

Exercise 8.4

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1. Express the trigonometric ratios $\sin A$, $\sec A$ and $\tan A$ in terms of $\cot A$.

Solution:

To convert the given trigonometric ratios in terms of \cot functions, use trigonometric formulas

We know that,

$$\operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

Since cosec function is the inverse of \sin function, it is written as

$$1/\sin^2 A = 1 + \cot^2 A$$

Now, rearrange the terms, it becomes

$$\sin^2 A = 1/(1 + \cot^2 A)$$

Now, take square roots on both sides, we get

$$\sin A = \pm 1/(\sqrt{1 + \cot^2 A})$$

The above equation defines the \sin function in terms of \cot function

Now, to express \sec function in terms of \cot function, use this formula

$$\sin^2 A = 1/(1 + \cot^2 A)$$

Now, represent the \sin function as \cos function

$$1 - \cos^2 A = 1/(1 + \cot^2 A)$$

Rearrange the terms,

$$\cos^2 A = 1 - 1/(1 + \cot^2 A)$$

$$\Rightarrow \cos^2 A = (1 - 1 + \cot^2 A)/(1 + \cot^2 A)$$

Since \sec function is the inverse of \cos function,

$$\Rightarrow 1/\sec^2 A = \cot^2 A/(1 + \cot^2 A)$$

Take the reciprocal and square roots on both sides, we get

$$\Rightarrow \sec A = \pm \sqrt{(1 + \cot^2 A)/\cot A}$$

Now, to express \tan function in terms of \cot function

$$\tan A = \sin A/\cos A \text{ and } \cot A = \cos A/\sin A$$

Since \cot function is the inverse of \tan function, it is rewritten as

$$\tan A = 1/\cot A$$

2. Write all the other trigonometric ratios of $\angle A$ in terms of $\sec A$.

Solution:

Cos A function in terms of sec A:

$$\sec A = 1/\cos A$$

$$\Rightarrow \cos A = 1/\sec A$$

sec A function in terms of sec A:

$$\cos^2 A + \sin^2 A = 1$$

Rearrange the terms

$$\sin^2 A = 1 - \cos^2 A$$

$$\sin^2 A = 1 - (1/\sec^2 A)$$

$$\sin^2 A = (\sec^2 A - 1)/\sec^2 A$$

$$\sin A = \pm \sqrt{(\sec^2 A - 1)/\sec A}$$

cosec A function in terms of sec A:

$$\sin A = 1/\text{cosec } A$$

$$\Rightarrow \text{cosec } A = 1/\sin A$$

$$\text{cosec } A = \pm \sec A/\sqrt{(\sec^2 A - 1)}$$

Now, tan A function in terms of sec A:

$$\sec^2 A - \tan^2 A = 1$$

Rearrange the terms

$$\Rightarrow \tan^2 A = \sec^2 A - 1$$

$$\tan A = \sqrt{(\sec^2 A - 1)}$$

cot A function in terms of sec A:

$$\tan A = 1/\cot A$$

$$\Rightarrow \cot A = 1/\tan A$$

$$\cot A = \pm 1/\sqrt{(\sec^2 A - 1)}$$

3. Evaluate:

(i) $(\sin^2 63^\circ + \sin^2 27^\circ)/(\cos^2 17^\circ + \cos^2 73^\circ)$

(ii) $\sin 25^\circ \cos 65^\circ + \cos 25^\circ \sin 65^\circ$

Solution:

(i) $(\sin^2 63^\circ + \sin^2 27^\circ)/(\cos^2 17^\circ + \cos^2 73^\circ)$

To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,

$$= [\sin^2(90^\circ - 27^\circ) + \sin^2 27^\circ] / [\cos^2(90^\circ - 73^\circ) + \cos^2 73^\circ]$$

$$= (\cos^2 27^\circ + \sin^2 27^\circ)/(\sin^2 27^\circ + \cos^2 73^\circ)$$

$$= 1/1 = 1 \quad (\text{since } \sin^2 A + \cos^2 A = 1)$$

Therefore, $(\sin^2 63^\circ + \sin^2 27^\circ) / (\cos^2 17^\circ + \cos^2 73^\circ) = 1$

(ii) $\sin 25^\circ \cos 65^\circ + \cos 25^\circ \sin 65^\circ$

To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,

$$\begin{aligned} &= \sin(90^\circ - 25^\circ) \cos 65^\circ + \cos(90^\circ - 65^\circ) \sin 65^\circ \\ &= \cos 65^\circ \cos 65^\circ + \sin 65^\circ \sin 65^\circ \\ &= \cos^2 65^\circ + \sin^2 65^\circ = 1 \quad (\text{since } \sin^2 A + \cos^2 A = 1) \end{aligned}$$

Therefore, $\sin 25^\circ \cos 65^\circ + \cos 25^\circ \sin 65^\circ = 1$

4. Choose the correct option. Justify your choice.

(i) $9 \sec^2 A - 9 \tan^2 A =$

- (A) 1 (B) 9 (C) 8 (D) 0

(ii) $(1 + \tan \theta + \sec \theta)(1 + \cot \theta - \operatorname{cosec} \theta)$

- (A) 0 (B) 1 (C) 2 (D) - 1

(iii) $(\sec A + \tan A)(1 - \sin A) =$

- (A) $\sec A$ (B) $\sin A$ (C) $\operatorname{cosec} A$ (D) $\cos A$

(iv) $1 + \tan^2 A / 1 + \cot^2 A =$

- (A) $\sec^2 A$ (B) -1 (C) $\cot^2 A$ (D) $\tan^2 A$

Solution:

(i) (B) is correct.

Justification:

Take 9 outside, and it becomes

$$\begin{aligned} &9 \sec^2 A - 9 \tan^2 A \\ &= 9 (\sec^2 A - \tan^2 A) \\ &= 9 \times 1 = 9 \quad (\because \sec^2 A - \tan^2 A = 1) \end{aligned}$$

Therefore, $9 \sec^2 A - 9 \tan^2 A = 9$

(ii) (C) is correct

Justification:

$$(1 + \tan \theta + \sec \theta)(1 + \cot \theta - \operatorname{cosec} \theta)$$

We know that, $\tan \theta = \sin \theta / \cos \theta$

$$\sec \theta = 1 / \cos \theta$$

$$\cot \theta = \cos \theta / \sin \theta$$

$$\operatorname{cosec} \theta = 1 / \sin \theta$$

Now, substitute the above values in the given problem, we get

$$= (1 + \sin \theta / \cos \theta + 1 / \cos \theta)(1 + \cos \theta / \sin \theta - 1 / \sin \theta)$$

Simplify the above equation,

$$= (\cos \theta + \sin \theta + 1) / \cos \theta \times (\sin \theta + \cos \theta - 1) / \sin \theta$$

$$= (\cos \theta + \sin \theta)^2 - 1^2 / (\cos \theta \sin \theta)$$

$$= (\cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta - 1) / (\cos \theta \sin \theta)$$

$$= (1 + 2\cos \theta \sin \theta - 1) / (\cos \theta \sin \theta) \text{ (Since } \cos^2 \theta + \sin^2 \theta = 1)$$

$$= (2\cos \theta \sin \theta) / (\cos \theta \sin \theta) = 2$$

$$\text{Therefore, } (1 + \tan \theta + \sec \theta) (1 + \cot \theta - \operatorname{cosec} \theta) = 2$$

(iii) (D) is correct.

Justification:

We know that,

$$\sec A = 1/\cos A$$

$$\tan A = \sin A / \cos A$$

Now, substitute the above values in the given problem, we get

$$(\sec A + \tan A) (1 - \sin A)$$

$$= (1/\cos A + \sin A/\cos A) (1 - \sin A)$$

$$= (1 + \sin A/\cos A) (1 - \sin A)$$

$$= (1 - \sin^2 A) / \cos A$$

$$= \cos^2 A / \cos A = \cos A$$

$$\text{Therefore, } (\sec A + \tan A) (1 - \sin A) = \cos A$$

(iv) (D) is correct.

Justification:

We know that,

$$\tan^2 A = 1/\cot^2 A$$

Now, substitute this in the given problem, we get

$$1 + \tan^2 A / 1 + \cot^2 A$$

$$= (1 + 1/\cot^2 A) / 1 + \cot^2 A$$

$$= (\cot^2 A + 1/\cot^2 A) \times (1/1 + \cot^2 A)$$

$$= 1/\cot^2 A = \tan^2 A$$

$$\text{So, } 1 + \tan^2 A / 1 + \cot^2 A = \tan^2 A$$

5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\operatorname{cosec} \theta - \cot \theta)^2 = (1 - \cos \theta) / (1 + \cos \theta)$

(ii) $\cos A / (1 + \sin A) + (1 + \sin A) / \cos A = 2 \sec A$

(iii) $\tan \theta / (1 - \cot \theta) + \cot \theta / (1 - \tan \theta) = 1 + \sec \theta \operatorname{cosec} \theta$

[Hint : Write the expression in terms of $\sin \theta$ and $\cos \theta$]

(iv) $(1 + \sec A) / \sec A = \sin^2 A / (1 - \cos A)$

[Hint : Simplify LHS and RHS separately]

(v) $(\cos A - \sin A + 1) / (\cos A + \sin A - 1) = \operatorname{cosec} A + \cot A$, using the identity $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

(vi) $\sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$

$$(vii) (\sin \theta - 2\sin^3\theta)/(2\cos^3\theta - \cos \theta) = \tan \theta$$

$$(viii) (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$$

$$(ix) (\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1/(\tan A + \cot A)$$

[Hint : Simplify LHS and RHS separately]

$$(x) (1 + \tan^2 A / 1 + \cot^2 A) = (1 - \tan A / 1 - \cot A)^2 = \tan^2 A$$

Solution:

$$(i) (\operatorname{cosec} \theta - \cot \theta)^2 = (1 - \cos \theta)/(1 + \cos \theta)$$

To prove this, first take the Left-Hand side (L.H.S) of the given equation, to prove the Right Hand Side (R.H.S)

$$\text{L.H.S.} = (\operatorname{cosec} \theta - \cot \theta)^2$$

The above equation is in the form of $(a-b)^2$, and expand it

$$\text{Since } (a-b)^2 = a^2 + b^2 - 2ab$$

Here $a = \operatorname{cosec} \theta$ and $b = \cot \theta$

$$= (\operatorname{cosec}^2 \theta + \cot^2 \theta - 2\operatorname{cosec} \theta \cot \theta)$$

Now, apply the corresponding inverse functions and equivalent ratios to simplify

$$= (1/\sin^2 \theta + \cos^2 \theta / \sin^2 \theta - 2\cos \theta / \sin^2 \theta)$$

$$= (1 + \cos^2 \theta - 2\cos \theta)/(1 - \cos^2 \theta)$$

$$= (1 - \cos \theta)^2 / (1 - \cos \theta)(1 + \cos \theta)$$

$$= (1 - \cos \theta)/(1 + \cos \theta) = \text{R.H.S.}$$

$$\text{Therefore, } (\operatorname{cosec} \theta - \cot \theta)^2 = (1 - \cos \theta)/(1 + \cos \theta)$$

Hence proved.

$$(ii) (\cos A/(1 + \sin A)) + ((1 + \sin A)/\cos A) = 2 \sec A$$

Now, take the L.H.S of the given equation.

$$\text{L.H.S.} = (\cos A/(1 + \sin A)) + ((1 + \sin A)/\cos A)$$

$$= [\cos^2 A + (1 + \sin A)^2] / (1 + \sin A) \cos A$$

$$= (\cos^2 A + \sin^2 A + 1 + 2\sin A) / (1 + \sin A) \cos A$$

Since $\cos^2 A + \sin^2 A = 1$, we can write it as

$$= (1 + 1 + 2\sin A) / (1 + \sin A) \cos A$$

$$= (2 + 2\sin A) / (1 + \sin A) \cos A$$

$$= 2(1 + \sin A) / (1 + \sin A) \cos A$$

$$= 2/\cos A = 2 \sec A = \text{R.H.S.}$$

L.H.S. = R.H.S.

$$(\cos A/(1 + \sin A)) + ((1 + \sin A)/\cos A) = 2 \sec A$$

Hence proved.

$$(iii) \tan \theta / (1 - \cot \theta) + \cot \theta / (1 - \tan \theta) = 1 + \sec \theta \operatorname{cosec} \theta$$

$$\text{L.H.S.} = \tan \theta / (1 - \cot \theta) + \cot \theta / (1 - \tan \theta)$$

$$\text{We know that } \tan \theta = \sin \theta / \cos \theta$$

$$\cot \theta = \cos \theta / \sin \theta$$

Now, substitute it in the given equation, to convert it in a simplified form

$$\begin{aligned} &= [(\sin \theta / \cos \theta) / (1 - (\cos \theta / \sin \theta))] + [(\cos \theta / \sin \theta) / (1 - (\sin \theta / \cos \theta))] \\ &= [(\sin \theta / \cos \theta) / ((\sin \theta - \cos \theta) / \sin \theta)] + [(\cos \theta / \sin \theta) / ((\cos \theta - \sin \theta) / \cos \theta)] \\ &= \sin^2 \theta / [\cos \theta (\sin \theta - \cos \theta)] + \cos^2 \theta / [\sin \theta (\cos \theta - \sin \theta)] \\ &= \sin^2 \theta / [\cos \theta (\sin \theta - \cos \theta)] - \cos^2 \theta / [\sin \theta (\sin \theta - \cos \theta)] \\ &= 1 / (\sin \theta - \cos \theta) [(\sin^2 \theta / \cos \theta) - (\cos^2 \theta / \sin \theta)] \\ &= 1 / (\sin \theta - \cos \theta) \times [(\sin^3 \theta - \cos^3 \theta) / \sin \theta \cos \theta] \\ &= [(\sin \theta - \cos \theta)(\sin^2 \theta + \cos^2 \theta + \sin \theta \cos \theta)] / [(\sin \theta - \cos \theta) \sin \theta \cos \theta] \\ &= (1 + \sin \theta \cos \theta) / \sin \theta \cos \theta \\ &= 1 / \sin \theta \cos \theta + 1 \\ &= 1 + \sec \theta \operatorname{cosec} \theta = \text{R.H.S.} \end{aligned}$$

Therefore, L.H.S. = R.H.S.

Hence proved

$$(iv) (1 + \sec A) / \sec A = \sin^2 A / (1 - \cos A)$$

First find the simplified form of L.H.S

$$\text{L.H.S.} = (1 + \sec A) / \sec A$$

Since secant function is the inverse function of cos function and it is written as

$$\begin{aligned} &= (1 + 1 / \cos A) / (1 / \cos A) \\ &= (\cos A + 1) / \cos A \times \cos A \end{aligned}$$

$$\text{Therefore, } (1 + \sec A) / \sec A = \cos A + 1$$

$$\text{R.H.S.} = \sin^2 A / (1 - \cos A)$$

We know that $\sin^2 A = (1 - \cos^2 A)$, we get

$$\begin{aligned} &= (1 - \cos^2 A) / (1 - \cos A) \\ &= (1 - \cos A)(1 + \cos A) / (1 - \cos A) \end{aligned}$$

$$\text{Therefore, } \sin^2 A / (1 - \cos A) = \cos A + 1$$

L.H.S. = R.H.S.

Hence proved

$$(v) (\cos A - \sin A + 1) / (\cos A + \sin A - 1) = \operatorname{cosec} A + \cot A, \text{ using the identity } \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

With the help of identity function, $\operatorname{cosec}^2 A = 1 + \cot^2 A$, let us prove the above equation.

$$\text{L.H.S.} = (\cos A - \sin A + 1) / (\cos A + \sin A - 1)$$

Divide the numerator and denominator by $\sin A$, we get

$$= (\cos A - \sin A + 1) / \sin A / (\cos A + \sin A - 1) / \sin A$$

We know that $\cos A / \sin A = \cot A$ and $1 / \sin A = \operatorname{cosec} A$

$$= (\cot A - 1 + \operatorname{cosec} A) / (\cot A + 1 - \operatorname{cosec} A)$$

$$= (\cot A - \operatorname{cosec}^2 A + \cot^2 A + \operatorname{cosec} A) / (\cot A + 1 - \operatorname{cosec} A) \text{ (using } \operatorname{cosec}^2 A - \cot^2 A = 1)$$

$$= [(\cot A + \operatorname{cosec} A) - (\operatorname{cosec}^2 A - \cot^2 A)] / (\cot A + 1 - \operatorname{cosec} A)$$

$$= [(\cot A + \operatorname{cosec} A) - (\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)] / (1 - \operatorname{cosec} A + \cot A)$$

$$= (\cot A + \operatorname{cosec} A)(1 - \operatorname{cosec} A + \cot A) / (1 - \operatorname{cosec} A + \cot A)$$

$$= \cot A + \operatorname{cosec} A = \text{R.H.S.}$$

Therefore, $(\cos A - \sin A + 1) / (\cos A + \sin A - 1) = \operatorname{cosec} A + \cot A$

Hence Proved

$$(vi) \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$$

$$\text{L.H.S.} = \sqrt{\frac{1 + \sin A}{1 - \sin A}}$$

First divide the numerator and denominator of L.H.S. by $\cos A$,

$$= \sqrt{\frac{\frac{1}{\cos A} + \frac{\sin A}{\cos A}}{\frac{1}{\cos A} - \frac{\sin A}{\cos A}}}$$

We know that $1 / \cos A = \sec A$ and $\sin A / \cos A = \tan A$ and it becomes,

$$= \sqrt{(\sec A + \tan A) / (\sec A - \tan A)}$$

Now using rationalization, we get

$$= \sqrt{\frac{\sec A + \tan A}{\sec A - \tan A}} \times \sqrt{\frac{\sec A + \tan A}{\sec A + \tan A}}$$

$$= \sqrt{\frac{(\sec A + \tan A)^2}{\sec^2 A - \tan^2 A}}$$

$$= (\sec A + \tan A) / 1$$

$$= \sec A + \tan A = \text{R.H.S.}$$

Hence proved

$$(vii) (\sin \theta - 2\sin^3\theta)/(2\cos^3\theta - \cos \theta) = \tan \theta$$

$$\text{L.H.S.} = (\sin \theta - 2\sin^3\theta)/(2\cos^3\theta - \cos \theta)$$

Take $\sin \theta$ as in numerator and $\cos \theta$ in denominator as outside, it becomes

$$= [\sin \theta(1 - 2\sin^2\theta)]/[\cos \theta(2\cos^2\theta - 1)]$$

We know that $\sin^2\theta = 1 - \cos^2\theta$

$$= \sin \theta[1 - 2(1 - \cos^2\theta)]/[\cos \theta(2\cos^2\theta - 1)]$$

$$= [\sin \theta(2\cos^2\theta - 1)]/[\cos \theta(2\cos^2\theta - 1)]$$

$$= \tan \theta = \text{R.H.S.}$$

Hence proved

$$(viii) (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$$

$$\text{L.H.S.} = (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$$

It is of the form $(a+b)^2$, expand it

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$= (\sin^2 A + \operatorname{cosec}^2 A + 2 \sin A \operatorname{cosec} A) + (\cos^2 A + \sec^2 A + 2 \cos A \sec A)$$

$$= (\sin^2 A + \cos^2 A) + 2 \sin A(1/\sin A) + 2 \cos A(1/\cos A) + 1 + \tan^2 A + 1 + \cot^2 A$$

$$= 1 + 2 + 2 + 2 + \tan^2 A + \cot^2 A$$

$$= 7 + \tan^2 A + \cot^2 A = \text{R.H.S.}$$

Therefore, $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

Hence proved.

$$(ix) (\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1/(\tan A + \cot A)$$

First, find the simplified form of L.H.S

$$\text{L.H.S.} = (\operatorname{cosec} A - \sin A)(\sec A - \cos A)$$

Now, substitute the inverse and equivalent trigonometric ratio forms

$$= (1/\sin A - \sin A)(1/\cos A - \cos A)$$

$$= [(1 - \sin^2 A)/\sin A][(1 - \cos^2 A)/\cos A]$$

$$= (\cos^2 A/\sin A) \times (\sin^2 A/\cos A)$$

$$= \cos A \sin A$$

Now, simplify the R.H.S

$$\text{R.H.S.} = 1/(\tan A + \cot A)$$

$$= 1/(\sin A/\cos A + \cos A/\sin A)$$

$$= 1/[(\sin^2 A + \cos^2 A)/\sin A \cos A]$$

$$= \cos A \sin A$$

L.H.S. = R.H.S.

$$(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1/(\tan A + \cot A)$$

Hence proved

$$(x) (1+\tan^2 A/1+\cot^2 A) = (1-\tan A/1-\cot A)^2 = \tan^2 A$$

$$\text{L.H.S.} = (1+\tan^2 A/1+\cot^2 A)$$

Since cot function is the inverse of tan function,

$$= (1+\tan^2 A/1+1/\tan^2 A)$$

$$= 1+\tan^2 A/[(1+\tan^2 A)/\tan^2 A]$$

Now cancel the $1+\tan^2 A$ terms, we get

$$= \tan^2 A$$

$$(1+\tan^2 A/1+\cot^2 A) = \tan^2 A$$

Similarly,

$$(1-\tan A/1-\cot A)^2 = \tan^2 A$$

Hence proved