$

(c) $\frac{dR}{dx} > 0, \frac{d^2R}{dx^2} < 0$ (d) $\frac{dR}{dx} > 0, \frac{d^2R}{dx^2} > 0$

## Section B

**Answer the following questions in one sentence:**

**1.** Define differentiation.

**2.** Find $f'(x)$ for the function $f(x) = 50$.

**3.** Find $\frac{dy}{dx}$ if $y = a^n$, $a$ is constant.

**4.** State the rule for derivative for product of two functions of $x$.

**5.** How will be the first order derivative of a function at $x = a$ if function is decreasing at $x = a$ ?

**6.** How will be the second order derivative of a function at $x = a$ if function is maximum at $x = a$ ?

**7.** What are the stationary points of a function ?

**8.** What is marginal revenue ?

**9.** Define marginal cost.

**10.** State the formula of elasticity of demand.

**11.** Find $f'(x)$ if $f(x) = 7x^2 - 6x + 5$.

**12.** Find $\frac{dy}{dx}$ if $y = 6x^3 + \frac{7}{2}x^2 + \frac{6}{5}x - 8$.

## Section C

**Answer the following questions :**

**1.** Define derivative.

**2.** State the division rule of derivative.

**3.** State necessary and sufficient conditions for a function to be maximum at $x = a$.

**4.** Explain marginal cost and give its formula.

**5.** Define elasticity of demand.

**6.** What are the conditions for profit function $P$ to be maximum ?

**7.** State the conditions for production cost function $C$ to be minimum.

**8.** Find $f''(x)$ if $f(x) = \sqrt[4]{x}$.

**9.** Write the chain rule of differentiation.

**10.** Find $f''(0)$ if $f(x) = x^4 - 4x^3 + 3x^2 + x + 1$.

**11.** Find marginal revenue if revenue function is $90x - \frac{x^2}{2}$.

12. What is maximum value of a function ?

13. When can it be said that a function is decreasing at a point ?

14. Determine whether the function $y = 12 + 4x - 7x^2$ is increasing or decreasing at $x = 2$.

15. Find the derivative of $y = 4x^2 + 4x + 8$. For which value of $x$ will the derivative be zero ?

16. $f(x) = x^3 + 5x^2 + 3x + 7$, prove that $f'(2) = 35$.

17. If $f(x) = 3x^2 + 3$ then for which value of $x$, $f'(x) = f(x)$ ?

18. Find $\frac{d^2y}{dx^2}$ if $y = 2x^3 + 5x^2 - 3 + \frac{4}{x^2} - \frac{5}{x^3}$.

19. Find $\frac{d^2y}{dx^2}$ if $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

20. Obtain marginal cost if the production cost function is $C = 0.0012x^2 - 0.18x + 25$.

## Section D

**Answer the following questions :**

1. Find derivative of $y = ax + b$ ($a$ and $b$ are constants) using definition.

2. Find derivative of $f(x) = x^{10}$ using definition.

3. Find derivative of $\frac{2}{3 + 4x}$ using definition.

4. $y = x^3 - 3x^2 - 3x + 80$. For which value of $x$, $\frac{dy}{dx} = -6$ ?

5. Find $f'(2)$ if $f(x) = \frac{4x^5 + 3x^3 + 2x^2 + 24}{x^2}$.

6. Find the derivative of $y = (3x^2 + 4x - 2)(3x + 2)$ with respect to $x$.

7. Find $\frac{dy}{dx}$ if $y = \frac{ax + b}{bx + a}$ ($a$ and $b$ are constants).

8. Find the derivative of $y = 1 + \frac{1}{1 + \frac{1}{x}}$ with respect to $x$.

9. Find $\frac{dy}{dx}$ if $(2x + 3)(y + 2) = 15$.

10. Find $\frac{dy}{dx}$ if $y = 5 + \frac{6}{7x + 8}$.

11. Find $f'(x)$ if $f(x) = \sqrt{x^2 + 5}$.

12. Find the derivative of $(3x^3 - 2x^2 + 1)^{\frac{5}{2}}$ with respect to $x$.

13. Find $f'(x)$ if $f(x) = \left(x^2 + 3x + 4\right)^7$.

14. If $f(x) = 3x^2 + 4x + 5$ then for which value of $x$, $f'(x) = f''(x)$ ?

15. Find marginal revenue if demand function is $p = \frac{2500 - x^2}{100}$.

16. Determine whether the function $y = 3x^2 - 10x + 7$ is increasing or decreasing at $x = 1$ and $x = 2$.

17. Determine whether the function $y = 2x^3 - 7x^2 - 11x + 5$ is increasing or decreasing at $x = \frac{1}{2}$ and $x = 3$.

18. Determine whether the function $y = 3 + 2x - 7x^2$ is increasing or decreasing at $x = -4$ and $x = 4$.

19. Production cost of a factory producing sugar is $C = \frac{x^2}{10} + 5x + 200$. Find the marginal cost if the production is 100 units and interpret it.

20. The cost function of producing $x$ units of a commodity is $C = 50 + 2x + \sqrt{x}$. Find the marginal cost if the production is 100 units and interpret it.

21. State the method of obtaining maximum or minimum value of a function.

## Section E

**Answer the following questions :**

1. Give working rules for differentiation.
2. How can it be decided using derivative that the function is increasing or decreasing at a point ?
3. What is maximum value of a function ? State the conditions for maximum value.
4. What is minimum value of a function ? State the conditions for minimum value.
5. In a factory, production cost per hundred tons of steel is $\frac{1}{10}x^3 - 4x^2 + 50x + 300$. Determine the production for minimum cost.
6. The cost of producing $x$ units of an item is $C = 1000 + 8x + \frac{5000}{x}$. What should be the production for minimum cost ? Also find the minimum cost.
7. Production cost function of a commodity is $C = 1500 + 0.05x - 2\sqrt{x}$. Prove that production is minimum when 400 units are produced.
8. The demand function of an item is $p = 30 - \frac{x^2}{10}$. Find the demand and price for maximum revenue.
9. In a market, demand law of rice is $x = 3(60 - p)$. Find the demand for maximum revenue. Also find the price and revenue for that demand.
10. If the demand function is $p = 75 - \frac{x^2}{2500}$ then at which demand is revenue maximum ? Also find the price for maximum revenue.

11. The profit function of a producer is $40x + 10000 - 0.1x^2$. At what production is the profit maximum ? Also find this maximum profit.

12. The profit function of a merchant is $5x - 100 - 0.01x^2$. How many units should be produced for maximum profit ?

## Section F

**Solve the following :**

1. Find the values of $x$ which maximize or minimize $y = 2x^3 - 15x^2 + 36x + 12$. Also find the maximum and minimum values of $y$.

2. Find the values of $x$ which maximize or minimize $f(x) = 2x^3 + 3x^2 - 36x + 10$. Also find the maximum and minimum values of $f(x)$.

3. Find the maximum and minimum values of $f(x) = x^3 - x^2 - x + 2$.

4. A producer produces $x$ units at cost $200x + 15x^2$. The demand function is $p = 1200 - 10x$. Find the profit function and how many units should be produced for maximum profit ?

5. The selling price of a refrigerator as determined by the company is ₹ 10,000. The total cost of the production for $x$ refrigerator is $C = 0.1x^2 + 9000x + 100$ rupees. How many refrigerators should be manufactured for maximum profit ?

6. A toy is sold at ₹ 20. Total cost of producing $x$ such toys is $C = 1000 + 16.5x + 0.001x^2$ rupees. How many toys should be produced for maximum profit ?

**Gottfried Wilhelm Leibniz**
**(1646 - 1716)**

Gottfried Leibniz was a German polymath and philosopher who occupies a prominent place in the history of mathematics and the history of philosophy, having developed differential and integral calculus independently of Isaac Newton. It was only in the 20th century that his Law of Continuity and Transcendental Law of Homogeneity found mathematical implementation (by means of non-standard analysis). He became one of the most prolific inventors in the field of mechanical calculators.

Leibniz made major contributions to physics and technology, and anticipated notions that surfaced much later in philosophy, probability theory, biology, medicine, geology, psychology, linguistics, and computer science.

# Answers

## Exercise 1.1

**1.** (1) $U = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(2) $U = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H),$
$(1, T), (2, T), (3, T), (4, T), (5, T), (6, T)\}$

(3) $U = \{(a, b), (a, c), (a, d), (a, e), (b, c),$
$(b, d), (b, e), (c, d), (c, e), (d, e)\}$

**2.** (1) U = {0, 1, 2, ....., 100}, No. of sample points = 101

**3.** Denoting four persons by a, b, c, d

U = {(a, b), (a, c), (a, d), (b, c), (b, d), (c, d), (b, a), (c, a), (d, a), (c, b), (d, b), (d, c)}

The first place in the bracket shows minister and the second place shows deputy minister.

**4.** U = {H, TH, TTH, TTTH, .....}, infinite sample space

**5.** U = {(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 3, 5), (1, 4, 5), (2, 3, 4), (2, 3, 5), (2, 4, 5), (3, 4, 5) }

**6.** (1) A = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

(2) B = {3, 6, 9, 12, 15, 18}

(3) C = {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20}

**7.** U= {BB, BG, GB, GG}

(1) $A_1$ = {BG, GB}

(2) $A_2$ = {BG, GB, GG}

**8.** U= {(i, j); i, j = 1, 2, 3, 4, 5, 6}

(1) $A_1$ = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

(2) $A_2$ = {(1, 1), (1, 2), (2, 1),}

(3) $A_3$ = {(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}

(4) $A_4$ = { }

**9.** (1) U = {(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

(2) A = {(1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

(3) B = {(1, 3), (1, 5), (2, 4), (3, 5)}

(4) $A \cup B$ = {(1, 3), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

(5) $A \cap B$ = {(1, 5), (2, 4), (3, 5)}

(6) $A'$ = {(1, 2), (1, 3), (1, 4), (2, 3)}

(7) $A - B$ = {(2, 5), (3, 4), (4, 5)}

(8) $A' \cap B$ = {(1, 3)}

(9) Events A and B are not mutually exclusive as $A \cap B \neq \phi$

(10) No. of sample points = 10

**10.** Denoting three females by *a, b, c* and two males by *x, y*.

(1) U = {*a, b, c, x, y*}

(2) A = {*a, b, c*}

(3) B = {*x, y*}

(4) $A \cup B$ = {*a, b, c, x, y*}

(5) $A \cap B$ = { }

(6) $A' \cap B$ = {*x, y*}

(7) Events A and B are mutually exclusive as $A \cap B = \phi$

(8) Events A and B are exhaustive as $A \cup B = U$

**11.** (1) $U = \{S_A, S_2, S_3, ....., S_K, D_A, D_2, D_3, ....., D_K,$
$C_A, C_2, C_3, ....., C_K, H_A, H_2, H_3, ....., H_K\}$

(2) $A = \{S_A, S_2, S_3, ....., S_K\}$

(3) $B = \{S_A, S_2, S_3, ....., S_{10}, D_A, D_2, D_3, ....., D_{10},$
$C_A, C_2, C_3, ....., C_{10}, H_A, H_2, H_3, ....., H_{10}\}$

(4) $A \cup B = \{S_A, S_2, S_3, ....., S_k, D_A, D_2, D_3, ....., D_{10},$
$C_A, C_2, C_3, ....., C_{10}, H_A, H_2, H_3, ....., H_{10}\}$

(5) $A \cap B = \{S_1, S_2, S_3, ....., S_{10}\}$

(6) $B' = \{S_J, S_Q, S_K, D_J, D_Q, D_K, C_J, C_Q, C_K, H_J, H_Q, H_K\}$

**12.** $A_1 \cup A_2 = \{x \mid 0 \le x < 5\}$

$A_1 \cap A_2 = \{x \mid x = 1, 2\}$

**13.** $A_1 \cup A_2 = \{x \mid 2 \le x \le 8, x \in N\}$

$A_1 \cap A_2 = \{x \mid x = 4, 5\}$

**14.** $A' = \{x \mid x = 0, 1, 3, 5, 7, 8, 9, 10\}$

**15.** $A' = \{x \mid 0 < x < \frac{1}{2}\}$

## Exercise 1.2

**1.** (1) $\frac{1}{8}$ (2) $\frac{1}{8}$ (3) $\frac{7}{8}$ (4) $\frac{1}{2}$ (5) $\frac{1}{2}$

(6) $\frac{1}{2}$ (7) $\frac{1}{4}$ (8) $\frac{1}{2}$ **2.** (1) $\frac{5}{36}$ (2) $\frac{11}{12}$

(3) $\frac{1}{3}$ (4) $\frac{1}{9}$ **3.** (1) $\frac{1}{2}$ (2) $\frac{3}{4}$ **4.** $\frac{7}{50}$

**5.** (1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) $\frac{1}{4}$ (4) $\frac{3}{4}$ (5) $\frac{1}{12}$

**6.** $\frac{1}{30}$ **7.** $\frac{1}{20}$ **8.** $\frac{2}{5}$ **9.** $\frac{1}{7}$ **10.** $\frac{1}{7}$

**11.** $\frac{1}{7}$ **12.** $\frac{3}{7}$ **13.** (1) $\frac{1}{7}$ (2) $\frac{4}{7}$ (3) $\frac{3}{7}$

**14.** (1) $\frac{3}{28}$ (2) $\frac{5}{14}$ (3) $\frac{15}{28}$ **15.** (1) $\frac{26}{51}$ (2) $\frac{11}{221}$

(3) $\frac{32}{221}$ **16.** $\frac{14}{15}$ **17.** (1) 0.4 (2) 0.35 (3) 0.45

(4) 0.85 **18.** $P(A-B) = 0.2,\ P(B-A) = 0.5$

## Exercise 1.3

**1.** (1) $\frac{4}{17}$ (2) $\frac{25}{51}$ **2.** $\frac{3}{7}$ **3.** (1) $\frac{4}{13}$ (2) $\frac{9}{13}$

**4.** $\frac{47}{100}$ **5.** $\frac{2}{3}$ **6.** 0.9 **7.** 0.97 **8.** 0.59

**9.** $P(A) = \frac{2}{3}$ **10.** $P(A \cup C) = \frac{35}{47}$, $P(B \cup C) = \frac{32}{47}$ **11.** $P(A \cup B \cup C) = 0.8$

**12.** $P(A) = 0.4$, $P(B) = 0.4$

## Exercise 1.4

**1.** $\frac{1}{2}$ **2.** $\frac{1}{5}$ **3.** (1) $\frac{5}{8}$ (2) $\frac{5}{6}$ **4.** $\frac{4}{5}$

**5.** $P(A/B) = \frac{5}{6}$ **6.** $P(A \cap M) = \frac{1}{20}$, $P(A \cap F) = \frac{1}{4}$ **7.** $\frac{11}{24}$ **8.** $\frac{8}{15}$

**9.** $\frac{9}{100}$ **10.** $\frac{4}{15}$ **11.** (1) $\frac{11}{36}$ (2) $\frac{1}{6}$ **12.** $\frac{23}{24}$

**13.** $\frac{14}{15}$ **14.** 0.26 **15.** 0.72

## Exercise 1.5

**1.** (1) $\frac{29}{357}$ (2) $\frac{125}{357}$ (3) $\frac{275}{357}$ **2.** (1) $\frac{1319}{2536}$ (2) $\frac{1319}{2437}$

## Exercise 1

### Section A

**1.** (d) **2.** (b) **3.** (c) **4.** (a) **5.** (a)

**6.** (c) **7.** (c) **8.** (b) **9.** (a) **10.** (b)

**11.** (d) **12.** (b) **13.** (a) **14.** (a) **15.** (c)

**16.** (c) **17.** (b)

### Section B

**13.** $P(A \cap B) = 0$, $P(A \cup B) = 1$ **15.** $A \cap B = \{x \mid \frac{1}{4} \le x < 1\}$ **16.** 0.15

**17.** 0.55 **18.** 0.1 **19.** 2K **20.** 0.45 **21.** $\frac{2}{3}$

**22.** 0.98 **23.** $2^5 = 32$ **24.** $2 \times 6 \times 6 = 72$ **25.** Not possible. As $P(A \cup B) > P(A)$

**26.** 2704 **27.** $\frac{2}{5}$ **28.** $\frac{1}{1000}$

## Section C

**12.** $\frac{1}{6}$ **13.** 0.08 **14.** 0.0024 **15.** $\frac{19}{20}$ **16.** 0.58

**17.** $A \cup B = \{x \mid \frac{1}{2} \le x < 3\}$, $A \cap B = \{x \mid 1 < x < 2\}$ **18.** $\frac{1}{20}$ **19.** $\frac{1}{6}$

**20.** (1) $\frac{1}{6}$ (2) $\frac{3}{10}$ **21.** (1) 0.44 (2) 0.09 **22.** $\frac{17}{20}$

**23.** 0.976

## Section D

**1.** $\frac{31}{80}$ **2.** $\frac{4}{25}$ **3.** $\frac{1}{2}$ **4.** $p(3-3p+p^2)$

**5.** (1) $\frac{9}{10}$ (2) $\frac{4}{5}$

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## Exercise 2.1

**1.** Given distribution is a probability distribution of variable $x$.

**2.** $k = 30$ **3.** $k = \frac{24}{17}$, $P(1 < x < 4) = \frac{5}{17}$ **4.** $k = \frac{1}{7}$, Mean $= \frac{9}{7}$

**5.** Mean $= \frac{-1}{8}$, Variance $= \frac{135}{64}$

**6.** Probability distribution of sum

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | કુલ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 1 |

Expected value of sum = 7

**7.** Probability distribution of $x$

| $x$ | 1 | 2 | 3 | Total |
|---|---|---|---|---|
| $p(x)$ | $\frac{4}{20}$ | $\frac{12}{20}$ | $\frac{4}{20}$ | 1 |

Expected number of red balls = 2

**8.** Probability distribution of $x$

| $x$ | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | 1 |

Mean $= \frac{31}{16}$, Variance $= \frac{367}{256}$

**9.** Expected value of the prize = ₹ $\frac{20}{3}$

## Exercise 2.2

**1.** $p(X \leq 1) = \frac{9}{256}$ **2.** $n = 6,\ p = \frac{5}{6}$; $\frac{15625}{46656}$ **3.** 0.6912 **4.** 0.0729

**5.** 0.1382

## Exercise 2

### Section A

**1.** (d) **2.** (c) **3.** (c) **4.** (a) **5.** (d)

**6.** (d) **7.** (b) **8.** (b) **9.** (b) **10.** (b)

### Section B

**6.** 14 **7.** $\frac{12}{5}$ **8.** $p + q = 1$ **9.** Mean > Variance **10.** 0.4

### Section C

**1.** $C = 0.1$ **2.** $\frac{10}{3}$ **3.** 2.8 **4.** $\frac{1}{16}$ **6.** $\frac{8}{9}$

**7.** $\frac{4}{3}$ **8.** $n = 8,\ p = \frac{1}{2}$ **9.** 2 **10.** 0.96

### Section D

**1.** $k = \frac{1}{5}$; $\frac{2}{5}$ **2.** $c = \frac{1}{10}$ **3.** $k = \frac{1}{326}$, Mean = $\frac{1305}{326}$ **6.** $\frac{54}{125}$, Mean = $\frac{9}{5}$

**7.** $\frac{1620}{16807}$ **8.** $p(1) = \frac{162}{625}$, $p(2) = \frac{216}{625}$ **9.** 0.0146, Variance = 0.54

### Section E

**1.** Expected demand = 3.62, Variance = 2.2156

**2.**

| $x$ | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| $p(x)$ | $\frac{25}{36}$ | $\frac{10}{36}$ | $\frac{1}{36}$ | 1 |

**3.** (i) 0.9510 (ii) 0.0490 **4.** 0.1631 **5.** 0.3447 **6.** 0.5443

**7.** (i) 0.3599 (ii) 0.1066

## Section F

(1)

| $x$ | 0 | 1 | 2 | 3 | Total |
|---|---|---|---|---|---|
| $p(x)$ | $\frac{84}{220}$ | $\frac{108}{220}$ | $\frac{27}{220}$ | $\frac{1}{220}$ | 1 |

Expected value $= \frac{165}{220}$, Variance $= 0.4602$

(2) 56

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## Exercise 3

### Section A

**1.** (c) **2.** (d) **3.** (b) **4.** (a) **5.** (c)
**6.** (b) **7.** (d) **8.** (c) **9.** (a) **10.** (b)
**11.** (c) **12.** (d) **13.** (a) **14.** (b) **15.** (c)

### Section B

**2.** 0 **4.** 0 **5.** Yes **6.** $z = 0$ **7.** Mean
**8.** 95.45 % **9.** 20 **10.** 4 **11.** 10 **12.** 18
**13.** (–0.675, 0.675) **14.** 25 **15.** No **16.** 1.5 **17.** 50

### Section C

**7.** $Q_1 = 6.63$ **8.** 12 **9.** $Q_1 = 40$ **10.** (90, 110) **11.** $\pm 2.575$

### Section D

**4.** (1) 0.3413 (2) 0.6826 **5.** (1) 84.13 % (2) 97.72 % **6.** (1) 409 approximately
(2) 11 approximately **7.** (1) $Z_1 = 2.445$ (2) $Z_1 = 1.96$ **8.** (1) $Z_1 = -1.035$ (2) $Z_1 = -0.675$

**9.** (1) $Z_1 = -0.5$ (2) $Z_1 = 1.08$ **10.** $\mu = 2000$ $\sigma = 400$

**11.** $Q_1 = 95.95$, $Q_3 = 104.05$ **12.** (5, 95) **13.** $\mu = 15.75$

**14.** (1) $Q_1 = 193.25$, $Q_3 = 206.75$ (2) 6.67 (3) 8

### Section E

**1.** (1) 0.3785 (2) 0.2426

**2.** No. of fat persons = 33, No. of healthy persons = 192, No. of physically weak persons = 11

**3.** (1) 6.06 % (2) 65.54 % (3) 78.81 % **4.** ₹ 8320 and ₹ 12,560

**5.** (49.44, 54.56) **6.** $x_1 = 55$, 8 weeks (approximately) **7.** $\sigma^2 = 1361.61$, 0.2148

**8.** $\mu = 21.15$ mm, 80.27 % **9.** $D_4 = 392.35$, $P_{90} = 438.4$

**10.** (1) 150 (2) 140

## Section F

**1.** (1) 250 children (2) 30.54 % (3) 96 marks (approximately)

**2.** (i) 30.13 years (ii) 33.79 years (iii) 47.68 years

**3.** (a) 456 (b) 1846 (c) 16362 (d) 1336

**4.** N = 5051, ₹ 8350 **5.** $\mu = 62.12$, $\sigma = 17.28$, $Q_3 = 73.78\%$

**6.** $\mu = 4300$, $\sigma = 500$, (3320, 5280)

**7.** (1) $x_2 = 157$ (2) $x_1 = 68$ (3) 0.1401

**8.** (1) 50 (2) $Q_1 = 43.25$, $Q_3 = 56.75$ (3) $\frac{20}{3}$ (4) 8

●

## Exercise 4.1

**1.** (1) Modulus form : $|x-4| < 0.4$

Interval form : (3.6, 4.4)

(2) Modulus form : $|x-2| < 0.02$

Interval form : (1.98, 2.02)

(3) Modulus form : $|x| < 0.05$

Interval form : (–0.05, 0.05)

(4) Modulus form : $|x+1| < 0.001$

Interval form : (–1.001, –0.999)

**2.** (1) Interval form : (1.99, 2.01)

Neighbourhood form : N (2, 0.01)

(2) Interval form : $(-5.1, -4.9)$

Neighbourhood form : N $(-5, 0.1)$

(3) Interval form : $\left(-\frac{1}{3}, \frac{1}{3}\right)$

Neighbourhood form : N $\left(0, \frac{1}{3}\right)$

(4) Interval form : $(-3.15, -2.85)$

Neighbourhood form : N $(-3, 0.15)$

**3.** (1) Modulus form : $|x-4.3|<0.5$

Neighbourhood form : N $(4.3, 0.5)$

(2) Modulus form : $|x-2|<0.05$

Neighbourhood form : N $(2, 0.05)$

(3) Modulus form : $|x-0.5|<0.9$

Neighbourhood form : $N$ $(0.5, 0.9)$

(4) Modulus form : $|x-2|<0.002$

Neighbourhood form : $N$ $(2, 0.002)$

**4.** Interval form : $(15.5, 16.5)$

Modulus form : $|x-16|<0.5$

**5.** $b = 0.05,\ k = 3.05$

**6.** $K_1 = 0.01,\ K_2 = 9.99$

## Exercise 4.2

**1.** (1) 3 (2) 4 (3) 11 (4) –3 (5) 2

## Exercise 4

### Section A

**1.** (b) **2.** (c) **3.** (a) **4.** (b) **5.** (d)

**6.** (c) **7.** (d) **8.** (a) **9.** (a) **10.** (d)

**11.** (b) **12.** (c)

## Section B

**1.** (–0.09, 0.09) **2.** $|x+5|<0.001$ **3.** N(10, $\frac{1}{10}$) **4.** $\left(-\frac{1}{4}, \frac{1}{4}\right)$

**5.** $|x-50|<0.8$ **6.** $a = 7$ **7.** $k = -4.04$ **8.** 20 **9.** 2

**10.** 2 **11.** 80 **12.** $ma^{m-1}$ **13.** $k = 10$ **14.** $k = 5$

## Section C

**4.** $|x|<0.5$ **5.** $N$ (–8, 0.75) **6.** $K_1 = 20,\ K_2 = 20.5$

**7.** Neighbourhood form : $N\left(-\frac{1}{3}, \frac{2}{3}\right)$ Interval form : $\left(-1, \frac{1}{3}\right)$ **8.** $A_1 = 4$ $A_2 = 3.91$

## Section D

**1.** 2 **2.** $\frac{1}{9}$ **3.** –8 **4.** 2 **5.** $\frac{7}{4}$

**6.** $\frac{3}{5}$ **7.** $-\frac{1}{3}$ **8.** $\frac{31}{3}$ **9.** $-\frac{1}{7}$ **10.** 1

**11.** 1 **12.** $-\frac{4}{3}p$ **13.** $\frac{27}{2}$ **14.** –64 **15.** $-\frac{2017}{2018}$

**16.** $\frac{7}{3}$ **17.** $\frac{2}{3}$

## Section E

**II.** (1) 7 (2) 7 (3) –3 (4) –1

**III.** (1) $7x^6$ (2) $\frac{1}{10}$ (3) $n$ (4) 1 (5) $3x^2$

(6) $7x^6$ (7) $\frac{1}{6}$ (8) 8 (9) 4 (10) 5

•

## Exercise 5.1

**1.** 2 **2.** $2x$ **3.** $7x^6$ **4.** $\frac{-1}{(x+1)^2}$ **5.** $\frac{1}{3x^{\frac{2}{3}}}$

**6.** $\frac{-6}{(3x-4)^2}$ **7.** 0

## Exercise 5

### Section A

| | | | | |
|---|---|---|---|---|
| **1.** (c) | **2.** (b) | **3.** (a) | **4.** (b) | **5.** (d) |
| **6.** (d) | **7.** (b) | **8.** (c) | **9.** (a) | **10.** (a) |

### Section B

**2.** 0 **3.** 0 **5.** Negative **6.** Negative **11.** $14x-6$

**12.** $18x^2+7x+\frac{6}{5}$

### Section C

**8.** $-\frac{3}{16x^{\frac{7}{4}}}$ **10.** 6 **11.** $90-x$ **14.** Decreasing **15.** $-\frac{1}{2}$

**17.** 1 **18.** $12x+10+\frac{24}{x^4}-\frac{60}{x^5}$ **19.** $\frac{-1}{4x^{\frac{3}{2}}}+\frac{3}{4x^{\frac{5}{2}}}$

**20.** $0.0024x - 0.18$

### Section D

**1.** a **2.** $10x^9$ **3.** $\frac{-8}{(3+4x)^2}$ **4.** 1 **5.** 45

**6.** $27x^2+36x+2$ **7.** $\frac{a^2-b^2}{(bx+a)^2}$ **8.** $\frac{1}{(x+1)^2}$ **9.** $\frac{-30}{(2x+3)^2}$

**10.** $\frac{-42}{(7x+8)^2}$ **11.** $\frac{x}{\sqrt{x^2+5}}$ **12.** $\frac{5}{2}(3x^3-2x^2+1)^{\frac{3}{2}}(9x^2-4x)$

**13.** $7(x^2+3x+4)^6(2x+3)$ **14.** $\frac{1}{3}$ **15.** $25-\frac{3x^2}{100}$

**16.** Function is decreasing at $x=1$

Function is decreasing at $x=2$

**17.** Function is decreasing at $x=\frac{1}{2}$

Function is decreasing at $x=3$

**18.** Function is increasing at $x = -4$

Function is decreasing at $x = 4$

**19.** Marginal cost $= \frac{x}{5} + 5$

$x = 100$, Marginal cost $= 25$

**20.** Marginal cost $= 2 + \frac{1}{2\sqrt{x}}$

$x = 100$ Marginal cost $= 2.05$

## Section E

**5.** $\frac{50}{3}$ hundred tons **6.** $x = 25$, Minimum cost = 1400 **8.** $x = 10$, $p = 20$

**9.** $p = 30$, $x = 90$, $R = 2700$ **10.** $x = 250$, $p = 50$ **11.** $x = 200$, Maximum profit = 14,000

**12.** $x = 250$

## Section F

**1.** $y$ is maximum at $x = 2$, Maximum value of $y$ is 40

$y$ is minimum at $x = 3$, Minimum value of $y$ is 39

**2.** $f(x)$ is maximum at $x = -3$, Maximum value of $f(x)$ is 91

$f(x)$ is minimum at $x = 2$, Minimum value of $f(x)$ is $-34$

**3.** $f(x)$ is maximum at $x = \frac{-1}{3}$, Maximum value of $f(x)$ is $\frac{59}{27}$

$f(x)$ is minimum at $x = 1$, Minimum value of $f(x)$ is 1

**4.** Profit function $= 1000x - 25x^2$

Profit is maximum at $x = 20$

**5.** 5000 refrigerators

**6.** 1750 toys

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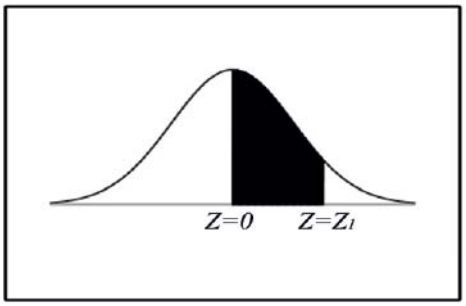
# *Table of Standard Normal Curve*




***Area Under the Standard Normal Curve***

*$Z = 0$ to $Z = Z_l$ , z being standard normal variate*

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| 0.6 | .2257 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2549 |
| 0.7 | .2580 | .2611 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4762 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |
| 3.1 | .4990 | .4991 | .4991 | .4991 | .4992 | .4992 | .4992 | .4992 | .4993 | .4993 |
| 3.2 | .4993 | .4993 | .4994 | .4994 | .4994 | .4994 | .4994 | .4995 | .4995 | .4995 |
| 3.3 | .4995 | .4995 | .4995 | .4996 | .4996 | .4996 | .4996 | .4996 | .4996 | .4997 |
| 3.4 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4998 |
| 3.5 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 |