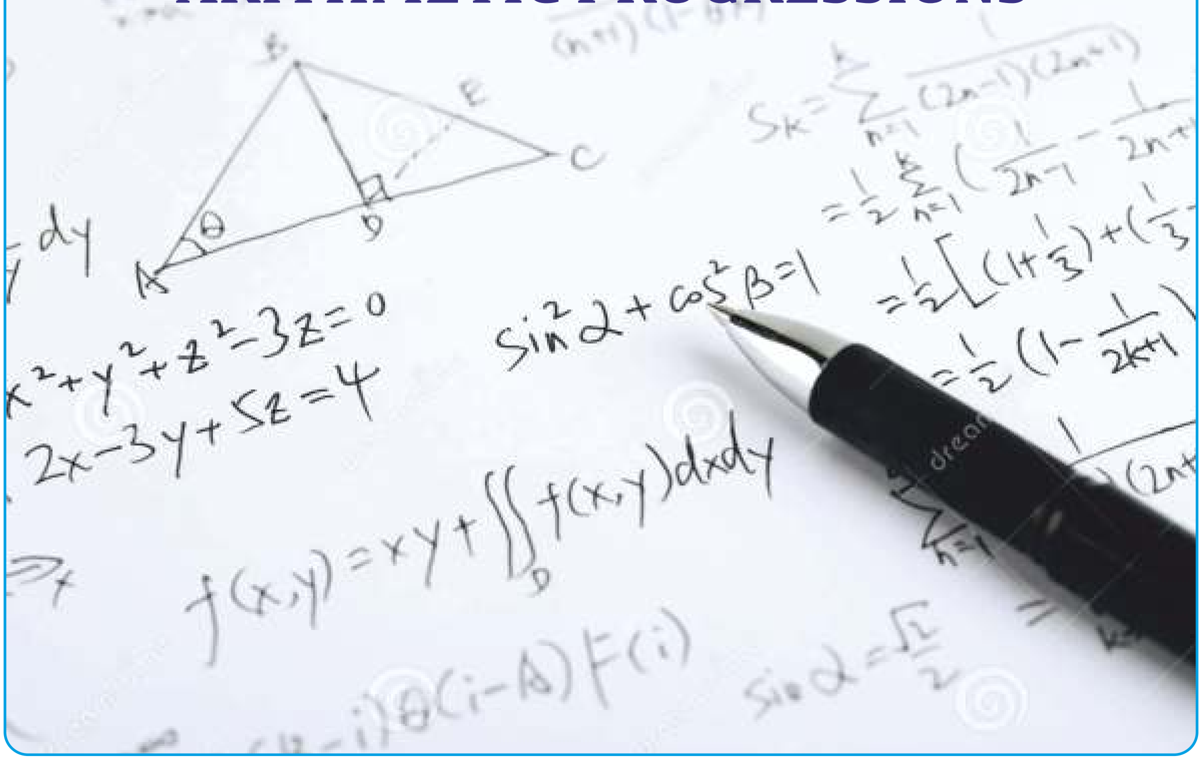
05

ARITHMETIC PROGRESSIONS



ARITHMETIC PROGRESSIONS 5

5.1 Introduction

You must have observed that in nature, many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple and on a pine cone etc.

We now look for some patterns which occur in our day-to-day life. Some such examples are :

(i) Reena applied for a job and got selected. She has been offered a job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500 in her salary. Her salary (in ₹) for the 1st, 2nd, 3rd, . . . years will be, respectively

$$8000, \quad 8500, \quad 9000, \ldots .$$

(ii) The lengths of the rungs of a ladder decrease uniformly by 2 cm from bottom to top (see Fig. 5.1). The bottom rung is 45 cm in length. The lengths (in cm) of the 1st, 2nd, 3rd, . . ., 8th rung from the bottom to the top are, respectively

$$45, 43, 41, 39, 37, 35, 33, 31$$

Fig. 5.1

(iii) In a savings scheme, the amount becomes $\frac{5}{4}$ times of itself after every 3 years. The maturity amount (in ₹) of an investment of ₹ 8000 after 3, 6, 9 and 12 years will be, respectively :

$$10000, \quad 12500, \quad 15625, \quad 19531.25$$

(iv) The number of unit squares in squares with side 1, 2, 3, . . . units (see Fig. 5.2) are, respectively

$1^2, 2^2, 3^2, \ldots .$

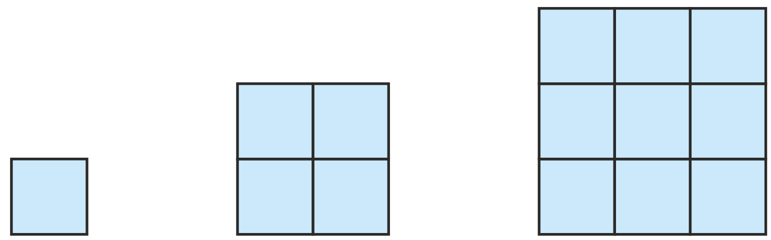


Fig. 5.2

(v) Shakila puts ₹ 100 into her daughter's money box when she was one year old and increased the amount by ₹ 50 every year. The amounts of money (in ₹) in the box on the 1st, 2nd, 3rd, 4th, . . . birthday were

100, 150, 200, 250, . . ., respectively.

(vi) A pair of rabbits are too young to produce in their first month. In the second, and every subsequent month, they produce a new pair. Each new pair of rabbits produce a new pair in their second month and in every subsequent month (see Fig. 5.3). Assuming no rabbit dies, the number of pairs of rabbits at the start of the 1st, 2nd, 3rd, . . ., 6th month, respectively are :

1, 1, 2, 3, 5, 8

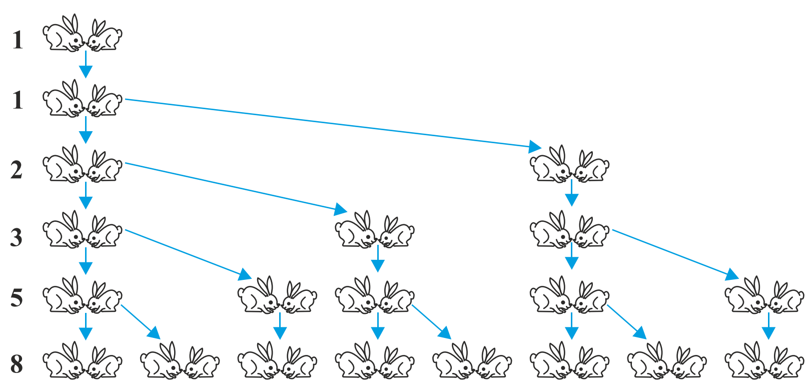


Fig. 5.3

In the examples above, we observe some patterns. In some, we find that the succeeding terms are obtained by adding a fixed number, in other by multiplying with a fixed number, in another we find that they are squares of consecutive numbers, and so on.

In this chapter, we shall discuss one of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms. We shall also see how to find their nth terms and the sum of n consecutive terms, and use this knowledge in solving some daily life problems.

5.2 Arithmetic Progressions

Consider the following lists of numbers :

(i) 1, 2, 3, 4, . . .

(ii) 100, 70, 40, 10, . . .

(iii) –3, –2, –1, 0, . . .

(iv) 3, 3, 3, 3, . . .

(v) –1.0, –1.5, –2.0, –2.5, . . .

Each of the numbers in the list is called a **term**.

Given a term, can you write the next term in each of the lists above? If so, how will you write it? Perhaps by following a pattern or rule. Let us observe and write the rule.

In (i), each term is 1 more than the term preceding it.

In (ii), each term is 30 less than the term preceding it.

In (iii), each term is obtained by adding 1 to the term preceding it.

In (iv), all the terms in the list are 3 , i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.

In (v), each term is obtained by adding – 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

In all the lists above, we see that successive terms are obtained by adding a fixed number to the preceding terms. Such list of numbers is said to form an **Arithmetic Progression (AP)**.

So, **an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.**

This fixed number is called the **common difference** of the AP. Remember that it **can be positive, negative or zero**.

Let us denote the first term of an AP by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes $a_1, a_2, a_3, \ldots, a_n$.

So, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$.

Some more examples of AP are:

(a) The heights (in cm) of some students of a school standing in a queue in the morning assembly are 147 , 148, 149, . . ., 157.

(b) The minimum temperatures (in degree celsius) recorded for a week in the month of January in a city, arranged in ascending order are

$$-3.1, -3.0, -2.9, -2.8, -2.7, -2.6, -2.5$$

(c) The balance money (in ₹) after paying 5 % of the total loan of ₹ 1000 every month is 950, 900, 850, 800, . . ., 50.

(d) The cash prizes (in ₹) given by a school to the toppers of Classes I to XII are, respectively, 200, 250, 300, 350, . . ., 750.

(e) The total savings (in ₹) after every month for 10 months when ₹ 50 are saved each month are 50, 100, 150, 200, 250, 300, 350, 400, 450, 500.

It is left as an exercise for you to explain why each of the lists above is an AP.

You can see that

$$a, a + d, a + 2d, a + 3d, \ldots$$

represents an arithmetic progression where a is the first term and d the common difference. This is called the **general form of an AP.**

Note that in examples (a) to (e) above, there are only a finite number of terms. Such an AP is called a **finite AP.** Also note that each of these Arithmetic Progressions (APs) has a last term. The APs in examples (i) to (v) in this section, are not finite APs and so they are called **infinite Arithmetic Progressions**. Such APs do not have a last term.

Now, to know about an AP, what is the minimum information that you need? Is it enough to know the first term? Or, is it enough to know only the common difference? You will find that you will need to know both – the first term a and the common difference d.

For instance if the first term a is 6 and the common difference d is 3, then the AP is

$$6, 9,12, 15, \ldots$$

and if a is 6 and d is – 3, then the AP is

$$6, 3, 0, -3, \ldots$$

Similarly, when

$a = -7, \quad d = -2, \quad$ the AP is $-7, -9, -11, -13, \ldots$

$a = 1.0, \quad d = 0.1, \quad$ the AP is $1.0, 1.1, 1.2, 1.3, \ldots$

$a = 0, \quad d = 1\frac{1}{2}, \quad$ the AP is $0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 6, \ldots$

$a = 2, \quad d = 0, \quad$ the AP is $2, 2, 2, 2, \ldots$

So, if you know what a and d are, you can list the AP. What about the other way round? That is, if you are given a list of numbers can you say that it is an AP and then find a and d? Since a is the first term, it can easily be written. We know that in an AP, every succeeding term is obtained by adding d to the preceding term. So, d found by subtracting any term from its succeeding term, i.e., the term which immediately follows it should be same for an AP.

For example, for the list of numbers :

$$6, 9, 12, 15, \ldots,$$

We have

$$a_2 - a_1 = 9 - 6 = 3,$$
$$a_3 - a_2 = 12 - 9 = 3,$$
$$a_4 - a_3 = 15 - 12 = 3$$

Here the difference of any two consecutive terms in each case is 3. So, the given list is an AP whose first term a is 6 and common difference d is 3.

For the list of numbers : $6, 3, 0, -3, \ldots$,

$$a_2 - a_1 = 3 - 6 = -3$$
$$a_3 - a_2 = 0 - 3 = -3$$
$$a_4 - a_3 = -3 - 0 = -3$$

Similarly this is also an AP whose first term is 6 and the common difference is -3.

In general, for an AP $a_1, a_2, \ldots, a_n$, we have

$$d = a_{k+1} - a_k$$

where a_{k+1} and a_k are the $(k + 1)$th and the kth terms respectively.

To obtain d in a given AP, we need not find all of $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$. It is enough to find only one of them.

Consider the list of numbers $1, 1, 2, 3, 5, \ldots$. By looking at it, you can tell that the difference between any two consecutive terms is not the same. So, this is not an AP.

Note that to find d in the AP : 6, 3, 0, – 3, . . ., we have subtracted 6 from 3 and not 3 from 6, i.e., we should subtract the kth term from the $(k + 1)$ th term even if the $(k + 1)$ th term is smaller.

Let us make the concept more clear through some examples.

Example 1 : For the AP : $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, . . ., write the first term a and the common difference d.

Solution : Here, $a = \frac{3}{2}$, $d = \frac{1}{2} - \frac{3}{2} = -1$.

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

Example 2 : Which of the following list of numbers form an AP? If they form an AP, write the next two terms :

(i) 4, 10, 16, 22, . . .
(ii) 1, – 1, – 3, – 5, . . .
(iii) – 2, 2, – 2, 2, – 2, . . .
(iv) 1, 1, 1, 2, 2, 2, 3, 3, 3, . . .

Solution : (i) We have

$$a_2 - a_1 = 10 - 4 = 6$$
$$a_3 - a_2 = 16 - 10 = 6$$
$$a_4 - a_3 = 22 - 16 = 6$$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = 6$.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii) $a_2 - a_1 = -1 - 1 = -2$

$a_3 - a_2 = -3 - (-1) = -3 + 1 = -2$

$a_4 - a_3 = -5 - (-3) = -5 + 3 = -2$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = -2$.

The next two terms are:

$$-5 + (-2) = -7 \quad \text{and} \quad -7 + (-2) = -9$$

(iii) $a_2 - a_1 = 2 - (-2) = 2 + 2 = 4$

$a_3 - a_2 = -2 - 2 = -4$

As $a_2 - a_1 \neq a_3 - a_2$, the given list of numbers does not form an AP.

(iv) $a_2 - a_1 = 1 - 1 = 0$

$a_3 - a_2 = 1 - 1 = 0$

$a_4 - a_3 = 2 - 1 = 1$

Here, $a_2 - a_1 = a_3 - a_2 \neq a_4 - a_3$.

So, the given list of numbers does not form an AP.

EXERCISE 5.1

1. In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

(i) The taxi fare after each km when the fare is ₹ 15 for the first km and ₹ 8 for each additional km.

(ii) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.

(iii) The cost of digging a well after every metre of digging, when it costs ₹ 150 for the first metre and rises by ₹ 50 for each subsequent metre.

(iv) The amount of money in the account every year, when ₹ 10000 is deposited at compound interest at 8 % per annum.

2. Write first four terms of the AP, when the first term a and the common difference d are given as follows:

(i) $a = 10,\quad d = 10$

(ii) $a = -2,\quad d = 0$

(iii) $a = 4,\quad d = -3$

(iv) $a = -1,\quad d = \frac{1}{2}$

(v) $a = -1.25,\quad d = -0.25$

3. For the following APs, write the first term and the common difference:

(i) $3, 1, -1, -3, \ldots$

(ii) $-5, -1, 3, 7, \ldots$

(iii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, \ldots$

(iv) $0.6, 1.7, 2.8, 3.9, \ldots$

4. Which of the following are APs ? If they form an AP, find the common difference d and write three more terms.

(i) $2, 4, 8, 16, \ldots$

(ii) $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

(iii) $-1.2, -3.2, -5.2, -7.2, \ldots$

(iv) $-10, -6, -2, 2, \ldots$

(v) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, \ldots$

(vi) $0.2, 0.22, 0.222, 0.2222, \ldots$

(vii) $0, -4, -8, -12, \ldots$

(viii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$

(ix) 1, 3, 9, 27, . . .

(x) $a, 2a, 3a, 4a, \ldots$

(xi) $a, a^2, a^3, a^4, \ldots$

(xii) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots$

(xiii) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots$

(xiv) $1^2, 3^2, 5^2, 7^2, \ldots$

(xv) $1^2, 5^2, 7^2, 73, \ldots$

5.3 *n*th Term of an AP

Let us consider the situation again, given in Section 5.1 in which Reena applied for a job and got selected. She has been offered the job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500. What would be her monthly salary for the fifth year?

To answer this, let us first see what her monthly salary for the second year would be.

It would be ₹ (8000 + 500) = ₹ 8500. In the same way, we can find the monthly salary for the 3rd, 4th and 5th year by adding ₹ 500 to the salary of the previous year.

So, the salary for the 3rd year = ₹ (8500 + 500)

= ₹ (8000 + 500 + 500)

= ₹ (8000 + 2 × 500)

= ₹ [8000 + **(3 – 1)** × 500] (for the **3rd year**)

= ₹ 9000

Salary for the 4th year = ₹ (9000 + 500)

= ₹ (8000 + 500 + 500 + 500)

= ₹ (8000 + 3 × 500)

= ₹ [8000 + **(4 – 1)** × 500] (for the **4th year**)

= ₹ 9500

Salary for the 5th year = ₹ (9500 + 500)

= ₹ (8000+500+500+500 + 500)

= ₹ (8000 + 4 × 500)

= ₹ [8000 + **(5 – 1)** × 500] (for the **5th year**)

= ₹ 10000

Observe that we are getting a list of numbers

8000, 8500, 9000, 9500, 10000, . . .

These numbers are in AP. (Why?)

Now, looking at the pattern formed above, can you find her monthly salary for the 6th year? The 15th year? And, assuming that she will still be working in the job, what about the monthly salary for the 25th year? You would calculate this by adding ₹ 500 each time to the salary of the previous year to give the answer. Can we make this process shorter? Let us see. You may have already got some idea from the way we have obtained the salaries above.

Salary for the 15th year

$$= \text{Salary for the 14th year} + ₹\ 500$$

$$= ₹\left[8000 + \underbrace{500+500+500+\ldots+500}_{13\text{ times}}\right] + ₹\ 500$$

$$= ₹\ [8000 + 14 \times 500]$$

$$= ₹\ [8000 + \mathbf{(15 - 1)} \times 500] = ₹\ 15000$$

i.e., First salary + (15 – 1) × Annual increment.

In the same way, her monthly salary for the 25th year would be

$$₹\ [8000 + \mathbf{(25 - 1)} \times 500] = ₹\ 20000$$

$$= \text{First salary} + \mathbf{(25 - 1)} \times \text{Annual increment}$$

This example would have given you some idea about how to write the 15th term, or the 25th term, and more generally, the *n*th term of the AP.

Let $a_1, a_2, a_3, \ldots$ be an AP whose first term a_1 is a and the common difference is d.

Then,

the **second term** $a_2 = a + d = a + \mathbf{(2 - 1)}\, d$

the **third** term $a_3 = a_2 + d = (a + d) + d = a + 2d = a + \mathbf{(3 - 1)}\, d$

the **fourth** term $a_4 = a_3 + d = (a + 2d) + d = a + 3d = a + \mathbf{(4 - 1)}\, d$

. . . .

. . . .

Looking at the pattern, we can say that the ***n*th** term $a_n = a + (n - 1)\, d$.

So, **the *n*th term a_n of the AP with first term a and common difference d is given by $a_n = a + (n - 1)\, d$.**

a_n is also called the **general term of the AP**. If there are m terms in the AP, then a_m represents the **last term which is sometimes also denoted by *l***.

Let us consider some examples.

Example 3 : Find the 10th term of the AP : 2, 7, 12, . . .

Solution : Here, $a = 2$, $d = 7 - 2 = 5$ and $n = 10$.

We have $a_n = a + (n - 1)\, d$

So, $a_{10} = 2 + (10 - 1) \times 5 = 2 + 45 = 47$

Therefore, the 10th term of the given AP is 47.

Example 4 : Which term of the AP : 21, 18, 15, . . . is -81? Also, is any term 0? Give reason for your answer.

Solution : Here, $a = 21$, $d = 18 - 21 = -3$ and $a_n = -81$, and we have to find n.

As $a_n = a + (n - 1)\, d$,

we have $-81 = 21 + (n - 1)(-3)$

$-81 = 24 - 3n$

$-105 = -3n$

So, $n = 35$

Therefore, the 35th term of the given AP is -81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)(-3) = 0,$$

i.e., $3(n - 1) = 21$

i.e., $n = 8$

So, the eighth term is 0.

Example 5 : Determine the AP whose 3rd term is 5 and the 7th term is 9.

Solution : We have

$$a_3 = a + (3 - 1)\, d = a + 2d = 5 \qquad (1)$$

and $a_7 = a + (7 - 1)\, d = a + 6d = 9 \qquad (2)$

Solving the pair of linear equations (1) and (2), we get

$$a = 3, \quad d = 1$$

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example 6 : Check whether 301 is a term of the list of numbers 5, 11, 17, 23, . . .

Solution : We have :

$$a_2 - a_1 = 11 - 5 = 6, \quad a_3 - a_2 = 17 - 11 = 6, \quad a_4 - a_3 = 23 - 17 = 6$$

As $a_{k+1} - a_k$ is the same for $k = 1, 2, 3$, etc., the given list of numbers is an AP.

Now, $a = 5$ and $d = 6$.

Let 301 be a term, say, the nth term of this AP.

We know that

$$a_n = a + (n - 1)\, d$$

So, $$301 = 5 + (n - 1) \times 6$$

i.e., $$301 = 6n - 1$$

So, $$n = \frac{302}{6} = \frac{151}{3}$$

But n should be a positive integer (Why?). So, 301 is not a term of the given list of numbers.

Example 7 : How many two-digit numbers are divisible by 3?

Solution : The list of two-digit numbers divisible by 3 is :

$$12, 15, 18, \ldots, 99$$

Is this an AP? Yes it is. Here, $a = 12$, $d = 3$, $a_n = 99$.

As $$a_n = a + (n - 1)\, d,$$

we have $$99 = 12 + (n - 1) \times 3$$

i.e., $$87 = (n - 1) \times 3$$

i.e., $$n - 1 = \frac{87}{3} = 29$$

i.e., $$n = 29 + 1 = 30$$

So, there are 30 two-digit numbers divisible by 3.

Example 8 : Find the 11th term from the last term (towards the first term) of the AP : 10, 7, 4, . . ., – 62.

Solution : Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,

where $$l = a + (n - 1)\, d$$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $$-62 = 10 + (n-1)(-3)$$

i.e., $$-72 = (n-1)(-3)$$

i.e., $$n - 1 = 24$$

or $$n = 25$$

So, there are 25 terms in the given AP.

The 11th term from the last term will be the 15th term. (Note that it will not be the 14th term. Why?)

So, $$a_{15} = 10 + (15-1)(-3) = 10 - 42 = -32$$

i.e., the 11th term from the last term is – 32.

Alternative Solution :

If we write the given AP in the reverse order, then $a = -62$ and $d = 3$ (Why?)

So, the question now becomes finding the 11th term with these a and d.

So, $$a_{11} = -62 + (11-1) \times 3 = -62 + 30 = -32$$

So, the 11th term, which is now the required term, is – 32.

Example 9 : A sum of ₹ 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years making use of this fact.

Solution : We know that the formula to calculate simple interest is given by

$$\text{Simple Interest} = \frac{\text{P} \times \text{R} \times \text{T}}{100}$$

So, the interest at the end of the 1st year $= ₹\ \frac{1000 \times 8 \times 1}{100} = ₹\ 80$

The interest at the end of the 2nd year $= ₹\ \frac{1000 \times 8 \times 2}{100} = ₹\ 160$

The interest at the end of the 3rd year $= ₹\ \frac{1000 \times 8 \times 3}{100} = ₹\ 240$

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on.

So, the interest (in ₹) at the end of the 1st, 2nd, 3rd, . . . years, respectively are

$$80, 160, 240, \ldots$$

It is an AP as the difference between the consecutive terms in the list is 80, i.e., $d = 80$. Also, $a = 80$.

So, to find the interest at the end of 30 years, we shall find a_{30}.

Now, $a_{30} = a + (30 - 1)\, d = 80 + 29 \times 80 = 2400$

So, the interest at the end of 30 years will be ₹ 2400.

Example 10 : In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

Solution : The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are :

$$23, 21, 19, \ldots, 5$$

It forms an AP (Why?). Let the number of rows in the flower bed be n.

Then $a = 23,\quad d = 21 - 23 = -2,\quad a_n = 5$

As, $a_n = a + (n - 1)\, d$

We have, $5 = 23 + (n - 1)(-2)$

i.e., $-18 = (n - 1)(-2)$

i.e., $n = 10$

So, there are 10 rows in the flower bed.

EXERCISE 5.2

1. Fill in the blanks in the following table, given that a is the first term, d the common difference and a_n the nth term of the AP:

	a	d	n	a_n
(i)	7	3	8	. . .
(ii)	– 18	. . .	10	0
(iii)	. . .	– 3	18	– 5
(iv)	– 18.9	2.5	. . .	3.6
(v)	3.5	0	105	. . .

2. Choose the correct choice in the following and justify :

(i) 30th term of the AP: 10, 7, 4, . . . , is

(A) 97 (B) 77 (C) -77 (D) -87

(ii) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots,$ is

(A) 28 (B) 22 (C) -38 (D) $-48\frac{1}{2}$

3. In the following APs, find the missing terms in the boxes :

(i) 2, $\square$, 26

(ii) $\square$, 13, $\square$, 3

(iii) 5, $\square$, $\square$, $9\frac{1}{2}$

(iv) -4, $\square$, $\square$, $\square$, $\square$, 6

(v) $\square$, 38, $\square$, $\square$, $\square$, -22

4. Which term of the AP : 3, 8, 13, 18, . . . ,is 78?

5. Find the number of terms in each of the following APs :

(i) 7, 13, 19, . . . , 205 (ii) $18, 15\frac{1}{2}, 13, \ldots, -47$

6. Check whether -150 is a term of the AP : 11, 8, 5, 2 . . .

7. Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

8. An AP consists of 50 terms of which 3rd term is 12 and the last term is 106. Find the 29th term.

9. If the 3rd and the 9th terms of an AP are 4 and -8 respectively, which term of this AP is zero?

10. The 17th term of an AP exceeds its 10th term by 7. Find the common difference.

11. Which term of the AP : 3, 15, 27, 39, . . . will be 132 more than its 54th term?

12. Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

13. How many three-digit numbers are divisible by 7?

14. How many multiples of 4 lie between 10 and 250?

15. For what value of n, are the nth terms of two APs: 63, 65, 67, . . . and 3, 10, 17, . . . equal?

16. Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

17. Find the 20th term from the last term of the AP : 3, 8, 13, . . ., 253.

18. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

19. Subba Rao started work in 1995 at an annual salary of ₹ 5000 and received an increment of ₹ 200 each year. In which year did his income reach ₹ 7000?

20. Ramkali saved ₹ 5 in the first week of a year and then increased her weekly savings by ₹ 1.75. If in the nth week, her weekly savings become ₹ 20.75, find n.

5.4 Sum of First n Terms of an AP

Let us consider the situation again given in Section 5.1 in which Shakila put ₹ 100 into her daughter's money box when she was one year old, ₹ 150 on her second birthday, ₹ 200 on her third birthday and will continue in the same way. How much money will be collected in the money box by the time her daughter is 21 years old?

Here, the amount of money (in ₹) put in the money box on her first, second, third, fourth . . . birthday were respectively 100, 150, 200, 250, . . . till her 21st birthday. To find the total amount in the money box on her 21st birthday, we will have to write each of the 21 numbers in the list above and then add them up. Don't you think it would be a tedious and time consuming process? Can we make the process shorter? This would be possible if we can find a method for getting this sum. Let us see.

We consider the problem given to Gauss (about whom you read in Chapter 1), to solve when he was just 10 years old. He was asked to find the sum of the positive integers from 1 to 100. He immediately replied that the sum is 5050. Can you guess how did he do? He wrote :

$$S = 1 + 2 + 3 + . . . + 99 + 100$$

And then, reversed the numbers to write

$$S = 100 + 99 + . . . + 3 + 2 + 1$$

Adding these two, he got

$$2S = (100 + 1) + (99 + 2) + . . . + (3 + 98) + (2 + 99) + (1 + 100)$$

$$= 101 + 101 + . . . + 101 + 101 \quad \text{(100 times)}$$

So, $$S = \frac{100 \times 101}{2} = 5050, \text{ i.e., the sum} = 5050.$$

We will now use the same technique to find the sum of the first n terms of an AP :

$$a, a + d, a + 2d, \ldots$$

The nth term of this AP is $a + (n - 1)\, d$. Let S denote the sum of the first n terms of the AP. We have

$$S = a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)\, d] \qquad (1)$$

Rewriting the terms in reverse order, we have

$$S = [a + (n - 1)\, d] + [a + (n - 2)\, d] + \ldots + (a + d) + a \qquad (2)$$

On adding (1) and (2), term-wise. we get

$$2S = \underbrace{[2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d] + [2a + (n - 1)d]}_{n \text{ times}}$$

or, $2S = n\, [2a + (n - 1)\, d]$ (Since, there are n terms)

or, $S = \dfrac{n}{2}\, [2a + (n - 1)\, d]$

So, **the sum of the first n terms of an AP is given by**

$$\mathbf{S = \frac{n}{2}\, [2a + (n - 1)\, d]}$$

We can also write this as $S = \dfrac{n}{2}\, [a + a + (n - 1)\, d]$

i.e.,
$$\mathbf{S = \frac{n}{2}\, (a + a_n)} \qquad (3)$$

Now, if there are only n terms in an AP, then $a_n = l$, the last term.
From (3), we see that

$$\mathbf{S = \frac{n}{2}\, (a + l)} \qquad (4)$$

This form of the result is useful when the first and the last terms of an AP are given and the common difference is not given.

Now we return to the question that was posed to us in the beginning. The amount of money (in Rs) in the money box of Shakila's daughter on 1st, 2nd, 3rd, 4th birthday, . . ., were 100, 150, 200, 250, . . ., respectively.

This is an AP. We have to find the total money collected on her 21st birthday, i.e., the sum of the first 21 terms of this AP.

Here, $a = 100$, $d = 50$ and $n = 21$. Using the formula :

$$S = \frac{n}{2}\left[2a + (n-1)\,d\right],$$

we have $$S = \frac{21}{2}\left[2 \times 100 + (21-1) \times 50\right] = \frac{21}{2}\left[200 + 1000\right]$$

$$= \frac{21}{2} \times 1200 = 12600$$

So, the amount of money collected on her 21st birthday is ₹ 12600.

Hasn't the use of the formula made it much easier to solve the problem?

We also use S_n in place of S to denote the sum of first n terms of the AP. We write S_{20} to denote the sum of the first 20 terms of an AP. The formula for the sum of the first n terms involves four quantities S, a, d and n. If we know any three of them, we can find the fourth.

Remark : The nth term of an AP is the difference of the sum to first n terms and the sum to first $(n-1)$ terms of it, i.e., $a_n = S_n - S_{n-1}$.

Let us consider some examples.

Example 11 : Find the sum of the first 22 terms of the AP : 8, 3, –2, . . .

Solution : Here, $a = 8$, $d = 3 - 8 = -5$, $n = 22$.

We know that

$$S = \frac{n}{2}\left[2a + (n-1)\,d\right]$$

Therefore, $$S = \frac{22}{2}\left[16 + 21\,(-5)\right] = 11(16 - 105) = 11(-89) = -979$$

So, the sum of the first 22 terms of the AP is – 979.

Example 12 : If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Solution : Here, $S_{14} = 1050$, $n = 14$, $a = 10$.

As $$S_n = \frac{n}{2}\left[2a + (n-1)d\right],$$

so, $$1050 = \frac{14}{2}\left[20 + 13d\right] = 140 + 91d$$

i.e., $910 = 91d$

or, $d = 10$

Therefore, $a_{20} = 10 + (20 - 1) \times 10 = 200$, i.e. 20th term is 200.

Example 13 : How many terms of the AP : 24, 21, 18, . . . must be taken so that their sum is 78?

Solution : Here, $a = 24$, $d = 21 - 24 = -3$, $S_n = 78$. We need to find n.

We know that $$S_n = \frac{n}{2}[2a + (n-1)d]$$

So, $$78 = \frac{n}{2}[48 + (n-1)(-3)] = \frac{n}{2}[51 - 3n]$$

or $3n^2 - 51n + 156 = 0$

or $n^2 - 17n + 52 = 0$

or $(n - 4)(n - 13) = 0$

or $n = 4$ or 13

Both values of n are admissible. So, the number of terms is either 4 or 13.

Remarks:

1. In this case, the sum of the first 4 terms = the sum of the first 13 terms = 78.
2. Two answers are possible because the sum of the terms from 5th to 13th will be zero. This is because a is positive and d is negative, so that some terms will be positive and some others negative, and will cancel out each other.

Example 14 : Find the sum of :

(i) the first 1000 positive integers (ii) the first n positive integers

Solution :

(i) Let $S = 1 + 2 + 3 + . . . + 1000$

Using the formula $S_n = \frac{n}{2}(a + l)$ for the sum of the first n terms of an AP, we have

$$S_{1000} = \frac{1000}{2}(1 + 1000) = 500 \times 1001 = 500500$$

So, the sum of the first 1000 positive integers is 500500.

(ii) Let $S_n = 1 + 2 + 3 + . . . + n$

Here $a = 1$ and the last term l is n.

Therefore, $\quad \mathrm{S}_n = \frac{n\,(1+n)}{2} \quad$ or $\quad \mathrm{S}_n = \frac{n\,(n+1)}{2}$

So, **the sum of first *n* positive integers is given by**

$$\mathbf{S}_n = \frac{\boldsymbol{n(n+1)}}{\mathbf{2}}$$

Example 15 : Find the sum of first 24 terms of the list of numbers whose nth term is given by

$$a_n = 3 + 2n$$

Solution :

As $\quad a_n = 3 + 2n,$

so, $\quad a_1 = 3 + 2 = 5$

$$a_2 = 3 + 2 \times 2 = 7$$

$$a_3 = 3 + 2 \times 3 = 9$$

$$\vdots$$

List of numbers becomes 5, 7, 9, 11, . . .

Here, $\quad 7 - 5 = 9 - 7 = 11 - 9 = 2$ and so on.

So, it forms an AP with common difference $d = 2$.

To find S_{24}, we have $n = 24$, $\quad a = 5$, $\quad d = 2$.

Therefore, $\quad \mathrm{S}_{24} = \frac{24}{2}\left[2\times 5 + (24-1)\times 2\right] = 12\left[10 + 46\right] = 672$

So, sum of first 24 terms of the list of numbers is 672.

Example 16 : A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find :

(i) the production in the 1st year

(ii) the production in the 10th year

(iii) the total production in first 7 years

Solution : (i) Since the production increases uniformly by a fixed number every year, the number of TV sets manufactured in 1st, 2nd, 3rd, . . ., years will form an AP.

Let us denote the number of TV sets manufactured in the nth year by a_n.

Then, $\quad a_3 = 600$ and $a_7 = 700$

or, $a + 2d = 600$

and $a + 6d = 700$

Solving these equations, we get $d = 25$ and $a = 550$.

Therefore, production of TV sets in the first year is 550.

(ii) Now $a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production of TV sets in the 10th year is 775.

(iii) Also,
$$S_7 = \frac{7}{2}[2 \times 550 + (7-1) \times 25]$$
$$= \frac{7}{2}[1100 + 150] = 4375$$

Thus, the total production of TV sets in first 7 years is 4375.

EXERCISE 5.3

1. Find the sum of the following APs:

(i) 2, 7, 12, . . ., to 10 terms.

(ii) –37, –33, –29, . . ., to 12 terms.

(iii) 0.6, 1.7, 2.8, . . ., to 100 terms.

(iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$, . . ., to 11 terms.

2. Find the sums given below :

(i) $7 + 10\frac{1}{2} + 14 + \ldots + 84$

(ii) $34 + 32 + 30 + \ldots + 10$

(iii) $-5 + (-8) + (-11) + \ldots + (-230)$

3. In an AP:

(i) given $a = 5$, $d = 3$, $a_n = 50$, find n and S_n.

(ii) given $a = 7$, $a_{13} = 35$, find d and S_{13}.

(iii) given $a_{12} = 37$, $d = 3$, find a and S_{12}.

(iv) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.

(v) given $d = 5$, $S_9 = 75$, find a and a_9.

(vi) given $a = 2$, $d = 8$, $S_n = 90$, find n and a_n.

(vii) given $a = 8$, $a_n = 62$, $S_n = 210$, find n and d.

(viii) given $a_n = 4$, $d = 2$, $S_n = -14$, find n and a.

(ix) given $a = 3$, $n = 8$, $S = 192$, find d.

(x) given $l = 28$, $S = 144$, and there are total 9 terms. Find a.

4. How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636?

5. The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

6. The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

7. Find the sum of first 22 terms of an AP in which $d = 7$ and 22nd term is 149.

8. Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

9. If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

10. Show that $a_1, a_2, \ldots, a_n, \ldots$ form an AP where a_n is defined as below :

    (i) $a_n = 3 + 4n$ (ii) $a_n = 9 - 5n$

    Also find the sum of the first 15 terms in each case.

11. If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (that is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

12. Find the sum of the first 40 positive integers divisible by 6.

13. Find the sum of the first 15 multiples of 8.

14. Find the sum of the odd numbers between 0 and 50.

15. A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹ 200 for the first day, ₹ 250 for the second day, ₹ 300 for the third day, etc., the penalty for each succeeding day being ₹ 50 more than for the preceding day. How much money the contractor has to pay as penalty, if he has delayed the work by 30 days?

16. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each of the prizes.

17. In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

18. A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, . . . as shown in Fig. 5.4. What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)

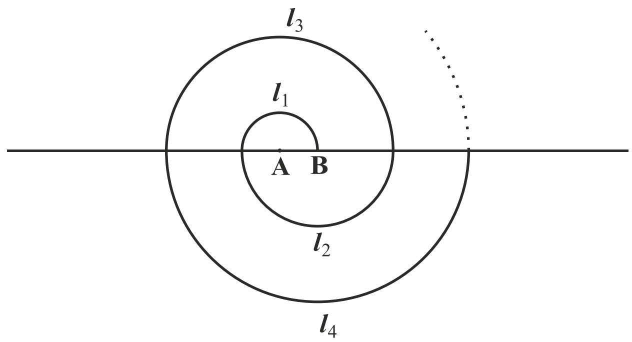


Fig. 5.4

[**Hint :** Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, . . ., respectively.]

19. 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see Fig. 5.5). In how many rows are the 200 logs placed and how many logs are in the top row?

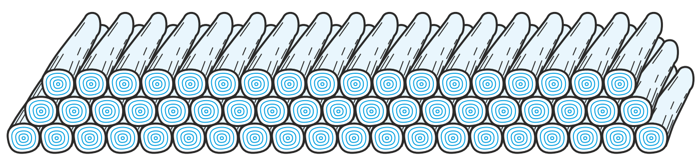


Fig. 5.5

20. In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato, and the other potatoes are placed 3 m apart in a straight line. There are ten potatoes in the line (see Fig. 5.6).

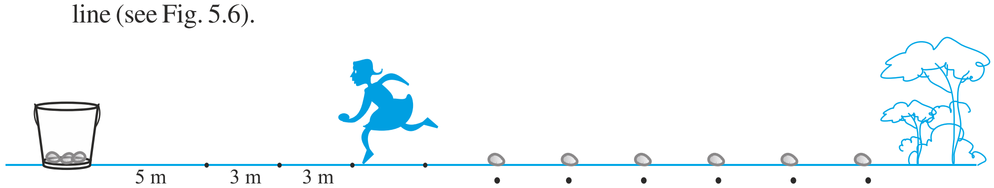


Fig. 5.6

A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?

[**Hint :** To pick up the first potato and the second potato, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$]

EXERCISE 5.4 (Optional)*

1. Which term of the AP : 121, 117, 113, . . ., is its first negative term?

   [**Hint :** Find n for $a_n < 0$]

2. The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

3. A ladder has rungs 25 cm apart. (see Fig. 5.7). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are $2\frac{1}{2}$ m apart, what is the length of the wood required for the rungs?

   [**Hint :** Number of rungs $= \frac{250}{25} + 1$]

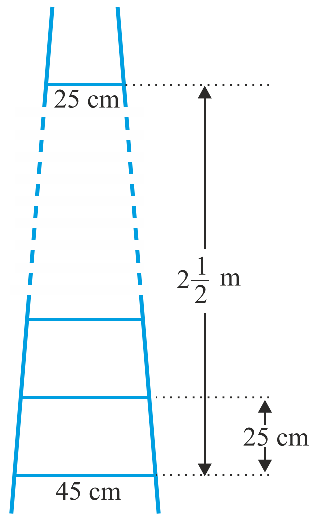


Fig. 5.7

4. The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find this value of x.

   [**Hint :** $S_{x-1} = S_{49} - S_x$]

5. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.

   Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. (see Fig. 5.8). Calculate the total volume of concrete required to build the terrace.

   [**Hint :** Volume of concrete required to build the first step $= \frac{1}{4} \times \frac{1}{2} \times 50 \text{ m}^3$]

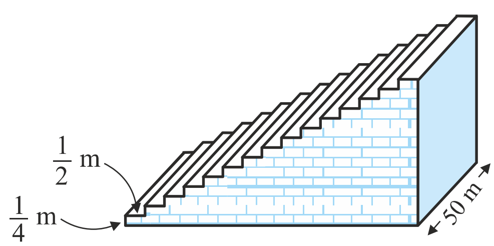


Fig. 5.8

* These exercises are not from the examination point of view.

5.5 Summary

In this chapter, you have studied the following points :

1. An **arithmetic progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number d to the preceding term, except the first term. The fixed number d is called the **common difference**.

   The general form of an AP is $a, \ a + d, \ a + 2d, \ a + 3d, \ldots$

2. A given list of numbers $a_1, a_2, a_3, \ldots$ is an AP, if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$, give the same value, i.e., if $a_{k+1} - a_k$ is the same for different values of k.

3. In an AP with first term a and common difference d, the nth term (or the general term) is given by $a_n = a + (n - 1)\, d.$

4. The sum of the first n terms of an AP is given by :

$$S = \frac{n}{2}[2a + (n-1)d]$$

5. If l is the last term of the finite AP, say the nth term, then the sum of all terms of the AP is given by :

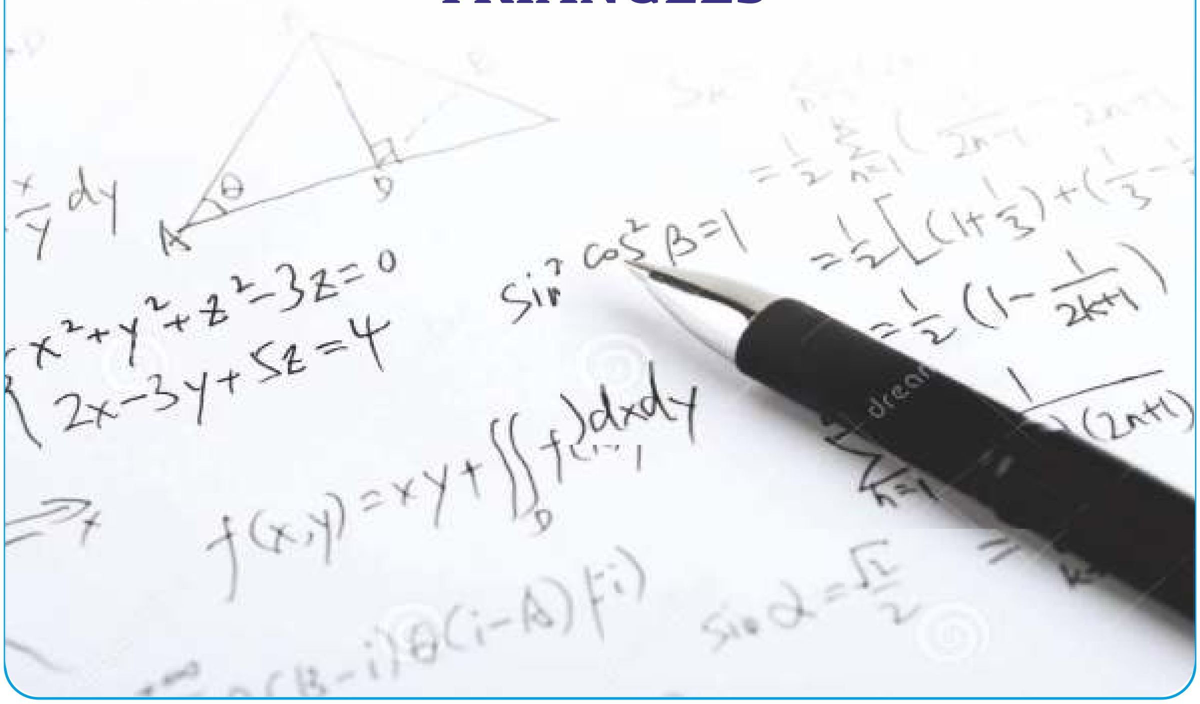
$$S = \frac{n}{2}(a + l)$$

A Note to the Reader

If a, b, c are in AP, then $b = \frac{a+c}{2}$ and b is called the arithmetic mean of a and c.

6

TRIANGLES

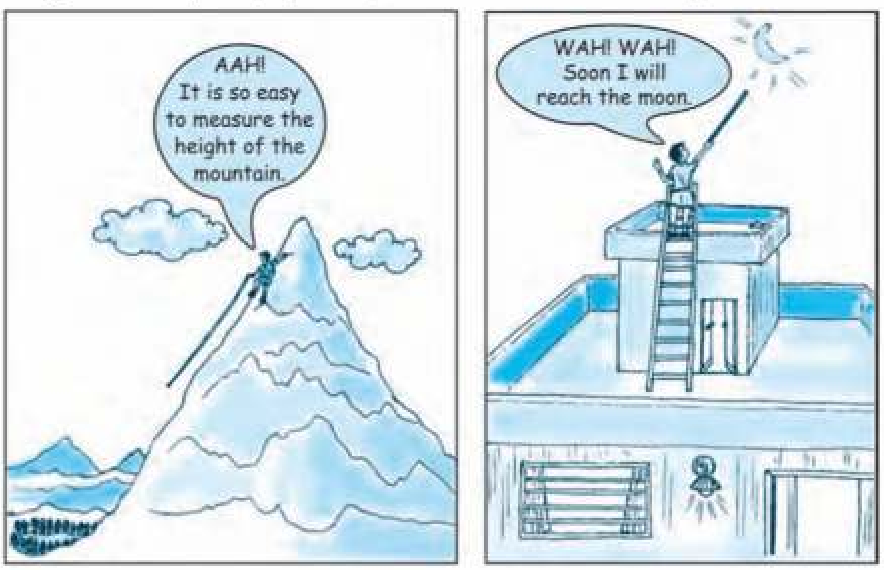


TRIANGLES 6

6.1 Introduction

You are familiar with triangles and many of their properties from your earlier classes. In Class IX, you have studied congruence of triangles in detail. Recall that two figures are said to be *congruent*, if they have the same shape and the same size. In this chapter, we shall study about those figures which have the same shape but not necessarily the same size. Two figures having the same shape (and not necessarily the same size) are called *similar figures.* In particular, we shall discuss the similarity of triangles and apply this knowledge in giving a simple proof of Pythagoras Theorem learnt earlier.

Can you guess how heights of mountains (say Mount Everest) or distances of some long distant objects (say moon) have been found out? Do you think these have



been measured directly with the help of a measuring tape? In fact, all these heights and distances have been found out using the idea of indirect measurements, which is based on the principle of similarity of figures (see Example 7, Q.15 of Exercise 6.3 and also Chapters 8 and 9 of this book).

6.2 Similar Figures

In Class IX, you have seen that all circles with the same radii are congruent, all squares with the same side lengths are congruent and all equilateral triangles with the same side lengths are congruent.

Now consider any two (or more) circles [see Fig. 6.1 (i)]. Are they congruent? Since all of them do not have the same radius, they are not congruent to each other. Note that some are congruent and some are not, but all of them have the same shape. So they all are, what we call, *similar*. Two similar figures have the same shape but not necessarily the same size. Therefore, all circles are similar. What about two (or more) squares or two (or more) equilateral triangles [see Fig. 6.1 (ii) and (iii)]? As observed in the case of circles, here also all squares are similar and all equilateral triangles are similar.

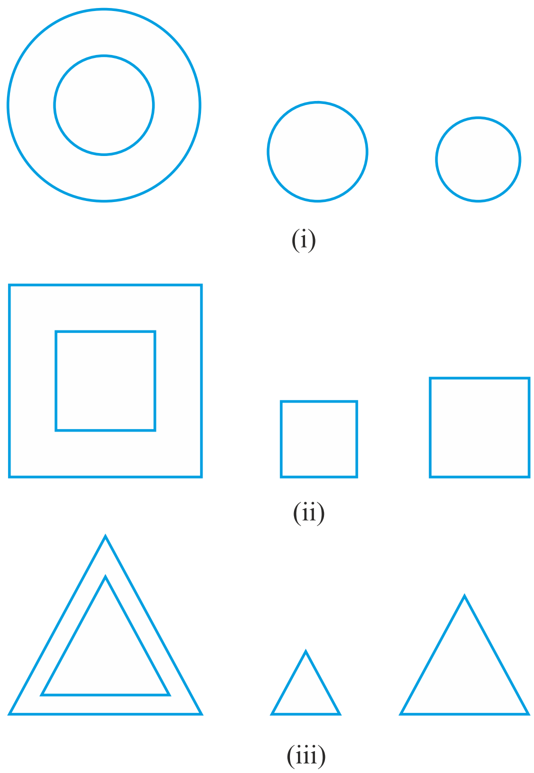


Fig. 6.1

From the above, we can say *that all congruent figures are similar but the similar figures need not be congruent.*

Can a circle and a square be similar? Can a triangle and a square be similar? These questions can be answered by just looking at the figures (see Fig. 6.1). Evidently these figures are not similar. (Why?)

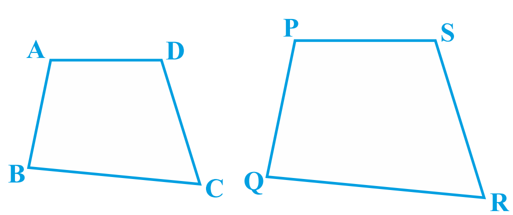


Fig. 6.2

What can you say about the two quadrilaterals ABCD and PQRS (see Fig 6.2)?Are they similar? These figures appear to be similar but we cannot be certain about it.Therefore, we must have some definition of similarity of figures and based on this definition some rules to decide whether the two given figures are similar or not. For this, let us look at the photographs given in Fig. 6.3:

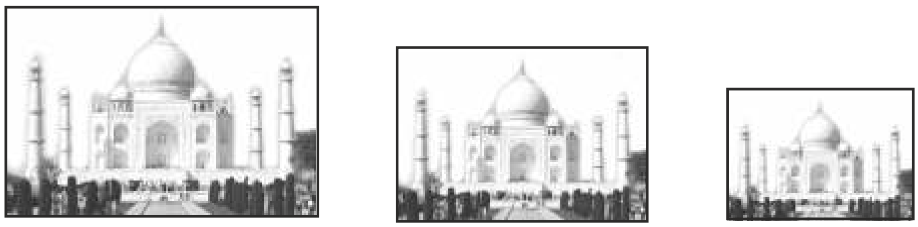


Fig. 6.3

You will at once say that they are the photographs of the same monument (Taj Mahal) but are in different sizes. Would you say that the three photographs are similar? Yes,they are.

What can you say about the two photographs of the same size of the same person one at the age of 10 years and the other at the age of 40 years? Are these photographs similar? These photographs are of the same size but certainly they are not of the same shape. So, they are not similar.

What does the photographer do when she prints photographs of different sizes from the same negative? You must have heard about the stamp size, passport size and postcard size photographs. She generally takes a photograph on a small size film, say of 35mm size and then enlarges it into a bigger size, say 45mm (or 55mm). Thus, if we consider any line segment in the smaller photograph (figure), its corresponding line segment in the bigger photograph (figure) will be $\frac{45}{35}\left(\text{or } \frac{55}{35}\right)$ of that of the line segment. This really means that every line segment of the smaller photograph is enlarged (increased) *in the ratio* 35:45 (or 35:55). It can also be said that every line segment of the bigger photograph is reduced (decreased) in the ratio 45:35 (or 55:35). Further, if you consider inclinations (or angles) between any pair of corresponding line segments in the two photographs of different sizes, you shall see that these inclinations(or angles) *are always equal.* This is the essence of the similarity of two figures and in particular of two polygons. We say that:

Two polygons of the same number of sides are similar, if (*i*) *their corresponding angles are equal and* (*ii*) *their corresponding sides are in the same ratio* (*or proportion*).

Note that the same ratio of the corresponding sides is referred to as *the scale factor* (or the *Representative Fraction*) for the polygons. You must have heard that world maps (i.e., global maps) and blue prints for the construction of a building are prepared using a suitable scale factor and observing certain conventions.

In order to understand similarity of figures more clearly, let us perform the following activity:

Activity 1 : Place a lighted bulb at a point O on the ceiling and directly below it a table in your classroom. Let us cut a polygon, say a quadrilateral ABCD, from a plane cardboard and place this cardboard parallel to the ground between the lighted bulb and the table. Then a shadow of ABCD is cast on the table. Mark the outline of this shadow as A′B′C′D′ (see Fig.6.4).

Fig. 6.4

Note that the quadrilateral A′B′C′D′ is an enlargement (or magnification) of the quadrilateral ABCD. This is because of the property of light that light propogates in a straight line. You may also note that A′ lies on ray OA, B′ lies on ray OB, C′ lies on OC and D′ lies on OD. Thus, quadrilaterals A′B′C′D′ and ABCD are of the same shape but of different sizes.

So, quadrilateral A′B′C′D′ is similiar to quadrilateral ABCD. We can also say that quadrilateral ABCD is similar to the quadrilateral A′B′C′D′.

Here, you can also note that vertex A′ corresponds to vertex A, vertex B′ corresponds to vertex B, vertex C′ corresponds to vertex C and vertex D′ corresponds to vertex D. Symbolically, these correspondences are represented as $A' \leftrightarrow A$, $B' \leftrightarrow B$, $C' \leftrightarrow C$ and $D' \leftrightarrow D$. By actually measuring the angles and the sides of the two quadrilaterals, you may verify that

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$.

This again emphasises that *two polygons of the same number of sides are similar, if* (*i*) *all the corresponding angles are equal and* (*ii*) *all the corresponding sides are in the same ratio* (*or proportion*).

From the above, you can easily say that quadrilaterals ABCD and PQRS of Fig. 6.5 are similar.

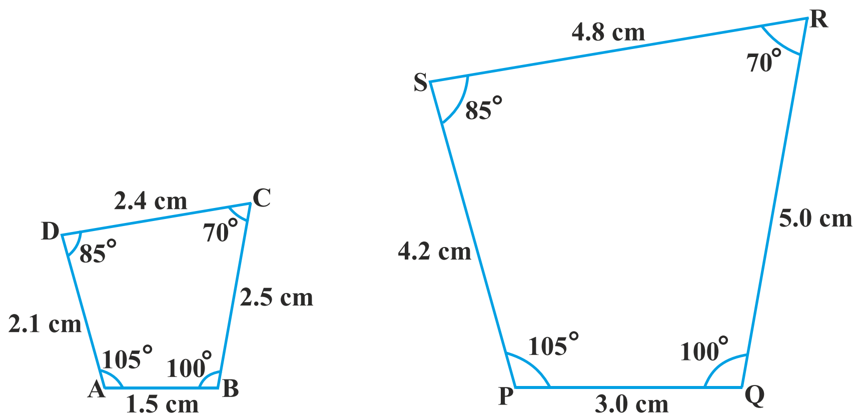


Fig. 6.5

Remark : You can verify that if one polygon is similar to another polygon and this second polygon is similar to a third polygon, then the first polygon is similar to the third polygon.

You may note that in the two quadrilaterals (a square and a rectangle) of Fig. 6.6, corresponding angles are equal, but their corresponding sides are not in the same ratio.

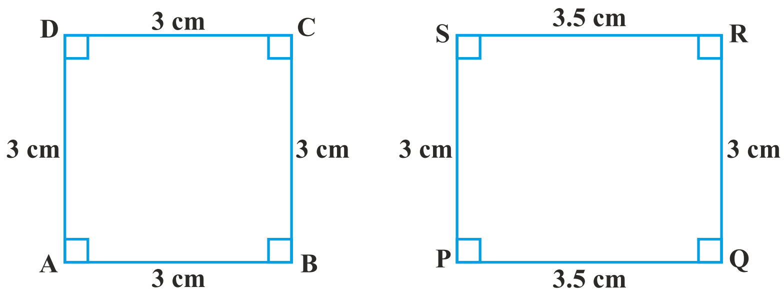


Fig. 6.6

So, the two quadrilaterals are not similar. Similarly, you may note that in the two quadrilaterals (a square and a rhombus) of Fig. 6.7, corresponding sides are in the same ratio, but their corresponding angles are not equal. Again, the two polygons (quadrilaterals) are not similar.

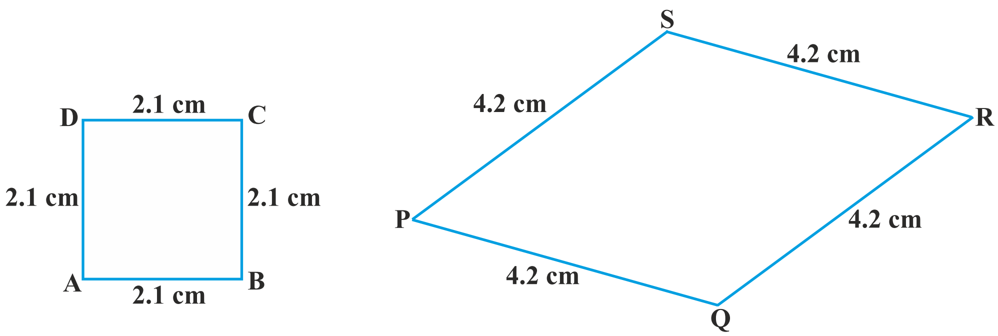


Fig. 6.7

Thus, *either of the above two conditions* (*i*) *and* (*ii*) *of similarity of two polygons is not sufficient for them to be similar.*

EXERCISE 6.1

1. Fill in the blanks using the correct word given in brackets :

(i) All circles are ________. (congruent, similar)

(ii) All squares are ________. (similar, congruent)

(iii) All ________ triangles are similar. (isosceles, equilateral)

(iv) Two polygons of the same number of sides are similar, if (a) their corresponding angles are ________ and (b) their corresponding sides are ________. (equal, proportional)

2. Give two different examples of pair of

(i) similar figures. (ii) non-similar figures.

3. State whether the following quadrilaterals are similar or not:

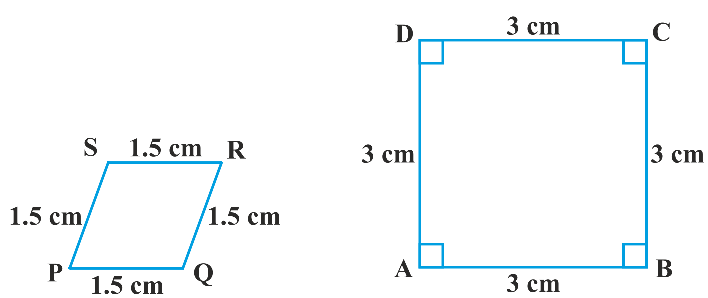


Fig. 6.8

6.3 Similarity of Triangles

What can you say about the similarity of two triangles?

You may recall that triangle is also a polygon. So, we can state the same conditions for the similarity of two triangles. That is:

Two triangles are similiar, if

(i) their corresponding angles are equal and

(ii) their corresponding sides are in the same ratio (or proportion).

Note that if corresponding angles of two triangles are equal, then they are known as *equiangular triangles.* A famous Greek mathematician Thales gave an important truth relating to two equiangular triangles which is as follows:

Thales
(640 – 546 B.C.)

The ratio of any two corresponding sides in two equiangular triangles is always the same.

It is believed that he had used a result called the *Basic Proportionality Theorem* (now known as the *Thales Theorem*) for the same.

To understand the Basic Proportionality Theorem, let us perform the following activity:

Activity 2 : Draw any angle XAY and on its one arm AX, mark points (say five points) P, Q, D, R and B such that AP = PQ = QD = DR = RB.

Now, through B, draw any line intersecting arm AY at C (see Fig. 6.9).

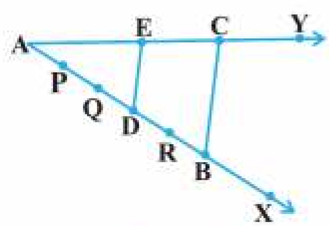


Fig. 6.9

Also, through the point D, draw a line parallel to BC to intersect AC at E. Do you observe from your constructions that $\frac{AD}{DB} = \frac{3}{2}$? Measure AE and EC. What about $\frac{AE}{EC}$? Observe that $\frac{AE}{EC}$ is also equal to $\frac{3}{2}$. Thus, you can see that in Δ ABC, DE || BC and $\frac{AD}{DB} = \frac{AE}{EC}$. Is it a coincidence? No, it is due to the following theorem (known as the Basic Proportionality Theorem):

Theorem 6.1 : *If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.*

Proof : We are given a triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively (see Fig. 6.10).

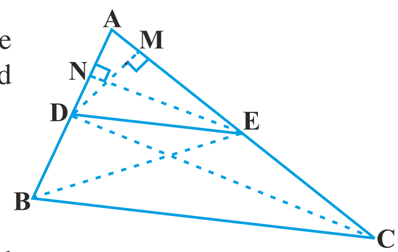


Fig. 6.10

We need to prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Let us join BE and CD and then draw DM $\perp$ AC and EN $\perp$ AB.

Now, area of Δ ADE (= $\frac{1}{2}$ base $\times$ height) = $\frac{1}{2}$ AD $\times$ EN.

Recall from Class IX, that area of Δ ADE is denoted as ar(ADE).

So, $$\text{ar(ADE)} = \frac{1}{2}\ \text{AD} \times \text{EN}$$

Similarly, $$\text{ar(BDE)} = \frac{1}{2}\ \text{DB} \times \text{EN},$$

$$\text{ar(ADE)} = \frac{1}{2}\ \text{AE} \times \text{DM} \quad \text{and} \quad \text{ar(DEC)} = \frac{1}{2}\ \text{EC} \times \text{DM}.$$

Therefore, $$\frac{\text{ar(ADE)}}{\text{ar(BDE)}} = \frac{\frac{1}{2}\ \text{AD} \times \text{EN}}{\frac{1}{2}\ \text{DB} \times \text{EN}} = \frac{\text{AD}}{\text{DB}} \qquad (1)$$

and $$\frac{\text{ar(ADE)}}{\text{ar(DEC)}} = \frac{\frac{1}{2}\ \text{AE} \times \text{DM}}{\frac{1}{2}\ \text{EC} \times \text{DM}} = \frac{\text{AE}}{\text{EC}} \qquad (2)$$

Note that Δ BDE and DEC are on the same base DE and between the same parallels BC and DE.

So, $$\text{ar(BDE)} = \text{ar(DEC)} \qquad (3)$$

Therefore, from (1), (2) and (3), we have :

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Is the converse of this theorem also true (For the meaning of converse, see Appendix 1)? To examine this, let us perform the following activity:

Activity 3 : Draw an angle XAY on your notebook and on ray AX, mark points B_1, B_2, B_3, B_4 and B such that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B$.

Similarly, on ray AY, mark points C_1, C_2, C_3, C_4 and C such that $AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C$. Then join B_1C_1 and BC (see Fig. 6.11).

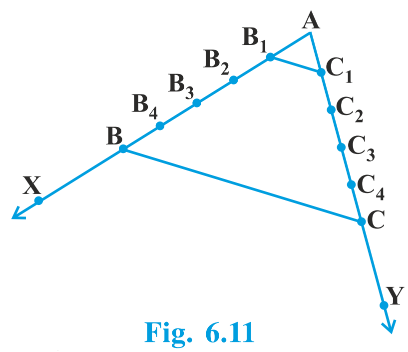


Fig. 6.11

Note that $\quad \dfrac{AB_1}{B_1B} = \dfrac{AC_1}{C_1C}$ (Each equal to $\dfrac{1}{4}$)

You can also see that lines B_1C_1 and BC are parallel to each other, i.e.,

$$B_1C_1 \parallel BC \qquad (1)$$

Similarly, by joining B_2C_2, B_3C_3 and B_4C_4, you can see that:

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C}\left(=\frac{2}{3}\right) \text{ and } B_2C_2 \parallel BC \qquad (2)$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C}\left(=\frac{3}{2}\right) \text{ and } B_3C_3 \parallel BC \qquad (3)$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C}\left(=\frac{4}{1}\right) \text{ and } B_4C_4 \parallel BC \qquad (4)$$

From (1), (2), (3) and (4), it can be observed that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

You can repeat this activity by drawing any angle XAY of different measure and taking any number of equal parts on arms AX and AY . Each time, you will arrive at the same result. Thus, we obtain the following theorem, which is the converse of Theorem 6.1:

Theorem 6.2 : *If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.*

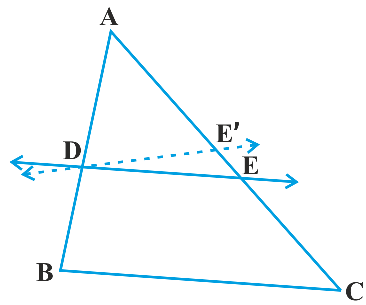


Fig. 6.12

This theorem can be proved by taking a line DE such that $\frac{AD}{DB} = \frac{AE}{EC}$ and assuming that DE is not parallel to BC (see Fig. 6.12).

If DE is not parallel to BC, draw a line DE′ parallel to BC.

So, $$\frac{AD}{DB} = \frac{AE'}{E'C} \quad \text{(Why ?)}$$

Therefore, $$\frac{AE}{EC} = \frac{AE'}{E'C} \quad \text{(Why ?)}$$

Adding 1 to both sides of above, you can see that E and E′ must coincide. (Why ?)

Let us take some examples to illustrate the use of the above theorems.

Example 1 : If a line intersects sides AB and AC of a Δ ABC at D and E respectively and is parallel to BC, prove that $\frac{AD}{AB} = \frac{AE}{AC}$ (see Fig. 6.13).

Solution : DE || BC (Given)

So, $$\frac{AD}{DB} = \frac{AE}{EC} \quad \text{(Theorem 6.1)}$$

or, $$\frac{DB}{AD} = \frac{EC}{AE}$$

or, $$\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1$$

or, $$\frac{AB}{AD} = \frac{AC}{AE}$$

So, $$\frac{AD}{AB} = \frac{AE}{AC}$$

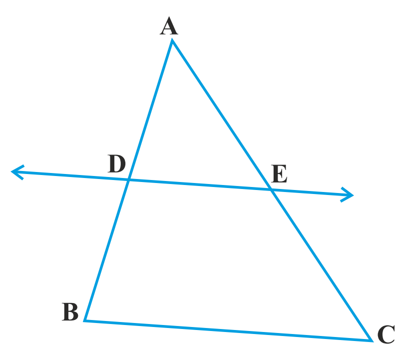


Fig. 6.13

Example 2 : ABCD is a trapezium with AB || DC. E and F are points on non-parallel sides AD and BC respectively such that EF is parallel to AB (see Fig. 6.14). Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

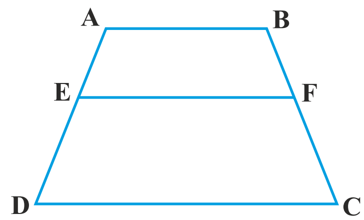


Fig. 6.14

Solution : Let us join AC to intersect EF at G (see Fig. 6.15).

AB || DC and EF || AB (Given)

So, EF || DC (Lines parallel to the same line are parallel to each other)

Now, in Δ ADC,

EG || DC (As EF || DC)

So, $\frac{AE}{ED} = \frac{AG}{GC}$ (Theorem 6.1) (1)

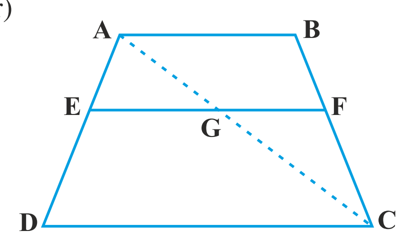


Fig. 6.15

Similarly, from Δ CAB,

$$\frac{CG}{AG} = \frac{CF}{BF}$$

i.e., $$\frac{AG}{GC} = \frac{BF}{FC} \quad (2)$$

Therefore, from (1) and (2),

$$\frac{AE}{ED} = \frac{BF}{FC}$$

Example 3 : In Fig. 6.16, $\frac{PS}{SQ} = \frac{PT}{TR}$ and ∠ PST = ∠ PRQ. Prove that PQR is an isosceles triangle.

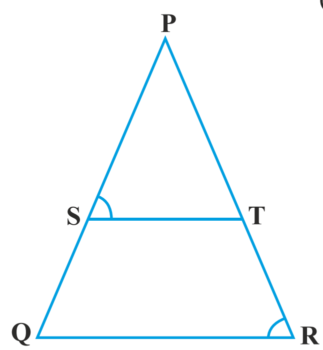


Fig. 6.16

Solution : It is given that $\frac{PS}{SQ} = \frac{PT}{TR}$.

So, ST || QR (Theorem 6.2)

Therefore, ∠ PST = ∠ PQR (Corresponding angles) (1)

Also, it is given that

$$\angle PST = \angle PRQ \qquad (2)$$

So, $\angle PRQ = \angle PQR$ [From (1) and (2)]

Therefore, PQ = PR (Sides opposite the equal angles)

i.e., PQR is an isosceles triangle.

EXERCISE 6.2

1. In Fig. 6.17, (i) and (ii), DE || BC. Find EC in (i) and AD in (ii).

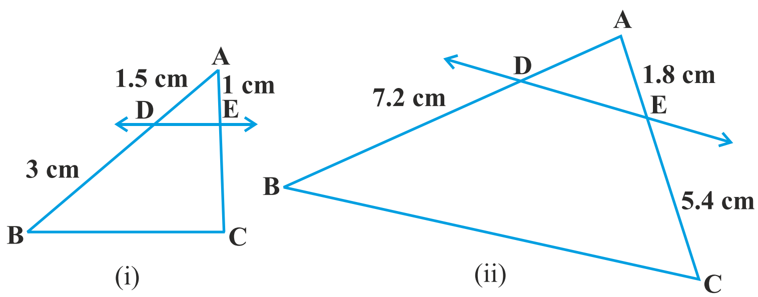


Fig. 6.17

2. E and F are points on the sides PQ and PR respectively of a Δ PQR. For each of the following cases, state whether EF || QR :

   (i) PE = 3.9 cm, EQ = 3 cm, PF = 3.6 cm and FR = 2.4 cm

   (ii) PE = 4 cm, QE = 4.5 cm, PF = 8 cm and RF = 9 cm

   (iii) PQ = 1.28 cm, PR = 2.56 cm, PE = 0.18 cm and PF = 0.36 cm

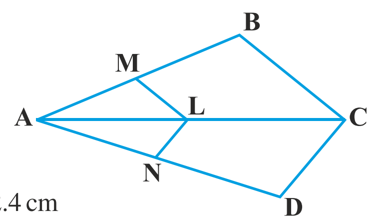


Fig. 6.18

3. In Fig. 6.18, if LM || CB and LN || CD, prove that

$$\frac{AM}{AB} = \frac{AN}{AD}.$$

4. In Fig. 6.19, DE || AC and DF || AE. Prove that

$$\frac{BF}{FE} = \frac{BE}{EC}.$$

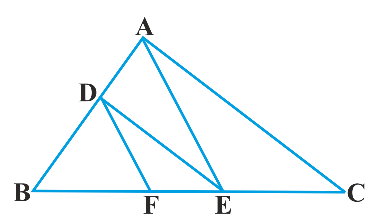


Fig. 6.19

5. In Fig. 6.20, DE || OQ and DF || OR. Show that EF || QR.

6. In Fig. 6.21, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and AC || PR. Show that BC || QR.

7. Using Theorem 6.1, prove that a line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side. (Recall that you have proved it in Class IX).

8. Using Theorem 6.2, prove that the line joining the mid-points of any two sides of a triangle is parallel to the third side. (Recall that you have done it in Class IX).

9. ABCD is a trapezium in which AB || DC and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$.

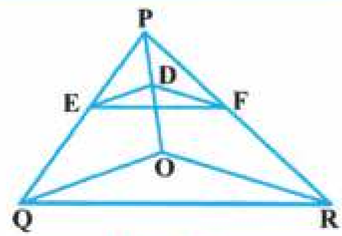


Fig. 6.20

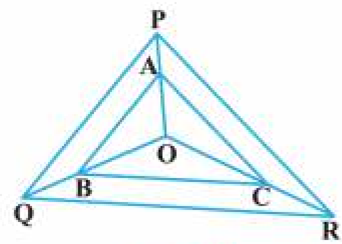


Fig. 6.21

10. The diagonals of a quadrilateral ABCD intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that ABCD is a trapezium.

6.4 Criteria for Similarity of Triangles

In the previous section, we stated that two triangles are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportion).

That is, in Δ ABC and Δ DEF, if

(i) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then the two triangles are similar (see Fig. 6.22).

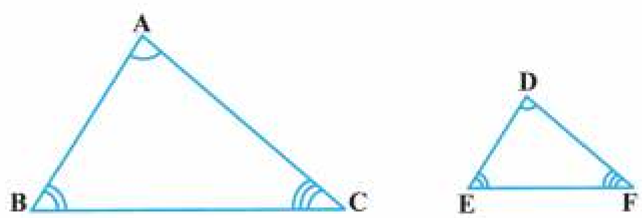


Fig. 6.22

Here, you can see that A corresponds to D, B corresponds to E and C corresponds to F. Symbolically, we write the similarity of these two triangles as 'Δ ABC ~ Δ DEF' and read it as 'triangle ABC is similar to triangle DEF'. The symbol '~' stands for 'is similar to'. Recall that you have used the symbol '≅' for 'is congruent to' in Class IX.

It must be noted that as done in the case of congruency of two triangles, the similarity of two triangles should also be expressed symbolically, using correct correspondence of their vertices. For example, for the triangles ABC and DEF of Fig. 6.22, we cannot write Δ ABC ~ Δ EDF or Δ ABC ~ Δ FED. However, we can write Δ BAC ~ Δ EDF.

Now a natural question arises : For checking the similarity of two triangles, say ABC and DEF, should we always look for all the equality relations of their corresponding angles ($\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$) and all the equality relations of the ratios of their corresponding sides $\left(\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\right)$? Let us examine. You may recall that in Class IX, you have obtained some criteria for congruency of two triangles involving only three pairs of corresponding parts (or elements) of the two triangles. Here also, let us make an attempt to arrive at certain criteria for similarity of two triangles involving relationship between less number of pairs of corresponding parts of the two triangles, instead of all the six pairs of corresponding parts. For this, let us perform the following activity:

Activity 4 : Draw two line segments BC and EF of two different lengths, say 3 cm and 5 cm respectively. Then, at the points B and C respectively, construct angles PBC and QCB of some measures, say, 60° and 40°. Also, at the points E and F, construct angles REF and SFE of 60° and 40° respectively (see Fig. 6.23).

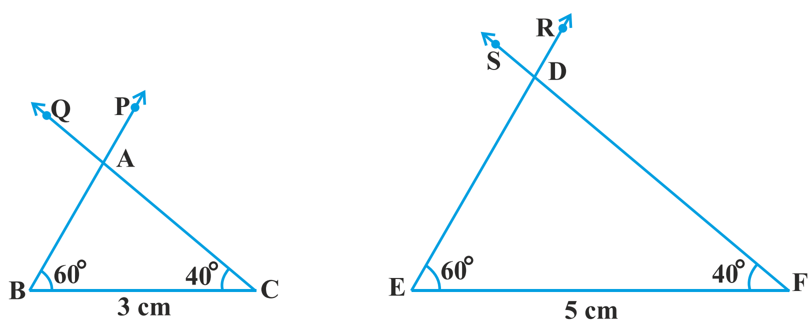


Fig. 6.23

Let rays BP and CQ intersect each other at A and rays ER and FS intersect each other at D. In the two triangles ABC and DEF, you can see that $\angle B = \angle E$, $\angle C = \angle F$ and $\angle A = \angle D$. That is, corresponding angles of these two triangles are equal. What can you say about their corresponding sides ? Note that $\frac{BC}{EF} = \frac{3}{5} = 0.6.$ What about $\frac{AB}{DE}$ and $\frac{CA}{FD}$? On measuring AB, DE, CA and FD, you will find that $\frac{AB}{DE}$ and $\frac{CA}{FD}$ are also equal to 0.6 (or nearly equal to 0.6, if there is some error in the measurement). Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\cdot$ You can repeat this activity by constructing several pairs of triangles having their corresponding angles equal. Every time, you will find that their corresponding sides are in the same ratio (or proportion). This activity leads us to the following criterion for similarity of two triangles.

Theorem 6.3 : *If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence the two triangles are similar.*

This criterion is referred to as the AAA (Angle–Angle–Angle) criterion of similarity of two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (see Fig. 6.24)

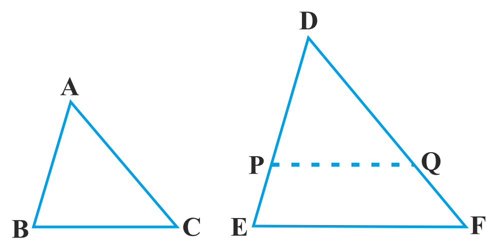


Fig. 6.24

Cut DP = AB and DQ = AC and join PQ.

So, $\Delta\, ABC \cong \Delta\, DPQ$ (Why ?)

This gives $\angle B = \angle P = \angle E$ and $PQ \parallel EF$ (How?)

Therefore, $$\frac{DP}{PE} = \frac{DQ}{QF} \quad \text{(Why?)}$$

i.e., $$\frac{AB}{DE} = \frac{AC}{DF} \quad \text{(Why?)}$$

Similarly, $\frac{AB}{DE} = \frac{BC}{EF}$ and so $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Remark : If two angles of a triangle are respectively equal to two angles of another triangle, then by the angle sum property of a triangle their third angles will also be equal. Therefore, AAA similarity criterion can also be stated as follows:

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

This may be referred to as the AA *similarity criterion* for two triangles.

You have seen above that if the three angles of one triangle are respectively equal to the three angles of another triangle, then their corresponding sides are proportional (i.e., in the same ratio). What about the converse of this statement? Is the converse true? In other words, if the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal? Let us examine it through an activity :

Activity 5 : Draw two triangles ABC and DEF such that AB = 3 cm, BC = 6 cm, CA = 8 cm, DE = 4.5 cm, EF = 9 cm and FD = 12 cm (see Fig. 6.25).

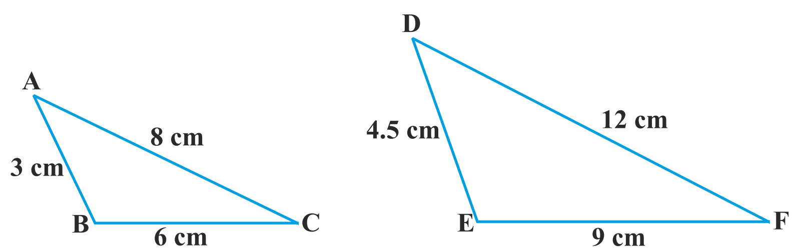


Fig. 6.25

So, you have : $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (each equal to $\frac{2}{3}$)

Now measure $\angle$ A, $\angle$ B, $\angle$ C, $\angle$ D, $\angle$ E and $\angle$ F. You will observe that $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F, i.e., the corresponding angles of the two triangles are equal.

You can repeat this activity by drawing several such triangles (having their sides in the same ratio). Everytime you shall see that their corresponding angles are equal. It is due to the following criterion of similarity of two triangles:

Theorem 6.4 : *If in two triangles, sides of one triangle are proportional to (i.e., in the same ratio of) the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similiar.*

This criterion is referred to as the SSS (Side–Side–Side) *similarity criterion for two triangles*.

This theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (< 1) (see Fig. 6.26):

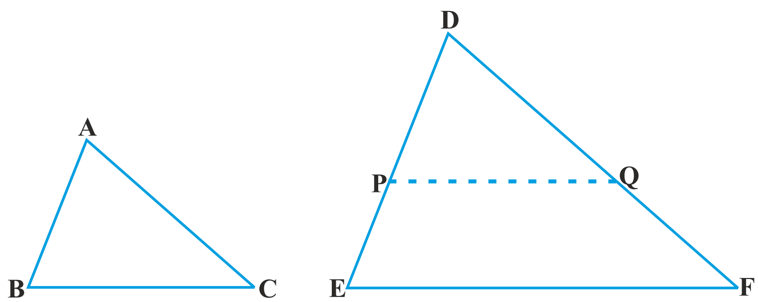


Fig. 6.26

Cut DP = AB and DQ = AC and join PQ.

It can be seen that $\frac{DP}{PE} = \frac{DQ}{QF}$ and PQ || EF (How?)

So, $\angle P = \angle E$ and $\angle Q = \angle F$.

Therefore, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$

So, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF}$ (Why?)

So, BC = PQ (Why?)

Thus, Δ ABC $\cong$ Δ DPQ (Why ?)

So, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (How ?)

Remark : You may recall that either of the two conditions namely, (i) corresponding angles are equal and (ii) corresponding sides are in the same ratio is not sufficient for two polygons to be similar. However, on the basis of Theorems 6.3 and 6.4, you can now say that in case of similarity of the two triangles, it is not necessary to check both the conditions as one condition implies the other.

Let us now recall the various criteria for congruency of two triangles learnt in Class IX. You may observe that SSS similarity criterion can be compared with the SSS congruency criterion.This suggests us to look for a similarity criterion comparable to SAS congruency criterion of triangles. For this, let us perform an activity.

Activity 6 : Draw two triangles ABC and DEF such that AB = 2 cm, $\angle A = 50°$, AC = 4 cm, DE = 3 cm, $\angle D = 50°$ and DF = 6 cm (see Fig.6.27).

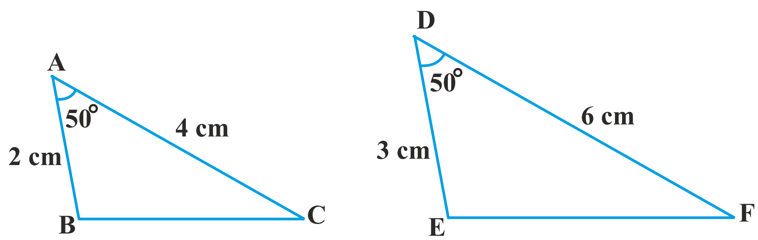


Fig. 6.27

Here, you may observe that $\frac{AB}{DE} = \frac{AC}{DF}$ (each equal to $\frac{2}{3}$) and $\angle$ A (included between the sides AB and AC) = $\angle$ D (included between the sides DE and DF). That is, one angle of a triangle is equal to one angle of another triangle and sides including these angles are in the same ratio (i.e., proportion). Now let us measure $\angle$ B, $\angle$ C, $\angle$ E and $\angle$ F.

You will find that $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. That is, $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. So, by AAA similarity criterion, Δ ABC ~ Δ DEF. You may repeat this activity by drawing several pairs of such triangles with one angle of a triangle equal to one angle of another triangle and the sides including these angles are proportional. Everytime, you will find that the triangles are similar. It is due to the following criterion of similarity of triangles:

Theorem 6.5 : *If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.*

This criterion is referred to as the SAS (Side–Angle–Side) similarity criterion for two triangles.

As before, this theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{AC}{DF}$ (< 1) and $\angle$ A = $\angle$ D (see Fig. 6.28). Cut DP = AB, DQ = AC and join PQ.

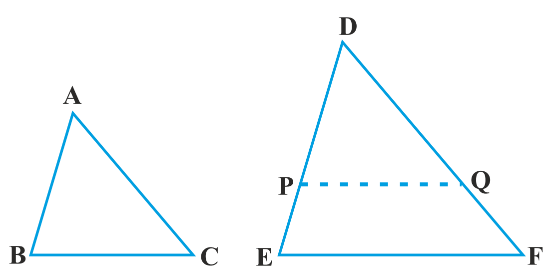


Fig. 6.28

Now, $\qquad PQ \parallel EF$ and $\Delta\ ABC \cong \Delta\ DPQ$ (How ?)

So, $\qquad \angle A = \angle D,\ \angle B = \angle P$ and $\angle C = \angle Q$

Therefore, $\qquad \Delta\ ABC \sim \Delta\ DEF$ (Why?)

We now take some examples to illustrate the use of these criteria.

Example 4 : In Fig. 6.29, if $PQ \parallel RS$, prove that $\Delta\ POQ \sim \Delta\ SOR$.

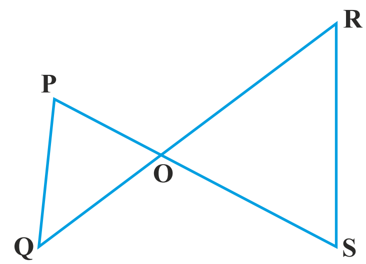


Fig. 6.29

Solution : $\qquad PQ \parallel RS$ (Given)

So, $\qquad \angle P = \angle S$ (Alternate angles)

and $\qquad \angle Q = \angle R$

Also, $\qquad \angle POQ = \angle SOR$ (Vertically opposite angles)

Therefore, $\qquad \Delta\ POQ \sim \Delta\ SOR$ (AAA similarity criterion)

Example 5 : Observe Fig. 6.30 and then find $\angle P$.

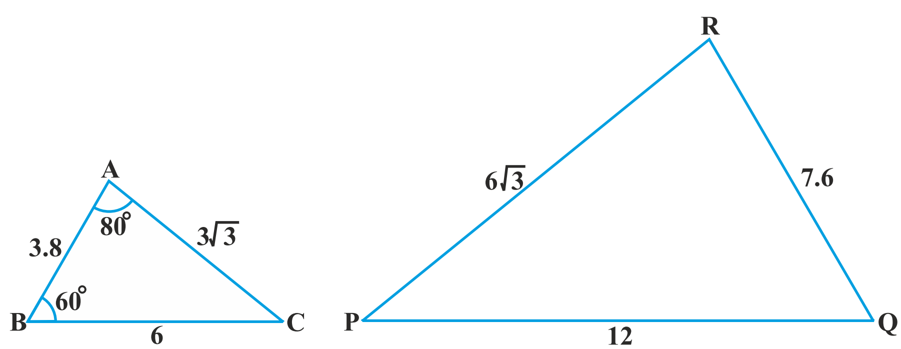


Fig. 6.30

Solution : In $\Delta\ ABC$ and $\Delta\ PQR$,

$$\frac{AB}{RQ} = \frac{3.8}{7.6} = \frac{1}{2}, \; \frac{BC}{QP} = \frac{6}{12} = \frac{1}{2} \text{ and } \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

That is, $\dfrac{AB}{RQ} = \dfrac{BC}{QP} = \dfrac{CA}{PR}$

So, $\Delta\, ABC \sim \Delta\, RQP$ (SSS similarity)

Therefore, $\angle C = \angle P$ (Corresponding angles of similar triangles)

But $\angle C = 180° - \angle A - \angle B$ (Angle sum property)

$= 180° - 80° - 60° = 40°$

So, $\angle P = 40°$

Example 6 : In Fig. 6.31,

$OA \,.\, OB = OC \,.\, OD.$

Show that $\angle A = \angle C$ and $\angle B = \angle D$.

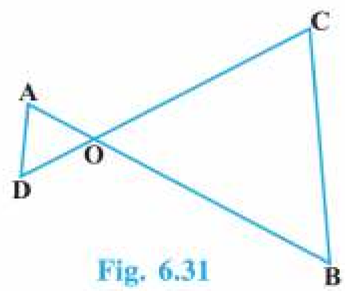


Fig. 6.31

Solution : $OA \,.\, OB = OC \,.\, OD$ (Given)

So, $\dfrac{OA}{OC} = \dfrac{OD}{OB}$ (1)

Also, we have $\angle AOD = \angle COB$ (Vertically opposite angles) (2)

Therefore, from (1) and (2), $\Delta\, AOD \sim \Delta\, COB$ (SAS similarity criterion)

So, $\angle A = \angle C$ and $\angle D = \angle B$

(Corresponding angles of similar triangles)

Example 7 : A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

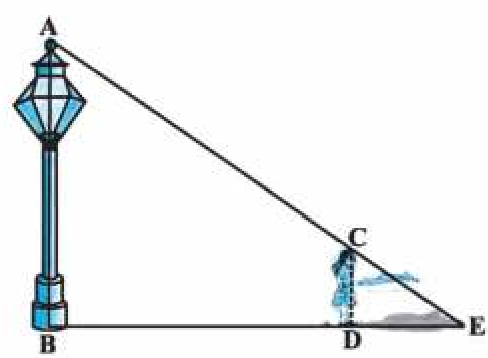


Fig. 6.32

Solution : Let AB denote the lamp-post and CD the girl after walking for 4 seconds away from the lamp-post (see Fig. 6.32).

From the figure, you can see that DE is the shadow of the girl. Let DE be x metres.

Now, BD = 1.2 m × 4 = 4.8 m.

Note that in Δ ABE and Δ CDE,

$\angle B = \angle D$ (Each is of 90° because lamp-post as well as the girl are standing vertical to the ground)

and $\angle E = \angle E$ (Same angle)

So, Δ ABE ~ Δ CDE (AA similarity criterion)

Therefore, $$\frac{BE}{DE} = \frac{AB}{CD}$$

i.e., $$\frac{4.8 + x}{x} = \frac{3.6}{0.9} \qquad \left(90 \text{ cm} = \frac{90}{100} \text{ m} = 0.9 \text{ m}\right)$$

i.e., $4.8 + x = 4x$

i.e., $3x = 4.8$

i.e., $x = 1.6$

So, the shadow of the girl after walking for 4 seconds is 1.6 m long.

Example 8 : In Fig. 6.33, CM and RN are respectively the medians of Δ ABC and Δ PQR. If Δ ABC ~ Δ PQR, prove that :

(i) Δ AMC ~ Δ PNR

(ii) $\frac{CM}{RN} = \frac{AB}{PQ}$

(iii) Δ CMB ~ Δ RNQ

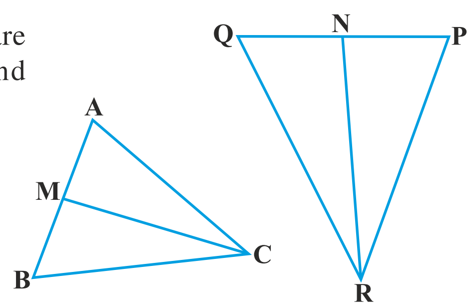


Fig. 6.33

Solution : (i) Δ ABC ~ Δ PQR (Given)

So, $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \qquad (1)$$

and $\angle A = \angle P, \angle B = \angle Q$ and $\angle C = \angle R$ (2)

But AB = 2 AM and PQ = 2 PN

(As CM and RN are medians)

So, from (1), $$\frac{2\,AM}{2\,PN} = \frac{CA}{RP}$$

i.e., $$\frac{AM}{PN} = \frac{CA}{RP} \quad (3)$$

Also, $$\angle MAC = \angle NPR \quad \text{[From (2)]} \quad (4)$$

So, from (3) and (4),

$$\Delta\, AMC \sim \Delta\, PNR \quad \text{(SAS similarity)} \quad (5)$$

(ii) From (5), $$\frac{CM}{RN} = \frac{CA}{RP} \quad (6)$$

But $$\frac{CA}{RP} = \frac{AB}{PQ} \quad \text{[From (1)]} \quad (7)$$

Therefore, $$\frac{CM}{RN} = \frac{AB}{PQ} \quad \text{[From (6) and (7)]} \quad (8)$$

(iii) Again, $$\frac{AB}{PQ} = \frac{BC}{QR} \quad \text{[From (1)]}$$

Therefore, $$\frac{CM}{RN} = \frac{BC}{QR} \quad \text{[From (8)]} \quad (9)$$

Also, $$\frac{CM}{RN} = \frac{AB}{PQ} = \frac{2\,BM}{2\,QN}$$

i.e., $$\frac{CM}{RN} = \frac{BM}{QN} \quad (10)$$

i.e., $$\frac{CM}{RN} = \frac{BC}{QR} = \frac{BM}{QN} \quad \text{[From (9) and (10)]}$$

Therefore, $$\Delta\, CMB \sim \Delta\, RNQ \quad \text{(SSS similarity)}$$

[**Note :** You can also prove part (iii) by following the same method as used for proving part (i).]

EXERCISE 6.3

1. State which pairs of triangles in Fig. 6.34 are similar. Write the similarity criterion used by you for answering the question and also write the pairs of similar triangles in the symbolic form :

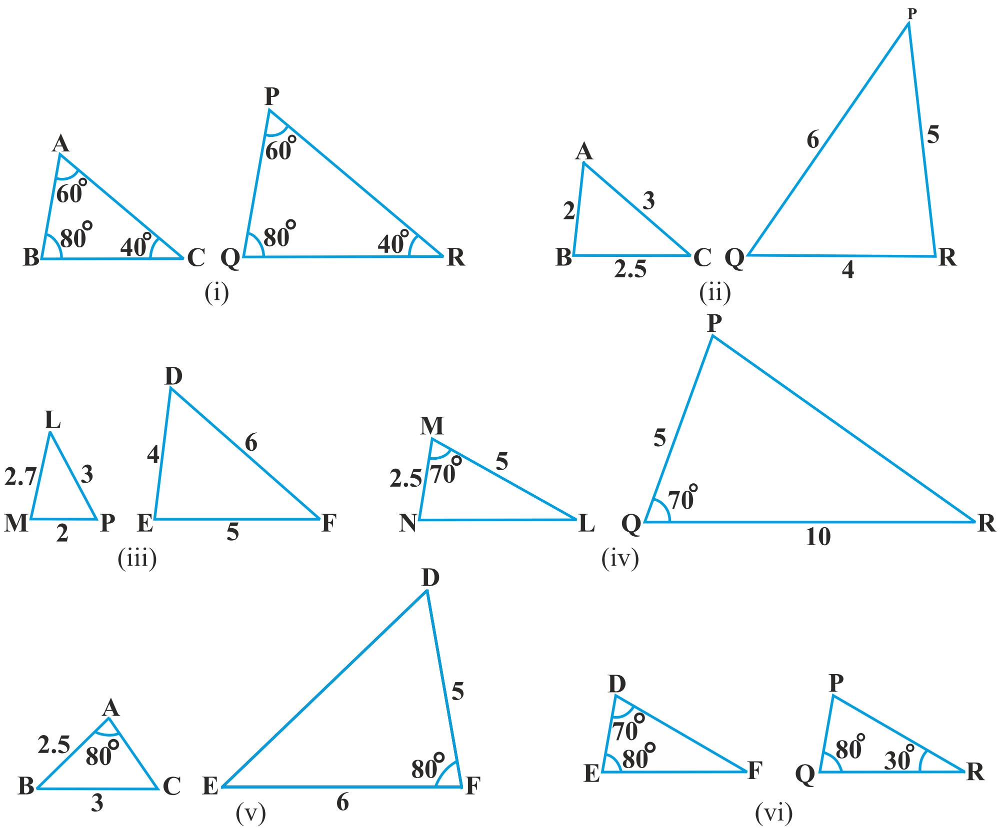


Fig. 6.34

2. In Fig. 6.35, $\Delta\, ODC \sim \Delta\, OBA$, $\angle\, BOC = 125°$ and $\angle\, CDO = 70°$. Find $\angle\, DOC$, $\angle\, DCO$ and $\angle\, OAB$.

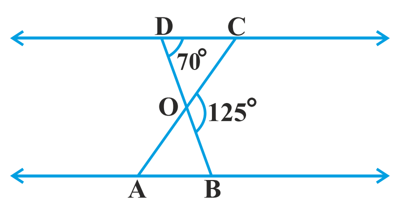


Fig. 6.35

3. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point O. Using a similarity criterion for two triangles, show that $\frac{OA}{OC} = \frac{OB}{OD}$.

4. In Fig. 6.36, $\frac{QR}{QS} = \frac{QT}{PR}$ and $\angle 1 = \angle 2$. Show that $\Delta PQS \sim \Delta TQR$.

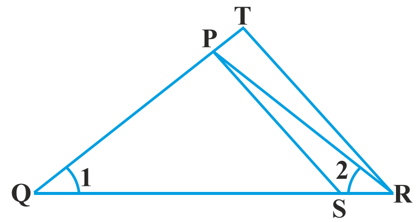


Fig. 6.36

5. S and T are points on sides PR and QR of ΔPQR such that $\angle P = \angle RTS$. Show that $\Delta RPQ \sim \Delta RTS$.

6. In Fig. 6.37, if $\Delta ABE \cong \Delta ACD$, show that $\Delta ADE \sim \Delta ABC$.

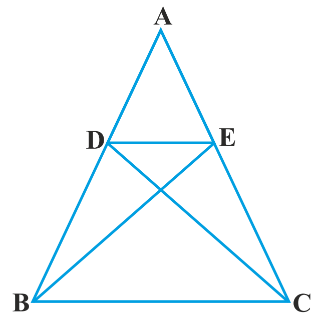


Fig. 6.37

7. In Fig. 6.38, altitudes AD and CE of ΔABC intersect each other at the point P. Show that:

   (i) $\Delta AEP \sim \Delta CDP$

   (ii) $\Delta ABD \sim \Delta CBE$

   (iii) $\Delta AEP \sim \Delta ADB$

   (iv) $\Delta PDC \sim \Delta BEC$

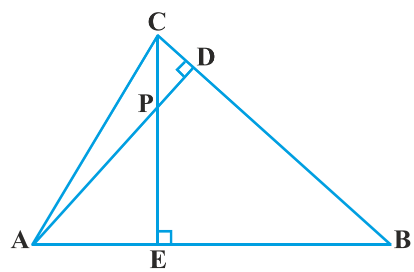


Fig. 6.38

8. E is a point on the side AD produced of a parallelogram ABCD and BE intersects CD at F. Show that $\Delta ABE \sim \Delta CFB$.

9. In Fig. 6.39, ABC and AMP are two right triangles, right angled at B and M respectively. Prove that:

   (i) $\Delta ABC \sim \Delta AMP$

   (ii) $\frac{CA}{PA} = \frac{BC}{MP}$

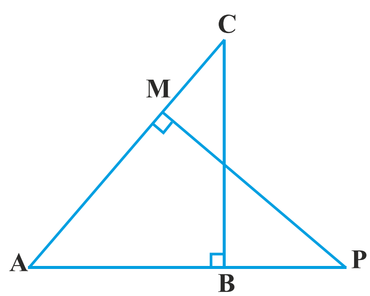


Fig. 6.39

10. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of ΔABC and ΔEFG respectively. If $\Delta ABC \sim \Delta FEG$, show that:

   (i) $\frac{CD}{GH} = \frac{AC}{FG}$

   (ii) $\Delta DCB \sim \Delta HGE$

   (iii) $\Delta DCA \sim \Delta HGF$

11. In Fig. 6.40, E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD ⊥ BC and EF ⊥ AC, prove that Δ ABD ~ Δ ECF.

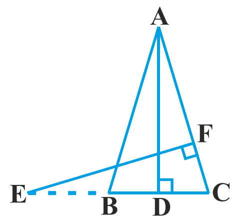


Fig. 6.40

12. Sides AB and BC and median AD of a triangle ABC are respectively proportional to sides PQ and QR and median PM of Δ PQR (see Fig. 6.41). Show that Δ ABC ~ Δ PQR.

13. D is a point on the side BC of a triangle ABC such that ∠ ADC = ∠ BAC. Show that $CA^2 = CB.CD$.

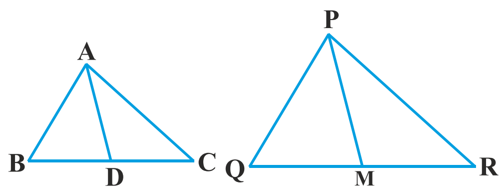


Fig. 6.41

14. Sides AB and AC and median AD of a triangle ABC are respectively proportional to sides PQ and PR and median PM of another triangle PQR. Show that Δ ABC ~ Δ PQR.

15. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

16. If AD and PM are medians of triangles ABC and PQR, respectively where Δ ABC ~ Δ PQR, prove that $\frac{AB}{PQ} = \frac{AD}{PM}$.

6.5 Areas of Similar Triangles

You have learnt that in two similar triangles, the ratio of their corresponding sides is the same. Do you think there is any relationship between the ratio of their areas and the ratio of the corresponding sides? You know that area is measured in square units. So, you may expect that this ratio is the square of the ratio of their corresponding sides. This is indeed true and we shall prove it in the next theorem.

Theorem 6.6 : *The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.*

Proof : We are given two triangles ABC and PQR such that Δ ABC ~ Δ PQR (see Fig. 6.42).

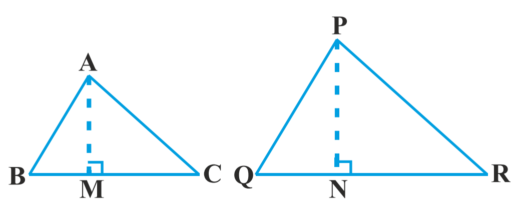


Fig. 6.42

We need to prove that $\frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$.

For finding the areas of the two triangles, we draw altitudes AM and PN of the triangles.

Now, $\qquad \text{ar (ABC)} = \frac{1}{2} BC \times AM$

and $\qquad \text{ar (PQR)} = \frac{1}{2} QR \times PN$

So, $\qquad \frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \frac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN} = \frac{BC \times AM}{QR \times PN} \qquad (1)$

Now, in Δ ABM and Δ PQN,

$\angle B = \angle Q$ (As Δ ABC ~ Δ PQR)

and $\qquad \angle M = \angle N$ (Each is of 90°)

So, $\qquad \Delta$ ABM ~ Δ PQN (AA similarity criterion)

Therefore, $\qquad \frac{AM}{PN} = \frac{AB}{PQ} \qquad (2)$

Also, $\qquad \Delta$ ABC ~ Δ PQR (Given)

So, $\qquad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \qquad (3)$

Therefore, $\qquad \frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \frac{AB}{PQ} \times \frac{AM}{PN}$ [From (1) and (3)]

$$= \frac{AB}{PQ} \times \frac{AB}{PQ} \qquad \text{[From (2)]}$$

$$= \left(\frac{AB}{PQ}\right)^2$$

Now using (3), we get

$$\frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

Let us take an example to illustrate the use of this theorem.

Example 9 : In Fig. 6.43, the line segment XY is parallel to side AC of Δ ABC and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$.

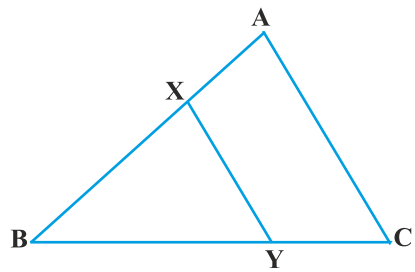


Fig. 6.43

Solution : We have $XY \parallel AC$ (Given)

So, $\angle BXY = \angle A$ and $\angle BYX = \angle C$ (Corresponding angles)

Therefore, $\Delta ABC \sim \Delta XBY$ (AA similarity criterion)

So, $$\frac{ar\,(ABC)}{ar\,(XBY)} = \left(\frac{AB}{XB}\right)^2 \qquad \text{(Theorem 6.6)} \quad (1)$$

Also, $ar\,(ABC) = 2\, ar\,(XBY)$ (Given)

So, $$\frac{ar\,(ABC)}{ar\,(XBY)} = \frac{2}{1} \qquad (2)$$

Therefore, from (1) and (2),

$$\left(\frac{AB}{XB}\right)^2 = \frac{2}{1}, \text{ i.e., } \frac{AB}{XB} = \frac{\sqrt{2}}{1}$$

or, $$\frac{XB}{AB} = \frac{1}{\sqrt{2}}$$

or, $$1 - \frac{XB}{AB} = 1 - \frac{1}{\sqrt{2}}$$

or, $$\frac{AB - XB}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}}, \text{ i.e., } \frac{AX}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}.$$

EXERCISE 6.4

1. Let $\Delta ABC \sim \Delta DEF$ and their areas be, respectively, 64 cm^2 and 121 cm^2. If EF = 15.4 cm, find BC.
2. Diagonals of a trapezium ABCD with AB || DC intersect each other at the point O. If AB = 2 CD, find the ratio of the areas of triangles AOB and COD.

3. In Fig. 6.44, ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, show that $\frac{ar\ (ABC)}{ar\ (DBC)} = \frac{AO}{DO}$.

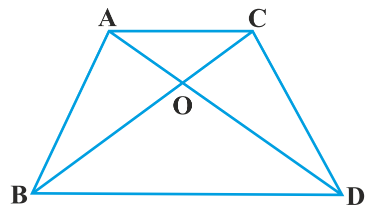


Fig. 6.44

4. If the areas of two similar triangles are equal, prove that they are congruent.

5. D, E and F are respectively the mid-points of sides AB, BC and CA of Δ ABC. Find the ratio of the areas of Δ DEF and Δ ABC.

6. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.

7. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.

Tick the correct answer and justify :

8. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. Ratio of the areas of triangles ABC and BDE is
   (A) 2 : 1 (B) 1 : 2 (C) 4 : 1 (D) 1 : 4

9. Sides of two similar triangles are in the ratio 4 : 9. Areas of these triangles are in the ratio
   (A) 2 : 3 (B) 4 : 9 (C) 81 : 16 (D) 16 : 81

6.6 Pythagoras Theorem

You are already familiar with the Pythagoras Theorem from your earlier classes. You had verified this theorem through some activities and made use of it in solving certain problems. You have also seen a proof of this theorem in Class IX. Now, we shall prove this theorem using the concept of similarity of triangles. In proving this, we shall make use of a result related to similarity of two triangles formed by the perpendicular to the hypotenuse from the opposite vertex of the right triangle.

Now, let us take a right triangle ABC, right angled at B. Let BD be the perpendicular to the hypotenuse AC (see Fig. 6.45).

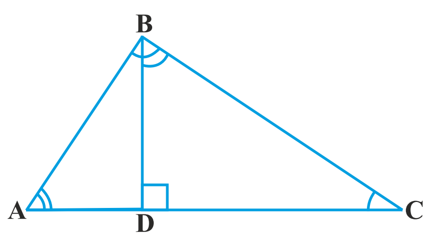


Fig. 6.45

You may note that in Δ ADB and Δ ABC

$$\angle A = \angle A$$

and $\angle ADB = \angle ABC$ (Why?)

So, $\Delta ADB \sim \Delta ABC$ (How?) (1)

Similarly, $\Delta BDC \sim \Delta ABC$ (How?) (2)

So, from (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.

Also, since $\Delta\, ADB \sim \Delta\, ABC$

and $\Delta\, BDC \sim \Delta\, ABC$

So, $\Delta\, ADB \sim \Delta\, BDC$ (From Remark in Section 6.2)

The above discussion leads to the following theorem :

Theorem 6.7 : *If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse then triangles on both sides of the perpendicular are similar to the whole triangle and to each other.*

Pythagoras
(569 – 479 B.C.E.)

Let us now apply this theorem in proving the Pythagoras Theorem:

Theorem 6.8 : *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

Proof : We are given a right triangle ABC right angled at B.

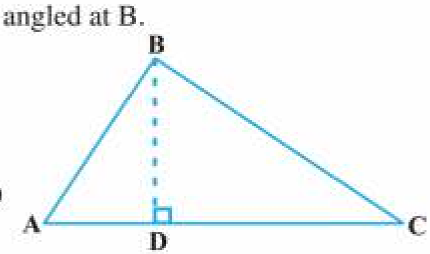


Fig. 6.46

We need to prove that $AC^2 = AB^2 + BC^2$

Let us draw $BD \perp AC$ (see Fig. 6.46).

Now, $\Delta\, ADB \sim \Delta\, ABC$ (Theorem 6.7)

So, $\frac{AD}{AB} = \frac{AB}{AC}$ (Sides are proportional)

or, $AD \,.\, AC = AB^2$ (1)

Also, $\Delta\, BDC \sim \Delta\, ABC$ (Theorem 6.7)

So, $\frac{CD}{BC} = \frac{BC}{AC}$

or, $CD \,.\, AC = BC^2$ (2)

Adding (1) and (2),

$$AD \,.\, AC + CD \,.\, AC = AB^2 + BC^2$$

or, $$AC\,(AD + CD) = AB^2 + BC^2$$

or, $$AC \,.\, AC = AB^2 + BC^2$$

or, $$AC^2 = AB^2 + BC^2$$

The above theorem was earlier given by an ancient Indian mathematician Baudhayan (about 800 B.C.E.) in the following form :

The diagonal of a rectangle produces by itself the same area as produced by its both sides (*i.e., length and breadth*).

For this reason, this theorem is sometimes also referred to as the *Baudhayan Theorem*.

What about the converse of the Pythagoras Theorem? You have already verified, in the earlier classes, that this is also true. We now prove it in the form of a theorem.

Theorem 6.9 : *In a triangle, if square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.*

Proof : Here, we are given a triangle ABC in which $AC^2 = AB^2 + BC^2$.

We need to prove that $\angle B = 90°$.

To start with, we construct a Δ PQR right angled at Q such that PQ = AB and QR = BC (see Fig. 6.47).

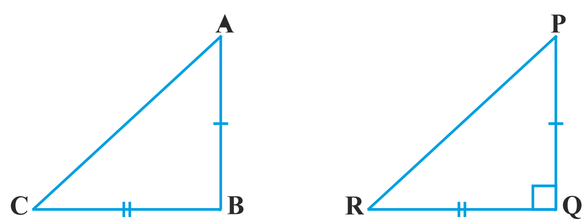


Fig. 6.47

Now, from Δ PQR, we have :

$$PR^2 = PQ^2 + QR^2$$ (Pythagoras Theorem, as $\angle Q = 90°$)

or, $$PR^2 = AB^2 + BC^2$$ (By construction) (1)

But	$AC^2 = AB^2 + BC^2$	(Given)	(2)
So,	$AC = PR$	[From (1) and (2)]	(3)

Now, in Δ ABC and Δ PQR,

	$AB = PQ$	(By construction)
	$BC = QR$	(By construction)
	$AC = PR$	[Proved in (3) above]
So,	$\Delta ABC \cong \Delta PQR$	(SSS congruence)
Therefore,	$\angle B = \angle Q$	(CPCT)
But	$\angle Q = 90°$	(By construction)
So,	$\angle B = 90°$	

Note : Also see Appendix 1 for another proof of this theorem.

Let us now take some examples to illustrate the use of these theorems.

Example 10 : In Fig. 6.48, $\angle ACB = 90°$ and $CD \perp AB$. Prove that $\dfrac{BC^2}{AC^2} = \dfrac{BD}{AD}$.

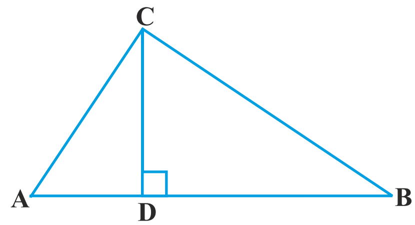


Fig. 6.48

Solution : $\Delta ACD \sim \Delta ABC$ (Theorem 6.7)

So, $$\frac{AC}{AB} = \frac{AD}{AC}$$

or, $$AC^2 = AB \cdot AD \qquad (1)$$

Similarly, $\Delta BCD \sim \Delta BAC$ (Theorem 6.7)

So, $$\frac{BC}{BA} = \frac{BD}{BC}$$

or, $$BC^2 = BA \cdot BD \qquad (2)$$

Therefore, from (1) and (2),

$$\frac{BC^2}{AC^2} = \frac{BA \cdot BD}{AB \cdot AD} = \frac{BD}{AD}$$

Example 11 : A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

Solution : Let AB be the ladder and CA be the wall with the window at A (see Fig. 6.49).

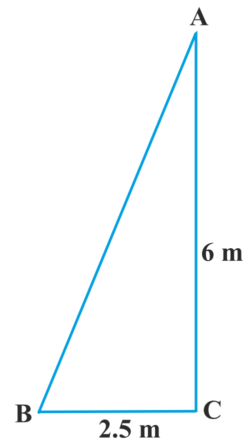


Fig. 6.49

Also, $\quad BC = 2.5 \text{ m and } CA = 6 \text{ m}$

From Pythagoras Theorem, we have:

$$AB^2 = BC^2 + CA^2$$
$$= (2.5)^2 + (6)^2$$
$$= 42.25$$

So, $\quad AB = 6.5$

Thus, length of the ladder is 6.5 m.

Example 12 : In Fig. 6.50, if $AD \perp BC$, prove that $AB^2 + CD^2 = BD^2 + AC^2$.

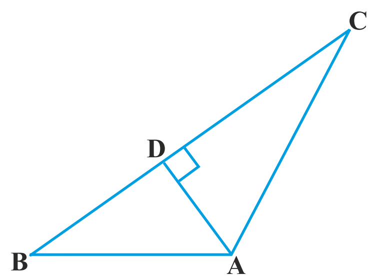


Fig. 6.50

Solution : From Δ ADC, we have

$$AC^2 = AD^2 + CD^2$$

(Pythagoras Theorem) (1)

From Δ ADB, we have

$$AB^2 = AD^2 + BD^2$$

(Pythagoras Theorem) (2)

Subtracting (1) from (2), we have

$$AB^2 - AC^2 = BD^2 - CD^2$$

or, $\quad AB^2 + CD^2 = BD^2 + AC^2$

Example 13 : BL and CM are medians of a triangle ABC right angled at A. Prove that $4 (BL^2 + CM^2) = 5 BC^2$.

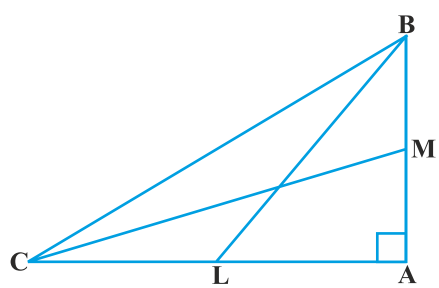


Fig. 6.51

Solution : BL and CM are medians of the Δ ABC in which ∠ A = 90° (see Fig. 6.51).

From Δ ABC,

$$BC^2 = AB^2 + AC^2 \quad \text{(Pythagoras Theorem)} \quad (1)$$

From Δ ABL,

$$BL^2 = AL^2 + AB^2$$

or, $$BL^2 = \left(\frac{AC}{2}\right)^2 + AB^2 \quad \text{(L is the mid-point of AC)}$$

or, $$BL^2 = \frac{AC^2}{4} + AB^2$$

or, $$4\,BL^2 = AC^2 + 4\,AB^2 \qquad (2)$$

From Δ CMA,

$$CM^2 = AC^2 + AM^2$$

or, $$CM^2 = AC^2 + \left(\frac{AB}{2}\right)^2 \quad \text{(M is the mid-point of AB)}$$

or, $$CM^2 = AC^2 + \frac{AB^2}{4}$$

or $$4\,CM^2 = 4\,AC^2 + AB^2 \qquad (3)$$

Adding (2) and (3), we have

$$4\,(BL^2 + CM^2) = 5\,(AC^2 + AB^2)$$

i.e., $$4\,(BL^2 + CM^2) = 5\,BC^2 \qquad \text{[From (1)]}$$

Example 14 : O is any point inside a rectangle ABCD (see Fig. 6.52). Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

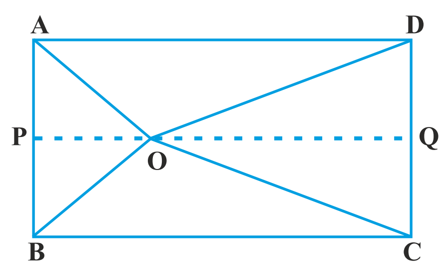


Fig. 6.52

Solution :

Through O, draw PQ || BC so that P lies on AB and Q lies on DC.

Now, $$PQ \parallel BC$$

Therefore, $$PQ \perp AB \text{ and } PQ \perp DC \ (\angle B = 90^\circ \text{ and } \angle C = 90^\circ)$$

So, $$\angle BPQ = 90^\circ \text{ and } \angle CQP = 90^\circ$$

Therefore, BPQC and APQD are both rectangles.

Now, from Δ OPB,

$$OB^2 = BP^2 + OP^2 \qquad (1)$$

Similarly, from Δ OQD,

$$OD^2 = OQ^2 + DQ^2 \quad (2)$$

From Δ OQC, we have

$$OC^2 = OQ^2 + CQ^2 \quad (3)$$

and from Δ OAP, we have

$$OA^2 = AP^2 + OP^2 \quad (4)$$

Adding (1) and (2),

$$\begin{aligned} OB^2 + OD^2 &= BP^2 + OP^2 + OQ^2 + DQ^2 \\ &= CQ^2 + OP^2 + OQ^2 + AP^2 \\ &\qquad\qquad \text{(As BP = CQ and DQ = AP)} \\ &= CQ^2 + OQ^2 + OP^2 + AP^2 \\ &= OC^2 + OA^2 \qquad \text{[From (3) and (4)]} \end{aligned}$$

EXERCISE 6.5

1. Sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse.
   (i) 7 cm, 24 cm, 25 cm
   (ii) 3 cm, 8 cm, 6 cm
   (iii) 50 cm, 80 cm, 100 cm
   (iv) 13 cm, 12 cm, 5 cm

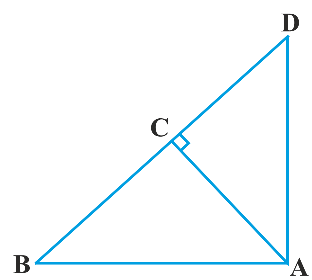


Fig. 6.53

2. PQR is a triangle right angled at P and M is a point on QR such that PM $\perp$ QR. Show that $PM^2 = QM \cdot MR$.
3. In Fig. 6.53, ABD is a triangle right angled at A and AC $\perp$ BD. Show that
   (i) $AB^2 = BC \cdot BD$
   (ii) $AC^2 = BC \cdot DC$
   (iii) $AD^2 = BD \cdot CD$
4. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.
5. ABC is an isosceles triangle with AC = BC. If $AB^2 = 2\,AC^2$, prove that ABC is a right triangle.
6. ABC is an equilateral triangle of side $2a$. Find each of its altitudes.
7. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

8. In Fig. 6.54, O is a point in the interior of a triangle ABC, OD $\perp$ BC, OE $\perp$ AC and OF $\perp$ AB. Show that

(i) $OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2 = AF^2 + BD^2 + CE^2$,

(ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

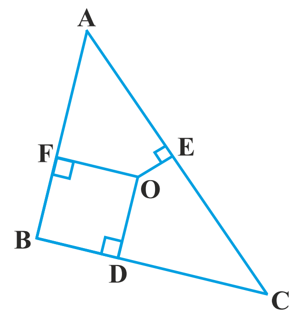


Fig. 6.54

9. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

10. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

11. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

12. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

13. D and E are points on the sides CA and CB respectively of a triangle ABC right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.

14. The perpendicular from A on side BC of a Δ ABC intersects BC at D such that DB = 3 CD (see Fig. 6.55). Prove that $2\,AB^2 = 2\,AC^2 + BC^2$.

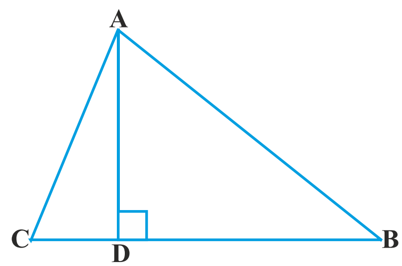


Fig. 6.55

15. In an equilateral triangle ABC, D is a point on side BC such that BD = $\frac{1}{3}$ BC. Prove that $9\,AD^2 = 7\,AB^2$.

16. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.

17. Tick the correct answer and justify : In ΔABC, AB = $6\sqrt{3}$ cm, AC = 12 cm and BC = 6 cm. The angle B is :

(A) 120° (B) 60°

(C) 90° (D) 45°

EXERCISE 6.6 (Optional)*

1. In Fig. 6.56, PS is the bisector of $\angle$ QPR of Δ PQR. Prove that $\frac{QS}{SR} = \frac{PQ}{PR}$.

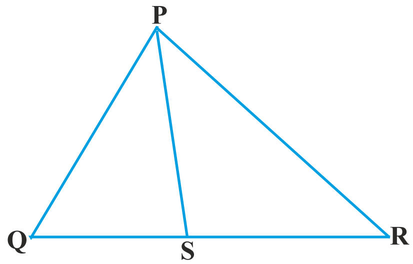


Fig. 6.56

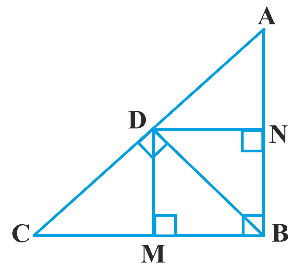


Fig. 6.57

2. In Fig. 6.57, D is a point on hypotenuse AC of Δ ABC, such that BD $\perp$ AC, DM $\perp$ BC and DN $\perp$ AB. Prove that :

(i) $DM^2 = DN . MC$ (ii) $DN^2 = DM . AN$

3. In Fig. 6.58, ABC is a triangle in which $\angle ABC > 90°$ and AD $\perp$ CB produced. Prove that $AC^2 = AB^2 + BC^2 + 2\,BC . BD$.

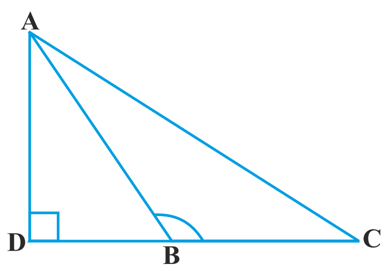


Fig. 6.58

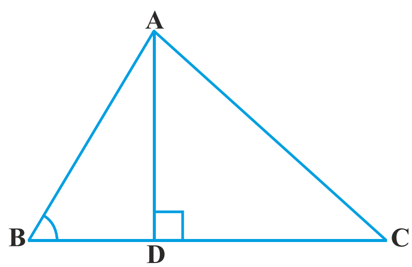


Fig. 6.59

4. In Fig. 6.59, ABC is a triangle in which $\angle ABC < 90°$ and AD $\perp$ BC. Prove that $AC^2 = AB^2 + BC^2 - 2\,BC . BD$.

5. In Fig. 6.60, AD is a median of a triangle ABC and AM $\perp$ BC. Prove that :

(i) $AC^2 = AD^2 + BC . DM + \left(\frac{BC}{2}\right)^2$

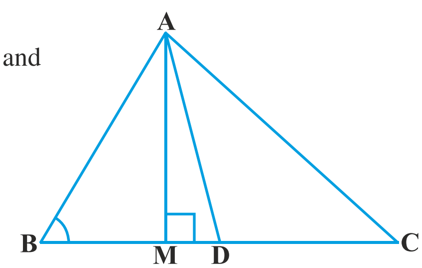


Fig. 6.60

* These exercises are not from examination point of view.

(ii) $AB^2 = AD^2 - BC \cdot DM + \left(\frac{BC}{2}\right)^2$ (iii) $AC^2 + AB^2 = 2\,AD^2 + \frac{1}{2}\,BC^2$

6. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.

7. In Fig. 6.61, two chords AB and CD intersect each other at the point P. Prove that :

(i) $\Delta APC \sim \Delta DPB$ (ii) $AP \cdot PB = CP \cdot DP$

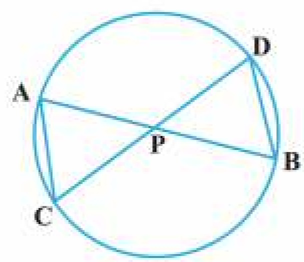


Fig. 6.61

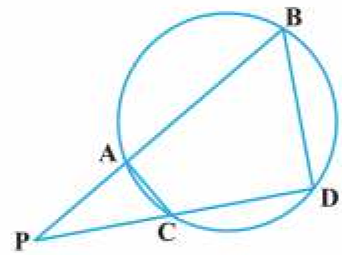


Fig. 6.62

8. In Fig. 6.62, two chords AB and CD of a circle intersect each other at the point P (when produced) outside the circle. Prove that

(i) $\Delta PAC \sim \Delta PDB$ (ii) $PA \cdot PB = PC \cdot PD$

9. In Fig. 6.63, D is a point on side BC of ΔABC such that $\frac{BD}{CD} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

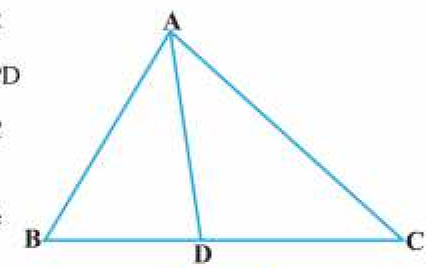


Fig. 6.63

10. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out (see Fig. 6.64)? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

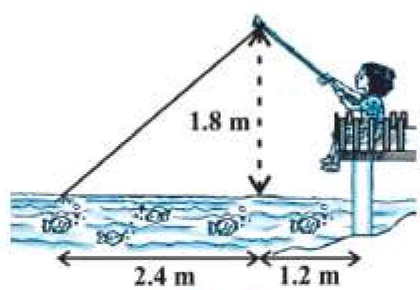


Fig. 6.64

6.7 Summary

In this chapter you have studied the following points :

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. All the congruent figures are similar but the converse is not true.
3. Two polygons of the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (i.e., proportion).
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.
6. If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio and hence the two triangles are similar (AAA similarity criterion).
7. If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar (AA similarity criterion).
8. If in two triangles, corresponding sides are in the same ratio, then their corresponding angles are equal and hence the triangles are similar (SSS similarity criterion).
9. If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio (proportional), then the triangles are similar (SAS similarity criterion).
10. The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and also to each other.
12. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides (Pythagoras Theorem).
13. If in a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.

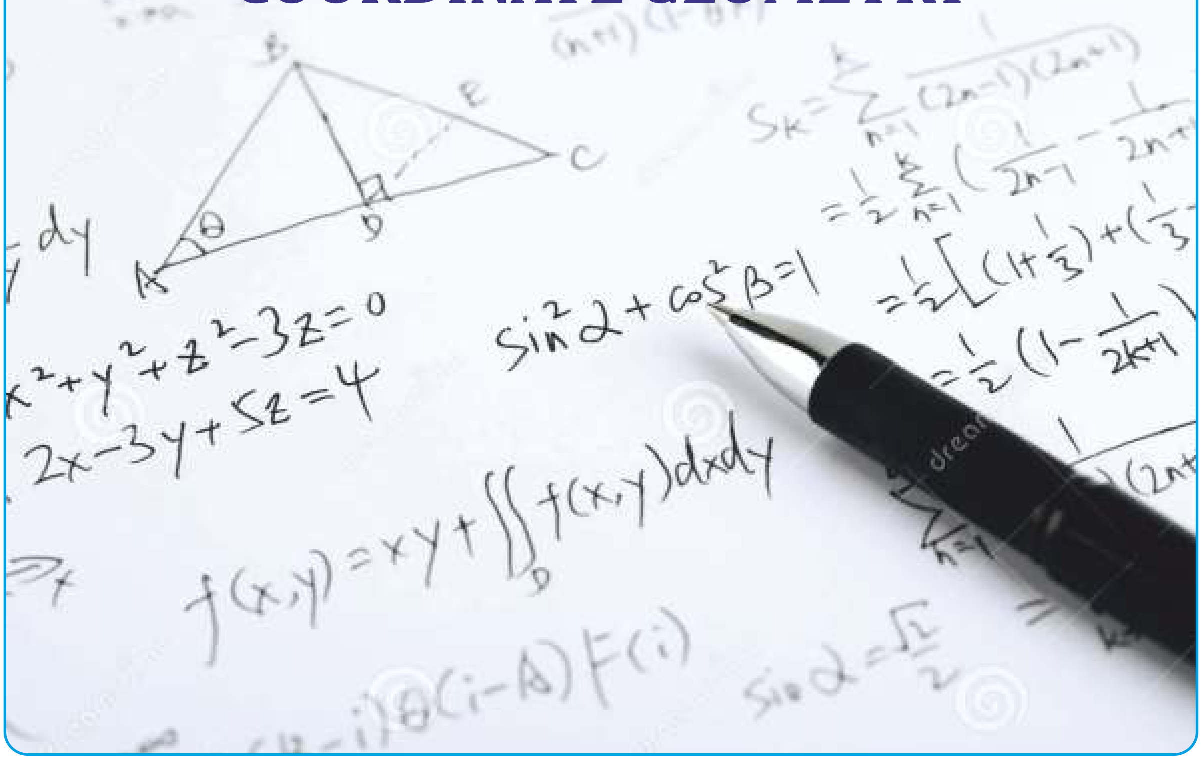
A Note to the Reader

If in two right triangles, hypotenuse and one side of one triangle are proportional to the hypotenuse and one side of the other triangle, then the two triangles are similar. This may be referred to as the RHS Similarity Criterion.

If you use this criterion in Example 2, Chapter 8, the proof will become simpler.

07

COORDINATE GEOMETRY



COORDINATE GEOMETRY 7

7.1 Introduction

In Class IX, you have studied that to locate the position of a point on a plane, we require a pair of coordinate axes. The distance of a point from the y-axis is called its ***x*-coordinate**, or **abscissa**. The distance of a point from the x-axis is called its ***y*-coordinate**, or **ordinate**. The coordinates of a point on the x-axis are of the form $(x, 0)$, and of a point on the y-axis are of the form $(0, y)$.

Here is a play for you. Draw a set of a pair of perpendicular axes on a graph paper. Now plot the following points and join them as directed: Join the point A(4, 8) to B(3, 9) to C(3, 8) to D(1, 6) to E(1, 5) to F(3, 3) to G(6, 3) to H(8, 5) to I(8, 6) to J(6, 8) to K(6, 9) to L(5, 8) to A. Then join the points P(3.5, 7), Q (3, 6) and R(4, 6) to form a triangle. Also join the points X(5.5, 7), Y(5, 6) and Z(6, 6) to form a triangle. Now join S(4, 5), T(4.5, 4) and U(5, 5) to form a triangle. Lastly join S to the points (0, 5) and (0, 6) and join U to the points (9, 5) and (9, 6). What picture have you got?

Also, you have seen that a linear equation in two variables of the form $ax + by + c = 0$, (a, b are not simultaneously zero), when represented graphically, gives a straight line. Further, in Chapter 2, you have seen the graph of $y = ax^2 + bx + c$ $(a \neq 0)$, is a parabola. In fact, coordinate geometry has been developed as an algebraic tool for studying geometry of figures. It helps us to study geometry using algebra, and understand algebra with the help of geometry. Because of this, coordinate geometry is widely applied in various fields such as physics, engineering, navigation, seismology and art!

In this chapter, you will learn how to find the distance between the two points whose coordinates are given, and to find the area of the triangle formed by three given points. You will also study how to find the coordinates of the point which divides a line segment joining two given points in a given ratio.

7.2 Distance Formula

Let us consider the following situation:

A town B is located 36 km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it. Let us see. This situation can be represented graphically as shown in Fig. 7.1. You may use the Pythagoras Theorem to calculate this distance.

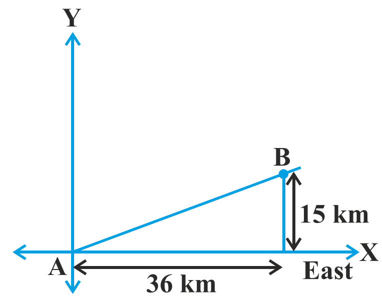


Fig. 7.1

Now, suppose two points lie on the x-axis. Can we find the distance between them? For instance, consider two points A(4, 0) and B(6, 0) in Fig. 7.2. The points A and B lie on the x-axis.

From the figure you can see that OA = 4 units and OB = 6 units.

Therefore, the distance of B from A, i.e., AB = OB – OA = 6 – 4 = 2 units.

So, if two points lie on the x-axis, we can easily find the distance between them.

Now, suppose we take two points lying on the y-axis. Can you find the distance between them. If the points C(0, 3) and D(0, 8) lie on the y-axis, similarly we find that CD = 8 – 3 = 5 units (see Fig. 7.2).

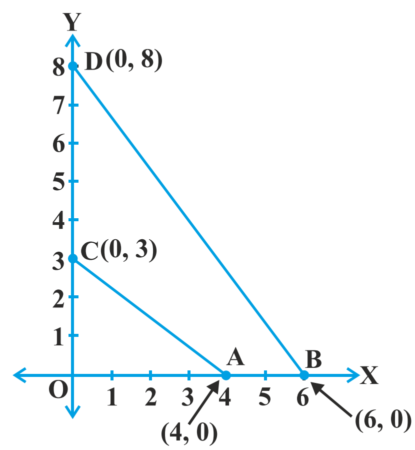


Fig. 7.2

Next, can you find the distance of A from C (in Fig. 7.2)? Since OA = 4 units and OC = 3 units, the distance of A from C, i.e., AC = $\sqrt{3^2 + 4^2}$ = 5 units. Similarly, you can find the distance of B from D = BD = 10 units.

Now, if we consider two points not lying on coordinate axis, can we find the distance between them? Yes! We shall use Pythagoras theorem to do so. Let us see an example.

In Fig. 7.3, the points P(4, 6) and Q(6, 8) lie in the first quadrant. How do we use Pythagoras theorem to find the distance between them? Let us draw PR and QS perpendicular to the x-axis from P and Q respectively. Also, draw a perpendicular from P on QS to meet QS at T. Then the coordinates of R and S are (4, 0) and (6, 0), respectively. So, RS = 2 units. Also, QS = 8 units and TS = PR = 6 units.

Therefore, QT = 2 units and PT = RS = 2 units.

Now, using the Pythagoras theorem, we have

$$PQ^2 = PT^2 + QT^2$$

$$= 2^2 + 2^2 = 8$$

So, $$PQ = 2\sqrt{2} \text{ units}$$

How will we find the distance between two points in two different quadrants?

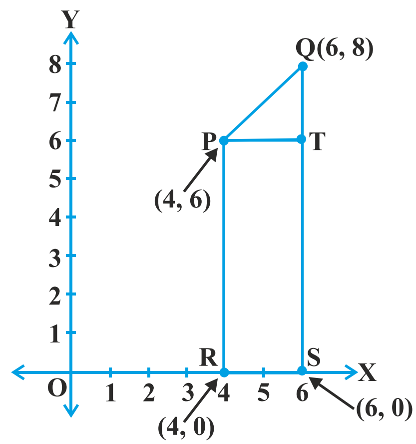


Fig. 7.3

Consider the points P(6, 4) and Q(–5, –3) (see Fig. 7.4). Draw QS perpendicular to the *x*-axis. Also draw a perpendicular PT from the point P on QS (extended) to meet *y*-axis at the point R.

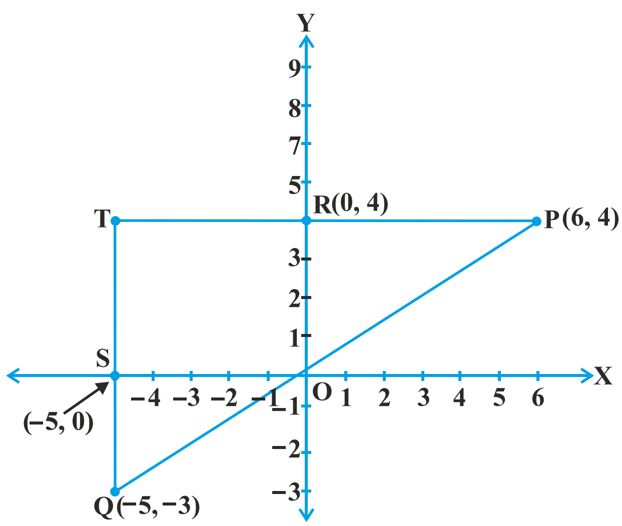


Fig. 7.4

Then PT = 11 units and QT = 7 units. (Why?)

Using the Pythagoras Theorem to the right triangle PTQ, we get $PQ = \sqrt{11^2 + 7^2} = \sqrt{170}$ units.

Let us now find the distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw PR and QS perpendicular to the x-axis. A perpendicular from the point P on QS is drawn to meet it at the point T (see Fig. 7.5).

Then, $OR = x_1$, $OS = x_2$. So, $RS = x_2 - x_1 = PT$.

Also, $SQ = y_2$, $ST = PR = y_1$. So, $QT = y_2 - y_1$.

Now, applying the Pythagoras theorem in Δ PTQ, we get

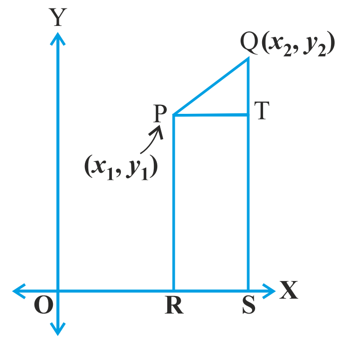


Fig. 7.5

$$PQ^2 = PT^2 + QT^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that since distance is always non-negative, we take only the positive square root. So, the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\mathbf{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}},$$

which is called the **distance formula**.

Remarks :

1. In particular, the distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{x^2 + y^2}.$$

2. We can also write, $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. (Why?)

Example 1 : Do the points (3, 2), (–2, –3) and (2, 3) form a triangle? If so, name the type of triangle formed.

Solution : Let us apply the distance formula to find the distances PQ, QR and PR, where P(3, 2), Q(–2, –3) and R(2, 3) are the given points. We have

$$PQ = \sqrt{(3+2)^2 + (2+3)^2} = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07 \text{ (approx.)}$$

$$QR = \sqrt{(-2-2)^2 + (-3-3)^2} = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52} = 7.21 \text{ (approx.)}$$

$$PR = \sqrt{(3-2)^2 + (2-3)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} = 1.41 \text{ (approx.)}$$

Since the sum of any two of these distances is greater than the third distance, therefore, the points P, Q and R form a triangle.

Also, $PQ^2 + PR^2 = QR^2$, by the converse of Pythagoras theorem, we have $\angle P = 90°$. Therefore, PQR is a right triangle.

Example 2 : Show that the points (1, 7), (4, 2), (–1, –1) and (– 4, 4) are the vertices of a square.

Solution : Let A(1, 7), B(4, 2), C(–1, –1) and D(– 4, 4) be the given points. One way of showing that ABCD is a square is to use the property that all its sides should be equal and both its digonals should also be equal. Now,

$$AB = \sqrt{(1-4)^2 + (7-2)^2} = \sqrt{9+25} = \sqrt{34}$$

$$BC = \sqrt{(4+1)^2 + (2+1)^2} = \sqrt{25+9} = \sqrt{34}$$

$$CD = \sqrt{(-1+4)^2 + (-1-4)^2} = \sqrt{9+25} = \sqrt{34}$$

$$DA = \sqrt{(1+4)^2 + (7-4)^2} = \sqrt{25+9} = \sqrt{34}$$

$$AC = \sqrt{(1+1)^2 + (7+1)^2} = \sqrt{4+64} = \sqrt{68}$$

$$BD = \sqrt{(4+4)^2 + (2-4)^2} = \sqrt{64+4} = \sqrt{68}$$

Since, AB = BC = CD = DA and AC = BD, all the four sides of the quadrilateral ABCD are equal and its diagonals AC and BD are also equal. Thereore, ABCD is a square.

Alternative Solution : We find the four sides and one diagonal, say, AC as above. Here $AD^2 + DC^2 = 34 + 34 = 68 = AC^2$. Therefore, by the converse of Pythagoras theorem, $\angle D = 90°$. A quadrilateral with all four sides equal and one angle 90° is a square. So, ABCD is a square.

Example 3 : Fig. 7.6 shows the arrangement of desks in a classroom. Ashima, Bharti and Camella are seated at A(3, 1), B(6, 4) and C(8, 6) respectively. Do you think they are seated in a line? Give reasons for your answer.

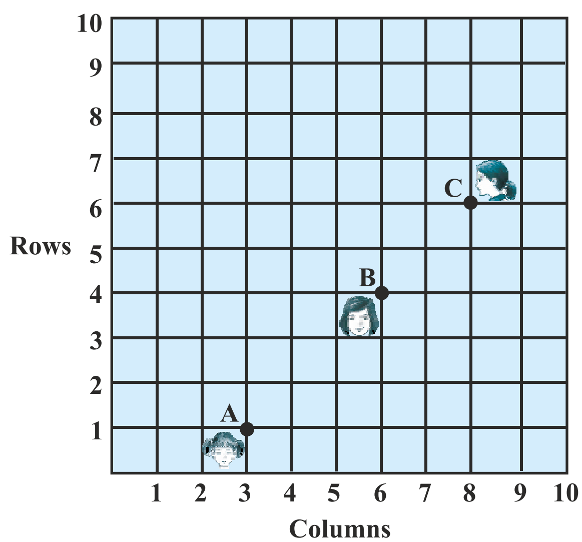


Fig. 7.6

Solution : Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Since, $AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC,$ we can say that the points A, B and C are collinear. Therefore, they are seated in a line.

Example 4 : Find a relation between x and y such that the point (x , y) is equidistant from the points (7, 1) and (3, 5).

Solution : Let P(x, y) be equidistant from the points A(7, 1) and B(3, 5).

We are given that AP = BP. So, $AP^2 = BP^2$

i.e., $\qquad (x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$

i.e., $\qquad x^2 - 14x + 49 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 10y + 25$

i.e., $\qquad x - y = 2$

which is the required relation.

Remark : Note that the graph of the equation $x - y = 2$ is a line. From your earlier studies, you know that a point which is equidistant from A and B lies on the perpendicular bisector of AB. Therefore, the graph of $x - y = 2$ is the perpendicular bisector of AB (see Fig. 7.7).

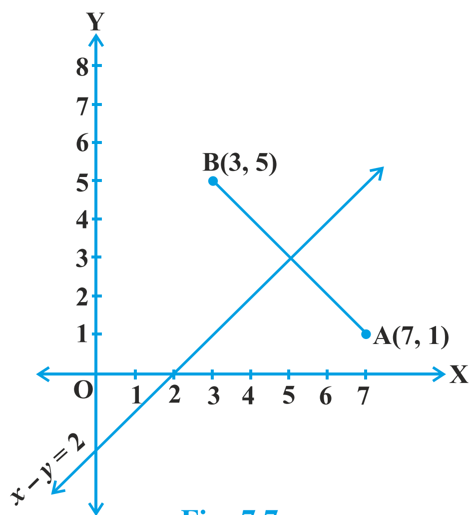


Fig. 7.7

Example 5 : Find a point on the y-axis which is equidistant from the points A(6, 5) and B(– 4, 3).

Solution : We know that a point on the y-axis is of the form $(0, y)$. So, let the point P(0, y) be equidistant from A and B. Then

$$(6-0)^2 + (5-y)^2 = (-4-0)^2 + (3-y)^2$$

i.e., $\qquad 36 + 25 + y^2 - 10y = 16 + 9 + y^2 - 6y$

i.e., $\qquad 4y = 36$

i.e., $\qquad y = 9$

So, the required point is (0, 9).

Let us check our solution : $\quad \text{AP} = \sqrt{(6-0)^2 + (5-9)^2} = \sqrt{36+16} = \sqrt{52}$

$$\text{BP} = \sqrt{(-4-0)^2 + (3-9)^2} = \sqrt{16+36} = \sqrt{52}$$

Note : Using the remark above, we see that (0, 9) is the intersection of the y-axis and the perpendicular bisector of AB.

EXERCISE 7.1

1. Find the distance between the following pairs of points :

   (i) (2, 3), (4, 1) (ii) (– 5, 7), (– 1, 3) (iii) $(a, b), (-a, -b)$

2. Find the distance between the points (0, 0) and (36, 15). Can you now find the distance between the two towns A and B discussed in Section 7.2.

3. Determine if the points (1, 5), (2, 3) and (– 2, – 11) are collinear.

4. Check whether (5, – 2), (6, 4) and (7, – 2) are the vertices of an isosceles triangle.

5. In a classroom, 4 friends are seated at the points A, B, C and D as shown in Fig. 7.8. Champa and Chameli walk into the class and after observing for a few minutes Champa asks Chameli, “Don’t you think ABCD is a square?” Chameli disagrees. Using distance formula, find which of them is correct.

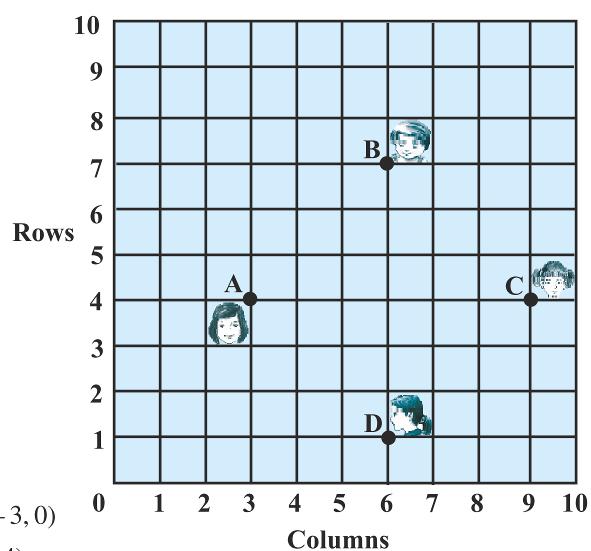


Fig. 7.8

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer:

   (i) (– 1, – 2), (1, 0), (– 1, 2), (– 3, 0)

   (ii) (–3, 5), (3, 1), (0, 3), (–1, – 4)

   (iii) (4, 5), (7, 6), (4, 3), (1, 2)

7. Find the point on the x-axis which is equidistant from (2, –5) and (–2, 9).

8. Find the values of y for which the distance between the points P(2, – 3) and Q(10, y) is 10 units.

9. If Q(0, 1) is equidistant from P(5, –3) and R(x, 6), find the values of x. Also find the distances QR and PR.

10. Find a relation between x and y such that the point (x, y) is equidistant from the point (3, 6) and (– 3, 4).

7.3 Section Formula

Let us recall the situation in Section 7.2. Suppose a telephone company wants to position a relay tower at P between A and B is such a way that the distance of the tower from B is twice its distance from A. If P lies on AB, it will divide AB in the ratio 1 : 2 (see Fig. 7.9). If we take A as the origin O, and 1 km as one unit on both the axis, the coordinates of B will be (36, 15). In order to know the position of the tower, we must know the coordinates of P. How do we find these coordinates?

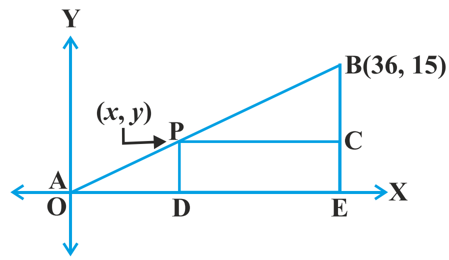


Fig. 7.9

Let the coordinates of P be (x, y). Draw perpendiculars from P and B to the x-axis, meeting it in D and E, respectively. Draw PC perpendicular to BE. Then, by the AA similarity criterion, studied in Chapter 6, Δ POD and Δ BPC are similar.

Therefore , $\dfrac{\text{OD}}{\text{PC}} = \dfrac{\text{OP}}{\text{PB}} = \dfrac{1}{2}$, and $\dfrac{\text{PD}}{\text{BC}} = \dfrac{\text{OP}}{\text{PB}} = \dfrac{1}{2}$

So, $\dfrac{x}{36 - x} = \dfrac{1}{2}$ and $\dfrac{y}{15 - y} = \dfrac{1}{2}$.

These equations give $x = 12$ and $y = 5$.

You can check that P(12, 5) meets the condition that OP : PB = 1 : 2.

Now let us use the understanding that you may have developed through this example to obtain the general formula.

Consider any two points A(x_1, y_1) and B(x_2, y_2) and assume that P (x, y) divides AB internally in the ratio $m_1 : m_2$, i.e., $\dfrac{\text{PA}}{\text{PB}} = \dfrac{m_1}{m_2}$ (see Fig. 7.10).

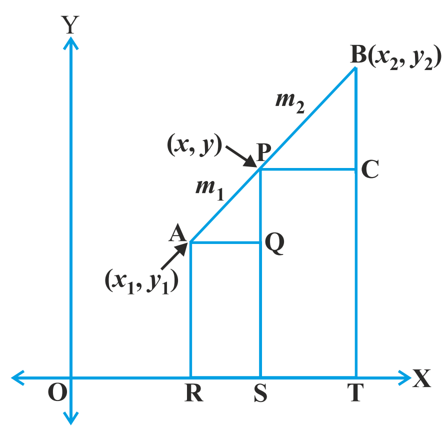


Fig. 7.10

Draw AR, PS and BT perpendicular to the x-axis. Draw AQ and PC parallel to the x-axis. Then, by the AA similarity criterion,

$$\Delta\, PAQ \sim \Delta\, BPC$$

Therefore, $$\frac{PA}{BP} = \frac{AQ}{PC} = \frac{PQ}{BC} \qquad (1)$$

Now, $$AQ = RS = OS - OR = x - x_1$$
$$PC = ST = OT - OS = x_2 - x$$
$$PQ = PS - QS = PS - AR = y - y_1$$
$$BC = BT - CT = BT - PS = y_2 - y$$

Substituting these values in (1), we get

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Taking $$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}, \text{ we get } x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

Similarly, taking $$\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}, \text{ we get } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

So, the coordinates of the point P(x, y) which divides the line segment joining the points A(x_1, y_1) and B(x_2, y_2), internally, in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \qquad (2)$$

This is known as the **section formula**.

This can also be derived by drawing perpendiculars from A, P and B on the y-axis and proceeding as above.

If the ratio in which P divides AB is $k : 1$, then the coordinates of the point P will be

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right).$$

Special Case : The mid-point of a line segment divides the line segment in the ratio 1 : 1. Therefore, the coordinates of the mid-point P of the join of the points A(x_1, y_1) and B(x_2, y_2) is

$$\left(\frac{1\cdot x_1 + 1\cdot x_2}{1+1}, \frac{1\cdot y_1 + 1\cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Let us solve a few examples based on the section formula.

Example 6 : Find the coordinates of the point which divides the line segment joining the points (4, – 3) and (8, 5) in the ratio 3 : 1 internally.

Solution : Let P(x, y) be the required point. Using the section formula, we get

$$x = \frac{3(8) + 1(4)}{3 + 1} = 7, \quad y = \frac{3(5) + 1(-3)}{3 + 1} = 3$$

Therefore, (7, 3) is the required point.

Example 7 : In what ratio does the point (– 4, 6) divide the line segment joining the points A(– 6, 10) and B(3, – 8)?

Solution : Let (– 4, 6) divide AB internally in the ratio $m_1 : m_2$. Using the section formula, we get

$$(-4, 6) = \left(\frac{3m_1 - 6m_2}{m_1 + m_2}, \frac{-8m_1 + 10m_2}{m_1 + m_2}\right) \qquad (1)$$

Recall that if (x, y) = (a, b) then $x = a$ and $y = b$.

So, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2} \text{ and } 6 = \frac{-8m_1 + 10m_2}{m_1 + m_2}$$

Now, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2} \quad \text{gives us}$$

$$-4m_1 - 4m_2 = 3m_1 - 6m_2$$

i.e., $$7m_1 = 2m_2$$

i.e., $$m_1 : m_2 = 2 : 7$$

You should verify that the ratio satisfies the y-coordinate also.

Now, $$\frac{-8m_1 + 10m_2}{m_1 + m_2} = \frac{-8\dfrac{m_1}{m_2} + 10}{\dfrac{m_1}{m_2} + 1} \quad \text{(Dividing throughout by } m_2\text{)}$$

$$= \frac{-8 \times \dfrac{2}{7} + 10}{\dfrac{2}{7} + 1} = 6$$

Therefore, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Alternatively : The ratio $m_1 : m_2$ can also be written as $\frac{m_1}{m_2} : 1$, or k : 1. Let (– 4, 6) divide AB internally in the ratio k : 1. Using the section formula, we get

$$(-4, 6) = \left(\frac{3k-6}{k+1}, \frac{-8k+10}{k+1}\right) \qquad (2)$$

So, $$-4 = \frac{3k-6}{k+1}$$

i.e., $$-4k - 4 = 3k - 6$$

i.e., $$7k = 2$$

i.e., $$k : 1 = 2 : 7$$

You can check for the y-coordinate also.

So, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Note : You can also find this ratio by calculating the distances PA and PB and taking their ratios provided you know that A, P and B are collinear.

Example 8 : Find the coordinates of the points of trisection (i.e., points dividing in three equal parts) of the line segment joining the points A(2, – 2) and B(– 7, 4).

Solution : Let P and Q be the points of trisection of AB i.e., AP = PQ = QB (see Fig. 7.11).

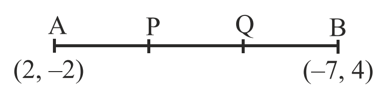


Fig. 7.11

Therefore, P divides AB internally in the ratio 1 : 2. Therefore, the coordinates of P, by applying the section formula, are

$$\left(\frac{1(-7)+2(2)}{1+2}, \frac{1(4)+2(-2)}{1+2}\right), \text{ i.e., } (-1, 0)$$

Now, Q also divides AB internally in the ratio 2 : 1. So, the coordinates of Q are

$$\left(\frac{2(-7)+1(2)}{2+1}, \frac{2(4)+1(-2)}{2+1}\right), \text{ i.e., } (-4, 2)$$

Therefore, the coordinates of the points of trisection of the line segment joining A and B are (–1, 0) and (– 4, 2).

Note : We could also have obtained Q by noting that it is the mid-point of PB. So, we could have obtained its coordinates using the mid-point formula.

Example 9 : Find the ratio in which the y-axis divides the line segment joining the points (5, – 6) and (–1, – 4). Also find the point of intersection.

Solution : Let the ratio be k : 1. Then by the section formula, the coordinates of the point which divides AB in the ratio k : 1 are $\left(\frac{-k+5}{k+1}, \frac{-4k-6}{k+1}\right)$.

This point lies on the y-axis, and we know that on the y-axis the abscissa is 0.

Therefore, $$\frac{-k+5}{k+1} = 0$$

So, $$k = 5$$

That is, the ratio is 5 : 1. Putting the value of $k = 5$, we get the point of intersection as $\left(0, \frac{-13}{3}\right)$.

Example 10 : If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken in order, find the value of p.

Solution : We know that diagonals of a parallelogram bisect each other.

So, the coordinates of the mid-point of AC = coordinates of the mid-point of BD

i.e., $$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

i.e., $$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

so, $$\frac{15}{2} = \frac{8+p}{2}$$

i.e., $$p = 7$$

EXERCISE 7.2

1. Find the coordinates of the point which divides the join of (–1, 7) and (4, –3) in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining (4, –1) and (–2, –3).

3. To conduct Sports Day activities, in your rectangular shaped school ground ABCD, lines have been drawn with chalk powder at a distance of 1m each. 100 flower pots have been placed at a distance of 1m from each other along AD, as shown in Fig. 7.12. Niharika runs $\frac{1}{4}$ th the distance AD on the 2nd line and posts a green flag. Preet runs $\frac{1}{5}$ th the distance AD on the eighth line and posts a red flag. What is the distance between both the flags? If Rashmi has to post a blue flag exactly halfway between the line segment joining the two flags, where should she post her flag?

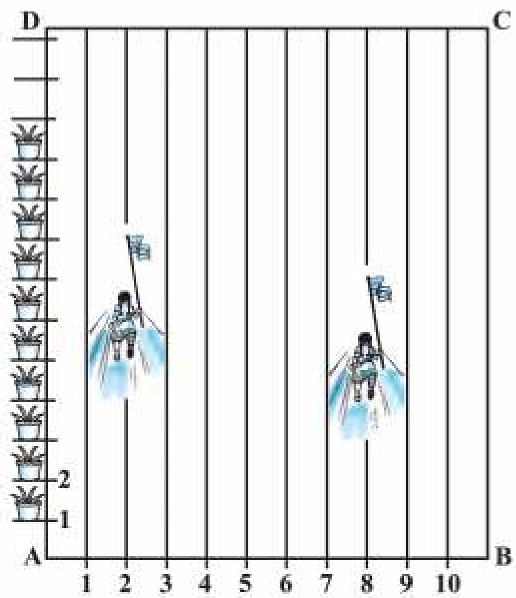


Fig. 7.12

4. Find the ratio in which the line segment joining the points (– 3, 10) and (6, – 8) is divided by (– 1, 6).

5. Find the ratio in which the line segment joining A(1, – 5) and B(– 4, 5) is divided by the x-axis. Also find the coordinates of the point of division.

6. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.

7. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is (2, – 3) and B is (1, 4).

8. If A and B are (– 2, – 2) and (2, – 4), respectively, find the coordinates of P such that $AP = \frac{3}{7} AB$ and P lies on the line segment AB.

9. Find the coordinates of the points which divide the line segment joining A(– 2, 2) and B(2, 8) into four equal parts.

10. Find the area of a rhombus if its vertices are (3, 0), (4, 5), (– 1, 4) and (– 2, – 1) taken in order. [**Hint :** Area of a rhombus = $\frac{1}{2}$ (product of its diagonals)]

7.4 Area of a Triangle

In your earlier classes, you have studied how to calculate the area of a triangle when its base and corresponding height (altitude) are given. You have used the formula :

$$\textbf{Area of a triangle} = \frac{1}{2} \times \textbf{base} \times \textbf{altitude}$$

In Class IX, you have also studied Heron's formula to find the area of a triangle. Now, if the coordinates of the vertices of a triangle are given, can you find its area? Well, you could find the lengths of the three sides using the distance formula and then use Heron's formula. But this could be tedious, particularly if the lengths of the sides are irrational numbers. Let us see if there is an easier way out.

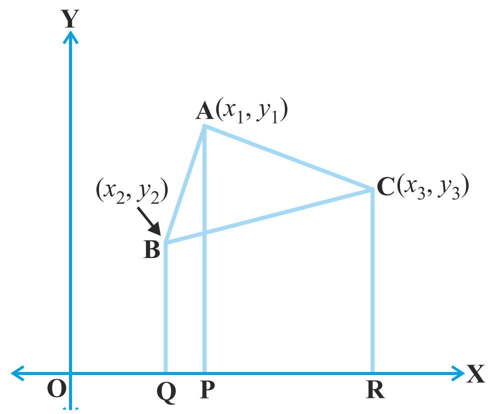


Fig. 7.13

Let ABC be any triangle whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3). Draw AP, BQ and CR perpendiculars from A, B and C, respectively, to the x-axis. Clearly ABQP, APRC and BQRC are all trapezia (see Fig. 7.13).

Now, from Fig. 7.13, it is clear that

area of Δ ABC = area of trapezium ABQP + area of trapezium APRC – area of trapezium BQRC.

You also know that the

$$\text{area of a trapezium} = \frac{1}{2}\text{ (sum of parallel sides)(distance between them)}$$

Therefore,

$$\text{Area of } \Delta \text{ ABC} = \frac{1}{2}(\text{BQ} + \text{AP})\text{ QP} + \frac{1}{2}(\text{AP} + \text{CR})\text{ PR} - \frac{1}{2}(\text{BQ} + \text{CR})\text{ QR}$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Thus, the area of Δ ABC is the numerical value of the expression

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Let us consider a few examples in which we make use of this formula.

Example 11 : Find the area of a triangle whose vertices are (1, –1), (– 4, 6) and (–3, –5).

Solution : The area of the triangle formed by the vertices A(1, –1), B(– 4, 6) and C (–3, –5), by using the formula above, is given by

$$\frac{1}{2}\left[1\,(6+5)+(-4)\,(-5+1)+(-3)\,(-1-6)\right]$$

$$= \frac{1}{2}(11+16+21) = 24$$

So, the area of the triangle is 24 square units.

Example 12 : Find the area of a triangle formed by the points A(5, 2), B(4, 7) and C (7, – 4).

Solution : The area of the triangle formed by the vertices A(5, 2), B(4, 7) and C (7, – 4) is given by

$$\frac{1}{2}\left[5\,(7+4)+4\,(-4-2)+7\,(2-7)\right]$$

$$= \frac{1}{2}(55-24-35) = \frac{-4}{2} = -2$$

Since area is a measure, which cannot be negative, we will take the numerical value of – 2, i.e., 2. Therefore, the area of the triangle = 2 square units.

Example 13 : Find the area of the triangle formed by the points P(–1.5, 3), Q(6, –2) and R(–3, 4).

Solution : The area of the triangle formed by the given points is equal to

$$\frac{1}{2}\left[-1.5(-2-4)+6(4-3)+(-3)(3+2)\right]$$

$$= \frac{1}{2}(9+6-15) = 0$$

Can we have a triangle of area 0 square units? What does this mean?

If the area of a triangle is 0 square units, then its vertices will be collinear.

Example 14 : Find the value of k if the points A(2, 3), B(4, k) and C(6, –3) are collinear.

Solution : Since the given points are collinear, the area of the triangle formed by them must be 0, i.e.,

$$\frac{1}{2}[2(k+3)+4(-3-3)+6(3-k)]=0$$

i.e., $$\frac{1}{2}(-4k)=0$$

Therefore, $$k=0$$

Let us verify our answer.

$$\text{area of } \Delta\text{ ABC} = \frac{1}{2}[2(0+3)+4(-3-3)+6(3-0)]=0$$

Example 15 : If A(–5, 7), B(– 4, –5), C(–1, –6) and D(4, 5) are the vertices of a quadrilateral, find the area of the quadrilateral ABCD.

Solution : By joining B to D, you will get two triangles ABD and BCD.

Now the area of Δ ABD $= \frac{1}{2}[-5(-5-5)+(-4)(5-7)+4(7+5)]$

$$= \frac{1}{2}(50+8+48) = \frac{106}{2} = 53 \text{ square units}$$

Also, the area of Δ BCD $= \frac{1}{2}[-4(-6-5)-1(5+5)+4(-5+6)]$

$$= \frac{1}{2}(44-10+4) = 19 \text{ square units}$$

So, the area of quadrilateral ABCD = 53 + 19 = 72 square units.

Note : To find the area of a polygon, we divide it into triangular regions, which have no common area, and add the areas of these regions.

EXERCISE 7.3

1. Find the area of the triangle whose vertices are :
   (i) (2, 3), (–1, 0), (2, – 4) (ii) (–5, –1), (3, –5), (5, 2)
2. In each of the following find the value of '*k*', for which the points are collinear.
   (i) (7, –2), (5, 1), (3, *k*) (ii) (8, 1), (*k*, – 4), (2, –5)
3. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, –1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle.
4. Find the area of the quadrilateral whose vertices, taken in order, are (– 4, – 2), (– 3, – 5), (3, – 2) and (2, 3).
5. You have studied in Class IX, (Chapter 9, Example 3), that a median of a triangle divides it into two triangles of equal areas. Verify this result for Δ ABC whose vertices are A(4, – 6), B(3, –2) and C(5, 2).

EXERCISE 7.4 (Optional)*

1. Determine the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points A(2, – 2) and B(3, 7).
2. Find a relation between x and y if the points (x, y), (1, 2) and (7, 0) are collinear.
3. Find the centre of a circle passing through the points (6, – 6), (3, – 7) and (3, 3).
4. The two opposite vertices of a square are (–1, 2) and (3, 2). Find the coordinates of the other two vertices.
5. The Class X students of a secondary school in Krishinagar have been allotted a rectangular plot of land for their gardening activity. Sapling of Gulmohar are planted on the boundary at a distance of 1m from each other. There is a triangular grassy lawn in the plot as shown in the Fig. 7.14. The students are to sow seeds of flowering plants on the remaining area of the plot.

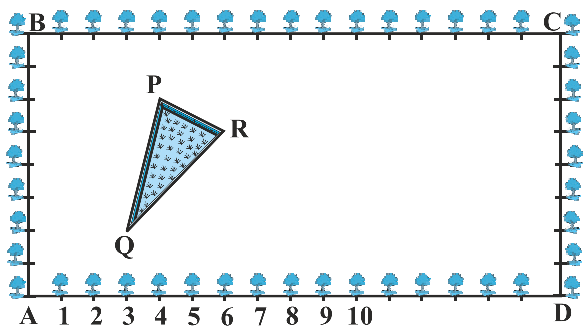


Fig. 7.14

   (i) Taking A as origin, find the coordinates of the vertices of the triangle.
   (ii) What will be the coordinates of the vertices of Δ PQR if C is the origin?
   Also calculate the areas of the triangles in these cases. What do you observe?
6. The vertices of a Δ ABC are A(4, 6), B(1, 5) and C(7, 2). A line is drawn to intersect sides AB and AC at D and E respectively, such that $\frac{\text{AD}}{\text{AB}} = \frac{\text{AE}}{\text{AC}} = \frac{1}{4}\cdot$ Calculate the area of the Δ ADE and compare it with the area of Δ ABC. (Recall Theorem 6.2 and Theorem 6.6).
7. Let A (4, 2), B(6, 5) and C(1, 4) be the vertices of Δ ABC.
   (i) The median from A meets BC at D. Find the coordinates of the point D.
   (ii) Find the coordinates of the point P on AD such that AP : PD = 2 : 1
   (iii) Find the coordinates of points Q and R on medians BE and CF respectively such that BQ : QE = 2 : 1 and CR : RF = 2 : 1.
   (iv) What do yo observe?
   [**Note :** The point which is common to all the three medians is called the *centroid* and this point divides each median in the ratio 2 : 1.]

* These exercises are not from the examination point of view.

(v) If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of Δ ABC, find the coordinates of the centroid of the triangle.

8. ABCD is a rectangle formed by the points A(–1, –1), B(– 1, 4), C(5, 4) and D(5, – 1). P, Q, R and S are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral PQRS a square? a rectangle? or a rhombus? Justify your answer.

7.5 Summary

In this chapter, you have studied the following points :

1. The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The distance of a point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

3. The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$.

4. The mid-point of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

5. The area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the numerical value of the expression

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

A Note to the Reader

Section 7.3 discusses the Section Formula for the coordinates (x, y) of a point P which divides internally the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2$ as follows :

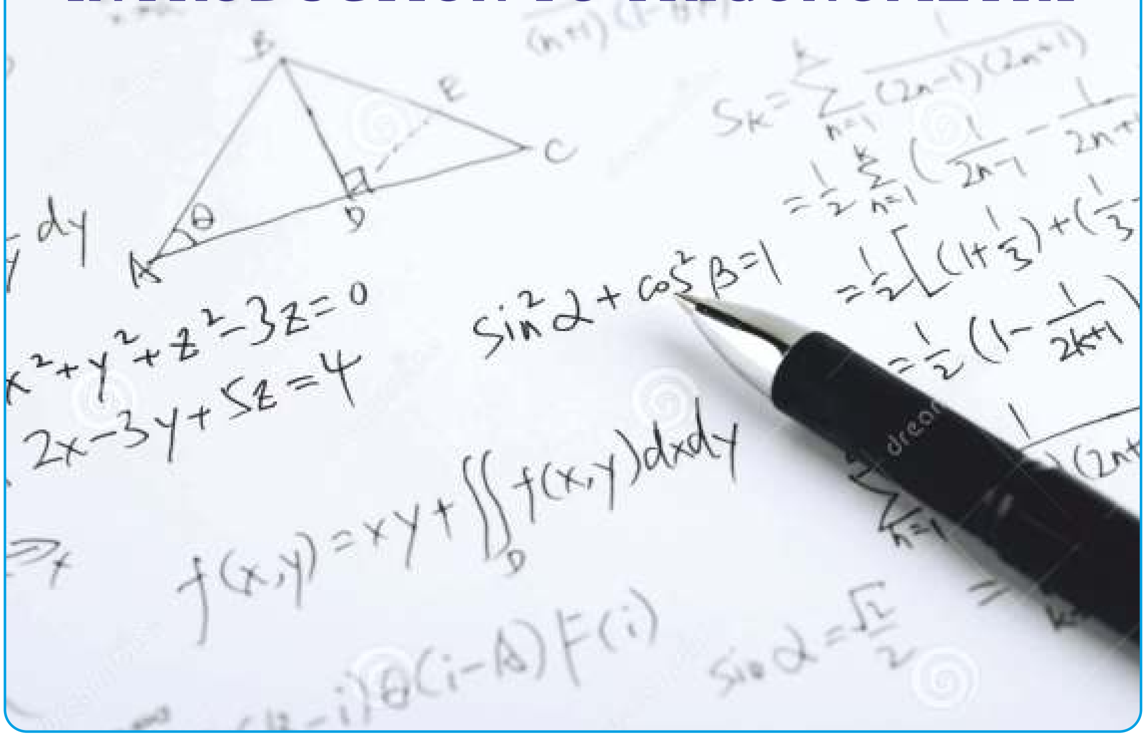
$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Note that, here, PA : PB = $m_1 : m_2$.

However, if P does not lie between A and B but lies on the line AB, outside the line segment AB, and PA : PB = $m_1 : m_2$, we say that P divides externally the line segment joining the points A and B. You will study Section Formula for such case in higher classes.

08

INTRODUCTION TO TRIGONOMETRY



Introduction to Trigonometry 8

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. Herbart (1890)

8.1 Introduction

You have already studied about triangles, and in particular, right triangles, in your earlier classes. Let us take some examples from our surroundings where right triangles can be imagined to be formed. For instance :

1. Suppose the students of a school are visiting Qutub Minar. Now, if a student is looking at the top of the Minar, a right triangle can be imagined to be made, as shown in Fig 8.1. Can the student find out the height of the Minar, without actually measuring it?

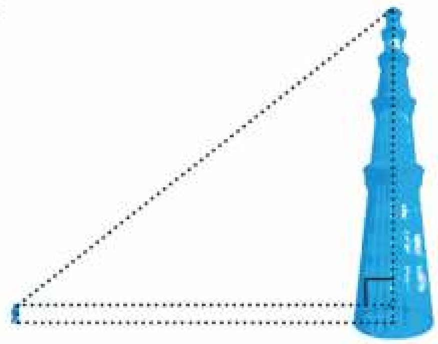


Fig. 8.1

2. Suppose a girl is sitting on the balcony of her house located on the bank of a river. She is looking down at a flower pot placed on a stair of a temple situated nearby on the other bank of the river. A right triangle is imagined to be made in this situation as shown in Fig.8.2. If you know the height at which the person is sitting, can you find the width of the river?

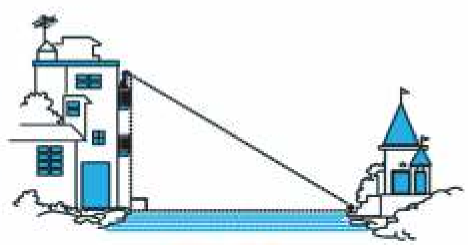


Fig. 8.2

3. Suppose a hot air balloon is flying in the air. A girl happens to spot the balloon in the sky and runs to her mother to tell her about it. Her mother rushes out of the house to look at the balloon.Now when the girl had spotted the balloon intially it was at point A. When both the mother and daughter came out to see it, it had already travelled to another point B. Can you find the altitude of B from the ground?

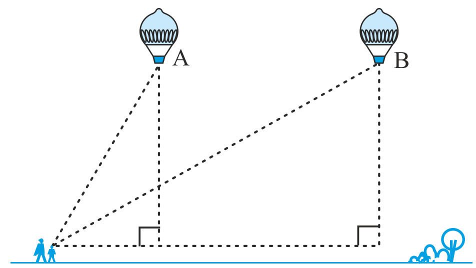


Fig. 8.3

In all the situations given above, the distances or heights can be found by using some mathematical techniques, which come under a branch of mathematics called 'trigonometry'. The word 'trigonometry' is derived from the Greek words 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, **trigonometry** is the study of relationships between the sides and angles of a triangle. The earliest known work on trigonometry was recorded in Egypt and Babylon. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometrical concepts.

In this chapter, we will study some ratios of the sides of a right triangle with respect to its acute angles, called **trigonometric ratios of the angle.** We will restrict our discussion to acute angles only. However, these ratios can be extended to other angles also. We will also define the trigonometric ratios for angles of measure 0° and 90°. We will calculate trigonometric ratios for some specific angles and establish some identities involving these ratios, called **trigonometric identities.**

8.2 Trigonometric Ratios

In Section 8.1, you have seen some right triangles imagined to be formed in different situations.

Let us take a right triangle ABC as shown in Fig. 8.4.

Here, $\angle$ CAB (or, in brief, angle A) is an acute angle. Note the position of the side BC with respect to angle A. It faces $\angle$ A. We call it the *side opposite* to angle A. AC is the *hypotenuse* of the right triangle and the side AB is a part of $\angle$ A. So, we call it the *side adjacent* to angle A.

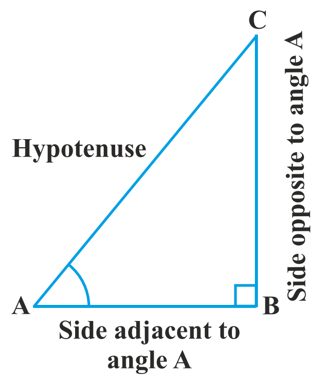


Fig. 8.4

Note that the position of sides change when you consider angle C in place of A (see Fig. 8.5).

You have studied the concept of 'ratio' in your earlier classes. We now define certain ratios involving the sides of a right triangle, and call them trigonometric ratios.

The trigonometric ratios of the angle A in right triangle ABC (see Fig. 8.4) are defined as follows :

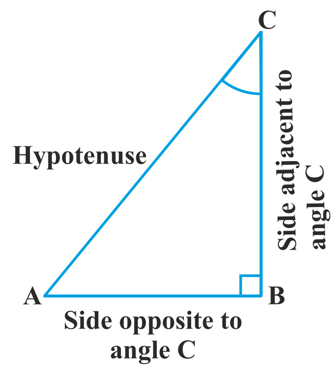


Fig. 8.5

$$\text{sine of } \angle A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{hypotenuse}}{\text{side adjacent to angle A}} = \frac{AC}{AB}$$

$$\text{cotangent of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$$

The ratios defined above are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Note that the ratios **cosec A, sec A and cot A** are respectively, the reciprocals of the ratios sin A, cos A and tan A.

Also, observe that $\tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$.

So, the **trigonometric ratios** of an acute angle in a right triangle express the relationship between the angle and the length of its sides.

Why don't you try to define the trigonometric ratios for angle C in the right triangle? (See Fig. 8.5)

The first use of the idea of **'sine'** in the way we use it today was in the work *Aryabhatiyam* by Aryabhata, in A.D. 500. Aryabhata used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation **'*sin*'.**

Aryabhata
C.E. 476 – 550

The origin of the terms **'cosine'** and **'tangent'** was much later. The cosine function arose from the need to compute the sine of the complementary angle. Aryabhatta called it **kotijya**. The name ***cosinus*** originated with Edmund Gunter. In 1674, the English Mathematician Sir Jonas Moore first used the abbreviated notation **'*cos*'**.

Remark : Note that the symbol sin A is used as an abbreviation for 'the sine of the angle A'. sin A is *not* the product of 'sin' and A. 'sin' separated from A has no meaning. Similarly, cos A is *not* the product of 'cos' and A. Similar interpretations follow for other trigonometric ratios also.

Now, if we take a point P on the hypotenuse AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended (see Fig. 8.6), how will the trigonometric ratios of $\angle$ A in Δ PAM differ from those of $\angle$ A in Δ CAB or from those of $\angle$ A in Δ QAN?

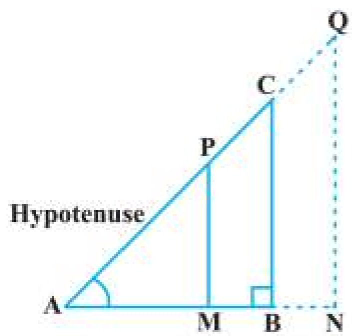


Fig. 8.6

To answer this, first look at these triangles. Is Δ PAM similar to Δ CAB? From Chapter 6, recall the AA similarity criterion. Using the criterion, you will see that the triangles PAM and CAB are similar. Therefore, by the property of similar triangles, the corresponding sides of the triangles are proportional.

So, we have
$$\frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC}.$$

From this, we find $$\frac{MP}{AP} = \frac{BC}{AC} = \sin A.$$

Similarly, $$\frac{AM}{AP} = \frac{AB}{AC} = \cos A, \quad \frac{MP}{AM} = \frac{BC}{AB} = \tan A \text{ and so on.}$$

This shows that the trigonometric ratios of angle A in Δ PAM not differ from those of angle A in Δ CAB.

In the same way, you should check that the value of sin A (and also of other trigonometric ratios) remains the same in Δ QAN also.

From our observations, it is now clear that **the values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.**

Note : For the sake of convenience, we may write $\sin^2 A$, $\cos^2 A$, etc., in place of $(\sin A)^2$, $(\cos A)^2$, etc., respectively. But cosec A = $(\sin A)^{-1} \neq \sin^{-1} A$ (it is called sine inverse A). $\sin^{-1} A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for the other trigonometric ratios as well. Sometimes, the Greek letter θ (theta) is also used to denote an angle.

We have defined six trigonometric ratios of an acute angle. If we know any one of the ratios, can we obtain the other ratios? Let us see.

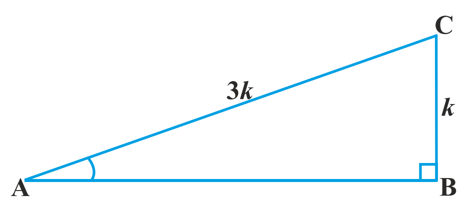


Fig. 8.7

If in a right triangle ABC, $\sin A = \frac{1}{3}$, then this means that $\frac{BC}{AC} = \frac{1}{3}$, i.e., the lengths of the sides BC and AC of the triangle ABC are in the ratio 1 : 3 (see Fig. 8.7). So if BC is equal to k, then AC will be $3k$, where k is any positive number. To determine other trigonometric ratios for the angle A, we need to find the length of the third side AB. Do you remember the Pythagoras theorem? Let us use it to determine the required length AB.

$$AB^2 = AC^2 - BC^2 = (3k)^2 - (k)^2 = 8k^2 = (2\sqrt{2}\,k)^2$$

Therefore, $$AB = \pm 2\sqrt{2}\,k$$

So, we get $$AB = 2\sqrt{2}\,k \quad \text{(Why is AB not } -2\sqrt{2}\,k\text{ ?)}$$

Now, $$\cos A = \frac{AB}{AC} = \frac{2\sqrt{2}\,k}{3k} = \frac{2\sqrt{2}}{3}$$

Similarly, you can obtain the other trigonometric ratios of the angle A.

Remark : Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).

Let us consider some examples.

Example 1 : Given $\tan A = \frac{4}{3}$, find the other trigonometric ratios of the angle A.

Solution : Let us first draw a right Δ ABC (see Fig 8.8).

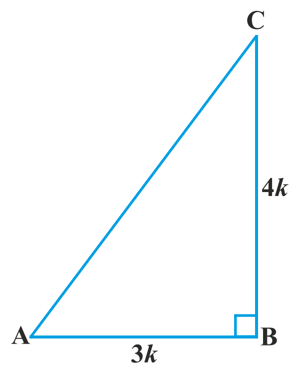


Fig. 8.8

Now, we know that $\tan A = \frac{BC}{AB} = \frac{4}{3}$.

Therefore, if BC = $4k$, then AB = $3k$, where k is a positive number.

Now, by using the Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2 = (4k)^2 + (3k)^2 = 25k^2$$

So, $\quad AC = 5k$

Now, we can write all the trigonometric ratios using their definitions.

$$\sin A = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{3k}{5k} = \frac{3}{5}$$

Therefore, $\cot A = \frac{1}{\tan A} = \frac{3}{4}$, $\text{cosec } A = \frac{1}{\sin A} = \frac{5}{4}$ and $\sec A = \frac{1}{\cos A} = \frac{5}{3}$.

Example 2 : If ∠ B and ∠ Q are acute angles such that sin B = sin Q, then prove that ∠ B = ∠ Q.

Solution : Let us consider two right triangles ABC and PQR where sin B = sin Q (see Fig. 8.9).

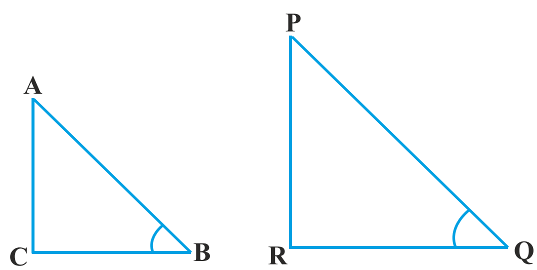


Fig. 8.9

We have $\quad \sin B = \frac{AC}{AB}$

and $\quad \sin Q = \frac{PR}{PQ}$

Then $$\frac{AC}{AB} = \frac{PR}{PQ}$$

Therefore, $$\frac{AC}{PR} = \frac{AB}{PQ} = k, \text{ say} \quad (1)$$

Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$

and $$QR = \sqrt{PQ^2 - PR^2}$$

So, $$\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \quad (2)$$

From (1) and (2), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Then, by using Theorem 6.4, Δ ACB ~ Δ PRQ and therefore, $\angle$ B = $\angle$ Q.

Example 3 : Consider Δ ACB, right-angled at C, in which AB = 29 units, BC = 21 units and $\angle$ ABC = θ (see Fig. 8.10). Determine the values of

(i) $\cos^2 \theta + \sin^2 \theta$,

(ii) $\cos^2 \theta - \sin^2 \theta$.

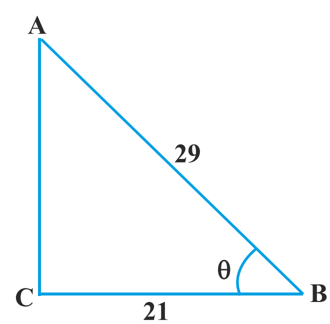


Fig. 8.10

Solution : In Δ ACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{(29 - 21)(29 + 21)} = \sqrt{(8)(50)} = \sqrt{400} = 20 \text{ units}$$

So, $$\sin \theta = \frac{AC}{AB} = \frac{20}{29}, \cos \theta = \frac{BC}{AB} = \frac{21}{29}.$$

Now, (i) $$\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{20^2 + 21^2}{29^2} = \frac{400 + 441}{841} = 1,$$

and (ii) $$\cos^2 \theta - \sin^2 \theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{(21 + 20)(21 - 20)}{29^2} = \frac{41}{841}.$$

Example 4 : In a right triangle ABC, right-angled at B, if tan A = 1, then verify that

2 sin A cos A = 1.

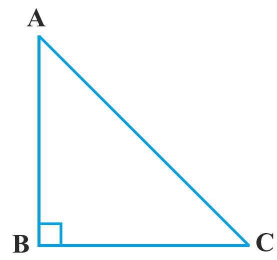


Fig. 8.11

Solution : In Δ ABC, tan A = $\frac{BC}{AB} = 1$ (see Fig 8.11)

i.e., $$BC = AB$$

Let AB = BC = k, where k is a positive number.

Now, $$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{(k)^2 + (k)^2} = k\sqrt{2}$$

Therefore, $$\sin A = \frac{BC}{AC} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos A = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

So, $$2 \sin A \cos A = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1,$$ which is the required value.

Example 5 : In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see Fig. 8.12). Determine the values of sin Q and cos Q.

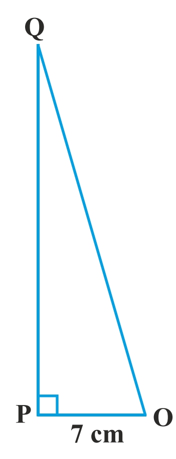


Fig. 8.12

Solution : In Δ OPQ, we have

$$OQ^2 = OP^2 + PQ^2$$

i.e., $$(1 + PQ)^2 = OP^2 + PQ^2 \quad \text{(Why?)}$$

i.e., $$1 + PQ^2 + 2PQ = OP^2 + PQ^2$$

i.e., $$1 + 2PQ = 7^2 \quad \text{(Why?)}$$

i.e., $$PQ = 24 \text{ cm and } OQ = 1 + PQ = 25 \text{ cm}$$

So, $$\sin Q = \frac{7}{25} \text{ and } \cos Q = \frac{24}{25}.$$

EXERCISE 8.1

1. In Δ ABC, right-angled at B, AB = 24 cm, BC = 7 cm. Determine :
   (i) sin A, cos A
   (ii) sin C, cos C

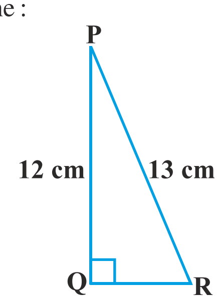


Fig. 8.13

2. In Fig. 8.13, find tan P – cot R.
3. If sin A = $\frac{3}{4}$, calculate cos A and tan A.
4. Given 15 cot A = 8, find sin A and sec A.
5. Given sec θ = $\frac{13}{12}$, calculate all other trigonometric ratios.
6. If $\angle$ A and $\angle$ B are acute angles such that cos A = cos B, then show that $\angle$ A = $\angle$ B.
7. If cot θ = $\frac{7}{8}$, evaluate : (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$, (ii) $\cot^2\theta$
8. If 3 cot A = 4, check whether $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A$ or not.
9. In triangle ABC, right-angled at B, if tan A = $\frac{1}{\sqrt{3}}$, find the value of:
   (i) sin A cos C + cos A sin C
   (ii) cos A cos C – sin A sin C
10. In Δ PQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P.
11. State whether the following are true or false. Justify your answer.
    (i) The value of tan A is always less than 1.
    (ii) sec A = $\frac{12}{5}$ for some value of angle A.
    (iii) cos A is the abbreviation used for the cosecant of angle A.
    (iv) cot A is the product of cot and A.
    (v) sin θ = $\frac{4}{3}$ for some angle θ.

8.3 Trigonometric Ratios of Some Specific Angles

From geometry, you are already familiar with the construction of angles of 30°, 45°, 60° and 90°. In this section, we will find the values of the trigonometric ratios for these angles and, of course, for 0°.

Trigonometric Ratios of 45°

In Δ ABC, right-angled at B, if one angle is 45°, then the other angle is also 45°, i.e., ∠ A = ∠ C = 45° (see Fig. 8.14).

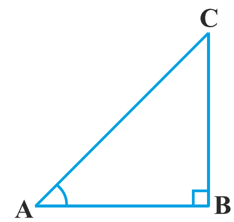


Fig. 8.14

So, BC = AB (Why?)

Now, Suppose BC = AB = a.

Then by Pythagoras Theorem, $AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$,

and, therefore, $AC = a\sqrt{2}$.

Using the definitions of the trigonometric ratios, we have :

$$\sin 45° = \frac{\text{side opposite to angle } 45°}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{side adjacent to angle } 45°}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{side opposite to angle } 45°}{\text{side adjacent to angle } 45°} = \frac{BC}{AB} = \frac{a}{a} = 1$$

Also, $\operatorname{cosec} 45° = \frac{1}{\sin 45°} = \sqrt{2}$, $\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$, $\cot 45° = \frac{1}{\tan 45°} = 1$.

Trigonometric Ratios of 30° and 60°

Let us now calculate the trigonometric ratios of 30° and 60°. Consider an equilateral triangle ABC. Since each angle in an equilateral triangle is 60°, therefore, ∠ A = ∠ B = ∠ C = 60°.

Draw the perpendicular AD from A to the side BC (see Fig. 8.15).

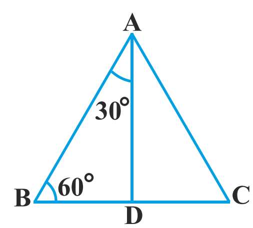


Fig. 8.15

Now Δ ABD ≅ Δ ACD (Why?)

Therefore, BD = DC

and ∠ BAD = ∠ CAD (CPCT)

Now observe that:

Δ ABD is a right triangle, right-angled at D with ∠ BAD = 30° and ∠ ABD = 60° (see Fig. 8.15).

As you know, for finding the trigonometric ratios, we need to know the lengths of the sides of the triangle. So, let us suppose that AB = $2a$.

Then, $$BD = \frac{1}{2}BC = a$$

and $$AD^2 = AB^2 - BD^2 = (2a)^2 - (a)^2 = 3a^2,$$

Therefore, $$AD = a\sqrt{3}$$

Now, we have :

$$\sin 30° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2},\ \cos 30° = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Also, $$\operatorname{cosec} 30° = \frac{1}{\sin 30°} = 2,\quad \sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$$

$$\cot 30° = \frac{1}{\tan 30°} = \sqrt{3}.$$

Similarly,

$$\sin 60° = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},\ \cos 60° = \frac{1}{2},\ \tan 60° = \sqrt{3},$$

$$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}},\ \sec 60° = 2 \text{ and } \cot 60° = \frac{1}{\sqrt{3}}.$$

Trigonometric Ratios of 0° and 90°

Let us see what happens to the trigonometric ratios of angle A, if it is made smaller and smaller in the right triangle ABC (see Fig. 8.16), till it becomes zero. As $\angle$ A gets smaller and smaller, the length of the side BC decreases. The point C gets closer to point B, and finally when $\angle$ A becomes very close to 0°, AC becomes almost the same as AB (see Fig. 8.17).

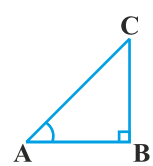


Fig. 8.16

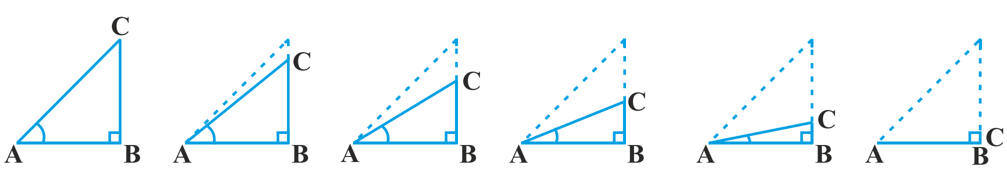


Fig. 8.17

When $\angle$ A is very close to 0°, BC gets very close to 0 and so the value of $\sin A = \frac{BC}{AC}$ is very close to 0. Also, when $\angle$ A is very close to 0°, AC is nearly the same as AB and so the value of $\cos A = \frac{AB}{AC}$ is very close to 1.

This helps us to see how we can define the values of sin A and cos A when A = 0°. We define : **sin 0° = 0 and cos 0° = 1.**

Using these, we have :

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = 0, \cot 0° = \frac{1}{\tan 0°}, \text{ which is not defined. (Why?)}$$

$$\sec 0° = \frac{1}{\cos 0°} = 1 \text{ and } \operatorname{cosec} 0° = \frac{1}{\sin 0°}, \text{ which is again not defined.(Why?)}$$

Now, let us see what happens to the trigonometric ratios of $\angle$ A, when it is made larger and larger in Δ ABC till it becomes 90°. As $\angle$ A gets larger and larger, $\angle$ C gets smaller and smaller. Therefore, as in the case above, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally when $\angle$ A is very close to 90°, $\angle$ C becomes very close to 0° and the side AC almost coincides with side BC (see Fig. 8.18).

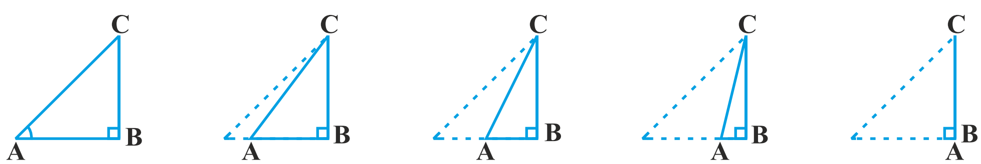


Fig. 8.18

When $\angle$ C is very close to 0°, $\angle$ A is very close to 90°, side AC is nearly the same as side BC, and so sin A is very close to 1. Also when $\angle$ A is very close to 90°, $\angle$ C is very close to 0°, and the side AB is nearly zero, so cos A is very close to 0.

So, we define : **sin 90° = 1 and cos 90° = 0.**

Now, why don't you find the other trigonometric ratios of 90°?

We shall now give the values of all the trigonometric ratios of 0°, 30°, 45°, 60° and 90° in Table 8.1, for ready reference.

Table 8.1

$\angle$ A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Remark : From the table above you can observe that as $\angle$ A increases from 0° to 90°, sin A increases from 0 to 1 and cos A decreases from 1 to 0.

Let us illustrate the use of the values in the table above through some examples.

Example 6 : In Δ ABC, right-angled at B, AB = 5 cm and $\angle$ ACB = 30° (see Fig. 8.19). Determine the lengths of the sides BC and AC.

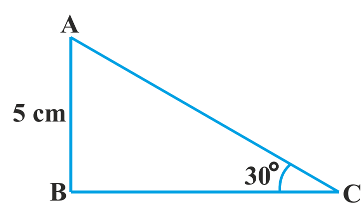


Fig. 8.19

Solution : To find the length of the side BC, we will choose the trigonometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C, therefore

$$\frac{AB}{BC} = \tan C$$

i.e., $$\frac{5}{BC} = \tan 30° = \frac{1}{\sqrt{3}}$$

which gives $$BC = 5\sqrt{3} \text{ cm}$$

To find the length of the side AC, we consider

$$\sin 30° = \frac{AB}{AC} \qquad \text{(Why?)}$$

i.e.,
$$\frac{1}{2} = \frac{5}{AC}$$

i.e., $\quad AC = 10 \text{ cm}$

Note that alternatively we could have used Pythagoras theorem to determine the third side in the example above,

i.e.,
$$AC = \sqrt{AB^2 + BC^2} = \sqrt{5^2 + (5\sqrt{3})^2} \text{ cm} = 10 \text{cm}.$$

Example 7 : In Δ PQR, right-angled at Q (see Fig. 8.20), PQ = 3 cm and PR = 6 cm. Determine ∠ QPR and ∠ PRQ.

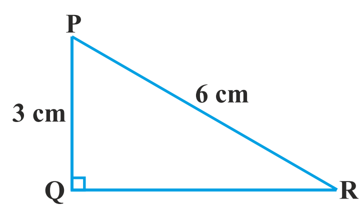


Fig. 8.20

Solution : Given PQ = 3 cm and PR = 6 cm.

Therefore, $\quad \frac{PQ}{PR} = \sin R$

or $\quad \sin R = \frac{3}{6} = \frac{1}{2}$

So, $\quad \angle PRQ = 30°$

and therefore, $\quad \angle QPR = 60°. \quad$ (Why?)

You may note that if one of the sides and any other part (either an acute angle or any side) of a right triangle is known, the remaining sides and angles of the triangle can be determined.

Example 8 : If $\sin (A - B) = \frac{1}{2}$, $\cos (A + B) = \frac{1}{2}$, $0° < A + B \le 90°$, $A > B$, find A and B.

Solution : Since, $\sin (A - B) = \frac{1}{2}$, therefore, $A - B = 30°$ (Why?) (1)

Also, since $\cos (A + B) = \frac{1}{2}$, therefore, $A + B = 60°$ (Why?) (2)

Solving (1) and (2), we get : $A = 45°$ and $B = 15°$.

EXERCISE 8.2

1. Evaluate the following :

(i) $\sin 60° \cos 30° + \sin 30° \cos 60°$

(ii) $2 \tan^2 45° + \cos^2 30° - \sin^2 60°$

(iii) $\dfrac{\cos 45°}{\sec 30° + \operatorname{cosec} 30°}$

(iv) $\dfrac{\sin 30° + \tan 45° - \operatorname{cosec} 60°}{\sec 30° + \cos 60° + \cot 45°}$

(v) $\dfrac{5 \cos^2 60° + 4 \sec^2 30° - \tan^2 45°}{\sin^2 30° + \cos^2 30°}$

2. Choose the correct option and justify your choice :

(i) $\dfrac{2 \tan 30°}{1 + \tan^2 30°} =$

(A) $\sin 60°$ (B) $\cos 60°$ (C) $\tan 60°$ (D) $\sin 30°$

(ii) $\dfrac{1 - \tan^2 45°}{1 + \tan^2 45°} =$

(A) $\tan 90°$ (B) 1 (C) $\sin 45°$ (D) 0

(iii) $\sin 2A = 2 \sin A$ is true when A =

(A) 0° (B) 30° (C) 45° (D) 60°

(iv) $\dfrac{2 \tan 30°}{1 - \tan^2 30°} =$

(A) $\cos 60°$ (B) $\sin 60°$ (C) $\tan 60°$ (D) $\sin 30°$

3. If $\tan (A + B) = \sqrt{3}$ and $\tan (A - B) = \dfrac{1}{\sqrt{3}}$; $0° < A + B \le 90°$; $A > B$, find A and B.

4. State whether the following are true or false. Justify your answer.

(i) $\sin (A + B) = \sin A + \sin B$.

(ii) The value of $\sin \theta$ increases as θ increases.

(iii) The value of $\cos \theta$ increases as θ increases.

(iv) $\sin \theta = \cos \theta$ for all values of θ.

(v) $\cot A$ is not defined for $A = 0°$.

8.4 Trigonometric Ratios of Complementary Angles

Recall that two angles are said to be complementary if their sum equals 90°. In Δ ABC, right-angled at B, do you see any pair of complementary angles? (See Fig. 8.21)

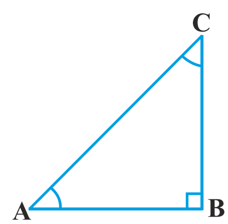


Fig. 8.21

Since $\angle A + \angle C = 90°$, they form such a pair. We have:

$$\left.\begin{array}{lll} \sin A = \dfrac{BC}{AC} & \cos A = \dfrac{AB}{AC} & \tan A = \dfrac{BC}{AB} \\[2ex] \operatorname{cosec} A = \dfrac{AC}{BC} & \sec A = \dfrac{AC}{AB} & \cot A = \dfrac{AB}{BC} \end{array}\right\} \quad (1)$$

Now let us write the trigonometric ratios for $\angle C = 90° - \angle A$.

For convenience, we shall write $90° - A$ instead of $90° - \angle A$.

What would be the side opposite and the side adjacent to the angle $90° - A$?

You will find that AB is the side opposite and BC is the side adjacent to the angle $90° - A$. Therefore,

$$\left.\begin{array}{lll} \sin (90° - A) = \dfrac{AB}{AC}, & \cos (90° - A) = \dfrac{BC}{AC}, & \tan (90° - A) = \dfrac{AB}{BC} \\[2ex] \operatorname{cosec} (90° - A) = \dfrac{AC}{AB}, & \sec (90° - A) = \dfrac{AC}{BC}, & \cot (90° - A) = \dfrac{BC}{AB} \end{array}\right\} \quad (2)$$

Now, compare the ratios in (1) and (2). Observe that :

$$\sin (90° - A) = \frac{AB}{AC} = \cos A \text{ and } \cos (90° - A) = \frac{BC}{AC} = \sin A$$

Also, $$\tan (90° - A) = \frac{AB}{BC} = \cot A, \quad \cot (90° - A) = \frac{BC}{AB} = \tan A$$

$$\sec (90° - A) = \frac{AC}{BC} = \operatorname{cosec} A, \quad \operatorname{cosec} (90° - A) = \frac{AC}{AB} = \sec A$$

So, **sin (90° – A) = cos A,** **cos (90° – A) = sin A,**

tan (90° – A) = cot A, **cot (90° – A) = tan A,**

sec (90° – A) = cosec A, **cosec (90° – A) = sec A,**

for all values of angle A lying between 0° and 90°. Check whether this holds for A = 0° or A = 90°.

Note : tan 0° = 0 = cot 90°, sec 0° = 1 = cosec 90° and sec 90°, cosec 0°, tan 90° and cot 0° are not defined.

Now, let us consider some examples.

Example 9 : Evaluate $\dfrac{\tan 65^\circ}{\cot 25^\circ}$.

Solution : We know : $\cot A = \tan (90^\circ - A)$

So, $\cot 25^\circ = \tan (90^\circ - 25^\circ) = \tan 65^\circ$

i.e., $$\frac{\tan 65^\circ}{\cot 25^\circ} = \frac{\tan 65^\circ}{\tan 65^\circ} = 1$$

Example 10 : If sin 3A = cos (A – 26°), where 3A is an acute angle, find the value of A.

Solution : We are given that sin 3A = cos (A – 26°). (1)

Since sin 3A = cos (90° – 3A), we can write (1) as

$$\cos (90^\circ - 3A) = \cos (A - 26^\circ)$$

Since 90° – 3A and A – 26° are both acute angles, therefore,

$$90^\circ - 3A = A - 26^\circ$$

which gives $$A = 29^\circ$$

Example 11 : Express cot 85° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

Solution : $$\cot 85^\circ + \cos 75^\circ = \cot (90^\circ - 5^\circ) + \cos (90^\circ - 15^\circ)$$
$$= \tan 5^\circ + \sin 15^\circ$$

EXERCISE 8.3

1. Evaluate :

(i) $\dfrac{\sin 18^\circ}{\cos 72^\circ}$ (ii) $\dfrac{\tan 26^\circ}{\cot 64^\circ}$ (iii) $\cos 48^\circ - \sin 42^\circ$ (iv) $\operatorname{cosec} 31^\circ - \sec 59^\circ$

2. Show that :

(i) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ = 1$

(ii) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ = 0$

3. If tan 2A = cot (A – 18°), where 2A is an acute angle, find the value of A.

4. If tan A = cot B, prove that A + B = 90°.

5. If sec 4A = cosec (A – 20°), where 4A is an acute angle, find the value of A.

6. If A, B and C are interior angles of a triangle ABC, then show that

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}.$$

7. Express sin 67° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

8.5 Trigonometric Identities

You may recall that an equation is called an identity when it is true for all values of the variables involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity,** if it is true for all values of the angle(s) involved.

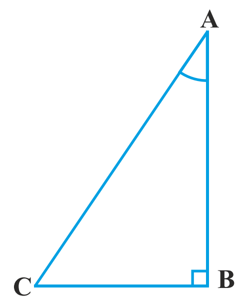


Fig. 8.22

In this section, we will prove one trigonometric identity, and use it further to prove other useful trigonometric identities.

In Δ ABC, right-angled at B (see Fig. 8.22), we have:

$$AB^2 + BC^2 = AC^2 \qquad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

i.e., $$(\cos A)^2 + (\sin A)^2 = 1$$

i.e., $$\mathbf{\cos^2 A + \sin^2 A = 1} \qquad (2)$$

This is true for all A such that 0° ≤ A ≤ 90°. So, this is a trigonometric identity.

Let us now divide (1) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

or, $$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $$\mathbf{1 + \tan^2 A = \sec^2 A} \qquad (3)$$

Is this equation true for A = 0°? Yes, it is. What about A = 90°? Well, tan A and sec A are not defined for A = 90°. So, (3) is true for all A such that $0° \le A < 90°$.

Let us see what we get on dividing (1) by BC^2. We get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

i.e.,
$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e.,
$$\mathbf{\cot^2 A + 1 = \operatorname{cosec}^2 A} \qquad (4)$$

Note that cosec A and cot A are not defined for A = 0°. Therefore (4) is true for all A such that $0° < A \le 90°$.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Let us see how we can do this using these identities. Suppose we know that $\tan A = \frac{1}{\sqrt{3}}$. Then, $\cot A = \sqrt{3}$.

Since, $\sec^2 A = 1 + \tan^2 A = 1 + \frac{1}{3} = \frac{4}{3}$, $\sec A = \frac{2}{\sqrt{3}}$, and $\cos A = \frac{\sqrt{3}}{2}$.

Again, $\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$. Therefore, cosec A = 2.

Example 12 : Express the ratios cos A, tan A and sec A in terms of sin A.

Solution : Since $\cos^2 A + \sin^2 A = 1$, therefore,

$$\cos^2 A = 1 - \sin^2 A, \text{ i.e., } \cos A = \pm\sqrt{1 - \sin^2 A}$$

This gives
$$\cos A = \sqrt{1 - \sin^2 A} \qquad \text{(Why?)}$$

Hence,
$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} \text{ and } \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}$$

Example 13 : Prove that sec A (1 – sin A)(sec A + tan A) = 1.

Solution :

$$\text{LHS} = \sec A\,(1 - \sin A)(\sec A + \tan A) = \left(\frac{1}{\cos A}\right)(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A} = \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A} = 1 = \text{RHS}$$

Example 14 : Prove that $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

Solution : $\text{LHS} = \dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\dfrac{\cos A}{\sin A} - \cos A}{\dfrac{\cos A}{\sin A} + \cos A}$

$$= \frac{\cos A\left(\dfrac{1}{\sin A} - 1\right)}{\cos A\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\left(\dfrac{1}{\sin A} - 1\right)}{\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \text{RHS}$$

Example 15 : Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Solution : Since we will apply the identity involving sec θ and tan θ, let us first convert the LHS (of the identity we need to prove) in terms of sec θ and tan θ by dividing numerator and denominator by cos θ.

$$\text{LHS} = \frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1} = \frac{\{(\tan\theta + \sec\theta) - 1\}\,(\tan\theta - \sec\theta)}{\{(\tan\theta - \sec\theta) + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{\{\tan\theta - \sec\theta + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1 - \tan\theta + \sec\theta}{(\tan\theta - \sec\theta + 1)\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta} = \frac{1}{\sec\theta - \tan\theta},$$

which is the RHS of the identity, we are required to prove.

EXERCISE 8.4

1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.
2. Write all the other trigonometric ratios of $\angle$A in terms of sec A.
3. Evaluate :

   (i) $\dfrac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$

   (ii) $\sin 25° \cos 65° + \cos 25° \sin 65°$
4. Choose the correct option. Justify your choice.

   (i) $9\sec^2 A - 9\tan^2 A =$

   (A) 1 (B) 9 (C) 8 (D) 0

   (ii) $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \operatorname{cosec}\theta) =$

   (A) 0 (B) 1 (C) 2 (D) –1

   (iii) $(\sec A + \tan A)(1 - \sin A) =$

   (A) sec A (B) sin A (C) cosec A (D) cos A

   (iv) $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} =$

   (A) $\sec^2 A$ (B) –1 (C) $\cot^2 A$ (D) $\tan^2 A$
5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

   (i) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$ (ii) $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 + \sin A}{\cos A} = 2\sec A$

(iii) $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\sec\theta\,\operatorname{cosec}\theta$

[**Hint :** Write the expression in terms of $\sin\theta$ and $\cos\theta$]

(iv) $\dfrac{1+\sec A}{\sec A}=\dfrac{\sin^2 A}{1-\cos A}$ [**Hint :** Simplify LHS and RHS separately]

(v) $\dfrac{\cos A-\sin A+1}{\cos A+\sin A-1}=\operatorname{cosec} A+\cot A$, using the identity $\operatorname{cosec}^2 A=1+\cot^2 A$.

(vi) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$

(vii) $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

(viii) $(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=7+\tan^2 A+\cot^2 A$

(ix) $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

[**Hint :** Simplify LHS and RHS separately]

(x) $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\dfrac{1-\tan A}{1-\cot A}\right)^2=\tan^2 A$

8.6 Summary

In this chapter, you have studied the following points :

1. In a right triangle ABC, right-angled at B,

   $\sin A=\dfrac{\text{side opposite to angle A}}{\text{hypotenuse}}$, $\cos A=\dfrac{\text{side adjacent to angle A}}{\text{hypotenuse}}$

   $\tan A=\dfrac{\text{side opposite to angle A}}{\text{side adjacent to angle A}}$.

2. $\operatorname{cosec} A=\dfrac{1}{\sin A}$; $\sec A=\dfrac{1}{\cos A}$; $\tan A=\dfrac{1}{\cot A}$, $\tan A=\dfrac{\sin A}{\cos A}$.
3. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be easily determined.
4. The values of trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°.
5. The value of sin A or cos A never exceeds 1, whereas the value of sec A or cosec A is always greater than or equal to 1.
6. $\sin(90°-A)=\cos A$, $\cos(90°-A)=\sin A$;

   $\tan(90°-A)=\cot A$, $\cot(90°-A)=\tan A$;

   $\sec(90°-A)=\operatorname{cosec} A$, $\operatorname{cosec}(90°-A)=\sec A$.
7. $\sin^2 A+\cos^2 A=1$,

   $\sec^2 A-\tan^2 A=1$ for $0°\le A<90°$,

   $\operatorname{cosec}^2 A=1+\cot^2 A$ for $0°<A\le 90°$.