# MATHEMATICS

## Textbook For Class X

Jammu and Kashmir
Board of School Education

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Professional Graphics Srinagar / 9419974394

# Foreword

The Jammu and Kashmir Board of School of Education is perpetually striving to upgrade the school Curriculum and courses of studies exclusively in light of National Curriculum Framework 2005. In order to provide comparative, comprehensive and holistic contents / course of studies for the budding learners in the contemporary atmosphere. The review / revision exercise of a book is a perpetual activity to imbibe the concept of modern needs and challenges with a view to harmonize and witness a competent, capable and diligent posterity.

The syllabus in the subject of Mathematics has undergone changes from time to time in accordance with the growth of the subject and the emerging needs of the society. As a matter of fact, Secondary / Higher Secondary stage is a fundamental feature for higher Academic opportunities and also for professional and other competitive course like, medicine, engineering, commerce and computer sciences etc. The present syllabus has been designed to meet the emerging needs of the students to have a link with the real life so that Mathematics is not presented only as a bundle of formulae and mechanical procedures but to remove fear psychosis regarding understanding of the subject of mathematics.

I am highly grateful to the Director NCERT for providing us the necessary textual material for the textbook of JKBOSE. We appreciate the hard work done by the textbook development committee responsible for its development. We place on record our thanks to the Chairperson of the Advisory group in Science and Mathematics, Professor J. V. Narlikar and the Chief Advisors for this book, Professor P. Sinclair of IGNOU, New Delhi and Professor G. P. Dikhshat (Retd.) of Lucknow University, for their valuable guidance. Our special thanks go to teachers and subject experts who contributed vehemently in the development and review of this textbook.

**Prof. Veena Pandita**
CHAIRPERSON

# Acknowledgement

The Jammu and Kashmir Board of School Education is grateful to all the following experts of Textbook Development Committee of National Council of Educational Research and Training (NCERT), for development of this textbook of Mathematics.

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I express my gratitude to Director NCERT for allowing us to use their textual material. I am grateful to Prof. Veena Pandita, Chairperson J&K Board of School Education for providing us all the support and help in completing the assignment of the textbook. I am also thankful to Mr. Reyaz Ahmad, Secretary JKBOSE, Mr Abdul Wahid Makhdoomi, Joint Secretary Publication, Dr Aaliya Qayoom, Assistant Director, Syed Fayaz Ahmad, Academic Officer, the Academic wing of JKBOSE and all the participants who participated in the NCERT Textbook Review workshop for their support in bringing out this NCERT based textbook.

**Dr. Farooq Ahmad Peer**
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# 01

# REAL NUMBERS

# REAL NUMBERS 1

## 1.1 Introduction

In Class IX, you began your exploration of the world of real numbers and encountered irrational numbers. We continue our discussion on real numbers in this chapter. We begin with two very important properties of positive integers in Sections 1.2 and 1.3, namely the Euclid's division algorithm and the Fundamental Theorem of Arithmetic.

Euclid's division algorithm, as the name suggests, has to do with divisibility of integers. Stated simply, it says any positive integer $a$ can be divided by another positive integer $b$ in such a way that it leaves a remainder $r$ that is smaller than $b$. Many of you probably recognise this as the usual long division process. Although this result is quite easy to state and understand, it has many applications related to the divisibility properties of integers. We touch upon a few of them, and use it mainly to compute the HCF of two positive integers.

The Fundamental Theorem of Arithmetic, on the other hand, has to do something with multiplication of positive integers. You already know that every composite number can be expressed as a product of primes in a unique way—this important fact is the Fundamental Theorem of Arithmetic. Again, while it is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. We use the Fundamental Theorem of Arithmetic for two main applications. First, we use it to prove the irrationality of many of the numbers you studied in Class IX, such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. Second, we apply this theorem to explore when exactly the decimal expansion of a rational number, say $\frac{p}{q} (q \neq 0)$, is terminating and when it is non-terminating repeating. We do so by looking at the prime factorisation of the denominator $q$ of $\frac{p}{q}$. You will see that the prime factorisation of $q$ will completely reveal the nature of the decimal expansion of $\frac{p}{q}$.

So let us begin our exploration.

## 1.2 Euclid's Division Lemma

Consider the following folk puzzle*.

*A trader was moving along a road selling eggs. An idler who didn't have much work to do, started to get the trader into a wordy duel. This grew into a fight, he pulled the basket with eggs and dashed it on the floor. The eggs broke. The trader requested the Panchayat to ask the idler to pay for the broken eggs. The Panchayat asked the trader how many eggs were broken. He gave the following response:*

*If counted in pairs, one will remain;*

*If counted in threes, two will remain;*

*If counted in fours, three will remain;*

*If counted in fives, four will remain;*

*If counted in sixes, five will remain;*

*If counted in sevens, nothing will remain;*

*My basket cannot accomodate more than 150 eggs.*

So, how many eggs were there? Let us try and solve the puzzle. Let the number of eggs be $a$. Then working backwards, we see that $a$ is less than or equal to 150:

If counted in sevens, nothing will remain, which translates to $a = 7p + 0$, for some natural number $p$. If counted in sixes, $a = 6q + 5$, for some natural number $q$.

If counted in fives, four will remain. It translates to $a = 5w + 4$, for some natural number $w$.

If counted in fours, three will remain. It translates to $a = 4s + 3$, for some natural number $s$.

If counted in threes, two will remain. It translates to $a = 3t + 2$, for some natural number $t$.

If counted in pairs, one will remain. It translates to $a = 2u + 1$, for some natural number $u$.

That is, in each case, we have $a$ and a positive integer $b$ (in our example, $b$ takes values 7, 6, 5, 4, 3 and 2, respectively) which divides $a$ and leaves a remainder $r$ (in our case, $r$ is 0, 5, 4, 3, 2 and 1, respectively), that is smaller than $b$. The

* This is modified form of a puzzle given in 'Numeracy Counts!' by A. Rampal, and others.

moment we write down such equations we are using Euclid's division lemma, which is given in Theorem 1.1.

Getting back to our puzzle, do you have any idea how you will solve it? Yes! You must look for the multiples of 7 which satisfy all the conditions. By trial and error (using the concept of LCM), you will find he had 119 eggs.

In order to get a feel for what Euclid's division lemma is, consider the following pairs of integers:

17, 6;   5, 12;   20, 4

Like we did in the example, we can write the following relations for each such pair:

$17 = 6 \times 2 + 5$ (6 goes into 17 twice and leaves a remainder 5)

$5 = 12 \times 0 + 5$ (This relation holds since 12 is larger than 5)

$20 = 4 \times 5 + 0$ (Here 4 goes into 20 five-times and leaves no remainder)

That is, for each pair of positive integers $a$ and $b$, we have found whole numbers $q$ and $r$, satisfying the relation:

$$a = bq + r,\ 0 \le r < b$$

**Note** that $q$ or $r$ can also be zero.

Why don't you now try finding integers $q$ and $r$ for the following pairs of positive integers $a$ and $b$?

(i) 10, 3;   (ii) 4, 19;   (iii) 81, 3

Did you notice that $q$ and $r$ are unique? These are the only integers satisfying the conditions $a = bq + r$, where $0 \le r < b$. You may have also realised that this is nothing but a restatement of the long division process you have been doing all these years, and that the integers $q$ and $r$ are called the *quotient* and *remainder,* respectively.

A formal statement of this result is as follows :

**Theorem 1.1 (Euclid's Division Lemma) :** *Given positive integers $a$ and $b$, there exist unique integers $q$ and $r$ satisfying $a = bq + r$, $0 \le r < b$.*

This result was perhaps known for a long time, but was first recorded in Book VII of Euclid's Elements. Euclid's division algorithm is based on this lemma.

An **algorithm** is a series of well defined steps which gives a procedure for solving a type of problem.

The word *algorithm* comes from the name of the 9th century Persian mathematician al-Khwarizmi. In fact, even the word 'algebra' is derived from a book, he wrote, called *Hisab al-jabr w'al-muqabala.*

A **lemma** is a proven statement used for proving another statement.

**Muhammad ibn Musa al-Khwarizmi (C.E. 780 – 850)**

Euclid's division algorithm is a technique to compute the Highest Common Factor (HCF) of two given positive integers. Recall that the HCF of two positive integers $a$ and $b$ is the largest positive integer $d$ that divides both $a$ and $b$.

Let us see how the algorithm works, through an example first. Suppose we need to find the HCF of the integers 455 and 42. We start with the larger integer, that is, 455. Then we use Euclid's lemma to get

$$455 = 42 \times 10 + 35$$

Now consider the divisor 42 and the remainder 35, and apply the division lemma to get

$$42 = 35 \times 1 + 7$$

Now consider the divisor 35 and the remainder 7, and apply the division lemma to get

$$35 = 7 \times 5 + 0$$

Notice that the remainder has become zero, and we cannot proceed any further. **We claim** that the HCF of 455 and 42 is the divisor at this stage, i.e., 7. You can easily verify this by listing all the factors of 455 and 42. Why does this method work? It works because of the following result.

So, let us state **Euclid's division algorithm** clearly.

*To obtain the HCF of two positive integers, say c and d, with c > d, follow the steps below:*

**Step 1 :** Apply Euclid's division lemma, to $c$ and $d$. So, we find whole numbers, $q$ and $r$ such that $c = dq + r$, $0 \leq r < d$.

**Step 2 :** If $r = 0$, $d$ is the HCF of $c$ and $d$. If $r \neq 0$, apply the division lemma to $d$ and $r$.

**Step 3 :** Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

This algorithm works because HCF ($c$, $d$) = HCF ($d$, $r$) where the symbol HCF ($c$, $d$) denotes the HCF of $c$ and $d$, etc.

**Example 1 :** Use Euclid's algorithm to find the HCF of 4052 and 12576.

**Solution :**

**Step 1 :** Since 12576 > 4052, we apply the division lemma to 12576 and 4052, to get

$$12576 = 4052 \times 3 + 420$$

**Step 2 :** Since the remainder $420 \neq 0$, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272$$

**Step 3 :** We consider the new divisor 420 and the new remainder 272, and apply the division lemma to get

$$420 = 272 \times 1 + 148$$

We consider the new divisor 272 and the new remainder 148, and apply the division lemma to get

$$272 = 148 \times 1 + 124$$

We consider the new divisor 148 and the new remainder 124, and apply the division lemma to get

$$148 = 124 \times 1 + 24$$

We consider the new divisor 124 and the new remainder 24, and apply the division lemma to get

$$124 = 24 \times 5 + 4$$

We consider the new divisor 24 and the new remainder 4, and apply the division lemma to get

$$24 = 4 \times 6 + 0$$

The remainder has now become zero, so our procedure stops. Since the divisor at this stage is 4, the HCF of 12576 and 4052 is 4.

Notice that 4 = HCF (24, 4) = HCF (124, 24) = HCF (148, 124) = HCF (272, 148) = HCF (420, 272) = HCF (4052, 420) = HCF (12576, 4052).

Euclid's division algorithm is not only useful for calculating the HCF of very large numbers, but also because it is one of the earliest examples of an algorithm that a computer had been programmed to carry out.

**Remarks :**

1. Euclid's division lemma and algorithm are so closely interlinked that people often call former as the division algorithm also.

2. Although Euclid's Division Algorithm is stated for only positive integers, it can be extended for all integers except zero, i.e., $b \neq 0$. However, we shall not discuss this aspect here.

Euclid's division lemma/algorithm has several applications related to finding properties of numbers. We give some examples of these applications below:

**Example 2 :** Show that every positive even integer is of the form $2q$, and that every positive odd integer is of the form $2q + 1$, where $q$ is some integer.

**Solution :** Let $a$ be any positive integer and $b = 2$. Then, by Euclid's algorithm, $a = 2q + r$, for some integer $q \geq 0$, and $r = 0$ or $r = 1$, because $0 \leq r < 2$. So, $a = 2q$ or $2q + 1$.

If $a$ is of the form $2q$, then $a$ is an even integer. Also, a positive integer can be either even or odd. Therefore, any positive odd integer is of the form $2q + 1$.

**Example 3 :** Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where $q$ is some integer.

**Solution :** Let us start with taking $a$, where $a$ is a positive odd integer. We apply the division algorithm with $a$ and $b = 4$.

Since $0 \leq r < 4$, the possible remainders are 0, 1, 2 and 3.

That is, $a$ can be $4q$, or $4q + 1$, or $4q + 2$, or $4q + 3$, where $q$ is the quotient. However, since $a$ is odd, $a$ cannot be $4q$ or $4q + 2$ (since they are both divisible by 2).

Therefore, any odd integer is of the form $4q + 1$ or $4q + 3$.

**Example 4 :** A sweetseller has 420 *kaju barfis* and 130 *badam barfis*. She wants to stack them in such a way that each stack has the same number, and they take up the least area of the tray. What is the number of that can be placed in each stack for this purpose?

**Solution :** This can be done by trial and error. But to do it systematically, we find HCF (420, 130). Then this number will give the maximum number of *barfis* in each stack and the number of stacks will then be the least. The area of the tray that is used up will be the least.

Now, let us use Euclid's algorithm to find their HCF. We have :

$$420 = 130 \times 3 + 30$$

$$130 = 30 \times 4 + 10$$

$$30 = 10 \times 3 + 0$$

So, the HCF of 420 and 130 is 10.

Therefore, the sweetseller can make stacks of 10 for both kinds of *barfi*.

## EXERCISE 1.1

1. Use Euclid's division algorithm to find the HCF of :

   (i) 135 and 225 (ii) 196 and 38220 (iii) 867 and 255

2. Show that any positive odd integer is of the form $6q + 1$, or $6q + 3$, or $6q + 5$, where $q$ is some integer.

3. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?

4. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer $m$.

   [**Hint :** Let $x$ be any positive integer then it is of the form $3q$, $3q + 1$ or $3q + 2$. Now square each of these and show that they can be rewritten in the form $3m$ or $3m + 1$.]

5. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

## 1.3 The Fundamental Theorem of Arithmetic

In your earlier classes, you have seen that any natural number can be written as a product of its prime factors. For instance, $2 = 2$, $4 = 2 \times 2$, $253 = 11 \times 23$, and so on. Now, let us try and look at natural numbers from the other direction. That is, can any natural number be obtained by multiplying prime numbers? Let us see.

Take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce a large collection of positive integers (In fact, infinitely many). Let us list a few :

$7 \times 11 \times 23 = 1771$ $\quad$ $3 \times 7 \times 11 \times 23 = 5313$

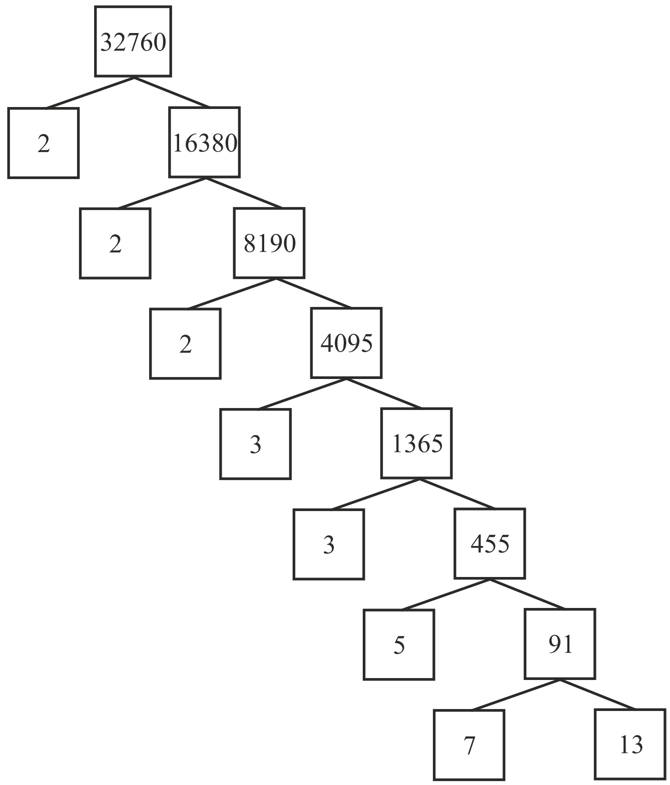
$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $\quad$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

and so on.

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of integers, or infinitely many? Infact, there are infinitely many primes. So, if we combine all these primes in all possible ways, we will get an infinite collection of numbers, all the primes and all possible products of primes. The question is – can we produce all the composite numbers this way? What do you think? Do you think that there may be a composite number which is not the product of powers of primes? Before we answer this, let us factorise positive integers, that is, do the opposite of what we have done so far.

We are going to use the factor tree with which you are all familiar. Let us take some large number, say, 32760, and factorise it as shown :




So we have factorised 32760 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$ as a product of primes, i.e., $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ as a product of powers of primes. Let us try another number, say, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that every composite number can be written as the product of powers of primes. In fact, this statement is true, and is called the **Fundamental Theorem of Arithmetic** because of its basic crucial importance to the study of integers. Let us now formally state this theorem.

**Theorem 1.2** **(Fundamental Theorem of Arithmetic) :** *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

An equivalent version of Theorem 1.2 was probably first recorded as Proposition 14 of Book IX in Euclid's Elements, before it came to be known as the Fundamental Theorem of Arithmetic. However, the first correct proof was given by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae*.

Carl Friedrich Gauss is often referred to as the 'Prince of Mathematicians' and is considered one of the three greatest mathematicians of all time, along with Archimedes and Newton. He has made fundamental contributions to both mathematics and science.

**Carl Friedrich Gauss**
**(1777 – 1855)**

The Fundamental Theorem of Arithmetic says that every composite number can be factorised as a product of primes. Actually it says more. It says that given any composite number it can be factorised as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. So, for example, we regard $2 \times 3 \times 5 \times 7$ as the same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. This fact is also stated in the following form:

*The prime factorisation of a natural number is unique, except for the order of its factors.*

In general, given a composite number $x$, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \le p_2 \le \ldots \le p_n$. If we combine the same primes, we will get powers of primes. For example,

$$32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Once we have decided that the order will be ascending, then the way the number is factorised, is unique.

The Fundamental Theorem of Arithmetic has many applications, both within mathematics and in other fields. Let us look at some examples.

**Example 5 :** Consider the numbers $4^n$, where $n$ is a natural number. Check whether there is any value of $n$ for which $4^n$ ends with the digit zero.

**Solution :** If the number $4^n$, for any $n$, were to end with the digit zero, then it would be divisible by 5. That is, the prime factorisation of $4^n$ would contain the prime 5. This is not possible because $4^n = (2)^{2n}$; so the only prime in the factorisation of $4^n$ is 2. So, the uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the factorisation of $4^n$. So, there is no natural number $n$ for which $4^n$ ends with the digit zero.

You have already learnt how to find the HCF and LCM of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realising it! This method is also called the *prime factorisation method*. Let us recall this method through an example.

**Example 6 :** Find the LCM and HCF of 6 and 20 by the prime factorisation method.

**Solution :** We have : $6 = 2^1 \times 3^1$ and $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$.

You can find HCF(6, 20) = 2 and LCM(6, 20) = $2 \times 2 \times 3 \times 5 = 60$, as done in your earlier classes.

Note that HCF(6, 20) = $2^1$ = **Product of the smallest power of each common prime factor in the numbers.**

LCM (6, 20) = $2^2 \times 3^1 \times 5^1$ = **Product of the greatest power of each prime factor**, **involved in the numbers**.

From the example above, you might have noticed that HCF(6, 20) × LCM(6, 20) = 6 × 20. In fact, we can verify that **for any two positive integers *a* and *b*, HCF (*a*, *b*) × LCM (*a*, *b*) = *a* × *b*.** We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

**Example 7 :** Find the HCF of 96 and 404 by the prime factorisation method. Hence, find their LCM.

**Solution :** The prime factorisation of 96 and 404 gives :

$$96 = 2^5 \times 3,\ 404 = 2^2 \times 101$$

Therefore, the HCF of these two integers is $2^2 = 4$.

Also, $$\text{LCM}(96, 404) = \frac{96 \times 404}{\text{HCF}(96, 404)} = \frac{96 \times 404}{4} = 9696$$

**Example 8 :** Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

**Solution :** We have :

$$6 = 2 \times 3,\ 72 = 2^3 \times 3^2,\ 120 = 2^3 \times 3 \times 5$$

Here, $2^1$ and $3^1$ are the smallest powers of the common factors 2 and 3, respectively.

So, HCF (6, 72, 120) = $2^1 \times 3^1 = 2 \times 3 = 6$

$2^3$, $3^2$ and $5^1$ are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, LCM (6, 72, 120) = $2^3 \times 3^2 \times 5^1 = 360$

**Remark :** Notice, $6 \times 72 \times 120 \neq$ HCF (6, 72, 120) $\times$ LCM (6, 72, 120). So, the product of three numbers is not equal to the product of their HCF and LCM.

## EXERCISE 1.2

**1.** Express each number as a product of its prime factors:

(i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

**2.** Find the LCM and HCF of the following pairs of integers and verify that LCM $\times$ HCF = product of the two numbers.

(i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54

**3.** Find the LCM and HCF of the following integers by applying the prime factorisation method.

(i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25

**4.** Given that HCF (306, 657) = 9, find LCM (306, 657).

**5.** Check whether $6^n$ can end with the digit 0 for any natural number $n$.

**6.** Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.

**7.** There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

## 1.4 Revisiting Irrational Numbers

In Class IX, you were introduced to irrational numbers and many of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and, in general, $\sqrt{p}$ is irrational, where $p$ is a prime. One of the theorems, we use in our proof, is the Fundamental Theorem of Arithmetic.

Recall, a number '$s$' is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.10110111011110\ldots, \text{etc.}$$

Before we prove that $\sqrt{2}$ is irrational, we need the following theorem, whose proof is based on the Fundamental Theorem of Arithmetic.

**Theorem 1.3 :** *Let p be a prime number. If p divides $a^2$, then p divides a, where a is a positive integer.*

***Proof :** Let the prime factorisation of $a$ be as follows :

$a = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore, $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p_1^2 p_2^2 \ldots p_n^2.$

Now, we are given that $p$ divides $a^2$. Therefore, from the Fundamental Theorem of Arithmetic, it follows that $p$ is one of the prime factors of $a^2$. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of $a^2$ are $p_1, p_2, \ldots, p_n$. So $p$ is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1\, p_2 \ldots p_n$, $p$ divides $a$.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'. (This technique is discussed in some detail in Appendix 1).

**Theorem 1.4 :** $\sqrt{2}$ *is irrational.*

**Proof :** Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers $r$ and $s$ ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose $r$ and $s$ have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where $a$ and $b$ are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides $a^2$.

Now, by Theorem 1.3, it follows that 2 divides $a$.

So, we can write $a = 2c$ for some integer $c$.

* Not from the examination point of view.

Substituting for $a$, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides $b^2$, and so 2 divides $b$ (again using Theorem 1.3 with $p = 2$).

Therefore, $a$ and $b$ have at least 2 as a common factor.

But this contradicts the fact that $a$ and $b$ have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

**Example 9 :** Prove that $\sqrt{3}$ is irrational.

**Solution :** Let us assume, to the contrary, that $\sqrt{3}$ is rational.

That is, we can find integers $a$ and $b$ ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}\cdot$

Suppose $a$ and $b$ have a common factor other than 1, then we can divide by the common factor, and assume that $a$ and $b$ are coprime.

So, $b\sqrt{3} = a\cdot$

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, $a^2$ is divisible by 3, and by Theorem 1.3, it follows that $a$ is also divisible by 3.

So, we can write $a = 3c$ for some integer $c$.

Substituting for $a$, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that $b^2$ is divisible by 3, and so $b$ is also divisible by 3 (using Theorem 1.3 with $p = 3$).

Therefore, $a$ and $b$ have at least 3 as a common factor.

But this contradicts the fact that $a$ and $b$ are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

In Class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product and quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

**Example 10 :** Show that $5 - \sqrt{3}$ is irrational.

**Solution :** Let us assume, to the contrary, that $5 - \sqrt{3}$ is rational.

That is, we can find coprime $a$ and $b$ ($b \neq 0$) such that $5 - \sqrt{3} = \frac{a}{b}$.

Therefore, $5 - \frac{a}{b} = \sqrt{3}$.

Rearranging this equation, we get $\sqrt{3} = 5 - \frac{a}{b} = \frac{5b - a}{b}$.

Since $a$ and $b$ are integers, we get $5 - \frac{a}{b}$ is rational, and so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5 - \sqrt{3}$ is rational.

So, we conclude that $5 - \sqrt{3}$ is irrational.

**Example 11 :** Show that $3\sqrt{2}$ is irrational.

**Solution :** Let us assume, to the contrary, that $3\sqrt{2}$ is rational.

That is, we can find coprime $a$ and $b$ ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, $a$ and $b$ are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

## EXERCISE 1.3

1. Prove that $\sqrt{5}$ is irrational.
2. Prove that $3 + 2\sqrt{5}$ is irrational.
3. Prove that the following are irrationals :

   (i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

## 1.5 Revisiting Rational Numbers and Their Decimal Expansions

In Class IX, you studied that rational numbers have either a terminating decimal expansion or a non-terminating repeating decimal expansion. In this section, we are going to consider a rational number, say $\frac{p}{q}(q \neq 0)$, and explore exactly when the decimal expansion of $\frac{p}{q}$ is terminating and when it is non-terminating repeating (or recurring). We do so by considering several examples.

Let us consider the following rational numbers :

(i) 0.375 (ii) 0.104 (iii) 0.0875 (iv) 23.3408.

Now (i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $0.104 = \frac{104}{1000} = \frac{104}{10^3}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $23.3408 = \frac{233408}{10000} = \frac{233408}{10^4}$

As one would expect, they can all be expressed as rational numbers whose denominators are powers of 10. Let us try and cancel the common factors between the numerator and denominator and see what we get :

(i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3}$ (ii) $0.104 = \frac{104}{10^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{13}{5^3}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{7}{2^4 \times 5}$ (iv) $23.3408 = \frac{233408}{10^4} = \frac{2^2 \times 7 \times 521}{5^4}$

Do you see any pattern? It appears that, we have converted a real number whose decimal expansion terminates into a rational number of the form $\frac{p}{q}$, where $p$ and $q$ are coprime, and the prime factorisation of the denominator (that is, $q$) has only powers of 2, or powers of 5, or both. We should expect the denominator to look like this, since powers of 10 can only have powers of 2 and 5 as factors.

Even though, we have worked only with a few examples, you can see that any real number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. Also the only prime factors of 10 are 2 and 5. So, cancelling out the common factors between the numerator and the denominator, we find that this real number is a rational number of the form $\frac{p}{q}$, where the prime factorisation of $q$ is of the form $2^n5^m$, and $n$, $m$ are some non-negative integers.

Let us write our result formally:

**Theorem 1.5 :** *Let $x$ be a rational number whose decimal expansion terminates. Then $x$ can be expressed in the form $\frac{p}{q}$, where $p$ and $q$ are coprime, and the prime factorisation of $q$ is of the form $2^n5^m$, where $n$, $m$ are non-negative integers.*

You are probably wondering what happens the other way round in Theorem 1.5. That is, if we have a rational number of the form $\frac{p}{q}$, and the prime factorisation of $q$ is of the form $2^n5^m$, where $n$, $m$ are non negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

Let us see if there is some obvious reason why this is true. You will surely agree that any rational number of the form $\frac{a}{b}$, where $b$ is a power of 10, will have a terminating decimal expansion. So it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where $q$ is of the form $2^n5^m$, to an equivalent rational number of the form $\frac{a}{b}$, where $b$ is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = 0.375$

(ii) $\frac{13}{125} = \frac{13}{5^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{104}{10^3} = 0.104$

(iii) $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5^4} = \frac{875}{10^4} = 0.0875$

(iv) $\frac{14588}{625} = \frac{2^2 \times 7 \times 521}{5^4} = \frac{2^6 \times 7 \times 521}{2^4 \times 5^4} = \frac{233408}{10^4} = 23.3408$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where $q$ is of the form $2^n5^m$, to an equivalent rational number of the form $\frac{a}{b}$, where $b$ is a power of 10. Therefore, the decimal expansion of such a rational number terminates. Let us write down our result formally.

**Theorem 1.6 :** *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of $q$ is of the form $2^n5^m$, where $n$, $m$ are non-negative integers. Then $x$ has a decimal expansion which terminates.*

We are now ready to move on to the rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on. We refer to Example 5, Chapter 1, from your Class IX textbook, namely, $\frac{1}{7}$. Here, remainders are 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, . . . and divisor is 7.

$$\begin{array}{r|l} & 0.1428571 \\ \hline 7 & 10 \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 10 \\ & \underline{7} \\ & 30 \end{array}$$

Notice that the denominator here, i.e., 7 is clearly not of the form $2^n5^m$. Therefore, from Theorems 1.5 and 1.6, we know that $\frac{1}{7}$ will not have a terminating decimal expansion. Hence, 0 will not show up as a remainder (Why?), and the remainders will start repeating after a certain stage. So, we will have a block of digits, namely, 142857, repeating in the quotient of $\frac{1}{7}$.

What we have seen, in the case of $\frac{1}{7}$, is true for any rational number not covered by Theorems 1.5 and 1.6. For such numbers we have :

**Theorem 1.7 :** *Let $x = \frac{p}{q}$, where p and q are coprimes, be a rational number, such that the prime factorisation of q is not of the form $2^n5^m$, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).*

From the discussion above, we can conclude that *the decimal expansion of every rational number is either terminating or non-terminating repeating*.

## EXERCISE 1.4

**1.** Without actually performing the long division, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating repeating decimal expansion:

(i) $\frac{13}{3125}$ (ii) $\frac{17}{8}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$

(v) $\frac{29}{343}$ (vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{6}{15}$

(ix) $\frac{35}{50}$ (x) $\frac{77}{210}$

2. Write down the decimal expansions of those rational numbers in Question 1 above which have terminating decimal expansions.

3. The following real numbers have decimal expansions as given below. In each case, decide whether they are rational or not. If they are rational, and of the form $\frac{p}{q}$, what can you say about the prime factors of $q$?

   (i) 43.123456789  (ii) 0.120120012000120000. . .  (iii) $43.\overline{123456789}$

## 1.6 Summary

In this chapter, you have studied the following points:

1. Euclid's division lemma :

   Given positive integers $a$ and $b$, there exist whole numbers $q$ and $r$ satisfying $a = bq + r$, $0 \le r < b$.

2. Euclid's division algorithm : This is based on Euclid's division lemma. According to this, the HCF of any two positive integers $a$ and $b$, with $a > b$, is obtained as follows:

   **Step 1 :** Apply the division lemma to find $q$ and $r$ where $a = bq + r$, $0 \le r < b$.

   **Step 2 :** If $r = 0$, the HCF is $b$. If $r \ne 0$, apply Euclid's lemma to $b$ and $r$.

   **Step 3 :** Continue the process till the remainder is zero. The divisor at this stage will be HCF $(a, b)$. Also, HCF$(a, b)$ = HCF$(b, r)$.

3. The Fundamental Theorem of Arithmetic :

   Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.

4. If $p$ is a prime and $p$ divides $a^2$, then $p$ divides $a$, where $a$ is a positive integer.

5. To prove that $\sqrt{2}, \sqrt{3}$ are irrationals.

6. Let $x$ be a rational number whose decimal expansion terminates. Then we can express $x$ in the form $\frac{p}{q}$, where $p$ and $q$ are coprime, and the prime factorisation of $q$ is of the form $2^n 5^m$, where $n$, $m$ are non-negative integers.

7. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of $q$ is of the form $2^n 5^m$, where $n$, $m$ are non-negative integers. Then $x$ has a decimal expansion which terminates.

8. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of $q$ is not of the form $2^n 5^m$, where $n$, $m$ are non-negative integers. Then $x$ has a decimal expansion which is non-terminating repeating (recurring).

# 02

# POLYNOMIALS

# POLYNOMIALS 2

## 2.1 Introduction

In Class IX, you have studied polynomials in one variable and their degrees. Recall that if $p(x)$ is a polynomial in $x$, the highest power of $x$ in $p(x)$ is called **the degree of the polynomial** $p(x)$. For example, $4x + 2$ is a polynomial in the variable $x$ of degree 1, $2y^2 - 3y + 4$ is a polynomial in the variable $y$ of degree 2, $5x^3 - 4x^2 + x - \sqrt{2}$ is a polynomial in the variable $x$ of degree 3 and $7u^6 - \frac{3}{2}u^4 + 4u^2 + u - 8$ is a polynomial in the variable $u$ of degree 6. Expressions like $\frac{1}{x-1}$, $\sqrt{x} + 2$, $\frac{1}{x^2 + 2x + 3}$ etc., are not polynomials.

A polynomial of degree 1 is called a **linear polynomial**. For example, $2x - 3$, $\sqrt{3}x + 5$, $y + \sqrt{2}$, $x - \frac{2}{11}$, $3z + 4$, $\frac{2}{3}u + 1$, etc., are all linear polynomials. Polynomials such as $2x + 5 - x^2$, $x^3 + 1$, etc., are not linear polynomials.

A polynomial of degree 2 is called a **quadratic polynomial**. The name 'quadratic' has been derived from the word 'quadrate', which means 'square'. $2x^2 + 3x - \frac{2}{5}$, $y^2 - 2$, $2 - x^2 + \sqrt{3}x$, $\frac{u}{3} - 2u^2 + 5$, $\sqrt{5}v^2 - \frac{2}{3}v$, $4z^2 + \frac{1}{7}$ are some examples of quadratic polynomials (whose coefficients are real numbers). More generally, any quadratic polynomial in $x$ is of the form $ax^2 + bx + c$, where $a$, $b$, $c$ are real numbers and $a \neq 0$. A polynomial of degree 3 is called a **cubic polynomial**. Some examples of a cubic polynomial are $2 - x^3$, $x^3$, $\sqrt{2}\,x^3$, $3 - x^2 + x^3$, $3x^3 - 2x^2 + x - 1$. In fact, the most general form of a cubic polynomial is

$$ax^3 + bx^2 + cx + d,$$

where, $a$, $b$, $c$, $d$ are real numbers and $a \neq 0$.

Now consider the polynomial $p(x) = x^2 - 3x - 4$. Then, putting $x = 2$ in the polynomial, we get $p(2) = 2^2 - 3 \times 2 - 4 = -6$. The value '$-6$', obtained by replacing $x$ by 2 in $x^2 - 3x - 4$, is the value of $x^2 - 3x - 4$ at $x = 2$. Similarly, $p(0)$ is the value of $p(x)$ at $x = 0$, which is $-4$.

If $p(x)$ is a polynomial in $x$, and if $k$ is any real number, then the value obtained by replacing $x$ by $k$ in $p(x)$, is called **the value of $p(x)$ at $x = k$**, and is denoted by $p(k)$.

What is the value of $p(x) = x^2 - 3x - 4$ at $x = -1$? We have :

$$p(-1) = (-1)^2 - \{3 \times (-1)\} - 4 = 0$$

Also, note that $\quad p(4) = 4^2 - (3 \times 4) - 4 = 0.$

As $p(-1) = 0$ and $p(4) = 0$, $-1$ and 4 are called the zeroes of the quadratic polynomial $x^2 - 3x - 4$. More generally, a real number $k$ is said to be a **zero of a polynomial $p(x)$**, if $p(k) = 0$.

You have already studied in Class IX, how to find the zeroes of a linear polynomial. For example, if $k$ is a zero of $p(x) = 2x + 3$, then $p(k) = 0$ gives us $2k + 3 = 0$, i.e., $k = -\frac{3}{2}\cdot$

In general, if $k$ is a zero of $p(x) = ax + b$, then $p(k) = ak + b = 0$, i.e., $k = \frac{-b}{a}\cdot$

So, the zero of the linear polynomial $ax + b$ is $\frac{-b}{a} = \frac{-\text{(Constant term)}}{\text{Coefficient of } x}$.

Thus, the zero of a linear polynomial is related to its coefficients. Does this happen in the case of other polynomials too? For example, are the zeroes of a quadratic polynomial also related to its coefficients?

In this chapter, we will try to answer these questions. We will also study the division algorithm for polynomials.

## 2.2 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number $k$ is a zero of the polynomial $p(x)$ if $p(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the **geometrical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line passing through the points $(-2, -1)$ and $(2, 7)$.

| $x$ | $-2$ | $2$ |
|---|---|---|
| $y = 2x + 3$ | $-1$ | $7$ |

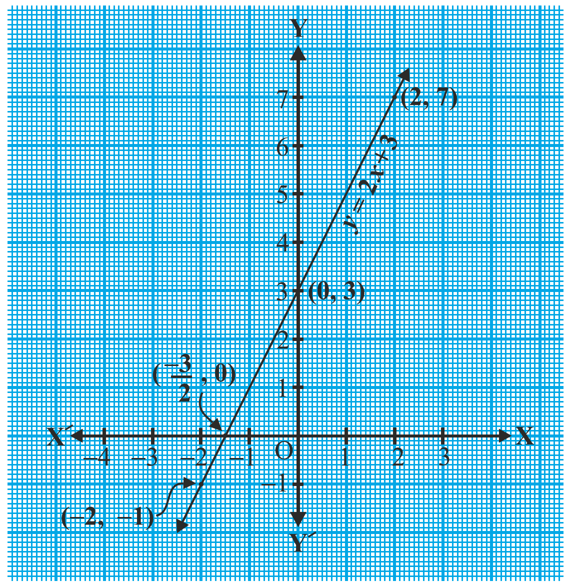



**Fig. 2.1**

From Fig. 2.1, you can see that the graph of $y = 2x + 3$ intersects the $x$-axis mid-way between $x = -1$ and $x = -2$, that is, at the point $\left(-\frac{3}{2}, 0\right)$. You also know that the zero of $2x + 3$ is $-\frac{3}{2}$. Thus, the zero of the polynomial $2x + 3$ is the $x$-coordinate of the point where the graph of $y = 2x + 3$ intersects the $x$-axis.

In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the $x$-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$. Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the $x$-coordinate of the point where the graph of $y = ax + b$ intersects the $x$-axis.

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see what the graph* of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for $x$ as given in Table 2.1.

---

* Plotting of graphs of quadratic or cubic polynomials is not meant to be done by the students, nor is to be evaluated.

**Table 2.1**

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|
| $y = x^2 - 3x - 4$ | $6$ | $0$ | $-4$ | $-6$ | $-6$ | $-4$ | $0$ | $6$ |

If we locate the points listed above on a graph paper and draw the graph, it will actually look like the one given in Fig. 2.2.

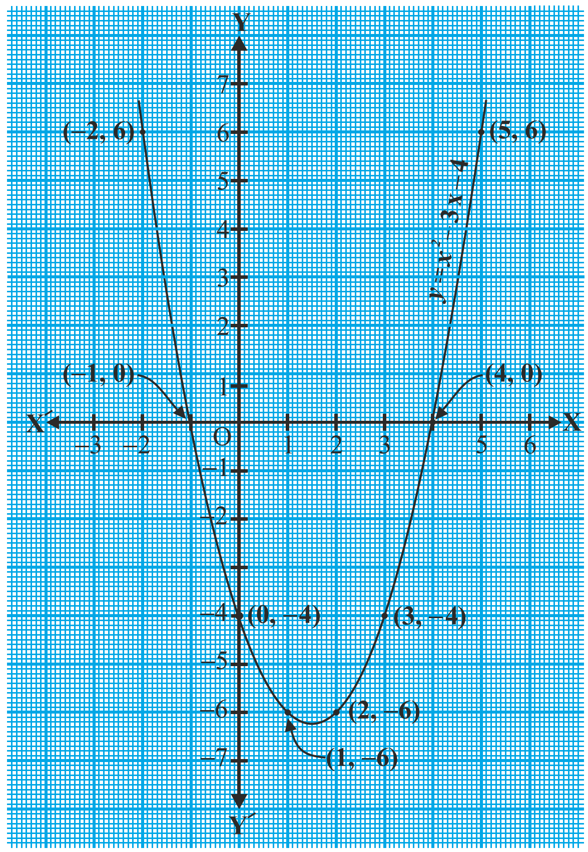



Fig. 2.2

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ has one of the two shapes either open upwards like ∪ or open downwards like ∩ depending on whether $a > 0$ or $a < 0$. (These curves are called **parabolas**.)

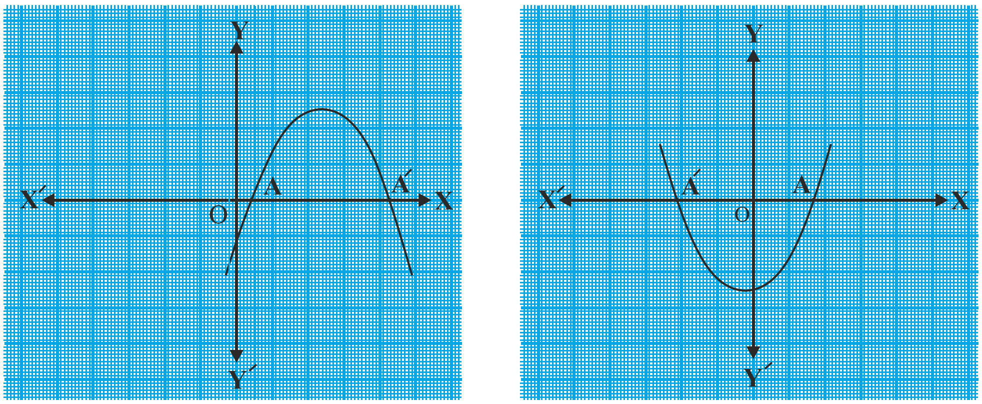
You can see from Table 2.1 that $-1$ and $4$ are zeroes of the quadratic polynomial. Also note from Fig. 2.2 that $-1$ and $4$ are the $x$-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the $x$-axis. Thus, the zeroes of the quadratic polynomial $x^2 - 3x - 4$ are $x$-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the $x$-axis.

This fact is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, are precisely the $x$-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the $x$-axis.

From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

**Case (i) :** Here, the graph cuts $x$-axis at two distinct points A and A′.

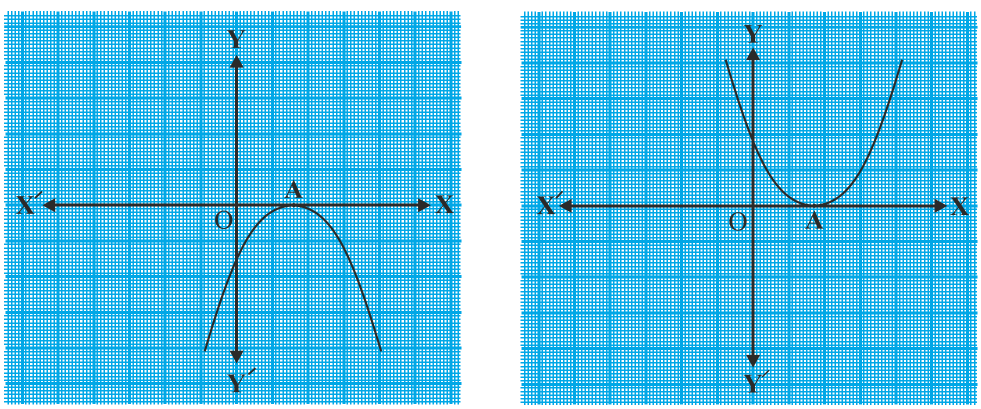
The $x$-coordinates of A and A′ are the **two zeroes** of the quadratic polynomial $ax^2 + bx + c$ in this case (see Fig. 2.3).



(i) (ii)

**Fig. 2.3**

**Case (ii) :** Here, the graph cuts the $x$-axis at exactly one point, i.e., at two coincident points. So, the two points A and A′ of Case (i) coincide here to become one point A (see Fig. 2.4).



(i) (ii)

**Fig. 2.4**

The $x$-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

**Case (iii) :** Here, the graph is either completely above the $x$-axis or completely below the $x$-axis. So, it does not cut the $x$-axis at any point (see Fig. 2.5).

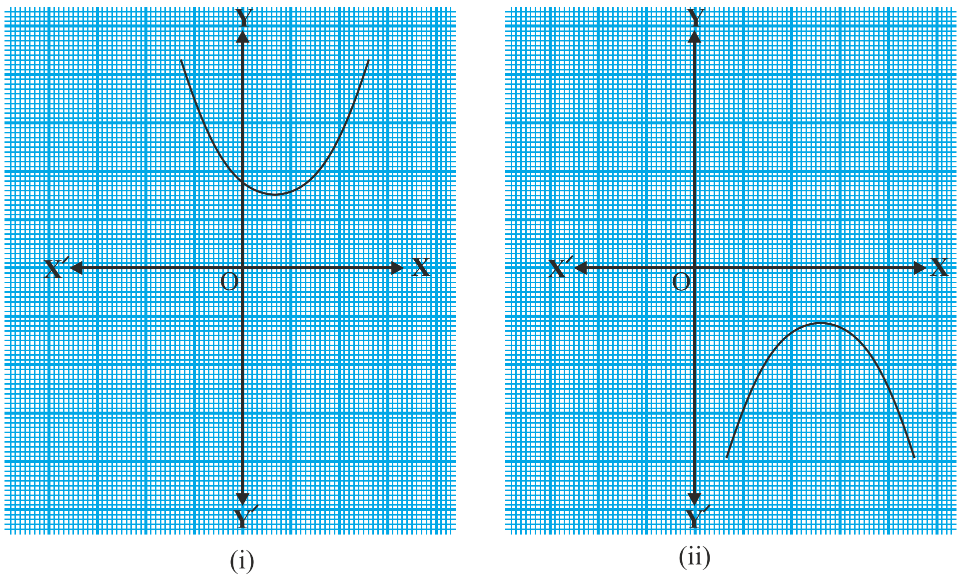



**Fig. 2.5**

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.

Now, what do you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see what the graph of $y = x^3 - 4x$ looks like, let us list a few values of $y$ corresponding to a few values for $x$ as shown in Table 2.2.

**Table 2.2**

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y = x^3 - 4x$ | $0$ | $3$ | $0$ | $-3$ | $0$ |

Locating the points of the table on a graph paper and drawing the graph, we see that the graph of $y = x^3 - 4x$ actually looks like the one given in Fig. 2.6.

We see from the table above that – 2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. Observe that –2, 0 and 2 are, in fact, the $x$-coordinates of the only points where the graph of $y = x^3 - 4x$ intersects the $x$-axis. Since the curve meets the $x$-axis in only these 3 points, their $x$-coordinates are the only zeroes of the polynomial.

Let us take a few more examples. Consider the cubic polynomials $x^3$ and $x^3 - x^2$. We draw the graphs of $y = x^3$ and $y = x^3 - x^2$ in Fig. 2.7 and Fig. 2.8 respectively.

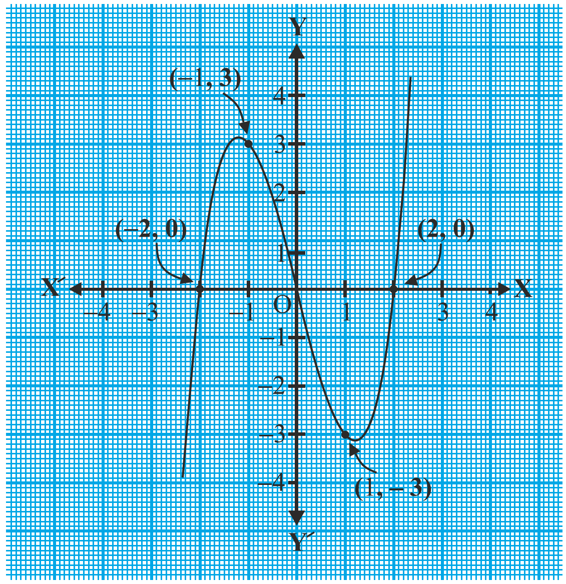



**Fig. 2.6**

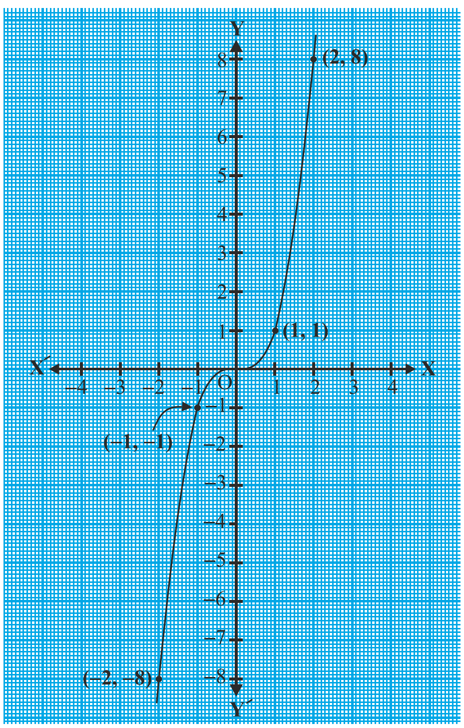



**Fig. 2.7**

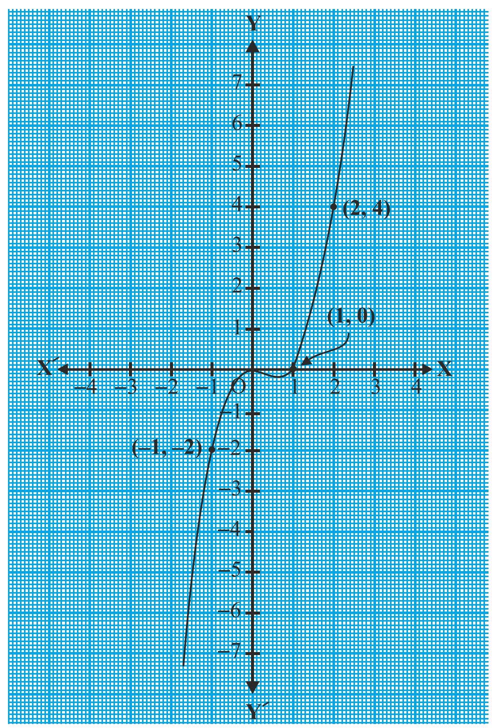



**Fig. 2.8**

Note that 0 is the only zero of the polynomial $x^3$. Also, from Fig. 2.7, you can see that 0 is the $x$-coordinate of the only point where the graph of $y = x^3$ intersects the $x$-axis. Similarly, since $x^3 - x^2 = x^2(x - 1)$, 0 and 1 are the only zeroes of the polynomial $x^3 - x^2$. Also, from Fig. 2.8, these values are the $x$-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the $x$-axis.

From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

**Remark :** In general, given a polynomial $p(x)$ of degree $n$, the graph of $y = p(x)$ intersects the $x$-axis at atmost $n$ points. Therefore, a polynomial $p(x)$ of degree $n$ has at most $n$ zeroes.

**Example 1 :** Look at the graphs in Fig. 2.9 given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. For each of the graphs, find the number of zeroes of $p(x)$.

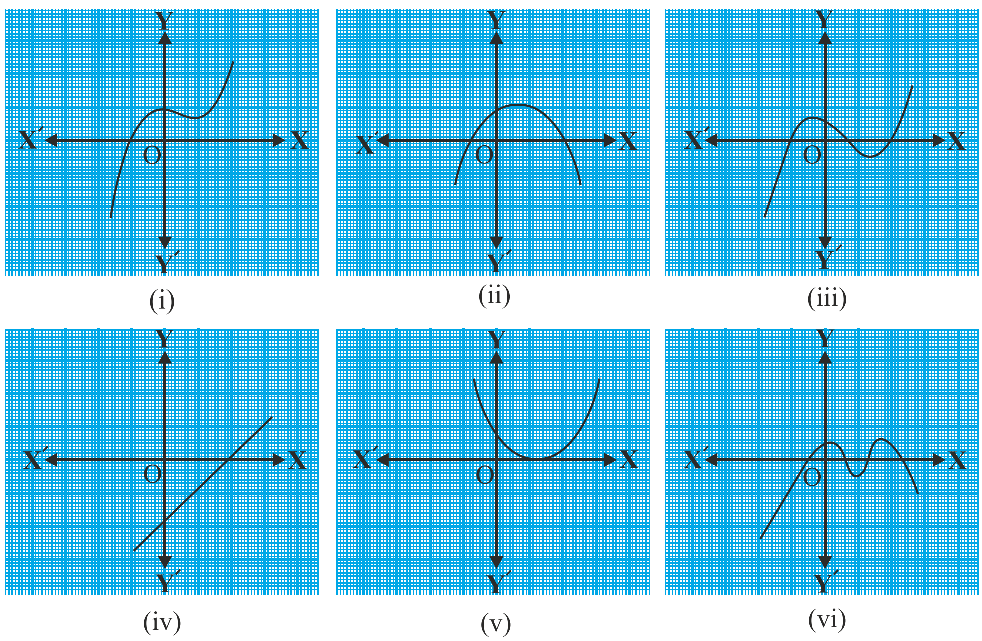



**Fig. 2.9**

**Solution :**

(i) The number of zeroes is 1 as the graph intersects the $x$-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the $x$-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

## EXERCISE 2.1

**1.** The graphs of $y = p(x)$ are given in Fig. 2.10 below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.

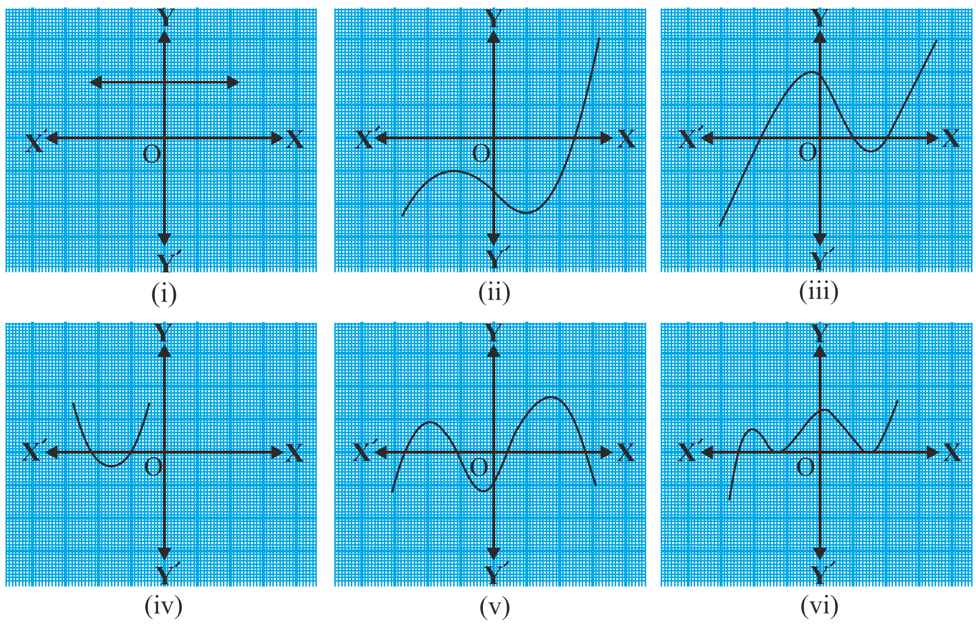



**Fig. 2.10**

## 2.3 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\frac{b}{a}$. We will now try to answer the question raised in Section 2.1 regarding the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take a quadratic polynomial, say $p(x) = 2x^2 - 8x + 6$. In Class IX, you have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we need to split the middle term '$- 8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write

$$2x^2 - 8x + 6 = 2x^2 - 6x - 2x + 6 = 2x(x - 3) - 2(x - 3)$$

$$= (2x - 2)(x - 3) = 2(x - 1)(x - 3)$$

So, the value of $p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. Observe that :

$$\text{Sum of its zeroes} = 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Let us take one more quadratic polynomial, say, $p(x) = 3x^2 + 5x - 2$. By the method of splitting the middle term,

$$\begin{aligned} 3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 = 3x(x + 2) - 1(x + 2) \\ &= (3x - 1)(x + 2) \end{aligned}$$

Hence, the value of $3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$, i.e., when $x = \frac{1}{3}$ or $x = -2$. So, the zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and $-2$. Observe that :

$$\text{Sum of its zeroes} = \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

In general, if $\alpha$* and $\beta$* are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then you know that $x - \alpha$ and $x - \beta$ are the factors of $p(x)$. Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant} \\ &= k[x^2 - (\alpha + \beta)x + \alpha\beta] \\ &= kx^2 - k(\alpha + \beta)x + k\alpha\beta \end{aligned}$$

Comparing the coefficients of $x^2$, $x$ and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta.$$

This gives
$$\boldsymbol{\alpha + \beta = \frac{-b}{a},}$$

$$\boldsymbol{\alpha\beta = \frac{c}{a}}$$

* $\alpha, \beta$ are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter '$\gamma$' pronounced as 'gamma'.

i.e., $$\text{sum of zeroes} = \alpha + \beta = -\frac{b}{a} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Let us consider some examples.

**Example 2 :** Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

**Solution :** We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$, i.e., when $x = -2$ or $x = -5$. Therefore, the zeroes of $x^2 + 7x + 10$ are $-2$ and $-5$. Now,

$$\text{sum of zeroes} = -2 + (-5) = -(7) = \frac{-(7)}{1} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = (-2) \times (-5) = 10 = \frac{10}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

**Example 3 :** Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

**Solution :** Recall the identity $a^2 - b^2 = (a - b)(a + b)$. Using it, we can write:

$$x^2 - 3 = \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Now,

$$\text{sum of zeroes} = \sqrt{3} - \sqrt{3} = 0 = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \left(\sqrt{3}\right)\left(-\sqrt{3}\right) = -3 = \frac{-3}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

**Example 4 :** Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.

**Solution :** Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be $\alpha$ and $\beta$. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and $$\alpha\beta = 2 = \frac{c}{a}.$$

If $a = 1$, then $b = 3$ and $c = 2$.

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

You can check that any other quadratic polynomial that fits these conditions will be of the form $k(x^2 + 3x + 2)$, where $k$ is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

You can check that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}\cdot$ Since $p(x)$ can have atmost three zeroes, these are the zeores of $2x^3 - 5x^2 - 14x + 8$. Now,

$$\text{sum of the zeroes} = 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3},$$

$$\text{product of the zeroes} = 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}.$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$\{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}.$$

In general, it can be proved that if $\alpha, \beta, \gamma$ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\,\beta\,\gamma = \frac{-d}{a}.$$

Let us consider an example.

**Example 5* :** Verify that 3, –1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

**Solution :** Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get

$a = 3$, $b = -5$, $c = -11$, $d = -3$. Further

$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$

$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$

$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, –1 and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So, we take $\alpha = 3$, $\beta = -1$ and $\gamma = -\frac{1}{3}$.

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

* Not from the examination point of view.

## EXERCISE 2.2

**1.** Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

(iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

**2.** Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

(iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

## 2.4 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two? For this, let us consider the cubic polynomial $x^3 - 3x^2 - x + 3$. If we tell you that one of its zeroes is 1, then you know that $x - 1$ is a factor of $x^3 - 3x^2 - x + 3$. So, you can divide $x^3 - 3x^2 - x + 3$ by $x - 1$, as you have learnt in Class IX, to get the quotient $x^2 - 2x - 3$.

Next, you could get the factors of $x^2 - 2x - 3$, by splitting the middle term, as $(x + 1)(x - 3)$. This would give you

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= (x - 1)(x^2 - 2x - 3) \\ &= (x - 1)(x + 1)(x - 3) \end{aligned}$$

So, all the three zeroes of the cubic polynomial are now known to you as $1, -1, 3$.

Let us discuss the method of dividing one polynomial by another in some detail. Before noting the steps formally, consider an example.

**Example 6 :** Divide $2x^2 + 3x + 1$ by $x + 2$.

$$\begin{array}{r|l} & 2x - 1 \\ \hline x + 2 & 2x^2 + 3x + 1 \\ & \underline{\overset{-}{2x^2} \overset{-}{+} 4x} \\ & -x + 1 \\ & \underline{\overset{+}{-x} \overset{+}{-} 2} \\ & 3 \end{array}$$

**Solution :** Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is $2x - 1$ and the remainder is 3. Also,

$$(2x - 1)(x + 2) + 3 = 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1$$

i.e., $2x^2 + 3x + 1 = (x + 2)(2x - 1) + 3$

Therefore, Dividend = Divisor × Quotient + Remainder

Let us now extend this process to divide a polynomial by a quadratic polynomial.

**Example 7 :** Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.

**Solution :** We first arrange the terms of the dividend and the divisor in the decreasing order of their degrees. Recall that arranging the terms in this order is called writing the polynomials in standard form. In this example, the dividend is already in standard form, and the divisor, in standard form, is $x^2 + 2x + 1$.

$$
\begin{array}{r|l}
 & \qquad\qquad 3x - 5 \\ \hline
x^2 + 2x + 1 & 3x^3 + x^2 + 2x + 5 \\
 & 3x^3 + 6x^2 + 3x \\
 & \underline{-\qquad -\qquad -\qquad\qquad} \\
 & \qquad -5x^2 - x + 5 \\
 & \qquad -5x^2 - 10x - 5 \\
 & \qquad \underline{+\qquad +\qquad +} \\
 & \qquad\qquad 9x + 10
\end{array}
$$

**Step 1 :** To obtain the first term of the quotient, divide the highest degree term of the dividend (i.e., $3x^3$) by the highest degree term of the divisor (i.e., $x^2$). This is $3x$. Then carry out the division process. What remains is $-5x^2 - x + 5$.

**Step 2 :** Now, to obtain the second term of the quotient, divide the highest degree term of the new dividend (i.e., $-5x^2$) by the highest degree term of the divisor (i.e., $x^2$). This gives $-5$. Again carry out the division process with $-5x^2 - x + 5$.

**Step 3 :** What remains is $9x + 10$. Now, the degree of $9x + 10$ is less than the degree of the divisor $x^2 + 2x + 1$. So, we cannot continue the division any further.

So, the quotient is $3x - 5$ and the remainder is $9x + 10$. Also,

$$
\begin{aligned}
(x^2 + 2x + 1) \times (3x - 5) + (9x + 10) &= 3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10 \\
&= 3x^3 + x^2 + 2x + 5
\end{aligned}
$$

Here again, we see that

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

What we are applying here is an algorithm which is similar to Euclid's division algorithm that you studied in Chapter 1.

This says that

**If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that**

$$\boldsymbol{p(x) = g(x) \times q(x) + r(x),}$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

This result is known as the **Division Algorithm** for polynomials.

Let us now take some examples to illustrate its use.

**Example 8 :** Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

**Solution :** Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

So, dividend $= -x^3 + 3x^2 - 3x + 5$ and divisor $= -x^2 + x - 1$.

Division process is shown on the right side.

$$\begin{array}{r|l} & x - 2 \\ \hline -x^2 + x - 1 & -x^3 + 3x^2 - 3x + 5 \\ & -x^3 + x^2 - x \\ & + \quad - \quad + \\ \hline & 2x^2 - 2x + 5 \\ & 2x^2 - 2x + 2 \\ & - \quad + \quad - \\ \hline & 3 \end{array}$$

We stop here since degree $(3) = 0 < 2 =$ degree $(-x^2 + x - 1)$.

So, quotient $= x - 2$, remainder $= 3$.

Now,

$$\begin{aligned} \text{Divisor} \times \text{Quotient} + \text{Remainder} &= (-x^2 + x - 1)(x - 2) + 3 \\ &= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3 \\ &= -x^3 + 3x^2 - 3x + 5 \\ &= \text{Dividend} \end{aligned}$$

In this way, the division algorithm is verified.

**Example 9 :** Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

**Solution :** Since two zeroes are $\sqrt{2}$ and $-\sqrt{2}$, $\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = x^2 - 2$ is a factor of the given polynomial. Now, we divide the given polynomial by $x^2 - 2$.

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ \hline x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\ & 2x^4 \qquad\quad - 4x^2 \\ & - \qquad\qquad + \\ \hline & -3x^3 + x^2 + 6x - 2 \\ & -3x^3 \qquad + 6x \\ & + \qquad\quad - \\ \hline & x^2 - 2 \\ & x^2 - 2 \\ & - \quad + \\ \hline & 0 \end{array}$$

First term of quotient is $\dfrac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\dfrac{-3x^3}{x^2} = -3x$

Third term of quotient is $\dfrac{x^2}{x^2} = 1$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.

Now, by splitting $-3x$, we factorise $2x^2 - 3x + 1$ as $(2x - 1)(x - 1)$. So, its zeroes are given by $x = \frac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2},\ -\sqrt{2},\ \frac{1}{2}$, and 1.

## EXERCISE 2.3

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :
   (i) $p(x) = x^3 - 3x^2 + 5x - 3, \quad g(x) = x^2 - 2$
   (ii) $p(x) = x^4 - 3x^2 + 4x + 5, \quad g(x) = x^2 + 1 - x$
   (iii) $p(x) = x^4 - 5x + 6, \quad g(x) = 2 - x^2$
2. Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:
   (i) $t^2 - 3,\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$
   (ii) $x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$
   (iii) $x^3 - 3x + 1,\ x^5 - 4x^3 + x^2 + 3x + 1$
3. Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.
4. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.
5. Give examples of polynomials $p(x)$, $g(x)$, $q(x)$ and $r(x)$, which satisfy the division algorithm and
   (i) $\deg p(x) = \deg q(x)$ (ii) $\deg q(x) = \deg r(x)$ (iii) $\deg r(x) = 0$

## EXERCISE 2.4 (Optional)*

1. Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:
   (i) $2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$ (ii) $x^3 - 4x^2 + 5x - 2;\ 2, 1, 1$
2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

*These exercises are not from the examination point of view.

3. If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b$, $a$, $a + b$, find $a$ and $b$.

4. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

5. If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find $k$ and $a$.

## 2.5 Summary

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in $x$ with real coefficients is of the form $ax^2 + bx + c$, where $a, b, c$ are real numbers with $a \neq 0$.
3. The zeroes of a polynomial $p(x)$ are precisely the $x$-coordinates of the points, where the graph of $y = p(x)$ intersects the $x$-axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at most 3 zeroes.
5. If $\alpha$ and $\beta$ are the zeroes of the quadratic polynomial $ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If $\alpha, \beta, \gamma$ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

7. The division algorithm states that given any polynomial $p(x)$ and any non-zero polynomial $g(x)$, there are polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x)\, q(x) + r(x),$$

where $$r(x) = 0 \text{ or degree } r(x) < \text{degree } g(x).$$

# 03

# PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

# PAIR OF LINEAR EQUATIONS IN TWO VARIABLES 3

## 3.1 Introduction

You must have come across situations like the one given below :

Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a ring on the items kept in a stall, and if the ring covers any object completely, you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. If each ride costs ₹ 3, and a game of Hoopla costs ₹ 4, how would you find out the number of rides she had and how many times she played Hoopla, provided she spent ₹ 20.

May be you will try it by considering different cases. If she has one ride, is it possible? Is it possible to have two rides? And so on. Or you may use the knowledge of Class IX, to represent such situations as linear equations in two variables.

Let us try this approach.

Denote the number of rides that Akhila had by $x$, and the number of times she played Hoopla by $y$. Now the situation can be represented by the two equations:

$$y = \frac{1}{2}x \qquad (1)$$

$$3x + 4y = 20 \qquad (2)$$

Can we find the solutions of this pair of equations? There are several ways of finding these, which we will study in this chapter.

## 3.2 Pair of Linear Equations in Two Variables

Recall, from Class IX, that the following are examples of linear equations in two variables:

$$2x + 3y = 5$$

$$x - 2y - 3 = 0$$

and $$x - 0y = 2, \text{ i.e., } x = 2$$

You also know that an equation which can be put in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are real numbers, and ***a* and *b* are not both zero**, is called a linear equation in two variables $x$ and $y$. (We often denote the condition $a$ and $b$ are not both zero by $a^2 + b^2 \neq 0$). You have also studied that a solution of such an equation is a pair of values, one for $x$ and the other for $y$, which makes the two sides of the equation equal.

For example, let us substitute $x = 1$ and $y = 1$ in the left hand side (LHS) of the equation $2x + 3y = 5$. Then

$$\text{LHS} = 2(1) + 3(1) = 2 + 3 = 5,$$

which is equal to the right hand side (RHS) of the equation.

Therefore, $x = 1$ and $y = 1$ is a solution of the equation $2x + 3y = 5$.

Now let us substitute $x = 1$ and $y = 7$ in the equation $2x + 3y = 5$. Then,

$$\text{LHS} = 2(1) + 3(7) = 2 + 21 = 23$$

which is not equal to the RHS.

Therefore, $x = 1$ and $y = 7$ is **not** a solution of the equation.

Geometrically, what does this mean? It means that the point (1, 1) lies on the line representing the equation $2x + 3y = 5$, and the point (1, 7) does not lie on it. So, **every solution of the equation is a point on the line representing it**.

In fact, this is true for any linear equation, that is, **each solution $(x, y)$ of a linear equation in two variables, $ax + by + c = 0$, corresponds to a point on the line representing the equation, and vice versa**.

Now, consider Equations (1) and (2) given above. These equations, **taken together**, represent the information we have about Akhila at the fair.

These two linear equations are **in the same two variables $x$ and $y$**. Equations like these are called a *pair of linear equations in two variables*.

Let us see what such pairs look like algebraically.

The general form for a pair of linear equations in two variables $x$ and $y$ is

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0,$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are all real numbers and $a_1^2 + b_1^2 \neq 0,\ a_2^2 + b_2^2 \neq 0$.

Some examples of pair of linear equations in two variables are:

$2x + 3y - 7 = 0$ and $9x - 2y + 8 = 0$

$5x = y$ and $-7x + 2y + 3 = 0$

$x + y = 7$ and $17 = y$

Do you know, what do they look like geometrically?

Recall, that you have studied in Class IX that the geometrical (i.e., graphical) representation of a linear equation in two variables is a straight line. Can you now suggest what a pair of linear equations in two variables will look like, geometrically? There will be two straight lines, both to be considered together.

You have also studied in Class IX that given two lines in a plane, only one of the following three possibilities can happen:

(i) The two lines will intersect at one point.

(ii) The two lines will not intersect, i.e., they are parallel.

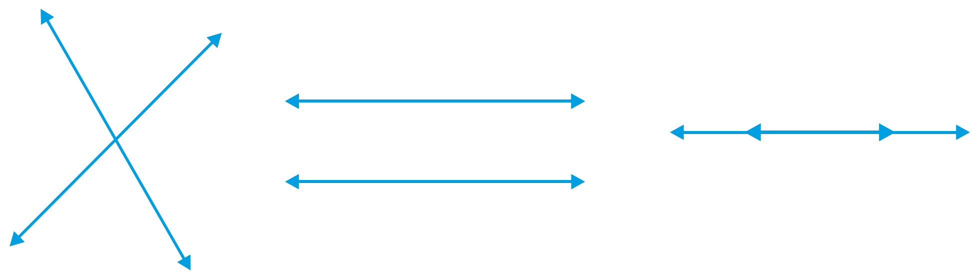
(iii) The two lines will be coincident.

We show all these possibilities in Fig. 3.1:

In Fig. 3.1 (a), they intersect.

In Fig. 3.1 (b), they are parallel.

In Fig. 3.1 (c), they are coincident.



(a) (b) (c)

**Fig. 3.1**

Both ways of representing a pair of linear equations go hand-in-hand — the algebraic and the geometric ways. Let us consider some examples.

**Example 1 :** Let us take the example given in Section 3.1. Akhila goes to a fair with ₹ 20 and wants to have rides on the Giant Wheel and play Hoopla. Represent this situation algebraically and graphically (geometrically).

**Solution :** The pair of equations formed is :

$$y = \frac{1}{2}x$$

i.e.,
$$x - 2y = 0 \qquad (1)$$

$$3x + 4y = 20 \qquad (2)$$

Let us represent these equations graphically. For this, we need at least two solutions for each equation. We give these solutions in Table 3.1.

**Table 3.1**

| $x$ | 0 | 2 |
|---|---|---|
| $y = \frac{x}{2}$ | 0 | 1 |

(i)

| $x$ | 0 | $\frac{20}{3}$ | 4 |
|---|---|---|---|
| $y = \frac{20-3x}{4}$ | 5 | 0 | 2 |

(ii)

Recall from Class IX that there are infinitely many solutions of each linear equation. So each of you can choose any two values, which may not be the ones we have chosen. Can you guess why we have chosen $x = 0$ in the first equation and in the second equation? When one of the variables is zero, the equation reduces to a linear equation in one variable, which can be solved easily. For instance, putting $x = 0$ in Equation (2), we get $4y = 20$, i.e., $y = 5$. Similarly, putting $y = 0$ in Equation (2), we get $3x = 20$, i.e., $x = \frac{20}{3}$. But as $\frac{20}{3}$ is not an integer, it will not be easy to plot exactly on the graph paper. So, we choose $y = 2$ which gives $x = 4$, an integral value.

Plot the points A(0, 0), B(2, 1) and P(0, 5), Q(4, 2), corresponding to the solutions in Table 3.1. Now draw the lines AB and PQ, representing the equations $x - 2y = 0$ and $3x + 4y = 20$, as shown in Fig. 3.2.

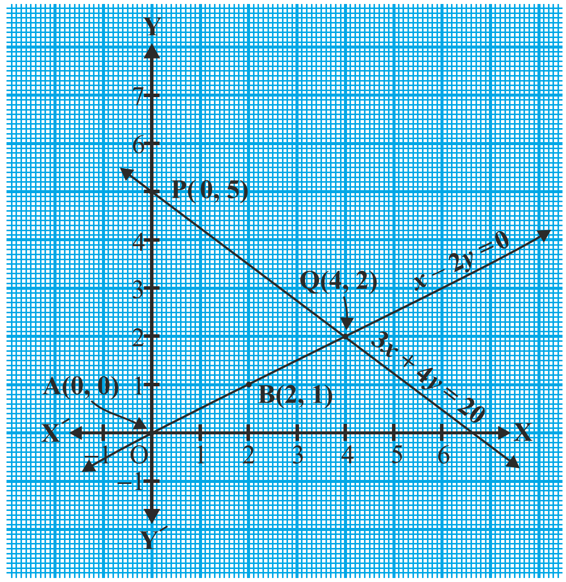


Fig. 3.2

In Fig. 3.2, observe that the two lines representing the two equations are intersecting at the point (4, 2). We shall discuss what this means in the next section.

**Example 2 :** Romila went to a stationery shop and purchased 2 pencils and 3 erasers for ₹ 9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹ 18. Represent this situation algebraically and graphically.

**Solution :** Let us denote the cost of 1 pencil by ₹ $x$ and one eraser by ₹ $y$. Then the algebraic representation is given by the following equations:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

To obtain the equivalent geometric representation, we find two points on the line representing each equation. That is, we find two solutions of each equation.

These solutions are given below in Table 3.2.

**Table 3.2**

| $x$ | 0 | 4.5 |
|---|---|---|
| $y = \frac{9-2x}{3}$ | 3 | 0 |

(i)

| $x$ | 0 | 3 |
|---|---|---|
| $y = \frac{18-4x}{6}$ | 3 | 1 |

(ii)

We plot these points in a graph paper and draw the lines. We find that both the lines coincide (see Fig. 3.3). This is so, because, both the equations are equivalent, i.e., one can be derived from the other.

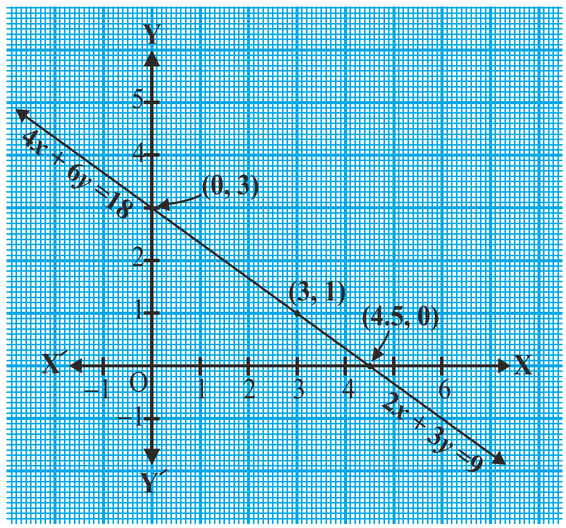



**Fig. 3.3**

**Example 3 :** Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation geometrically.

**Solution :** Two solutions of each of the equations :

$$x + 2y - 4 = 0 \tag{1}$$

$$2x + 4y - 12 = 0 \tag{2}$$

are given in Table 3.3

**Table 3.3**

| $x$ | 0 | 4 |
|---|---|---|
| $y = \frac{4-x}{2}$ | 2 | 0 |

(i)

| $x$ | 0 | 6 |
|---|---|---|
| $y = \frac{12-2x}{4}$ | 3 | 0 |

(ii)

To represent the equations graphically, we plot the points R(0, 2) and S(4, 0), to get the line RS and the points P(0, 3) and Q(6, 0) to get the line PQ.

We observe in Fig. 3.4, that the lines do not intersect anywhere, i.e., they are parallel.

So, we have seen several situations which can be represented by a pair of linear equations. We have seen their algebraic and geometric representations. In the next few sections, we will discuss how these representations can be used to look for solutions of the pair of linear equations.

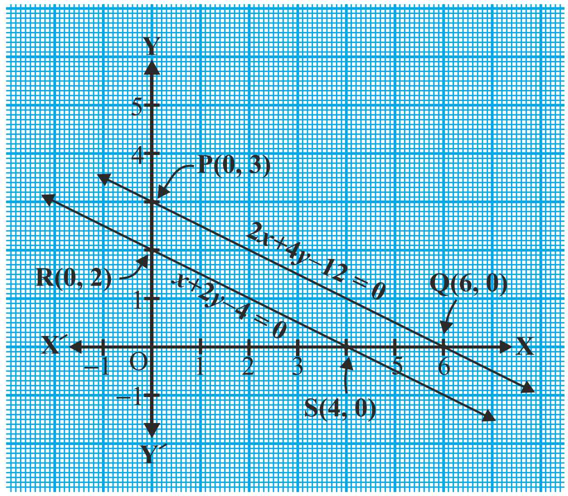



Fig. 3.4

## EXERCISE 3.1

1. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically.
2. The coach of a cricket team buys 3 bats and 6 balls for ₹ 3900. Later, she buys another bat and 3 more balls of the same kind for ₹ 1300. Represent this situation algebraically and geometrically.
3. The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹ 160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹ 300. Represent the situation algebraically and geometrically.

## 3.3 Graphical Method of Solution of a Pair of Linear Equations

In the previous section, you have seen how we can graphically represent a pair of linear equations as two lines. You have also seen that the lines may intersect, or may be parallel, or may coincide. Can we solve them in each case? And if so, how? We shall try and answer these questions from the geometrical point of view in this section.

Let us look at the earlier examples one by one.

- In the situation of Example 1, find out how many rides on the Giant Wheel Akhila had, and how many times she played Hoopla.

In Fig. 3.2, you noted that the equations representing the situation are geometrically shown by two lines intersecting at the point (4, 2). Therefore, the point (4, 2) lies on the lines represented by both the equations $x - 2y = 0$ and $3x + 4y = 20$. And this is the only common point.

Let us verify algebraically that $x = 4$, $y = 2$ is a solution of the given pair of equations. Substituting the values of $x$ and $y$ in each equation, we get $4 - 2 \times 2 = 0$ and $3(4) + 4(2) = 20$. So, we have verified that $x = 4$, $y = 2$ is a solution of both the equations. **Since (4, 2) is the only common point on both the lines, there is one and only one solution for this pair of linear equations in two variables.**

Thus, the number of rides Akhila had on Giant Wheel is 4 and the number of times she played Hoopla is 2.

- In the situation of Example 2, can you find the cost of each pencil and each eraser?

  In Fig. 3.3, the situation is geometrically shown by a pair of coincident lines. The solutions of the equations are given by the common points.

  Are there any common points on these lines? From the graph, we observe that every point on the line is a common solution to both the equations. So, the equations $2x + 3y = 9$ and $4x + 6y = 18$ have **infinitely many solutions.** This should not surprise us, because if we divide the equation $4x + 6y = 18$ by 2 , we get $2x + 3y = 9$, which is the same as Equation (1). That is, both the equations are equivalent. From the graph, we see that any point on the line gives us a possible cost of each pencil and eraser. For instance, each pencil and eraser can cost ₹ 3 and ₹ 1 respectively. Or, each pencil can cost ₹ 3.75 and eraser can cost ₹ 0.50, and so on.

- In the situation of Example 3, can the two rails cross each other?

  In Fig. 3.4, the situation is represented geometrically by two parallel lines. Since the lines do not intersect at all, the rails do not cross. This also means that the equations have no common solution.

A pair of linear equations which has no solution, is called an *inconsistent* pair of linear equations. A pair of linear equations in two variables, which has a solution, is called a *consistent* pair of linear equations. A pair of linear equations which are equivalent has infinitely many distinct common solutions. Such a pair is called a *dependent* pair of linear equations in two variables. Note that a dependent pair of linear equations is always consistent.

We can now summarise the behaviour of lines representing a pair of linear equations in two variables and the existence of solutions as follows:

(i) the lines may intersect in a single point. In this case, the pair of equations has a unique solution (consistent pair of equations).

(ii) the lines may be parallel. In this case, the equations have no solution (inconsistent pair of equations).

(iii) the lines may be coincident. In this case, the equations have infinitely many solutions [dependent (consistent) pair of equations].

Let us now go back to the pairs of linear equations formed in Examples 1, 2, and 3, and note down what kind of pair they are geometrically.

(i) $x - 2y = 0$ and $3x + 4y - 20 = 0$ (The lines intersect)

(ii) $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$ (The lines coincide)

(iii) $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ (The lines are parallel)

Let us now write down, and compare, the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ in all the three examples. Here, $a_1$, $b_1$, $c_1$ and $a_2$, $b_2$, $c_2$ denote the coefficents of equations given in the general form in Section 3.2.

**Table 3.4**

| Sl No. | Pair of lines | $\frac{a_1}{a_2}$ | $\frac{b_1}{b_2}$ | $\frac{c_1}{c_2}$ | Compare the ratios | Graphical representation | Algebraic interpretation |
|---|---|---|---|---|---|---|---|
| 1. | $x - 2y = 0$<br>$3x + 4y - 20 = 0$ | $\frac{1}{3}$ | $\frac{-2}{4}$ | $\frac{0}{-20}$ | $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ | Intersecting lines | Exactly one solution (unique) |
| 2. | $2x + 3y - 9 = 0$<br>$4x + 6y - 18 = 0$ | $\frac{2}{4}$ | $\frac{3}{6}$ | $\frac{-9}{-18}$ | $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ | Coincident lines | Infinitely many solutions |
| 3. | $x + 2y - 4 = 0$<br>$2x + 4y - 12 = 0$ | $\frac{1}{2}$ | $\frac{2}{4}$ | $\frac{-4}{-12}$ | $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ | Parallel lines | No solution |

From the table above, you can observe that if the lines represented by the equation

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0$$

are (i) intersecting, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

In fact, the converse is also true for any pair of lines. You can verify them by considering some more examples by yourself.

Let us now consider some more examples to illustrate it.

**Example 4 :** Check graphically whether the pair of equations

$$x + 3y = 6 \quad (1)$$

and

$$2x - 3y = 12 \quad (2)$$

is consistent. If so, solve them graphically.

**Solution :** Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table 3.5

**Table 3.5**

| $x$ | 0 | 6 |
|---|---|---|
| $y = \frac{6-x}{3}$ | 2 | 0 |

| $x$ | 0 | 3 |
|---|---|---|
| $y = \frac{2x-12}{3}$ | $-4$ | $-2$ |

Plot the points A(0, 2), B(6, 0), P(0, – 4) and Q(3, – 2) on graph paper, and join the points to form the lines AB and PQ as shown in Fig. 3.5.

We observe that there is a point B (6, 0) common to both the lines AB and PQ. So, the solution of the pair of linear equations is $x = 6$ and $y = 0$, i.e., the given pair of equations is consistent.

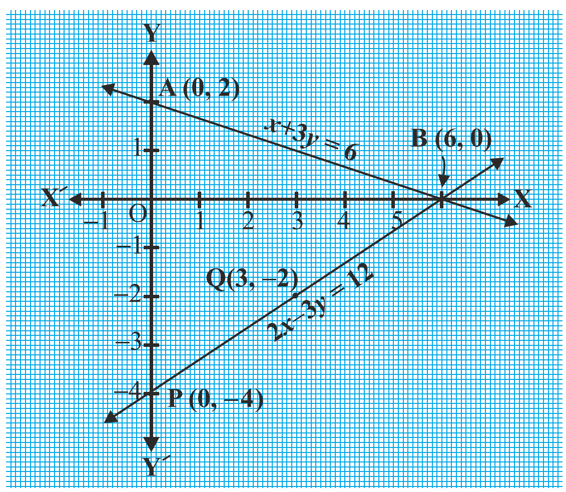


Fig. 3.5

**Example 5 :** Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$5x - 8y + 1 = 0 \quad (1)$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \quad (2)$$

**Solution :** Multiplying Equation (2) by $\frac{5}{3}$, we get

$$5x - 8y + 1 = 0$$

But, this is the same as Equation (1). Hence the lines represented by Equations (1) and (2) are coincident. Therefore, Equations (1) and (2) have infinitely many solutions.

Plot few points on the graph and verify it yourself.

**Example 6 :** Champa went to a 'Sale' to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased". Help her friends to find how many pants and skirts Champa bought.

**Solution :** Let us denote the number of pants by $x$ and the number of skirts by $y$. Then the equations formed are :

$$y = 2x - 2 \quad (1)$$

and

$$y = 4x - 4 \quad (2)$$

Let us draw the graphs of Equations (1) and (2) by finding two solutions for each of the equations. They are given in Table 3.6.

**Table 3.6**

| $x$ | 2 | 0 |
|---|---|---|
| $y = 2x - 2$ | 2 | – 2 |

| $x$ | 0 | 1 |
|---|---|---|
| $y = 4x - 4$ | – 4 | 0 |

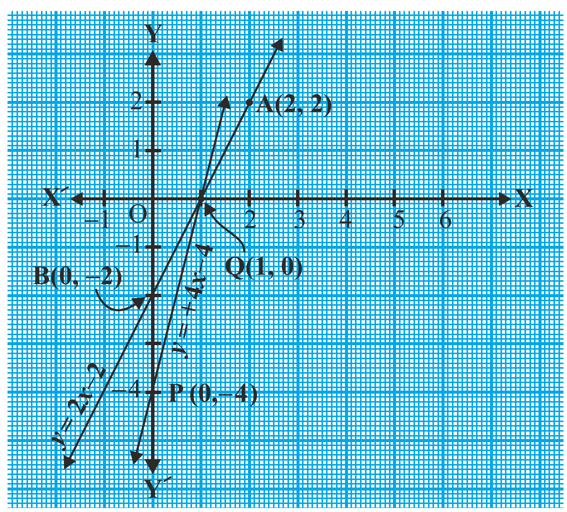



**Fig. 3.6**

Plot the points and draw the lines passing through them to represent the equations, as shown in Fig. 3.6.

The two lines intersect at the point (1, 0). So, $x = 1$, $y = 0$ is the required solution of the pair of linear equations, i.e., the number of pants she purchased is 1 and she did not buy any skirt.

**Verify** the answer by checking whether it satisfies the conditions of the given problem.

## EXERCISE 3.2

1. Form the pair of linear equations in the following problems, and find their solutions graphically.
   (i) 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
   (ii) 5 pencils and 7 pens together cost ₹ 50, whereas 7 pencils and 5 pens together cost ₹ 46. Find the cost of one pencil and that of one pen.

2. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:
   (i) $5x - 4y + 8 = 0$
   $7x + 6y - 9 = 0$
   (ii) $9x + 3y + 12 = 0$
   $18x + 6y + 24 = 0$
   (iii) $6x - 3y + 10 = 0$
   $2x - y + 9 = 0$

3. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent.
   (i) $3x + 2y = 5$ ; $2x - 3y = 7$
   (ii) $2x - 3y = 8$ ; $4x - 6y = 9$
   (iii) $\frac{3}{2}x + \frac{5}{3}y = 7$ ; $9x - 10y = 14$
   (iv) $5x - 3y = 11$ ; $-10x + 6y = -22$
   (v) $\frac{4}{3}x + 2y = 8$ ; $2x + 3y = 12$

4. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution graphically:

(i) $x + y = 5,\quad 2x + 2y = 10$

(ii) $x - y = 8,\quad 3x - 3y = 16$

(iii) $2x + y - 6 = 0,\quad 4x - 2y - 4 = 0$

(iv) $2x - 2y - 2 = 0,\quad 4x - 4y - 5 = 0$

**5.** Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

**6.** Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

(i) intersecting lines (ii) parallel lines

(iii) coincident lines

**7.** Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the $x$-axis, and shade the triangular region.

## 3.4 Algebraic Methods of Solving a Pair of Linear Equations

In the previous section, we discussed how to solve a pair of linear equations graphically. The graphical method is not convenient in cases when the point representing the solution of the linear equations has non-integral coordinates like $\left(\sqrt{3}, 2\sqrt{7}\right)$, $(-1.75, 3.3)$, $\left(\frac{4}{13}, \frac{1}{19}\right)$, etc. There is every possibility of making mistakes while reading such coordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall now discuss.

**3.4.1 Substitution Method :** We shall explain the method of substitution by taking some examples.

**Example 7 :** Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \qquad (1)$$

$$x + 2y = 3 \qquad (2)$$

**Solution :**

**Step 1 :** We pick either of the equations and write one variable in terms of the other. Let us consider the Equation (2) :

$$x + 2y = 3$$

and write it as

$$x = 3 - 2y \qquad (3)$$

**Step 2 :** Substitute the value of $x$ in Equation (1). We get

$$7(3-2y)-15y = 2$$

i.e., $$21-14y-15y = 2$$

i.e., $$-29y = -19$$

Therefore, $$y = \frac{19}{29}$$

**Step 3 :** Substituting this value of $y$ in Equation (3), we get

$$x = 3-2\left(\frac{19}{29}\right) = \frac{49}{29}$$

Therefore, the solution is $x = \frac{49}{29}$, $y = \frac{19}{29}$.

**Verification :** Substituting $x = \frac{49}{29}$ and $y = \frac{19}{29}$, you can verify that both the Equations (1) and (2) are satisfied.

To understand the substitution method more clearly, let us consider it stepwise:

**Step 1 :** Find the value of one variable, say $y$ in terms of the other variable, i.e., $x$ from either equation, whichever is convenient.

**Step 2 :** Substitute this value of $y$ in the other equation, and reduce it to an equation in one variable, i.e., in terms of $x$, which can be solved. Sometimes, as in Examples 9 and 10 below, you can get statements with no variable. If this statement is true, you can conclude that the pair of linear equations has infinitely many solutions. If the statement is false, then the pair of linear equations is inconsistent.

**Step 3 :** Substitute the value of $x$ (or $y$) obtained in Step 2 in the equation used in Step 1 to obtain the value of the other variable.

**Remark :** We have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why the method is known as the *substitution method*.

**Example 8 :** Solve Q.1 of Exercise 3.1 by the method of substitution.

**Solution :** Let $s$ and $t$ be the ages (in years) of Aftab and his daughter, respectively. Then, the pair of linear equations that represent the situation is

$$s-7 = 7\,(t-7), \text{ i.e., } s-7t+42 = 0 \qquad (1)$$

and $$s+3 = 3\,(t+3), \text{ i.e., } s-3t = 6 \qquad (2)$$

Using Equation (2), we get $s = 3t + 6$.

Putting this value of $s$ in Equation (1), we get

$$(3t + 6) - 7t + 42 = 0,$$

i.e., $$4t = 48, \text{ which gives } t = 12.$$

Putting this value of $t$ in Equation (2), we get

$$s = 3\,(12) + 6 = 42$$

So, Aftab and his daughter are 42 and 12 years old, respectively.

Verify this answer by checking if it satisfies the conditions of the given problems.

**Example 9 :** Let us consider Example 2 in Section 3.3, i.e., the cost of 2 pencils and 3 erasers is ₹ 9 and the cost of 4 pencils and 6 erasers is ₹ 18. Find the cost of each pencil and each eraser.

**Solution :** The pair of linear equations formed were:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

We first express the value of $x$ in terms of $y$ from the equation $2x + 3y = 9$, to get

$$x = \frac{9 - 3y}{2} \quad (3)$$

Now we substitute this value of $x$ in Equation (2), to get

$$\frac{4(9 - 3y)}{2} + 6y = 18$$

i.e., $$18 - 6y + 6y = 18$$

i.e., $$18 = 18$$

This statement is true for all values of $y$. However, we do not get a specific value of $y$ as a solution. Therefore, we cannot obtain a specific value of $x$. This situation has arisen because both the given equations are the same. Therefore, Equations (1) and (2) have *infinitely many solutions.* Observe that we have obtained the same solution graphically also. (Refer to Fig. 3.3, Section 3.2.) We cannot find a unique cost of a pencil and an eraser, because there are many common solutions, to the given situation.

**Example 10 :** Let us consider the Example 3 of Section 3.2. Will the rails cross each other?

**Solution :** The pair of linear equations formed were:

$$x + 2y - 4 = 0 \qquad (1)$$

$$2x + 4y - 12 = 0 \qquad (2)$$

We express $x$ in terms of $y$ from Equation (1) to get

$$x = 4 - 2y$$

Now, we substitute this value of $x$ in Equation (2) to get

$$2(4 - 2y) + 4y - 12 = 0$$

i.e., $\quad 8 - 12 = 0$

i.e., $\quad -4 = 0$

which is a false statement.

Therefore, the equations do not have a common solution. So, the two rails will not cross each other.

## EXERCISE 3.3

**1.** Solve the following pair of linear equations by the substitution method.

(i) $x + y = 14$
$x - y = 4$

(ii) $s - t = 3$
$\frac{s}{3} + \frac{t}{2} = 6$

(iii) $3x - y = 3$
$9x - 3y = 9$

(iv) $0.2x + 0.3y = 1.3$
$0.4x + 0.5y = 2.3$

(v) $\sqrt{2}\,x + \sqrt{3}\,y = 0$
$\sqrt{3}\,x - \sqrt{8}\,y = 0$

(vi) $\frac{3x}{2} - \frac{5y}{3} = -2$
$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

**2.** Solve $2x + 3y = 11$ and $2x - 4y = -24$ and hence find the value of '$m$' for which $y = mx + 3$.

**3.** Form the pair of linear equations for the following problems and find their solution by substitution method.

(i) The difference between two numbers is 26 and one number is three times the other. Find them.

(ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

(iii) The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

(iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a journey of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

(v) A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. Find the fraction.

(vi) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

### 3.4.2 Elimination Method

Now let us consider another method of eliminating (i.e., removing) one variable. This is sometimes more convenient than the substitution method. Let us see how this method works.

**Example 11 :** The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

**Solution :** Let us denote the incomes of the two person by ₹ $9x$ and ₹ $7x$ and their expenditures by ₹ $4y$ and ₹ $3y$ respectively. Then the equations formed in the situation is given by :

$$9x - 4y = 2000 \quad (1)$$

and

$$7x - 3y = 2000 \quad (2)$$

**Step 1 :** Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of $y$ equal. Then we get the equations:

$$27x - 12y = 6000 \quad (3)$$

$$28x - 12y = 8000 \quad (4)$$

**Step 2 :** Subtract Equation (3) from Equation (4) to *eliminate* $y$, because the coefficients of $y$ are the same. So, we get

$$(28x - 27x) - (12y - 12y) = 8000 - 6000$$

i.e.,

$$x = 2000$$

**Step 3 :** Substituting this value of $x$ in (1), we get

$$9(2000) - 4y = 2000$$

i.e.,

$$y = 4000$$

So, the solution of the equations is $x = 2000$, $y = 4000$. Therefore, the monthly incomes of the persons are ₹ 18,000 and ₹ 14,000, respectively.

**Verification :** 18000 : 14000 = 9 : 7. Also, the ratio of their expenditures = 18000 – 2000 : 14000 – 2000 = 16000 : 12000 = 4 : 3

**Remarks :**

1. The method used in solving the example above is called the *elimination* method, because we eliminate one variable first, to get a linear equation in one variable. In the example above, we eliminated $y$. We could also have eliminated $x$. Try doing it that way.
2. You could also have used the substitution, or graphical method, to solve this problem. Try doing so, and see which method is more convenient.

Let us now note down these steps in the elimination method :

**Step 1 :** First multiply both the equations by some suitable non-zero constants to make the coefficients of one variable (either $x$ or $y$) numerically equal.

**Step 2 :** Then add or subtract one equation from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If in Step 2, we obtain a true statement involving no variable, then the original pair of equations has infinitely many solutions.

If in Step 2, we obtain a false statement involving no variable, then the original pair of equations has no solution, i.e., it is inconsistent.

**Step 3 :** Solve the equation in one variable ($x$ or $y$) so obtained to get its value.

**Step 4 :** Substitute this value of $x$ (or $y$) in either of the original equations to get the value of the other variable.

Now to illustrate it, we shall solve few more examples.

**Example 12 :** Use elimination method to find all possible solutions of the following pair of linear equations :

$$2x + 3y = 8 \qquad (1)$$

$$4x + 6y = 7 \qquad (2)$$

**Solution :**

**Step 1 :** Multiply Equation (1) by 2 and Equation (2) by 1 to make the coefficients of $x$ equal. Then we get the equations as :

$$4x + 6y = 16 \qquad (3)$$

$$4x + 6y = 7 \qquad (4)$$

**Step 2 :** Subtracting Equation (4) from Equation (3),

$$(4x - 4x) + (6y - 6y) = 16 - 7$$

i.e., $$0 = 9, \text{ which is a false statement.}$$

Therefore, the pair of equations has no solution.

**Example 13 :** The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

**Solution :** Let the ten's and the unit's digits in the first number be $x$ and $y$, respectively. So, the first number may be written as $10x + y$ in the expanded form (for example, $56 = 10(5) + 6$).

When the digits are reversed, $x$ becomes the unit's digit and $y$ becomes the ten's digit. This number, in the expanded notation is $10y + x$ (for example, when 56 is reversed, we get $65 = 10(6) + 5$).

According to the given condition.

$$(10x + y) + (10y + x) = 66$$

i.e., $$11(x + y) = 66$$

i.e., $$x + y = 6 \tag{1}$$

We are also given that the digits differ by 2, therefore,

either $$x - y = 2 \tag{2}$$

or $$y - x = 2 \tag{3}$$

If $x - y = 2$, then solving (1) and (2) by elimination, we get $x = 4$ and $y = 2$. In this case, we get the number 42.

If $y - x = 2$, then solving (1) and (3) by elimination, we get $x = 2$ and $y = 4$. In this case, we get the number 24.

Thus, there are two such numbers 42 and 24.

**Verification :** Here $42 + 24 = 66$ and $4 - 2 = 2$. Also $24 + 42 = 66$ and $4 - 2 = 2$.

## EXERCISE 3.4

**1.** Solve the following pair of linear equations by the elimination method and the substitution method :

(i) $x + y = 5$ and $2x - 3y = 4$

(ii) $3x + 4y = 10$ and $2x - 2y = 2$

(iii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$

(iv) $\frac{x}{2} + \frac{2y}{3} = -1$ and $x - \frac{y}{3} = 3$

**2.** Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :

(i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

(ii) Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

(iii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

(iv) Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes of ₹ 50 and ₹ 100 she received.

(v) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

### 3.4.3 Cross - Multiplication Method

So far, you have learnt how to solve a pair of linear equations in two variables by graphical, substitution and elimination methods. Here, we introduce one more algebraic method to solve a pair of linear equations which for many reasons is a very useful method of solving these equations. Before we proceed further, let us consider the following situation.

The cost of 5 oranges and 3 apples is ₹ 35 and the cost of 2 oranges and 4 apples is ₹ 28. Let us find the cost of an orange and an apple.

Let us denote the cost of an orange by ₹ $x$ and the cost of an apple by ₹ $y$. Then, the equations formed are :

$$5x + 3y = 35, \text{ i.e., } 5x + 3y - 35 = 0 \qquad (1)$$

$$2x + 4y = 28, \text{ i.e., } 2x + 4y - 28 = 0 \qquad (2)$$

Let us use the elimination method to solve these equations.

Multiply Equation (1) by 4 and Equation (2) by 3. We get

$$(4)(5)x + (4)(3)y + (4)(-35) = 0 \qquad (3)$$

$$(3)(2)x + (3)(4)y + (3)(-28) = 0 \qquad (4)$$

Subtracting Equation (4) from Equation (3), we get

$$[(5)(4) - (3)(2)]x + [(4)(3) - (3)(4)]y + [4(-35) - (3)(-28)] = 0$$

Therefore, $$x = \frac{-[(4)(-35) - (3)(-28)]}{(5)(4) - (3)(2)}$$

i.e., $$x = \frac{(3)(-28) - (4)(-35)}{(5)(4) - (2)(3)} \qquad (5)$$

If Equations (1) and (2) are written as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then we have

$a_1 = 5, b_1 = 3, c_1 = -35, a_2 = 2, b_2 = 4, c_2 = -28.$

Then Equation (5) can be written as $x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$,

Similarly, you can get $y = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

By simplyfing Equation (5), we get

$$x = \frac{-84 + 140}{20 - 6} = 4$$

Similarly, $$y = \frac{(-35)(2) - (5)(-28)}{20 - 6} = \frac{-70 + 140}{14} = 5$$

Therefore, $x = 4$, $y = 5$ is the solution of the given pair of equations.

Then, the cost of an orange is ₹ 4 and that of an apple is ₹ 5.

**Verification :** Cost of 5 oranges + Cost of 3 apples = ₹ 20 + ₹ 15 = ₹ 35. Cost of 2 oranges + Cost of 4 apples = ₹ 8 + ₹ 20 = ₹ 28.

Let us now see how this method works for any pair of linear equations in two variables of the form

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

and $$a_2x + b_2y + c_2 = 0 \qquad (2)$$

To obtain the values of $x$ and $y$ as shown above, we follow the following steps:

**Step 1 :** Multiply Equation (1) by $b_2$ and Equation (2) by $b_1$, to get

$$b_2a_1x + b_2b_1y + b_2c_1 = 0 \qquad (3)$$

$$b_1a_2x + b_1b_2y + b_1c_2 = 0 \qquad (4)$$

**Step 2 :** Subtracting Equation (4) from (3), we get:

$$(b_2a_1 - b_1a_2)\,x + (b_2b_1 - b_1b_2)\,y + (b_2c_1 - b_1c_2) = 0$$

i.e., $(b_2a_1 - b_1a_2)\, x = b_1c_2 - b_2c_1$

So, $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0 \qquad (5)$$

**Step 3 :** Substituting this value of $x$ in (1) or (2), we get

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \qquad (6)$$

Now, two cases arise :

**Case 1 :** $a_1b_2 - a_2b_1 \neq 0$. In this case $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. Then the pair of linear equations has a unique solution.

**Case 2 :** $a_1b_2 - a_2b_1 = 0$. If we write $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, then $a_1 = k\, a_2$, $b_1 = k\, b_2$.

Substituting the values of $a_1$ and $b_1$ in the Equation (1), we get

$$k\,(a_2x + b_2y) + c_1 = 0. \qquad (7)$$

It can be observed that the Equations (7) and (2) can both be satisfied only if

$c_1 = k\, c_2$, i.e., $\frac{c_1}{c_2} = k$.

If $c_1 = k\, c_2$, any solution of Equation (2) will satisfy the Equation (1), and vice versa. So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq k\, c_2$, then any solution of Equation (1) will not satisfy Equation (2) and vice versa. Therefore the pair has no solution.

We can summarise the discussion above for the pair of linear equations given by (1) and (2) as follows:

(i) When $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we get a unique solution.

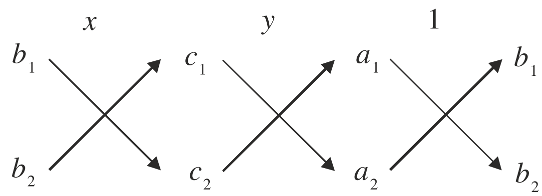
(ii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, there are infinitely many solutions.

(iii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, there is no solution.

Note that you can write the solution given by Equations (5) and (6) in the following form :

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \qquad (8)$$

In remembering the above result, the following diagram may be helpful to you :




The arrows between the two numbers indicate that they are to be multiplied and the second product is to be subtracted from the first.

For solving a pair of linear equations by this method, we will follow the following steps :

**Step 1 :** Write the given equations in the form (1) and (2).

**Step 2 :** Taking the help of the diagram above, write Equations as given in (8).

**Step 3 :** Find $x$ and $y$, provided $a_1b_2 - a_2b_1 \neq 0$

Step 2 above gives you an indication of why this method is called the **cross-multiplication method**.

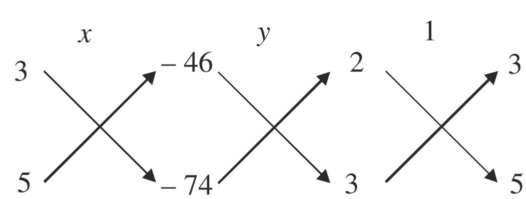
**Example 14 :** From a bus stand in Bangalore , if we buy 2 tickets to Malleswaram and 3 tickets to Yeshwanthpur, the total cost is ₹ 46; but if we buy 3 tickets to Malleswaram and 5 tickets to Yeshwanthpur the total cost is ₹ 74. Find the fares from the bus stand to Malleswaram, and to Yeshwanthpur.

**Solution :** Let ₹ $x$ be the fare from the bus stand in Bangalore to Malleswaram, and ₹ $y$ to Yeshwanthpur. From the given information, we have

$$2x + 3y = 46, \text{ i.e., } 2x + 3y - 46 = 0 \qquad (1)$$

$$3x + 5y = 74, \text{ i.e., } 3x + 5y - 74 = 0 \qquad (2)$$

To solve the equations by the cross-multiplication method, we draw the diagram as given below.



Then $$\frac{x}{(3)(-74)-(5)(-46)} = \frac{y}{(-46)(3)-(-74)(2)} = \frac{1}{(2)(5)-(3)(3)}$$

i.e., $$\frac{x}{-222+230} = \frac{y}{-138+148} = \frac{1}{10-9}$$

i.e., $$\frac{x}{8} = \frac{y}{10} = \frac{1}{1}$$

i.e., $$\frac{x}{8} = \frac{1}{1} \text{ and } \frac{y}{10} = \frac{1}{1}$$

i.e., $$x = 8 \text{ and } y = 10$$

Hence, the fare from the bus stand in Bangalore to Malleswaram is ₹ 8 and the fare to Yeshwanthpur is ₹ 10.

**Verification :** You can check from the problem that the solution we have got is correct.

**Example 15 :** For which values of $p$ does the pair of equations given below has unique solution?

$$4x + py + 8 = 0$$

$$2x + 2y + 2 = 0$$

**Solution :** Here $a_1 = 4$, $a_2 = 2$, $b_1 = p$, $b_2 = 2$.

Now for the given pair to have a unique solution : $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

i.e., $$\frac{4}{2} \neq \frac{p}{2}$$

i.e., $$p \neq 4$$

Therefore, for all values of $p$, except 4, the given pair of equations will have a unique solution.

**Example 16 :** For what values of $k$ will the following pair of linear equations have infinitely many solutions?

$$kx + 3y - (k - 3) = 0$$

$$12x + ky - k = 0$$

**Solution :** Here, $\frac{a_1}{a_2} = \frac{k}{12}$, $\frac{b_1}{b_2} = \frac{3}{k}$, $\frac{c_1}{c_2} = \frac{k-3}{k}$

For a pair of linear equations to have infinitely many solutions : $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

So, we need $\frac{k}{12} = \frac{3}{k} = \frac{k-3}{k}$

or, $\frac{k}{12} = \frac{3}{k}$

which gives $k^2 = 36$, i.e., $k = \pm 6$.

Also, $\frac{3}{k} = \frac{k-3}{k}$

gives $3k = k^2 - 3k$, i.e., $6k = k^2$, which means $k = 0$ or $k = 6$.

Therefore, the value of $k$, that satisfies both the conditions, is $k = 6$. For this value, the pair of linear equations has infinitely many solutions.

## EXERCISE 3.5

**1.** Which of the following pairs of linear equations has unique solution, no solution, or infinitely many solutions. In case there is a unique solution, find it by using cross multiplication method.

(i) $x - 3y - 3 = 0$
$3x - 9y - 2 = 0$

(ii) $2x + y = 5$
$3x + 2y = 8$

(iii) $3x - 5y = 20$
$6x - 10y = 40$

(iv) $x - 3y - 7 = 0$
$3x - 3y - 15 = 0$

**2.** (i) For which values of $a$ and $b$ does the following pair of linear equations have an infinite number of solutions?

$2x + 3y = 7$

$(a - b)\,x + (a + b)\,y = 3a + b - 2$

(ii) For which value of $k$ will the following pair of linear equations have no solution?

$3x + y = 1$

$(2k - 1)\,x + (k - 1)\,y = 2k + 1$

**3.** Solve the following pair of linear equations by the substitution and cross-multiplication methods :

$8x + 5y = 9$

$3x + 2y = 4$

**4.** Form the pair of linear equations in the following problems and find their solutions (if they exist) by any algebraic method :

(i) A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹ 1000 as hostel charges whereas a student B, who takes food for 26 days, pays ₹ 1180 as hostel charges. Find the fixed charges and the cost of food per day.

(ii) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

(iii) Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

(iv) Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?

(v) The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.

## 3.5 Equations Reducible to a Pair of Linear Equations in Two Variables

In this section, we shall discuss the solution of such pairs of equations which are not linear but can be reduced to linear form by making some suitable substitutions. We now explain this process through some examples.

**Example 17 :** Solve the pair of equations:

$$\frac{2}{x} + \frac{3}{y} = 13$$

$$\frac{5}{x} - \frac{4}{y} = -2$$

**Solution :** Let us write the given pair of equations as

$$2\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 13 \qquad (1)$$

$$5\left(\frac{1}{x}\right) - 4\left(\frac{1}{y}\right) = -2 \qquad (2)$$

These equations are not in the form $ax + by + c = 0$. However, if we substitute

$\frac{1}{x} = p$ and $\frac{1}{y} = q$ in Equations (1) and (2), we get

$$2p + 3q = 13 \quad (3)$$

$$5p - 4q = -2 \quad (4)$$

So, we have expressed the equations as a pair of linear equations. Now, you can use any method to solve these equations, and get $p = 2$, $q = 3$.

You know that $p = \frac{1}{x}$ and $q = \frac{1}{y}$.

Substitute the values of $p$ and $q$ to get

$$\frac{1}{x} = 2, \text{ i.e., } x = \frac{1}{2} \text{ and } \frac{1}{y} = 3, \text{ i.e., } y = \frac{1}{3}.$$

**Verification :** By substituting $x = \frac{1}{2}$ and $y = \frac{1}{3}$ in the given equations, we find that both the equations are satisfied.

**Example 18 :** Solve the following pair of equations by reducing them to a pair of linear equations :

$$\frac{5}{x-1} + \frac{1}{y-2} = 2$$

$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

**Solution :** Let us put $\frac{1}{x-1} = p$ and $\frac{1}{y-2} = q$. Then the given equations

$$5\left(\frac{1}{x-1}\right) + \frac{1}{y-2} = 2 \quad (1)$$

$$6\left(\frac{1}{x-1}\right) - 3\left(\frac{1}{y-2}\right) = 1 \quad (2)$$

can be written as :

$$5p + q = 2 \quad (3)$$

$$6p - 3q = 1 \quad (4)$$

Equations (3) and (4) form a pair of linear equations in the general form. Now, you can use any method to solve these equations. We get $p = \frac{1}{3}$ and $q = \frac{1}{3}$.

Now, substituting $\frac{1}{x-1}$ for $p$, we have

$$\frac{1}{x-1} = \frac{1}{3},$$

i.e., $$x - 1 = 3, \text{ i.e., } x = 4.$$

Similarly, substituting $\frac{1}{y-2}$ for $q$, we get

$$\frac{1}{y-2} = \frac{1}{3}$$

i.e., $$3 = y - 2, \text{ i.e., } y = 5$$

Hence, $x = 4$, $y = 5$ is the required solution of the given pair of equations.

**Verification :** Substitute $x = 4$ and $y = 5$ in (1) and (2) to check whether they are satisfied.

**Example 19 :** A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.

**Solution :** Let the speed of the boat in still water be $x$ km/h and speed of the stream be $y$ km/h. Then the speed of the boat downstream $= (x + y)$ km/h,

and the speed of the boat upstream $= (x - y)$ km/h

Also, $$\text{time} = \frac{\text{distance}}{\text{speed}}$$

In the first case, when the boat goes 30 km upstream, let the time taken, in hour, be $t_1$. Then

$$t_1 = \frac{30}{x-y}$$

Let $t_2$ be the time, in hours, taken by the boat to go 44 km downstream. Then $t_2 = \frac{44}{x+y}$. The total time taken, $t_1 + t_2$, is 10 hours. Therefore, we get the equation

$$\frac{30}{x-y} + \frac{44}{x+y} = 10 \qquad (1)$$

In the second case, in 13 hours it can go 40 km upstream and 55 km downstream. We get the equation

$$\frac{40}{x-y} + \frac{55}{x+y} = 13 \qquad (2)$$

Put $$\frac{1}{x-y} = u \text{ and } \frac{1}{x+y} = v \qquad (3)$$

On substituting these values in Equations (1) and (2), we get the pair of linear equations:

$$30u + 44v = 10 \quad \text{or} \quad 30u + 44v - 10 = 0 \qquad (4)$$

$$40u + 55v = 13 \quad \text{or} \quad 40u + 55v - 13 = 0 \qquad (5)$$

Using Cross-multiplication method, we get

$$\frac{u}{44(-13) - 55(-10)} = \frac{v}{40(-10) - 30(-13)} = \frac{1}{30(55) - 44(40)}$$

i.e., $$\frac{u}{-22} = \frac{v}{-10} = \frac{1}{-110}$$

i.e., $$u = \frac{1}{5},\quad v = \frac{1}{11}$$

Now put these values of $u$ and $v$ in Equations (3), we get

$$\frac{1}{x-y} = \frac{1}{5} \text{ and } \frac{1}{x+y} = \frac{1}{11}$$

i.e., $$x - y = 5 \text{ and } x + y = 11 \qquad (6)$$

Adding these equations, we get

$$2x = 16$$

i.e., $$x = 8$$

Subtracting the equations in (6), we get

$$2y = 6$$

i.e., $$y = 3$$

Hence, the speed of the boat in still water is 8 km/h and the speed of the stream is 3 km/h.

**Verification :** Verify that the solution satisfies the conditions of the problem.

## EXERCISE 3.6

**1.** Solve the following pairs of equations by reducing them to a pair of linear equations:

(i) $\frac{1}{2x} + \frac{1}{3y} = 2$

$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6}$

(ii) $\frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2$

$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1$

(iii) $\frac{4}{x} + 3y = 14$

$\frac{3}{x} - 4y = 23$

(iv) $\frac{5}{x-1} + \frac{1}{y-2} = 2$

$\frac{6}{x-1} - \frac{3}{y-2} = 1$

(v) $\frac{7x-2y}{xy} = 5$

$\frac{8x+7y}{xy} = 15$

(vi) $6x + 3y = 6xy$

$2x + 4y = 5xy$

(vii) $\frac{10}{x+y} + \frac{2}{x-y} = 4$

$\frac{15}{x+y} - \frac{5}{x-y} = -2$

(viii) $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$

$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$

**2.** Formulate the following problems as a pair of equations, and hence find their solutions:

(i) Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.

(ii) 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.

(iii) Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

## EXERCISE 3.7 (Optional)*

1. The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju is twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju.

2. One says, "Give me a hundred, friend! I shall then become twice as rich as you". The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their (respective) capital? [From the Bijaganita of Bhaskara II]
[**Hint :** $x + 100 = 2(y - 100)$, $y + 10 = 6(x - 10)$].

3. A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.

4. The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.

5. In a $\Delta$ ABC, $\angle C = 3 \angle B = 2 (\angle A + \angle B)$. Find the three angles.

6. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the $y$ axis.

7. Solve the following pair of linear equations:

   (i) $px + qy = p - q$
   $qx - py = p + q$

   (ii) $ax + by = c$
   $bx + ay = 1 + c$

   (iii) $\dfrac{x}{a} - \dfrac{y}{b} = 0$
   $ax + by = a^2 + b^2.$

   (iv) $(a - b)x + (a + b)y = a^2 - 2ab - b^2$
   $(a + b)(x + y) = a^2 + b^2$

   (v) $152x - 378y = -74$
   $-378x + 152y = -604$

8. ABCD is a cyclic quadrilateral (see Fig. 3.7). Find the angles of the cyclic quadrilateral.

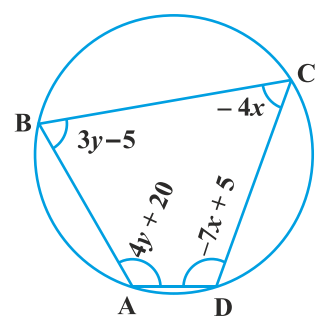



Fig. 3.7

* These exercises are not from the examination point of view.

## 3.6 Summary

In this chapter, you have studied the following points:

1. Two linear equations in the same two variables are called a pair of linear equations in two variables. The most general form of a pair of linear equations is

   $$a_1x + b_1y + c_1 = 0$$
   $$a_2x + b_2y + c_2 = 0$$

   where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers, such that $a_1^2 + b_1^2 \neq 0,\ a_2^2 + b_2^2 \neq 0$.

2. A pair of linear equations in two variables can be represented, and solved, by the:
   (i) graphical method
   (ii) algebraic method

3. Graphical Method :

   The graph of a pair of linear equations in two variables is represented by two lines.

   (i) If the lines intersect at a point, then that point gives the unique solution of the two equations. In this case, the pair of equations is **consistent**.

   (ii) If the lines coincide, then there are infinitely many solutions — each point on the line being a solution. In this case, the pair of equations is **dependent (consistent)**.

   (iii) If the lines are parallel, then the pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

4. Algebraic Methods : We have discussed the following methods for finding the solution(s) of a pair of linear equations :
   (i) Substitution Method
   (ii) Elimination Method
   (iii) Cross-multiplication Method

5. If a pair of linear equations is given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the following situations can arise :

   (i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_1}$ : In this case, the pair of linear equations is consistent.

   (ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ : In this case, the pair of linear equations is inconsistent.

   (iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ : In this case, the pair of linear equations is dependent and consistent.

6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we alter them so that they are reduced to a pair of linear equations.

# 04

# QUADRATIC EQUATIONS

# QUADRATIC EQUATIONS 4

## 4.1 Introduction

In Chapter 2, you have studied different types of polynomials. One type was the quadratic polynomial of the form $ax^2 + bx + c, a \neq 0$. When we equate this polynomial to zero, we get a quadratic equation. Quadratic equations come up when we deal with many real-life situations. For instance, suppose a charity trust decides to build a prayer hall having a carpet area of 300 square metres with its length one metre more than twice its breadth. What should be the length and breadth of the hall? Suppose the breadth of the hall is x metres. Then, its length should be $(2x + 1)$ metres. We can depict this information pictorially as shown in Fig. 4.1.

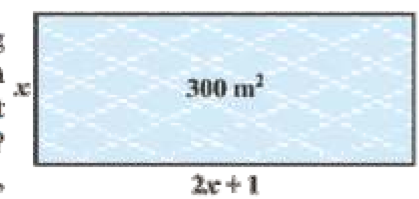



Fig. 4.1

Now, area of the hall $= (2x + 1).\ x\ m^2 = (2x^2 + x)\ m^2$

So, $2x^2 + x = 300$ (Given)

Therefore, $2x^2 + x - 300 = 0$

So, the breadth of the hall should satisfy the equation $2x^2 + x - 300 = 0$ which is a quadratic equation.

Many people believe that Babylonians were the first to solve quadratic equations. For instance, they knew how to find two positive numbers with a given positive sum and a given positive product, and this problem is equivalent to solving a quadratic equation of the form $x^2 - px + q = 0$. Greek mathematician Euclid developed a geometrical approach for finding out lengths which, in our present day terminology, are solutions of quadratic equations. Solving of quadratic equations, in general form, is often credited to ancient Indian mathematicians. In fact, Brahmagupta (C.E.598–665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx = c$. Later, Sridharacharya (C.E. 1025) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square. An Arab mathematician Al-Khwarizmi (about C.E. 800) also studied quadratic equations of different types. Abraham bar Hiyya Ha-Nasi, in his book 'Liber embadorum' published in Europe in C.E. 1145 gave complete solutions of different quadratic equations.

In this chapter, you will study quadratic equations, and various ways of finding their roots. You will also see some applications of quadratic equations in daily life situations.

## 4.2 Quadratic Equations

A quadratic equation in the variable $x$ is an equation of the form $ax^2 + bx + c = 0$, where $a$, $b$, $c$ are real numbers, $a \neq 0$. For example, $2x^2 + x - 300 = 0$ is a quadratic equation. Similarly, $2x^2 - 3x + 1 = 0$, $4x - 3x^2 + 2 = 0$ and $1 - x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the **standard form of a quadratic equation**.

Quadratic equations arise in several situations in the world around us and in different fields of mathematics. Let us consider a few examples.

**Example 1 :** Represent the following situations mathematically:

(i) John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. We would like to find out how many marbles they had to start with.

(ii) A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹ 750. We would like to find out the number of toys produced on that day.

**Solution :**

(i) Let the number of marbles John had be $x$.

Then the number of marbles Jivanti had = $45 - x$ (Why?).

The number of marbles left with John, when he lost 5 marbles = $x - 5$

The number of marbles left with Jivanti, when she lost 5 marbles = $45 - x - 5$

$$= 40 - x$$

Therefore, their product $= (x - 5)(40 - x)$

$$= 40x - x^2 - 200 + 5x$$

$$= -x^2 + 45x - 200$$

So, $-x^2 + 45x - 200 = 124$ (Given that product = 124)

i.e., $-x^2 + 45x - 324 = 0$

i.e., $x^2 - 45x + 324 = 0$

Therefore, the number of marbles John had, satisfies the quadratic equation

$$x^2 - 45x + 324 = 0$$

which is the required representation of the problem mathematically.

(ii) Let the number of toys produced on that day be $x$.

Therefore, the cost of production (in rupees) of each toy that day = $55 - x$

So, the total cost of production (in rupees) that day = $x(55 - x)$

Therefore, $x(55 - x) = 750$

i.e., $55x - x^2 = 750$

i.e., $-x^2 + 55x - 750 = 0$

i.e., $x^2 - 55x + 750 = 0$

Therefore, the number of toys produced that day satisfies the quadratic equation

$$x^2 - 55x + 750 = 0$$

which is the required representation of the problem mathematically.

**Example 2 :** Check whether the following are quadratic equations:

(i) $(x - 2)^2 + 1 = 2x - 3$ (ii) $x(x + 1) + 8 = (x + 2)(x - 2)$

(iii) $x(2x + 3) = x^2 + 1$ (iv) $(x + 2)^3 = x^3 - 4$

**Solution :**

(i) LHS $= (x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be rewritten as

$$x^2 - 4x + 5 = 2x - 3$$

i.e., $x^2 - 6x + 8 = 0$

It is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

(ii) Since $x(x + 1) + 8 = x^2 + x + 8$ and $(x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

i.e., $x + 12 = 0$

It is not of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

(iii) Here, LHS $= x(2x + 3) = 2x^2 + 3x$

So, $x(2x + 3) = x^2 + 1$ can be rewritten as

$$2x^2 + 3x = x^2 + 1$$

Therefore, we get $x^2 + 3x - 1 = 0$

It is of the form $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

(iv) Here, LHS $= (x + 2)^3 = x^3 + 6x^2 + 12x + 8$

Therefore, $(x + 2)^3 = x^3 - 4$ can be rewritten as

$$x^3 + 6x^2 + 12x + 8 = x^3 - 4$$

i.e., $6x^2 + 12x + 12 = 0$ or, $x^2 + 2x + 2 = 0$

It is of the form $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

**Remark :** Be careful! In (ii) above, the given equation appears to be a quadratic equation, but it is not a quadratic equation.

In (iv) above, the given equation appears to be a cubic equation (an equation of degree 3) and not a quadratic equation. But it turns out to be a quadratic equation. As you can see, often we need to simplify the given equation before deciding whether it is quadratic or not.

## EXERCISE 4.1

1. Check whether the following are quadratic equations :

(i) $(x + 1)^2 = 2(x - 3)$

(ii) $x^2 - 2x = (-2)(3 - x)$

(iii) $(x - 2)(x + 1) = (x - 1)(x + 3)$

(iv) $(x - 3)(2x + 1) = x(x + 5)$

(v) $(2x - 1)(x - 3) = (x + 5)(x - 1)$

(vi) $x^2 + 3x + 1 = (x - 2)^2$

(vii) $(x + 2)^3 = 2x(x^2 - 1)$

(viii) $x^3 - 4x^2 - x + 1 = (x - 2)^3$

2. Represent the following situations in the form of quadratic equations :

(i) The area of a rectangular plot is 528 $m^2$. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

(ii) The product of two consecutive positive integers is 306. We need to find the integers.

(iii) Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Rohan's present age.

(iv) A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

## 4.3 Solution of a Quadratic Equation by Factorisation

Consider the quadratic equation $2x^2 - 3x + 1 = 0$. If we replace $x$ by 1 on the LHS of this equation, we get $(2 \times 1^2) - (3 \times 1) + 1 = 0$ = RHS of the equation. We say that 1 is a root of the quadratic equation $2x^2 - 3x + 1 = 0$. This also means that 1 is a zero of the quadratic polynomial $2x^2 - 3x + 1$.

In general, a real number $\alpha$ is called a root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$. We also say that **$x = \alpha$ is a solution of the quadratic equation**, or that **$\alpha$ satisfies the quadratic equation**. Note that **the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same**.

You have observed, in Chapter 2, that a quadratic polynomial can have at most two zeroes. So, any quadratic equation can have atmost two roots.

You have learnt in Class IX, how to factorise quadratic polynomials by splitting their middle terms. We shall use this knowledge for finding the roots of a quadratic equation. Let us see how.

**Example 3 :** Find the roots of the equation $2x^2 - 5x + 3 = 0$, by factorisation.

**Solution :** Let us first split the middle term $-5x$ as $-2x -3x$ [because $(-2x) \times (-3x) = 6x^2 = (2x^2) \times 3$].

So, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3 = 2x(x - 1) - 3(x - 1) = (2x - 3)(x - 1)$

Now, $2x^2 - 5x + 3 = 0$ can be rewritten as $(2x - 3)(x - 1) = 0$.

So, the values of $x$ for which $2x^2 - 5x + 3 = 0$ are the same for which $(2x - 3)(x - 1) = 0$, i.e., either $2x - 3 = 0$ or $x - 1 = 0$.

Now, $2x - 3 = 0$ gives $x = \frac{3}{2}$ and $x - 1 = 0$ gives $x = 1$.

So, $x = \frac{3}{2}$ and $x = 1$ are the solutions of the equation.

In other words, 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

**Verify** that these are the roots of the given equation.

**Note that we have found the roots of** $2x^2 - 5x + 3 = 0$ **by factorising** $2x^2 - 5x + 3$ **into two linear factors and equating each factor to zero**.

**Example 4 :** Find the roots of the quadratic equation $6x^2 - x - 2 = 0$.

**Solution :** We have

$$\begin{aligned} 6x^2 - x - 2 &= 6x^2 + 3x - 4x - 2 \\ &= 3x\,(2x + 1) - 2\,(2x + 1) \\ &= (3x - 2)(2x + 1) \end{aligned}$$

The roots of $6x^2 - x - 2 = 0$ are the values of $x$ for which $(3x - 2)(2x + 1) = 0$

Therefore, $3x - 2 = 0$ or $2x + 1 = 0$,

i.e., $$x = \frac{2}{3} \quad \text{or} \quad x = -\frac{1}{2}$$

Therefore, the roots of $6x^2 - x - 2 = 0$ are $\frac{2}{3}$ and $-\frac{1}{2}$.

We verify the roots, by checking that $\frac{2}{3}$ and $-\frac{1}{2}$ satisfy $6x^2 - x - 2 = 0$.

**Example 5 :** Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.

**Solution :** $$\begin{aligned} 3x^2 - 2\sqrt{6}x + 2 &= 3x^2 - \sqrt{6}x - \sqrt{6}x + 2 \\ &= \sqrt{3}x\left(\sqrt{3}x - \sqrt{2}\right) - \sqrt{2}\left(\sqrt{3}x - \sqrt{2}\right) \\ &= \left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) \end{aligned}$$

So, the roots of the equation are the values of $x$ for which

$$\left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) = 0$$

Now, $\sqrt{3}x - \sqrt{2} = 0$ for $x = \sqrt{\frac{2}{3}}$.

So, this root is repeated twice, one for each repeated factor $\sqrt{3}x - \sqrt{2}$.

Therefore, the roots of $3x^2 - 2\sqrt{6}x + 2 = 0$ are $\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}$.

**Example 6 :** Find the dimensions of the prayer hall discussed in Section 4.1.

**Solution :** In Section 4.1, we found that if the breadth of the hall is $x$ m, then $x$ satisfies the equation $2x^2 + x - 300 = 0$. Applying the factorisation method, we write this equation as

$$2x^2 - 24x + 25x - 300 = 0$$

$$2x(x - 12) + 25(x - 12) = 0$$

i.e.,
$$(x - 12)(2x + 25) = 0$$

So, the roots of the given equation are $x = 12$ or $x = -12.5$. Since $x$ is the breadth of the hall, it cannot be negative.

Thus, the breadth of the hall is 12 m. Its length $= 2x + 1 = 25$ m.

## EXERCISE 4.2

1. Find the roots of the following quadratic equations by factorisation:

   (i) $x^2 - 3x - 10 = 0$ (ii) $2x^2 + x - 6 = 0$

   (iii) $\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0$ (iv) $2x^2 - x + \frac{1}{8} = 0$

   (v) $100x^2 - 20x + 1 = 0$

2. Solve the problems given in Example 1.
3. Find two numbers whose sum is 27 and product is 182.
4. Find two consecutive positive integers, sum of whose squares is 365.
5. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.
6. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article.

## 4.4 Solution of a Quadratic Equation by Completing the Square

In the previous section, you have learnt one method of obtaining the roots of a quadratic equation. In this section, we shall study another method.

Consider the following situation:

The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?

To answer this, let her present age (in years) be $x$. Then the product of her ages two years ago and four years from now is $(x - 2)(x + 4)$.

Therefore, $(x - 2)(x + 4) = 2x + 1$

i.e., $x^2 + 2x - 8 = 2x + 1$

i.e., $x^2 - 9 = 0$

So, Sunita's present age satisfies the quadratic equation $x^2 - 9 = 0$.

We can write this as $x^2 = 9$. Taking square roots, we get $x = 3$ or $x = -3$. Since the age is a positive number, $x = 3$.

So, Sunita's present age is 3 years.

Now consider the quadratic equation $(x + 2)^2 - 9 = 0$. To solve it, we can write it as $(x + 2)^2 = 9$. Taking square roots, we get $x + 2 = 3$ or $x + 2 = -3$.

Therefore, $x = 1$ or $x = -5$

So, the roots of the equation $(x + 2)^2 - 9 = 0$ are 1 and $-5$.

In both the examples above, the term containing $x$ is completely inside a square, and we found the roots easily by taking the square roots. But, what happens if we are asked to solve the equation $x^2 + 4x - 5 = 0$? We would probably apply factorisation to do so, unless we realise (somehow!) that $x^2 + 4x - 5 = (x + 2)^2 - 9$.

So, solving $x^2 + 4x - 5 = 0$ is equivalent to solving $(x + 2)^2 - 9 = 0$, which we have seen is very quick to do. In fact, we can convert any quadratic equation to the form $(x + a)^2 - b^2 = 0$ and then we can easily find its roots. Let us see if this is possible. Look at Fig. 4.2.

In this figure, we can see how $x^2 + 4x$ is being converted to $(x + 2)^2 - 4$.

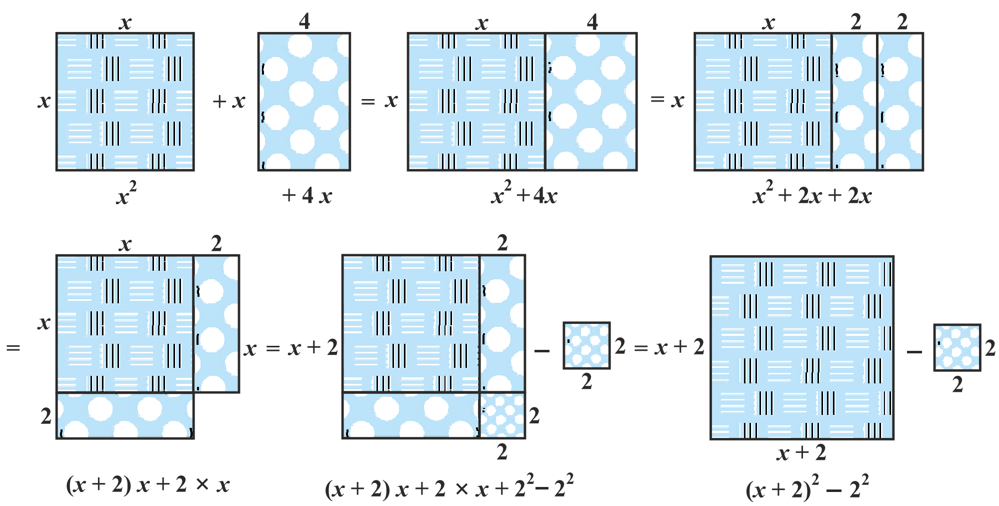



**Fig. 4.2**

The process is as follows:

$$x^2 + 4x = (x^2 + \frac{4}{2}x) + \frac{4}{2}x$$

$$= x^2 + 2x + 2x$$

$$= (x + 2)\, x + 2 \times x$$

$$= (x + 2)\, x + 2 \times x + 2 \times 2 - 2 \times 2$$

$$= (x + 2)\, x + (x + 2) \times 2 - 2 \times 2$$

$$= (x + 2)\,(x + 2) - 2^2$$

$$= (x + 2)^2 - 4$$

So, $\quad x^2 + 4x - 5 = (x + 2)^2 - 4 - 5 = (x + 2)^2 - 9$

So, $x^2 + 4x - 5 = 0$ can be written as $(x + 2)^2 - 9 = 0$ by this process of completing the square. This is known as the **method of completing the square.**

In brief, this can be shown as follows:

$$x^2 + 4x = \left(x + \frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 = \left(x + \frac{4}{2}\right)^2 - 4$$

So, $\quad x^2 + 4x - 5 = 0$ can be rewritten as

$$\left(x + \frac{4}{2}\right)^2 - 4 - 5 = 0$$

i.e., $\quad (x + 2)^2 - 9 = 0$

Consider now the equation $3x^2 - 5x + 2 = 0$. Note that the coefficient of $x^2$ is not a perfect square. So, we multiply the equation throughout by 3 to get

$$9x^2 - 15x + 6 = 0$$

Now, $\quad 9x^2 - 15x + 6 = (3x)^2 - 2 \times 3x \times \frac{5}{2} + 6$

$$= (3x)^2 - 2 \times 3x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$$

$$= \left(3x - \frac{5}{2}\right)^2 - \frac{25}{4} + 6 = \left(3x - \frac{5}{2}\right)^2 - \frac{1}{4}$$

So, $9x^2 - 15x + 6 = 0$ can be written as

$$\left(3x - \frac{5}{2}\right)^2 - \frac{1}{4} = 0$$

i.e.,
$$\left(3x - \frac{5}{2}\right)^2 = \frac{1}{4}$$

So, the solutions of $9x^2 - 15x + 6 = 0$ are the same as those of $\left(3x - \frac{5}{2}\right)^2 = \frac{1}{4}$.

i.e.,
$$3x - \frac{5}{2} = \frac{1}{2} \quad \text{or} \quad 3x - \frac{5}{2} = -\frac{1}{2}$$

(We can also write this as $3x - \frac{5}{2} = \pm \frac{1}{2}$, where '±' denotes 'plus minus'.)

Thus,
$$3x = \frac{5}{2} + \frac{1}{2} \quad \text{or} \quad 3x = \frac{5}{2} - \frac{1}{2}$$

So,
$$x = \frac{5}{6} + \frac{1}{6} \quad \text{or} \quad x = \frac{5}{6} - \frac{1}{6}$$

Therefore,
$$x = 1 \text{ or } x = \frac{4}{6}$$

i.e.,
$$x = 1 \text{ or } x = \frac{2}{3}$$

Therefore, the roots of the given equation are 1 and $\frac{2}{3}$.

**Remark :** Another way of showing this process is as follows :

The equation $3x^2 - 5x + 2 = 0$

is the same as

$$x^2 - \frac{5}{3}x + \frac{2}{3} = 0$$

Now,
$$x^2 - \frac{5}{3}x + \frac{2}{3} = \left\{x - \frac{1}{2}\left(\frac{5}{3}\right)\right\}^2 - \left\{\frac{1}{2}\left(\frac{5}{3}\right)\right\}^2 + \frac{2}{3}$$

$$= \left(x - \frac{5}{6}\right)^2 + \frac{2}{3} - \frac{25}{36}$$

$$= \left(x - \frac{5}{6}\right)^2 - \frac{1}{36} = \left(x - \frac{5}{6}\right)^2 - \left(\frac{1}{6}\right)^2$$

So, the solutions of $3x^2 - 5x + 2 = 0$ are the same as those of $\left(x - \frac{5}{6}\right)^2 - \left(\frac{1}{6}\right)^2 = 0$, which are $x - \frac{5}{6} = \pm \frac{1}{6}$, i.e., $x = \frac{5}{6} + \frac{1}{6} = 1$ and $x = \frac{5}{6} - \frac{1}{6} = \frac{2}{3}$.

Let us consider some examples to illustrate the above process.

**Example 7 :** Solve the equation given in Example 3 by the method of completing the square.

**Solution :** The equation $2x^2 - 5x + 3 = 0$ is the same as $x^2 - \frac{5}{2}x + \frac{3}{2} = 0$.

Now, $$x^2 - \frac{5}{2}x + \frac{3}{2} = \left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + \frac{3}{2} = \left(x - \frac{5}{4}\right)^2 - \frac{1}{16}$$

Therefore, $2x^2 - 5x + 3 = 0$ can be written as $\left(x - \frac{5}{4}\right)^2 - \frac{1}{16} = 0$.

So, the roots of the equation $2x^2 - 5x + 3 = 0$ are exactly the same as those of $\left(x - \frac{5}{4}\right)^2 - \frac{1}{16} = 0$. Now, $\left(x - \frac{5}{4}\right)^2 - \frac{1}{16} = 0$ is the same as $\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$

Therefore, $$x - \frac{5}{4} = \pm \frac{1}{4}$$

i.e., $$x = \frac{5}{4} \pm \frac{1}{4}$$

i.e., $$x = \frac{5}{4} + \frac{1}{4} \text{ or } x = \frac{5}{4} - \frac{1}{4}$$

i.e., $$x = \frac{3}{2} \text{ or } x = 1$$

Therefore, the solutions of the equations are $x = \frac{3}{2}$ and 1.

Let us **verify** our solutions.

Putting $x = \frac{3}{2}$ in $2x^2 - 5x + 3 = 0$, we get $2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 = 0$, which is correct. Similarly, you can verify that $x = 1$ also satisfies the given equation.

In Example 7, we divided the equation $2x^2 - 5x + 3 = 0$ throughout by 2 to get $x^2 - \frac{5}{2}x + \frac{3}{2} = 0$ to make the first term a perfect square and then completed the square. Instead, we can multiply throughout by 2 to make the first term as $4x^2 = (2x)^2$ and then complete the square.

This method is illustrated in the next example.

**Example 8 :** Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.

**Solution :** Multiplying the equation throughout by 5, we get

$$25x^2 - 30x - 10 = 0$$

This is the same as

$$(5x)^2 - 2 \times (5x) \times 3 + 3^2 - 3^2 - 10 = 0$$

i.e., $$(5x - 3)^2 - 9 - 10 = 0$$

i.e., $$(5x - 3)^2 - 19 = 0$$

i.e., $$(5x - 3)^2 = 19$$

i.e., $$5x - 3 = \pm\sqrt{19}$$

i.e., $$5x = 3 \pm \sqrt{19}$$

So, $$x = \frac{3 \pm \sqrt{19}}{5}$$

Therefore, the roots are $\frac{3 + \sqrt{19}}{5}$ and $\frac{3 - \sqrt{19}}{5}$.

Verify that the roots are $\frac{3 + \sqrt{19}}{5}$ and $\frac{3 - \sqrt{19}}{5}$.

**Example 9 :** Find the roots of $4x^2 + 3x + 5 = 0$ by the method of completing the square.

**Solution :** Note that $4x^2 + 3x + 5 = 0$ is the same as

$$(2x)^2 + 2 \times (2x) \times \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + 5 = 0$$

i.e.,
$$\left(2x + \frac{3}{4}\right)^2 - \frac{9}{16} + 5 = 0$$

i.e.,
$$\left(2x + \frac{3}{4}\right)^2 + \frac{71}{16} = 0$$

i.e.,
$$\left(2x + \frac{3}{4}\right)^2 = \frac{-71}{6} < 0$$

But $\left(2x + \frac{3}{4}\right)^2$ cannot be negative for any real value of $x$ (Why?). So, there is no real value of $x$ satisfying the given equation. Therefore, the given equation has no *real roots*.

Now, you have seen several examples of the use of the method of completing the square. So, let us give this method in general.

Consider the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$. Dividing throughout by $a$, we get
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

This is the same as
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

i.e.,
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

So, the roots of the given equation are the same as those of

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0, \text{ i.e., those of } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad (1)$$

If $b^2 - 4ac \geq 0$, then by taking the square roots in (1), we get

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Therefore,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, the roots of $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$, if $b^2 - 4ac \geq 0$. If $b^2 - 4ac < 0$, the equation will have no real roots. (Why?)

**Thus, if $b^2 - 4ac \geq 0$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$**

This formula for finding the roots of a quadratic equation is known as the **quadratic formula**.

Let us consider some examples for illustrating the use of the quadratic formula.

**Example 10 :** Solve Q. 2(i) of Exercise 4.1 by using the quadratic formula.

**Solution :** Let the breadth of the plot be $x$ metres. Then the length is $(2x + 1)$ metres. Then we are given that $x(2x + 1) = 528$, i.e., $2x^2 + x - 528 = 0$.

This is of the form $ax^2 + bx + c = 0$, where $a = 2$, $b = 1$, $c = -528$.

So, the quadratic formula gives us the solution as

$$x = \frac{-1 \pm \sqrt{1 + 4(2)(528)}}{4} = \frac{-1 \pm \sqrt{4225}}{4} = \frac{-1 \pm 65}{4}$$

i.e.,
$$x = \frac{64}{4} \text{ or } x = \frac{-66}{4}$$

i.e.,
$$x = 16 \quad \text{or} \quad x = -\frac{33}{2}$$

Since $x$ cannot be negative, being a dimension, the breadth of the plot is 16 metres and hence, the length of the plot is 33m.

You should verify that these values satisfy the conditions of the problem.

**Example 11 :** Find two consecutive odd positive integers, sum of whose squares is 290.

**Solution :** Let the smaller of the two consecutive odd positive integers be $x$. Then, the second integer will be $x + 2$. According to the question,

$$x^2 + (x + 2)^2 = 290$$

i.e., $$x^2 + x^2 + 4x + 4 = 290$$

i.e., $$2x^2 + 4x - 286 = 0$$

i.e., $$x^2 + 2x - 143 = 0$$

which is a quadratic equation in $x$.

Using the quadratic formula, we get

$$x = \frac{-2 \pm \sqrt{4 + 572}}{2} = \frac{-2 \pm \sqrt{576}}{2} = \frac{-2 \pm 24}{2}$$

i.e., $$x = 11 \quad \text{or} \quad x = -13$$

But $x$ is given to be an odd positive integer. Therefore, $x \neq -13$, $x = 11$.

Thus, the two consecutive odd integers are 11 and 13.

Check : $11^2 + 13^2 = 121 + 169 = 290$.

**Example 12 :** A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (see Fig. 4.3). Find its length and breadth.

**Solution :** Let the breadth of the rectangular park be $x$ m.

So, its length = $(x + 3)$ m.

Therefore, the area of the rectangular park = $x(x + 3)$ m$^2$ = $(x^2 + 3x)$ m$^2$.

Now, base of the isosceles triangle = $x$ m.

Therefore, its area = $\frac{1}{2} \times x \times 12 = 6x$ m$^2$.

According to our requirements,

$$x^2 + 3x = 6x + 4$$

i.e., $$x^2 - 3x - 4 = 0$$

Using the quadratic formula, we get

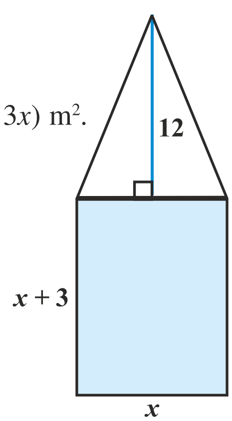



Fig. 4.3

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = 4 \text{ or } -1$$

But $x \neq -1$ (Why?). Therefore, $x = 4$.

So, the breadth of the park = 4m and its length will be 7m.

**Verification :** Area of rectangular park = $28 \text{ m}^2$,

area of triangular park = $24 \text{ m}^2 = (28 - 4) \text{ m}^2$

**Example 13 :** Find the roots of the following quadratic equations, if they exist, using the quadratic formula:

(i) $3x^2 - 5x + 2 = 0$ (ii) $x^2 + 4x + 5 = 0$ (iii) $2x^2 - 2\sqrt{2}\,x + 1 = 0$

**Solution :**

(i) $3x^2 - 5x + 2 = 0$. Here, $a = 3$, $b = -5$, $c = 2$. So, $b^2 - 4ac = 25 - 24 = 1 > 0$.

Therefore, $x = \frac{5 \pm \sqrt{1}}{6} = \frac{5 \pm 1}{6}$, i.e., $x = 1$ or $x = \frac{2}{3}$

So, the roots are $\frac{2}{3}$ and 1.

(ii) $x^2 + 4x + 5 = 0$. Here, $a = 1$, $b = 4$, $c = 5$. So, $b^2 - 4ac = 16 - 20 = -4 < 0$.

Since the square of a real number cannot be negative, therefore $\sqrt{b^2 - 4ac}$ will not have any real value.

So, there are no real roots for the given equation.

(iii) $2x^2 - 2\sqrt{2}\,x + 1 = 0$. Here, $a = 2$, $b = -2\sqrt{2}$, $c = 1$.

So, $b^2 - 4ac = 8 - 8 = 0$

Therefore, $x = \frac{2\sqrt{2} \pm \sqrt{0}}{4} = \frac{\sqrt{2}}{2} \pm 0$, i.e., $x = \frac{1}{\sqrt{2}}$.

So, the roots are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

**Example 14 :** Find the roots of the following equations:

(i) $x + \frac{1}{x} = 3, x \neq 0$ (ii) $\frac{1}{x} - \frac{1}{x-2} = 3, x \neq 0, 2$

**Solution :**

(i) $x + \frac{1}{x} = 3$. Multiplying throughout by $x$, we get

$$x^2 + 1 = 3x$$

i.e., $x^2 - 3x + 1 = 0$, which is a quadratic equation.

Here, $a = 1, b = -3, c = 1$

So, $b^2 - 4ac = 9 - 4 = 5 > 0$

Therefore, $x = \frac{3 \pm \sqrt{5}}{2}$ (Why?)

So, the roots are $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$.

(ii) $\frac{1}{x} - \frac{1}{x-2} = 3, x \neq 0, 2$.

As $x \neq 0, 2$, multiplying the equation by $x(x-2)$, we get

$$(x-2) - x = 3x(x-2)$$

$$= 3x^2 - 6x$$

So, the given equation reduces to $3x^2 - 6x + 2 = 0$, which is a quadratic equation.

Here, $a = 3, b = -6, c = 2$. So, $b^2 - 4ac = 36 - 24 = 12 > 0$

Therefore, $x = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6} = \frac{3 \pm \sqrt{3}}{3}$.

So, the roots are $\frac{3+\sqrt{3}}{3}$ and $\frac{3-\sqrt{3}}{3}$.

**Example 15 :** A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

**Solution :** Let the speed of the stream be $x$ km/h.

Therefore, the speed of the boat upstream = $(18 - x)$ km/h and the speed of the boat downstream = $(18 + x)$ km/h.

The time taken to go upstream = $\dfrac{\text{distance}}{\text{speed}} = \dfrac{24}{18-x}$ hours.

Similarly, the time taken to go downstream = $\dfrac{24}{18+x}$ hours.

According to the question,

$$\frac{24}{18-x} - \frac{24}{18+x} = 1$$

i.e., $$24(18 + x) - 24(18 - x) = (18 - x)(18 + x)$$

i.e., $$x^2 + 48x - 324 = 0$$

Using the quadratic formula, we get

$$x = \frac{-48 \pm \sqrt{48^2 + 1296}}{2} = \frac{-48 \pm \sqrt{3600}}{2}$$

$$= \frac{-48 \pm 60}{2} = 6 \text{ or } -54$$

Since $x$ is the speed of the stream, it cannot be negative. So, we ignore the root $x = -54$. Therefore, $x = 6$ gives the speed of the stream as 6 km/h.

## EXERCISE 4.3

**1.** Find the roots of the following quadratic equations, if they exist, by the method of completing the square:

(i) $2x^2 - 7x + 3 = 0$ (ii) $2x^2 + x - 4 = 0$

(iii) $4x^2 + 4\sqrt{3}x + 3 = 0$ (iv) $2x^2 + x + 4 = 0$

**2.** Find the roots of the quadratic equations given in Q.1 above by applying the quadratic formula.

3. Find the roots of the following equations:

(i) $x - \frac{1}{x} = 3, x \neq 0$ (ii) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq -4, 7$

4. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

5. In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

6. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

7. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

8. A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

10. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11km/h more than that of the passenger train, find the average speed of the two trains.

11. Sum of the areas of two squares is 468 m$^2$. If the difference of their perimeters is 24 m, find the sides of the two squares.

## 4.5 Nature of Roots

In the previous section, you have seen that the roots of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$, we get two distinct real roots $-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac = 0$, then $x = -\frac{b}{2a} \pm 0$, i.e., $x = -\frac{b}{2a}$ or $-\frac{b}{2a}$.

So, the roots of the equation $ax^2 + bx + c = 0$ are both $\frac{-b}{2a}$.

Therefore, we say that the quadratic equation $ax^2 + bx + c = 0$ has two equal real roots in this case.

If $b^2 - 4ac < 0$, then there is no real number whose square is $b^2 - 4ac$. Therefore, there are no real roots for the given quadratic equation in this case.

Since $b^2 - 4ac$ determines whether the quadratic equation $ax^2 + bx + c = 0$ has real roots or not, $b^2 - 4ac$ is called the **discriminant** of this quadratic equation.

So, a quadratic equation $ax^2 + bx + c = 0$ has

**(i) two distinct real roots, if $b^2 - 4ac > 0$,**

**(ii) two equal real roots, if $b^2 - 4ac = 0$,**

**(iii) no real roots, if $b^2 - 4ac < 0$.**

Let us consider some examples.

**Example 16 :** Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

**Solution :** The given equation is of the form $ax^2 + bx + c = 0$, where $a = 2$, $b = -4$ and $c = 3$. Therefore, the discriminant

$$b^2 - 4ac = (-4)^2 - (4 \times 2 \times 3) = 16 - 24 = -8 < 0$$

So, the given equation has no real roots.

**Example 17 :** A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

**Solution :** Let us first draw the diagram (see Fig. 4.4).

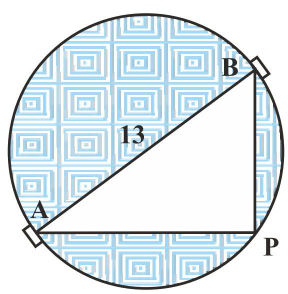



Fig. 4.4

Let P be the required location of the pole. Let the distance of the pole from the gate B be $x$ m, i.e., BP = $x$ m. Now the difference of the distances of the pole from the two gates = AP – BP (or, BP – AP) = 7 m. Therefore, AP = $(x + 7)$ m.

Now, AB = 13m, and since AB is a diameter,

$$\angle APB = 90° \quad \text{(Why?)}$$

Therefore, $\quad AP^2 + PB^2 = AB^2 \quad$ (By Pythagoras theorem)

i.e., $\quad (x + 7)^2 + x^2 = 13^2$

i.e., $\quad x^2 + 14x + 49 + x^2 = 169$

i.e., $\quad 2x^2 + 14x - 120 = 0$

So, the distance '$x$' of the pole from gate B satisfies the equation

$$x^2 + 7x - 60 = 0$$

So, it would be possible to place the pole if this equation has real roots. To see if this is so or not, let us consider its discriminant. The discriminant is

$$b^2 - 4ac = 7^2 - 4 \times 1 \times (-60) = 289 > 0.$$

So, the given quadratic equation has two real roots, and it is possible to erect the pole on the boundary of the park.

Solving the quadratic equation $x^2 + 7x - 60 = 0$, by the quadratic formula, we get

$$x = \frac{-7 \pm \sqrt{289}}{2} = \frac{-7 \pm 17}{2}$$

Therefore, $x = 5$ or $-12$.

Since $x$ is the distance between the pole and the gate B, it must be positive. Therefore, $x = -12$ will have to be ignored. So, $x = 5$.

Thus, the pole has to be erected on the boundary of the park at a distance of 5m from the gate B and 12m from the gate A.

**Example 18 :** Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

**Solution :** Here $a = 3$, $b = -2$ and $c = \frac{1}{3}$.

Therefore, discriminant $b^2 - 4ac = (-2)^2 - 4 \times 3 \times \frac{1}{3} = 4 - 4 = 0.$

Hence, the given quadratic equation has two equal real roots.

The roots are $\frac{-b}{2a}, \frac{-b}{2a}$, i.e., $\frac{2}{6}, \frac{2}{6}$, i.e., $\frac{1}{3}, \frac{1}{3}$.

## EXERCISE 4.4

**1.** Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:

(i) $2x^2 - 3x + 5 = 0$

(ii) $3x^2 - 4\sqrt{3}\,x + 4 = 0$

(iii) $2x^2 - 6x + 3 = 0$

**2.** Find the values of $k$ for each of the following quadratic equations, so that they have two equal roots.

(i) $2x^2 + kx + 3 = 0$

(ii) $kx\,(x - 2) + 6 = 0$

**3.** Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 $m^2$? If so, find its length and breadth.

**4.** Is the following situation possible? If so, determine their present ages.
The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48.

**5.** Is it possible to design a rectangular park of perimeter 80 m and area 400 $m^2$? If so, find its length and breadth.

## 4.6 Summary

In this chapter, you have studied the following points:

**1.** A quadratic equation in the variable $x$ is of the form $ax^2 + bx + c = 0$, where $a, b, c$ are real numbers and $a \neq 0$.

**2.** A real number $\alpha$ is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

**3.** If we can factorise $ax^2 + bx + c$, $a \neq 0$, into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.

**4.** A quadratic equation can also be solved by the method of completing the square.

**5.** Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided } b^2 - 4ac \geq 0.$$

**6.** A quadratic equation $ax^2 + bx + c = 0$ has

(i) two distinct real roots, if $b^2 - 4ac > 0$,

(ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

(iii) no real roots, if $b^2 - 4ac < 0$.

## A Note to the Reader

In case of word problems, the obtained solutions should always be verified with the conditions of the original problem and not in the equations formed (see Examples 11, 13, 19 of Chapter 3 and Examples 10, 11, 12 of Chapter 4).