MATHEMATICS

RELATIONS AND FUNCTIONS

INTRODUCTION TO RELATION

NOTES

What you already know

- Set theory

What you will learn

- Cartesian product of sets
- Relation
- Domain and range of a relation
- Inverse of a relation
- Types of relations

Definition of set: A set is a well-defined collection of objects.

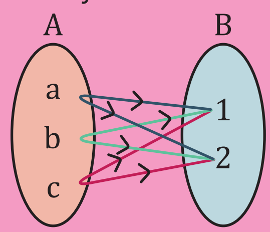
Let us consider the following examples

1. A collection of all the chapters in your grade 11 mathematics book.
2. A collection of all the difficult chapters in grade 11 mathematics book.

In the first case, we are getting a well-defined collection of objects, as it is the same for all the students. However, in the second case, we are not getting a well-defined collection of objects, as the difficult chapters will differ for each student. Hence it is not a set.

Cartesian Product of Sets

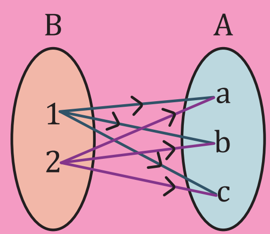
Let A and B be two non-empty sets, then the cartesian product denoted by $A \times B$ is the ordered pair of elements. $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$



Examples

Find $A \times B$, $B \times A$ and is $A \times B = B \times A$ for the sets ?

$A = \{a, b, c\}$
$B = \{1, 2\}$



Solution

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$A \times B \neq B \times A$, Because the orders are different.

Number of elements in the Cartesian product of sets

The total number of elements in the cartesian product can be calculated as follows:

If $n(A) = p$ and $n(B) = q$, then we get, cardinality of $A \times B = n(A \times B) = n(A) \times n(B) = pq$

Equivalent sets: Two sets are equivalent if they have the same number of elements.

Relations

Let A and B be two sets.
Then, relation R from A to B is a subset of A × B.

Examples

$A = \{a, b, c\}$ $B = \{1, 2\}$

Therefore,

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

R = Relation ending with an even number

$R = \{(a, 2), (b, 2), (c, 2)\}$

A relation from A to B is expressed as follows: $R : A \rightarrow B$
It is read as R is a relation such that it is from A to B.

Number of relations from A to B = Number of subsets of A × B
If $n(A) = p$, $n(B) = q$, and $R : A \rightarrow B$, then we get,
Number of relations from A to B = 2^{pq}

Examples

$n(A) = 3$ $n(B) = 2$

$\Rightarrow n(A \times B) = 3 \times 2 = 6$

Number of relations from A to B = $2^{n(A \times B)} = 2^6$

Domain and range of a relation

The domain of relation R is the set of the first element of all the ordered pairs of relation R.
The range of relation R is the set of the second element of all the ordered pairs of relation R.

If $A = \{1, 3, 5, 7\}$, $B = \{1, 4, 9, 16, 25\}$, and $R : A \rightarrow B$ such that $R = \{(a, b) : b = a^2, a \in A, b \in B\}$, then write relation R and also its domain and range.

Solution

We will map the relation to find the domain and range of R.
Given, $R = \{(a, b) : b = a^2, a \in A, b \in B\}$

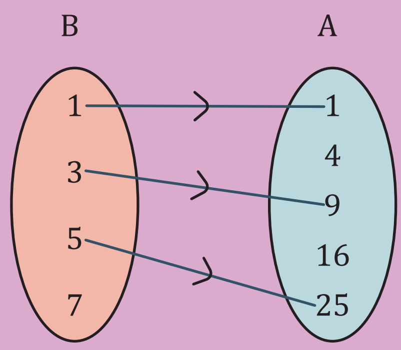
$a = 1 \Rightarrow b = 1^2 = 1 \quad (a, b) = (1, 1)$

$a = 3 \Rightarrow b = 3^2 = 9 \quad (a, b) = (3, 9)$

$a = 5 \Rightarrow b = 5^2 = 25 \quad (a, b) = (5, 25)$

$a = 7 \Rightarrow b = 7^2 = 49 \quad (a, b) = (7, 49)$

Therefore,
$R = \{(1, 1), (3, 9), (5, 25)\}$
$Domain(R) = \{1, 3, 5\}$
$Range(R) = \{1, 9, 25\}$



Inverse of a relation

Let A and B be two sets and R be a relation from A to B. Thus, the inverse of R is a relation from B to A, which is denoted by R^{-1}. It is defined as follows:

$R^{-1} = \{(b, a) : (a, b) \in R\}$

Basically, the position of elements in the ordered pair gets interchanged.
For example, if a relation R is $\{(a, b)\}$, then its inverse R^{-1} becomes $\{(b, a)\}$.

Examples

Let A and B be two non-empty sets: A = {a, b, c}, B = {1, 2}
R = Relation ending with an even number
R = {(a, 2), (b, 2), (c, 2)}
Domain of R = {a, b, c}
Range of R = {2}
R^{-1} = {(2, a), (2, b), (2, c)}
Domain of R^{-1} = {2}
Range of R^{-1} = {a, b ,c}
Domain of R^{-1} = Range of R
Range of R^{-1} = Domain of R

If $R = \{(x, y) : x, y \in \mathbb{Z}, x^2 + 3y^2 \le 8\}$ is a relation on a set of integers $\mathbb{Z}$, then find the domain of R^{-1}.

(a) {-2, -1, 1, 2} **(b) {-1, 0, 1}** **(c) {-2, -1, 0, 1, 2}** **(d) {0, 1}**

Solution

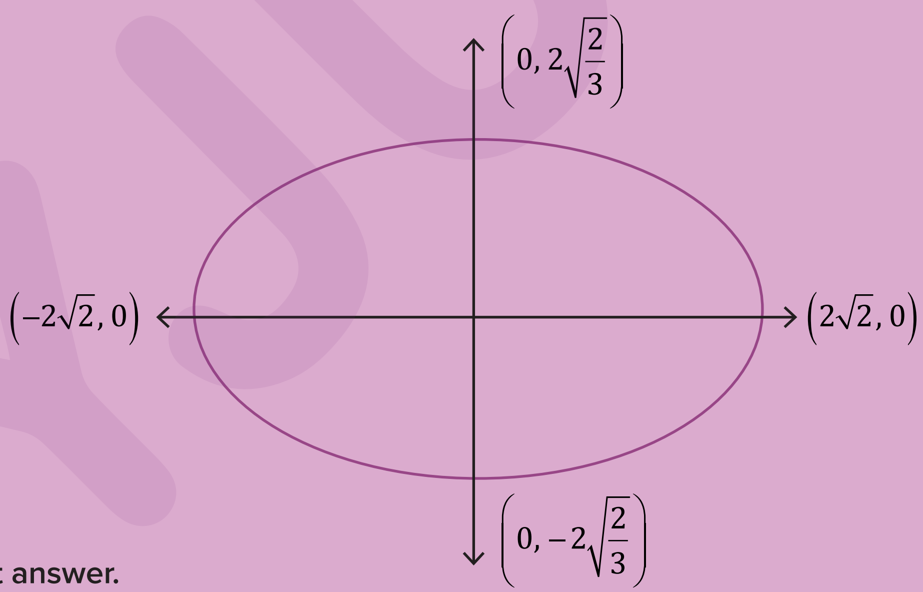
Method 1: Graphical method

$$x^2 + 3y^2 = 8$$

$$\Rightarrow \frac{x^2}{8} + \frac{y^2}{\frac{8}{3}} = 1$$

$$-2\sqrt{\frac{2}{3}} \le y \le 2\sqrt{\frac{2}{3}}$$

$$-1.6 \le y \le 1.6$$



Integral values of y = {-1, 0, 1}
Domain of R^{-1}= Range of R
Domain of R^{-1} = {-1, 0, 1}
Hence, option (b) is the correct answer.

Method 2: Algebraic method

Given, $R=\{(x, y):x, y \in \mathbb{Z}, x^2+3y^2 \le 8\}$

We know that the domain of an inverse relation is equal to the range of an original relation.
⇒ Domain of R^{-1} = Range of R (Values of y)

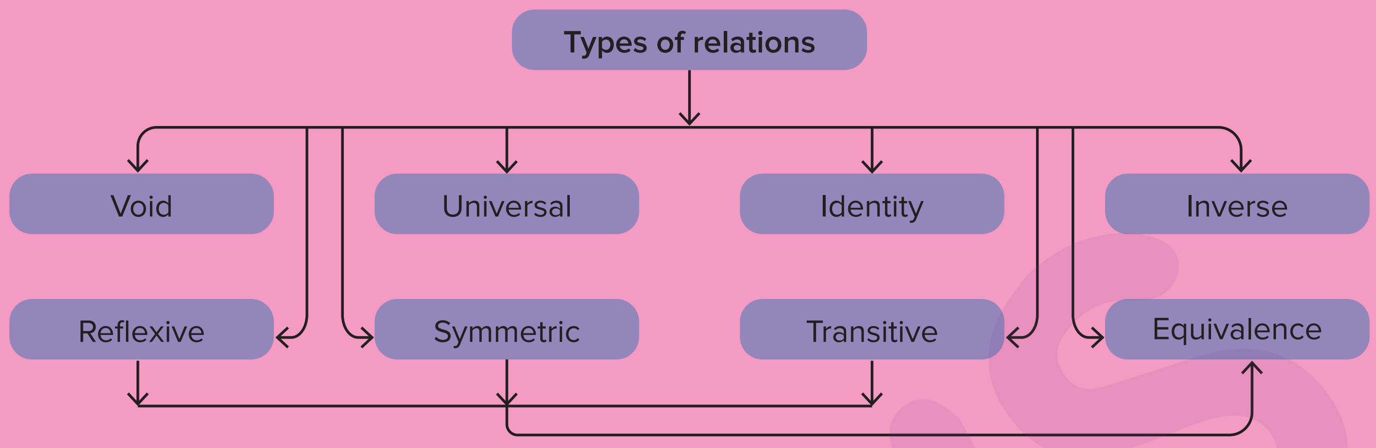
Case 1:
If x = 0
$3y^2 \le 8 \Rightarrow y^2 \le \frac{8}{3}$
$\Rightarrow y = \{-1, 0, 1\}$
$[\because y \in \mathbb{Z}]$

Case 2:
If x = ±1
$1 + 3y^2 \le 8$
$\Rightarrow y^2 \le \frac{7}{3}$
$\Rightarrow y = \{-1, 0, 1\}$
$[\because y \in \mathbb{Z}]$

Case 3:
If x = ±2
$4 + 3y^2 \le 8$
$\Rightarrow y^2 \le \frac{4}{3}$
$\Rightarrow y = \{-1, 0, 1\}$
$[\because y \in \mathbb{Z}]$

Case 4:
If x = ±3
$9 + 3y^2 \le 28$
⇒ $3y^2$ - 1, which is not possible.
$y \in \varphi$
Therefore,
Range of R = Domain of R^{-1} = {-1, 0, 1}

Types of Relations



Void relation

A relation R on set A is known as a void or empty relation if no element of set A is related to any element of set A.

$\Rightarrow R = \varphi$

Example

A= {Students in a boys school}

Let R = {(a, b) : b is a sister of a, a, b ∈ A}

Since A is a set of students in a boys school, there will be no girl. Therefore, relation R will not contain any element, and it will be a void relation.

$\Rightarrow R = \varphi$

Universal relation

It is a relation in which each element of set A is related to every element of set A.

$\Rightarrow R = A \times A$

Example

Let A = {Set of real numbers}

$R = \{(a, b) : |a - b| \geq 0, a, b \in A\}$

We know that a - b is a real number.

So, |a - b| will also be a non-negative real number.

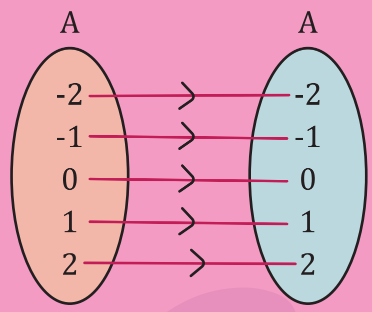
$\Rightarrow |a - b| \geq 0$ will hold true for every value of a and b.

Therefore, R will be a universal relation.

Identity relation

The relation on set A is an identity relation if each and every element of A is related to itself only.

Example



Let A = {Set of integers}
Relation R = {(a, b) : a = b, a, b ∈ A}
Both the elements of ordered pair (a, b) are equal.
Therefore, it is an identity relation.
$\Rightarrow R = \{(a, b): a = b,\ a, b \in A\} = I_A$

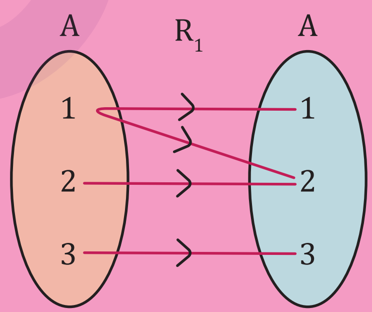
Reflexive relation

A relation R on set A is said to be reflexive if every element of A is related to itself, i.e., relation R is reflexive if $(a, a) \in R\ \forall\ a \in A$

Note

- $I \subseteq R$, where I is an identity relation on A.
- Every identity relation is a reflexive relation but every reflexive relation is not an identity relation.

Example



Consider a set, A={1, 2, 3}
Check if the following relations are reflexive.

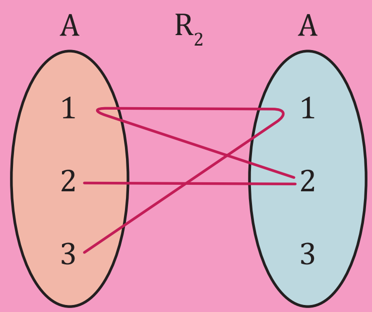
(1) $R_1 = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$

(2) $R_2 = \{(1, 1), (1, 2), (2, 2), (3, 1)\}$

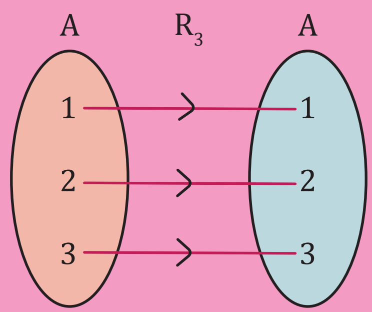
(3) $R_3 = \{(1, 1), (2, 2), (3, 3)\}$

Solution

(1) We have to check if every element of A is related to itself or not, i.e., are(1, 1), (2, 2), and (3, 3) present in relation R_1 or not.
As we can see that (1, 1), (2, 2), and (3, 3) are present in relation R_1, it is a reflexive relation.



(2) We have to check if every element of A is related to itself or not, i.e., if (1, 1), (2, 2),and (3, 3) are present in relation R_2 or not. As we can see that (3, 3) is not present in R_2, it is not a reflexive relation.



(3) We have to check if every element of A is related to itself or not, i.e., if (1, 1), (2, 2), and (3, 3) are present in relation R_3 or not. As we can see that every element of set A is related to itself only, it is an identity relation. Also, we know that every identity relation is a reflexive relation. Therefore, R_3 is a reflexive relation.

Note

In an identity relation, every element of a set is related to itself only. However, in a reflexive relation, every element is related to itself and to other elements as well.

Relation R is defined on a set of natural numbers.
$R = \{(a, b): a \text{ divides } b, a, b \in \mathbb{N}\}$.Check whether it is reflexive or not.

Solution

We know that a number divides itself. So, $(a, a)\ \forall\ a \in \mathbb{N}$ will be an element of relation R.
Therefore, $I_A = \{(1, 1), (2, 2), (3, 3), ...\}$ will be a subset of R, i.e., $I_A \subseteq R$
$\Rightarrow$ R is a reflexive relation.

Symmetric relation

Relation R on set A is said to be symmetric iff $(a, b) \in R \Rightarrow (b, a) \in R$
$a\ R\ b \Rightarrow b\ R\ a\ \forall\ (a, b) \in R\ (\ a, b \in A)$
It is read as 'a is related to b implies that b is related to a for all $(a, b) \in R$.'

Example:

Consider a set, $A = \{1, 2, 3\}$. Identify the symmetric relations among the following:

(1) $R_1 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$

(2) $R_2 = \{(1, 1), (1, 2), (2, 1), (1, 3)\}$

Solution

(1) Here, we have, $(1, 2)$ and $(2, 1)$;$(1, 3)$ and $(3, 1)$. For $(1, 1)$, we know that on interchanging the position of the first and the second element in the ordered pair $(1, 1)$, we will get $(1, 1)$. Thus, R_1 is a symmetric relation.

(2) R_2 is not a symmetric relation because it does not have the corresponding ordered pair $(3, 1)$ for $(1, 3)$.

Relation R is defined on the set of straight lines in a plane. $R = \{\ (L_1, L_2);\ L_1 \perp L_2\}$.
Check whether it is symmetric or not.

Solution

$(\ L_1, L_2) \in R \Rightarrow\ L_1 \perp L_2$
$\Rightarrow L_2 \perp L_1 \Rightarrow (\ L_2, L_1) \in R$
So, the given relation is a symmetric relation.

Transitive relation

Relation R on set A is said to be transitive, iff we have,
$(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$
Or $a R b$, $b R c \Rightarrow a R c$, $\forall\ a, b, c \in A$

Example

Consider a set, $A=\{1, 2, 3\}$. Identify the transitive relations among the following:
1. $R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
2. $R_2 = \{(1, 1), (1, 3), (3, 2)\}$

Solution

(1) Consider $(1, 2) \equiv (a, b)$ and $(2, 3) \equiv (b, c)$
$\Rightarrow (1, 3) \equiv (a, c) \in R_1$
If we consider $(1, 1) \equiv (a, b)$ and $(1, 2) \equiv (b, c)$, then $(1, 2) \equiv (a, c) \in R_1$
Similarly, we can check for different combinations of a, b, and c.
It comes out to be a transitive relation.

(2) Consider $(1, 3) \equiv (a, b)$ and $(3, 2) \equiv (b, c)$
However, $(1, 2) \equiv (a, c) \notin R_2$
Therefore, R_2 is not a transitive relation.

Show that relation R defined on the set of real numbers such that $R = \{(a, b) : a > b\}$ is transitive.

Solution

Let $(a, b) \in R$ and $(b, c) \in R$.
$\Rightarrow a > b$ and $b > c$
$\Rightarrow a > c$
$\Rightarrow (a, c) \in R$
Therefore, R is a transitive relation.

Equivalence Relation

Relation R on a set A is said to be an equivalence relation on A iff,

- It is reflexive, i.e., $(a, a) \in R\ \forall\ a \in A$
- It is symmetric, i.e., $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$
- It is transitive, i.e., $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$

Example:

An identity relation is an equivalence relation.
Since $(a, a) \in R\ \forall\ a \in A$
$\Rightarrow$ It is an equivalence relation

Let T be the set of all triangles in a plane with R as a relation given by $R = \{(T_1, T_2): T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution

(1) Every triangle is congruent to itself.
Therefore, R is reflexive.

(2) $(T_1, T_2) \in R : T_1$ is congruent to T_2.
$\Rightarrow T_2$ is congruent to T_1.
Therefore, R is symmetric.

(3) $(T_1, T_2) \in R : T_1$ is congruent to T_2.
$(T_2, T_3) \in R : T_2$ is congruent to T_3.
$\Rightarrow T_1$ is congruent to T_3.
Therefore, R is transitive.

Thus, the given relation R is an equivalence relation.

Concept Check

(1) Check whether relation R defined on the set of games in a cricket tournament such that $R = \{(A, B)$: Team A defeated Team B$\}$ is transitive.

(2) Check whether relation R defined on the set of real numbers such that $R=\{(a, b): 1 + ab > 0\}$ is transitive.

(3) Let $\mathbb{R}$ be the set of real numbers. **JEE MAIN 2011**
Statement 1: $A = \{(x, y)\ \ \mathbb{R} \times \mathbb{R}: y - x$ is an integer$\}$ is an equivalence relation on $\mathbb{R}$.
Statement 2: $B = \{(x, y)\ \ \mathbb{R} \times \mathbb{R}: x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation.
(a) Statement 1 is true, statement 2 is true, and statement 2 is the correct explanation of statement 1.
(b) Statement 1 is true, statement 2 is true, and statement 1 is not the correct explanation of statement 2.
(c) Statement 1 is true and statement 2 is false.
(d) Statement 1 is false and statement 2 is true.

(4) Let R_1 and R_2 be two relations defined as follows: **JEE MAIN SEP 2020**
$R_1 = \{(a, b) \in R \times R: a^2 + b^2 \in \mathbb{Q}\}$
$R_2 = \{(a, b) \in R \times R: a^2 + b^2 \notin \mathbb{Q}\}$
Which of the following is correct?
(a) Neither R_1 nor R_2 is transitive.
(b) R_1 and R_2 are transitive.
(c) R_1 is transitive but R_2 is not.
(d) R_2 is transitive but R_1 is not.

Summary Sheet

Key Takeaways

- A set is a well-defined collection of objects.
- Let A and B be two non-empty sets, then the cartesian product denoted by $A \times B$ is the ordered pair of elements. $A \times B = \{(a, b): a \in A$ and $b \in B\}$
- The domain of relation R is the set of the first element of all the ordered pairs of relation R.
- The range of relation R is the set of the second element of all the ordered pairs of relation R.

- If A and B are two sets and R is a relation from A to B, then the inverse of R is a relation from B to A. It is denoted by R^{-1}.
- In an identity relation, every element of a set is related to itself only. However, in a reflexive relation, every element is related to itself and to other elements as well.
- Relation R in set A is as follows:

Reflexive	Not Reflexive
$\forall a \in A, (a,a) \in R$	$\exists\, a \in A$ Such that $(a,a) \notin R$
Symmetric	**Not-symmetric**
For $a,b \in A, (a,b) \in R \Rightarrow (b,a) \in R$	$\exists\, a,b \in A, (a,b) \in R$ but $(b,a) \notin R$
Transitive	**Not-transitive**
For $a,b,c \in A, (a,b),(b,c) \in R \Rightarrow (a,c) \in R$	$\exists\, a,b,c \in A, (a,b),(b,c) \in R$ but $(a,c) \notin R$

Mind Map

- Relation
  - Domain and range of relation
  - Types of relation
    - Void relation
    - Universal relation
    - Identity relation
    - Reflexive relation
    - Symmetric relation
    - Transitive relation
    - Equivalence relation
  - Inverse of a relation

Self-Assessment

Let $R = \{(x, y) : 2x + y = 41, x, y \in \mathbb{N}\}$ be a relation defined on the set of natural numbers. Find the domain and range of R.

Answers

Concept Check

1. $(A, B) \in R \Rightarrow$ A defeated B
$(B, C) \in R \Rightarrow$ B defeated C
This does not imply that A defeated C.
Therefore, R is non-transitive.

2. For $a = -1$, $b = \frac{1}{2}$, we get,

$1 + ab > 0 \Rightarrow \left(-1, \frac{1}{2}\right) \in R$

For $b = \frac{1}{2}$, $c = 4$, we get,

$1 + bc > 0 \Rightarrow \left(\frac{1}{2}, 4\right) \in R$

However,
$1 + ac = 1 + (-1)4 = -3 < 0 \Rightarrow (-1, 4) \notin R$
Therefore, R is non-transitive.

3. Step 1:
Consider the following:
$(x, x) \in A$
$\Rightarrow y - x = x - x = 0$
The expression $y - x$ is an integer.
Thus, A is reflexive.

Step 2:
$(x, y) \in A \Rightarrow$ The expression $y - x$ is an integer.
$(y, x) \in A \Rightarrow$ The expression $x - y$ is also an integer.
Thus, A is symmetric.

Step 3:
$(x, y) \in A \Rightarrow$ The expression $y - x$ is an integer.
$(y, z) \in A \Rightarrow$ The expression $z - y$ is an integer.
$(x, z) \in A \Rightarrow$ The expression $z - x$ is also an integer.
Thus, A is transitive.
Hence, A is an equivalence relation.

Step 4:
Consider the following:
$(x, x) \in B$
$\Rightarrow x = \alpha x$
$\Rightarrow \alpha = 1$
Since α is a rational number, B is reflexive.
$(x, y) \in B \Rightarrow x = \alpha y$ (For some rational α)
$(y, x) \in B \not\Rightarrow y = \alpha x$ (For some rational α)
(**Example:** $x = 0$, $y = 1 : x = \alpha \times y \Rightarrow 0 = \alpha \times 1 = 0$;
$y = \alpha x \Rightarrow 1 = \alpha \times 0$ No rational exists)
Thus, B is not symmetric.
Hence, B is not an equivalence relation.
Thus, option (c) is the correct answer.

4. $R_1 = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 \in \mathbb{Q}\}$

$a = 2 + \sqrt{3}$, $b = 2 - \sqrt{3}$ & $c = 1 + 2\sqrt{3}$

$a^2 + b^2 = \left(2 + \sqrt{3}\right)^2 + \left(2 - \sqrt{3}\right)^2$

$= 4 + 3 + 4 + 3 = 14 \in \mathbb{Q}$

$b^2 + c^2 = \left(2 - \sqrt{3}\right)^2 + \left(1 + 2\sqrt{3}\right)^2$

$= 4 + 3 + 1 + 12 = 20 \in \mathbb{Q}$

$a^2 + c^2 = \left(2 + \sqrt{3}\right)^2 + \left(1 + 2\sqrt{3}\right)^2$

$= 4 + 3 + 1 + 12 + 4\sqrt{3} + 4\sqrt{3}$

$= 20 + 8\sqrt{3}$ is not an element of $\mathbb{Q}$

R_1 is a non-transitive relation.

$R_2 = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 \notin \mathbb{Q}\}$

Let

$a^2 = 1$, $b^2 = \sqrt{3}$, $c^2 = 2$

$a^2 + b^2 = 1 + \sqrt{3} \notin \mathbb{Q}$

$b^2 + c^2 = \sqrt{3} + 2 \notin \mathbb{Q}$

$a^2 + c^2 = 1 + 2 = 3 \in \mathbb{Q}$

$\Rightarrow R_2$ is a non-transitive relation.

Option (a) is the correct answer.

Self-Assessment

We know that $x, y \in \mathbb{N}$
For $x = 1$, $y = 39$
For $x = 2$, $y = 37$
. .
. .
. .
For $x = 20$, $y = 1$
For $x = 21$, $y = -1 \notin \mathbb{N}$

Therefore,
Domain of $R = \{1, 2, 3, \ldots, 20\}$
Range of $R = \{1, 3, 5, \ldots, 39\}$

M A T H E M A T I C S

RELATIONS AND FUNCTIONS

INTRODUCTION TO FUNCTION

NOTES

What you already know

- Set theory
- Cartesian product
- Relations

What you will learn

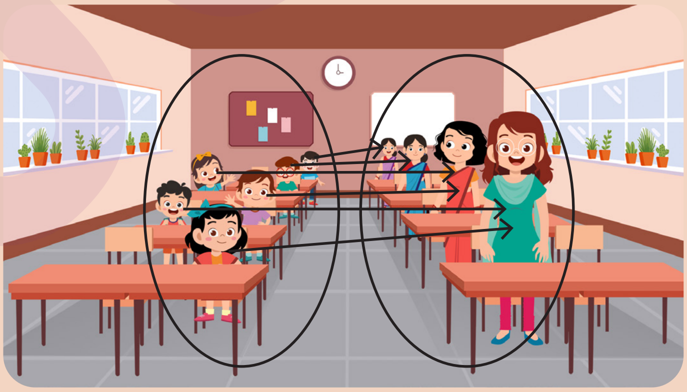
- Function
- Domain and Range
- Types of functions

Function

Sets → Cartesian product → Relations → Functions

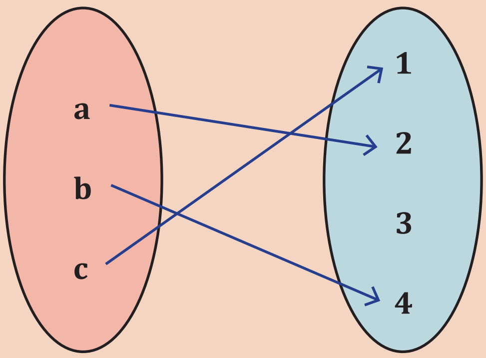
A function is a relation defined from set A to set B when it has the following properties:
(i) Each element in A is associated to some element in B.
(ii)The association is unique.
It is denoted by $f : A \to B$.

Let us consider two sets A and B.
Where A is a set of all the kids and B is a set of mothers of these kids as shown in the figure.
Here, the association from A to B is 'is the child of'.
Hence, all the elements of set A have to be connected to set B uniquely as more than one mother for a child is not possible.
∴ The association between A and B is a function.

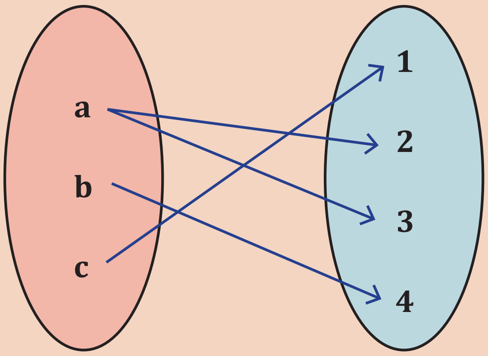


Examples

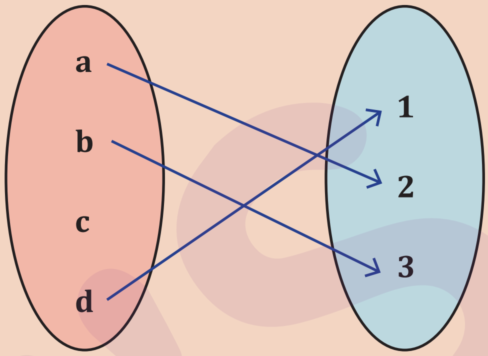
The given relation is a function. This is because all the elements of the first set are connected to the elements of the second set uniquely.



The given relation is not a function. This is because one element a of the first set is connected to two elements of the second set.



The given relation is not a function. This is because one element c of the first set is not connected to any element of the second set.



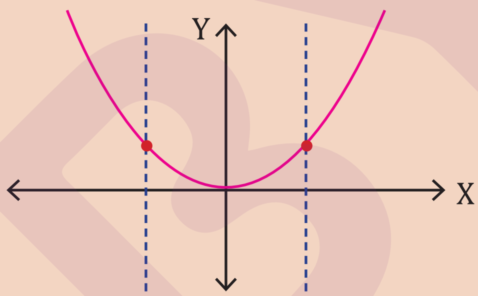
Graphical approach

Vertical line test

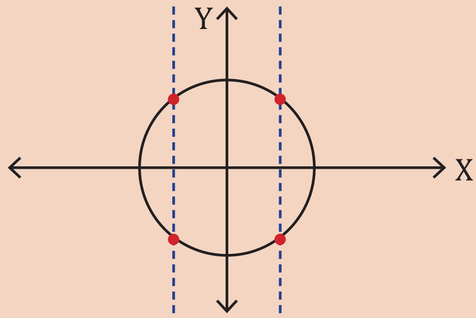
A graph is known to be a function if the vertical line drawn parallel to the Y-axis does not intersect the graph at more than one point in the domain.

Examples

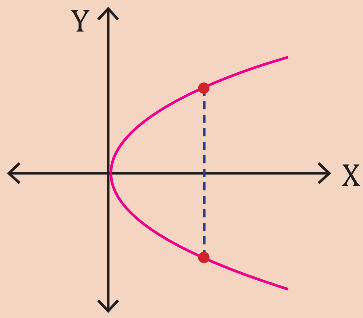
The vertical line intersects the graph of $x^2 = y$ at one point only.
Hence, $x^2 = y$ is a function.



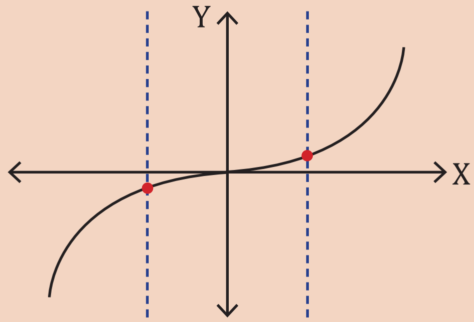
The vertical line intersects the graph of $x^2 + y^2 = 9$ at two points.
Hence, $x^2 + y^2 = 9$ is not a function.



The vertical line intersects the graph of $y^2 = x$ at two points.
Hence, $y^2 = x$ is not a function.



The vertical line intersects the graph of $x^3 = y$ at one point only.
Hence, $x^3 = y$ is a function.



Domain and Range

Let us try to understand the concept of domain and range of a function with the help of a juicer. In a juicer, we input fruits and get juice as the output. Similarly, a function can be compared to a juicer .

Here, all the permissible input to the function is known as domain. The outputs for the respective inputs of the function is known as range, and the possible set of outputs is known as codomain.

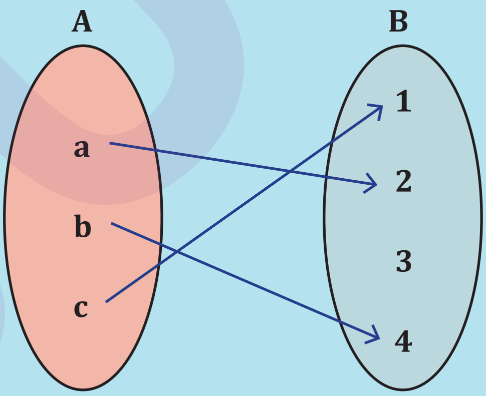
Examples

Let us consider a function $f : A \to B$

Domain: A

Range: $\{1, 2, 4\}$

Co-domain: $B = \{1, 2, 3, 4\}$



Domain: It is the value of set A for which the function is defined.

Range: It consists of all the values that the function gives.

Co-domain: It consists of the set of all the elements in set B. (Range $\subseteq$ Codomain)

Real valued function

If a function either has $\mathbb{R}$ or one of its subset as its range, then it is known as a real valued function. Further, if its domain is either $\mathbb{R}$ or a subset of $\mathbb{R}$, then it is known as a real function.

Types of function

Polynomial function

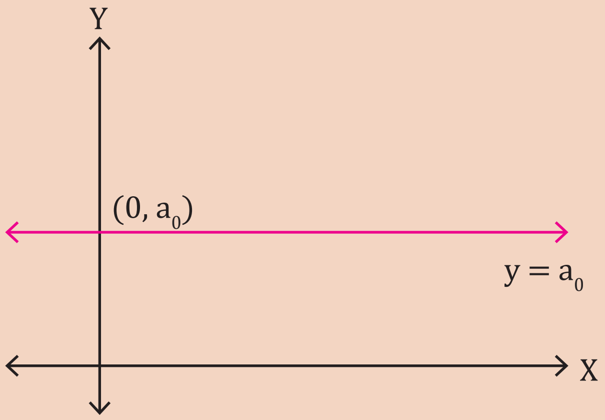
A polynomial function is given by,

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots\dots\dots + a_0$, where $a_0, a_1, \dots, a_n \in \mathbb{R}, a_n \neq 0,\ n \in \mathbb{W}$

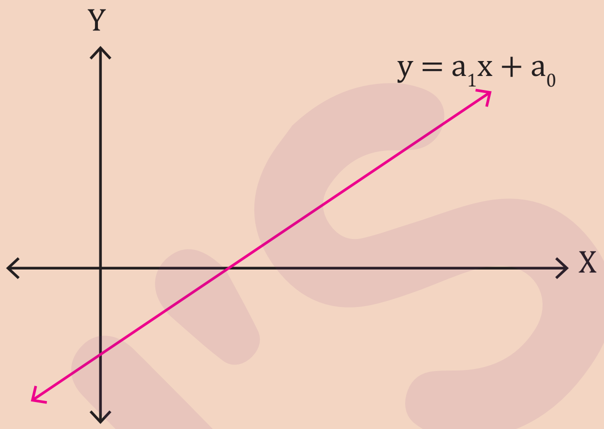
Domain: $x \in \mathbb{R}$

Example: $f(x) = x^{\frac{4}{3}}$ - 1 is not a polynomial function as $\frac{4}{3} \notin \mathbb{W}$

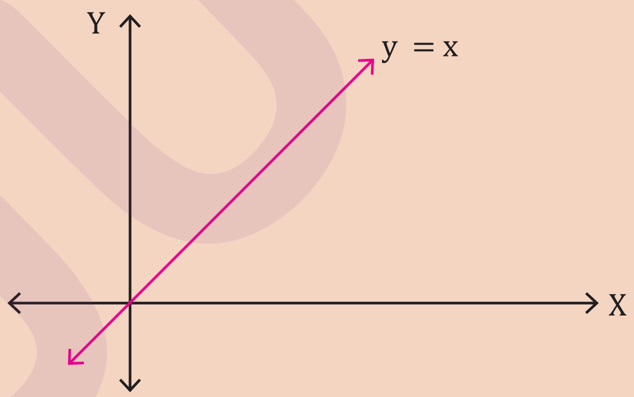
If $n = 0$, we get,
$f(x) = a_0$
So, in this case,
Range: $\{a_0\}$



If $n = 1$, we get,
$P(x) = a_1x + a_0$
Range: $\mathbb{R}$



If $a_1 = 1, a_0 = 0$
$P(x) = x \rightarrow$ Identity function
Range: $\mathbb{R}$



If $n = 2$, we get,
$P(x) = a_2x^2 + a_1x + a_0$
On considering the standard quadratic polynomial function, $f(x) = ax^2 + bx + c$, we get,
Let $f(x) = ax^2 + bx + c$

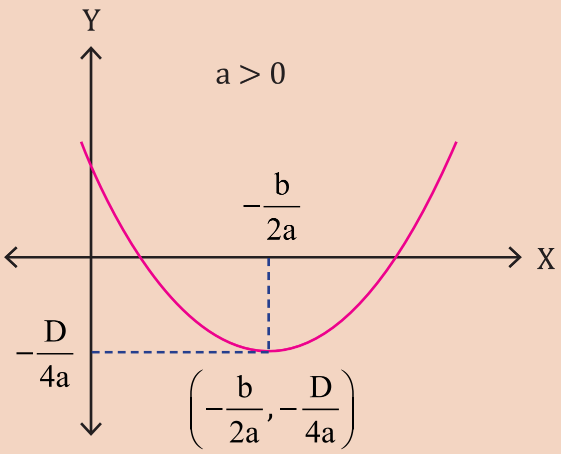
$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

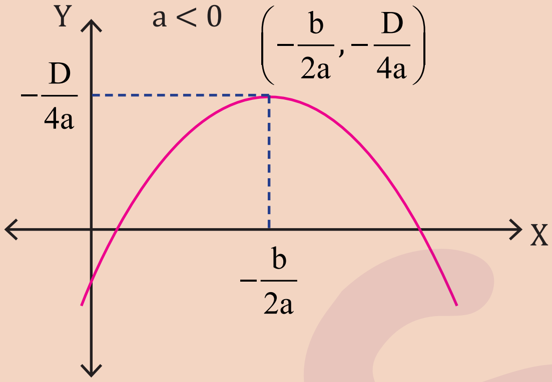


For $a > 0$, we get,

Range: $\left[-\dfrac{D}{4a}, \infty\right)$

For $a < 0$, we get,

Range: $\left(-\infty, -\dfrac{D}{4a}\right]$



Even degree polynomial

When the degree of each term in $P(x)$ is even, it is an even degree polynomial.

Examples:

$P(x) = x^2$, $P(x) = x^4 + 3x^2 + 7$, $P(x) = x^6 - 6x^4$

Odd degree polynomial

When the degree of each term in $P(x)$ is odd, it is an odd degree polynomial.

Examples:

$P(x) = x^3$, $P(x) = x^5 + 5x^3 + 2x$, $P(x) = x^7 - 6x^5 - x^3 + x$

Neither even nor odd

When the terms in $P(x)$ are a combination of both even and odd degrees, It is neither even nor odd.

Examples:

$P(x) = x^4 + 3x^3 - 7x^2 + 2x + 5$

Find the domain and range of the following: $f(x) = \sin^2x + \cos^2x$.

Solution

Here, $f(x)$ is defined for all the values of x.

$\therefore$ Domain: $x \in \mathbb{R}$

$\forall\, x \in \mathbb{R}$, $\sin^2x + \cos^2x = 1$

$\therefore$ Range: $f(x) \in \{1\}$

Find the domain and range of the following: $f(x) = x^2 + 3x + 1$

Solution

Here,
$f(x)$ is defined for all the values of x.
∴ The domain of $f(x)$ is $x \in \mathbb{R}$.
$f(x) = x^2 + 3x + 1$
$\Rightarrow f(x) = x^2 + 3x + \frac{9}{4} + 1 - \frac{9}{4}$ (To make it perfect square)
$\Rightarrow \left(x + \frac{3}{2}\right)^2 - \frac{5}{4} \geq -\frac{5}{4}$
$\Rightarrow$ The range of the function is $f(x) \in \left[-\frac{5}{4}, \infty\right)$.

Rational Function

A rational function is defined as $h(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the functions of x and $g(x) \neq 0$

Find the domain and range of the following: $f(x) = \frac{x+1}{3x-5}$

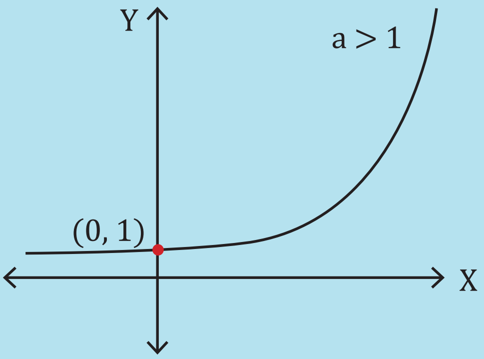
Solution

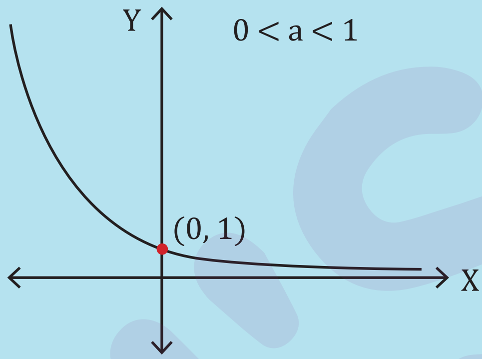
$f(x) = \frac{x+1}{3x-5}$ is not defined when $3x - 5 = 0 \Rightarrow x = \frac{5}{3}$
∴ Domain: $x \in \mathbb{R} - \left\{\frac{5}{3}\right\}$

Let $y = \frac{x+1}{3x-5}$
$\Rightarrow x + 1 = 3xy - 5y$
$\Rightarrow 1 + 5y = 3xy - x$
$\Rightarrow x = \frac{5y+1}{3y-1}$
When $y = \frac{1}{3}$, it is not defined.
∴ Range: $y \in \mathbb{R} - \left\{\frac{1}{3}\right\}$

Exponential function

$y = a^x, a > 0$

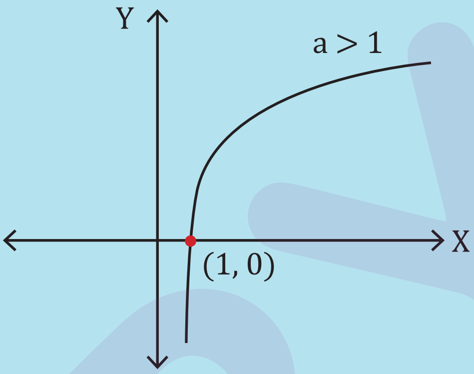


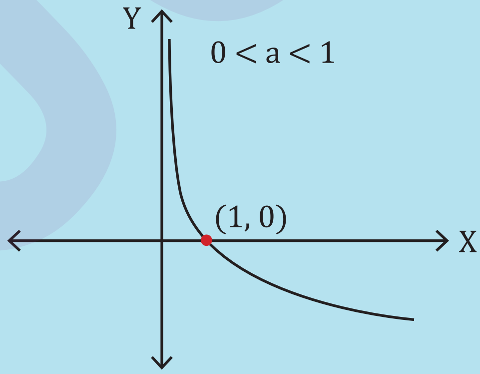


Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}^+$

Logarithmic function

$y = \log_a x, a > 0, a \neq 1$

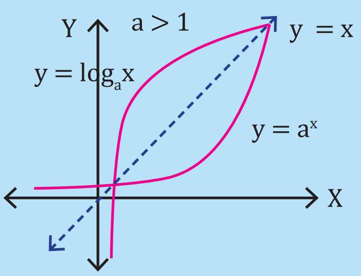


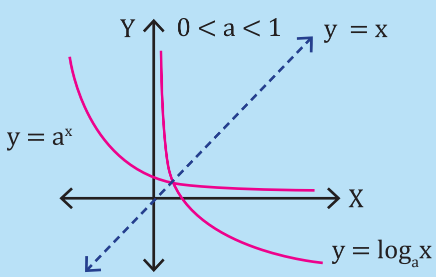


Domain: $x \in \mathbb{R}^+$
Range: $y \in \mathbb{R}$

Note

The graphs of logarithmic and exponential functions are the mirror images of each other about line $y = x$. Hence, both the functions are inverse of each other.





Find the domain and range of the following functions:
(i) $f(x) = e^{2x}$ (ii) $f(x) = e^x + 1$ (iii) $\log(x - 2)$

Solution

(i) $f(x)$ is defined for all the values of x.
∴ **Domain:** $x \in \mathbb{R}$
$f(x) > 0, \forall x \in \mathbb{R}$
∴ **Range:** $f(x) \in (0, \infty)$

(ii) $f(x)$ is defined for all the values of x.
∴ **Domain:** $x \in \mathbb{R}$
$e^x > 0$
$\Rightarrow e^x + 1 > 1$
∴ **Range:** $f(x) \in (1, \infty)$

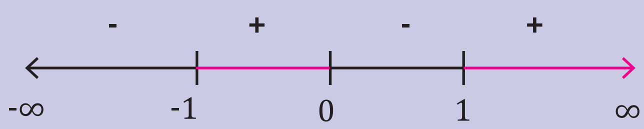
(iii) For a logarithmic function, the argument is always positive.
$\Rightarrow x - 2 > 0$
$\Rightarrow x > 2$
∴ **Domain:** $(2, \infty)$
∴ **Range:** $f(x) \in \mathbb{R}$

Find the domain of the function: $f(x) = \dfrac{1}{4 - x^2} + \log_{10}\left(x^3 - x\right)$

(a) $(1, 2) \cup (2, \infty)$ **(b) $(-2, -1) \cup (-1, 0) \cup (2, \infty)$**
(c) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ **(d) $(-1, 0) \cup (1, 2) \cup (3, \infty)$**

Solution

The expression $\dfrac{1}{4 - x^2}$ is defined only when it satisfies the following condition:
$4 - x^2 \neq 0 \Rightarrow x \neq \pm 2$.......(i)
The expression $\log_{10}(x^3 - x)$ is defined only when it satisfies the following condition:
$x^3 - x > 0$
$\Rightarrow x(x - 1)(x + 1) > 0$
By using the wavy curve method, we get,
$x \in (-1, 0) \cup (1, \infty)$........(ii)
From equations (i) and (ii), we get,
$x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$
∴ **Option (c) is the correct answer.**



Note

In general for $h(x) = f(x)^{g(x)}$, $f(x) > 0$ and normal condition for $g(x)$

Find the domain of the following function: $f(x) = \left(\frac{x-1}{x-2}\right)^{3x+4}$

Solution

$\left(\frac{x-1}{x-2}\right)^{3x+4}$ is in the form of $h(x)^{g(x)}$,

Here,

$g(x) = 3x + 4$ is defined for $\mathbb{R}$.....(1).

$h(x) = \frac{x-1}{x-2} > 0$

$\Rightarrow \frac{(x-1)(x-2)}{(x-2)^2} > 0$

$(x - 1)(x - 2) > 0$

By using the wavy curve method, we get the following:

$x \in (-\infty, 1) \cup (2, \infty)$.....(2)

On taking intersection of (1) and (2), we get

$x \in (-\infty, 1) \cup (2, \infty)$

Find the domain and range of the following: $f(x) = x^4 + x^2 + 4$

Solution

Given,

$f(x) = x^4 + x^2 + 4$

Since $f(x)$ is a polynomial function, its domain is $\mathbb{R}$.

Let $y = x^4 + x^2 + 4$

$\Rightarrow y = x^4 + x^2 + 4 + \frac{1}{4} - \frac{1}{4}$ (To make it perfect square)

$\Rightarrow y = \left(x^2 + \frac{1}{2}\right)^2 + \frac{15}{4}$

$\Rightarrow y \geq \frac{1}{4} + \frac{15}{4}$

$\Rightarrow y \geq 4$

$\therefore$ **Range:** $[4, \infty)$.

Concept Check

1. Find the domain and range of the following function: $f(x) = 3x + 5$

2. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x}{x^2+1}, x \in \mathbb{R}$. Find the range of f.

(a) $\mathbb{R} - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (b) $\mathbb{R}-[-1, 1]$ (c) $(-1, 1) - \{0\}$ (d) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

3. Find the domain of the function $f(x) = \left(1+\frac{3}{x}\right)^{\frac{1}{x-2}}$

4. Find the domain of the function $f(x) = \left(x^2+1\right)^{\frac{x-1}{2x-3}}$

Summary Sheet

Key Takeaways

- A function is a relation defined from set A to set B when it has the following properties:
  (i) Each element in A is associated to some element in B.
  (ii) The association is unique.
- A graph is known to be a function if the vertical line drawn parallel to the Y-axis does not intersect the graph at more than one point in the domain.
- Domain: It is the value of set A for which a function is defined.
- Range: It consists of all the values that the function gives.
- Co-domain: It consists of the set of all the elements in set B. (Range $\subseteq$ Co-domain)
- If a function either has $\mathbb{R}$ or one of its subset as its range, then it is known as a real valued function. Further if domain of a function is either $\mathbb{R}$ or a subset of $\mathbb{R}$, then it is known as a real function.

Function	Domain	Range
$f(x) = a_0$ (Constant function)	$\mathbb{R}$	$\{a_0\}$
$f(x) = a_1x + a_0$ (Linear function)	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x$ (Identity function)	$\mathbb{R}$	$\mathbb{R}$
$f(x) = a_2x^2 + a_1x + a_0$ (Quadratic function)	$\mathbb{R}$	When, $a_2 > 0$; $R \in \left[-\frac{D}{4a_2}, \infty\right)$ & When, $a_2 < 0$; $R \in \left(-\infty, -\frac{D}{4a_2}\right]$
$f(x) = a^x, a > 0$ (Exponential function)	$\mathbb{R}$	$(0, \infty)$
$f(x) = \log_a x, a > 0$ (Logarithmic function)	$\mathbb{R}^+$	$(-\infty, \infty)$

Mind Map

Function → Domain and Range → Types of functions
- Polynomial function
- Real valued function
- Rational function

Self-Assessment

Find the range of function $f(x) = \ln(x^2 - 2x + 3)$

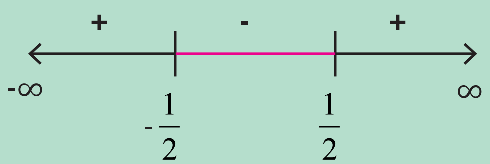
Answers

Concept Check

1. Here, $f(x)$ is defined for all the values of x.
   $\therefore$ **Domain:** $x \in \mathbb{R}$
   Range: $f(x) \in \mathbb{R}$

2. Let $f(x) = y = \dfrac{x}{x^2+1}$

   $\Rightarrow x = x^2y + y$
   $\Rightarrow x^2y - x + y = 0$
   $x \in \mathbb{R}$ (Given)
   $\Rightarrow (-1)^2 - 4 \times y \times y \geq 0$
   $\Rightarrow 4y^2 - 1 \leq 0$
   $\Rightarrow \left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq 0$



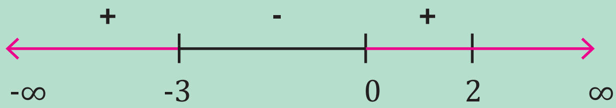
   By using the wavy curve method, we get,
   $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$
   $\therefore$ **Option (d) is the correct answer.**

3. Given, $f(x) = \left(1+\frac{3}{x}\right)^{\frac{1}{x-2}}$
   As the given $f(x)$ is in the form of $h(x)^{g(x)}$, it must satisfy following conditions: $h(x) = \left(1+\frac{3}{x}\right) > 0$

and $g(x) = \frac{1}{x-2} \Rightarrow x - 2 \neq 0$

$\Rightarrow \frac{(x+3)x}{x^2} > 0$ and

$x \neq 2$



By using the wavy curve method, we get,

$x \in (-\infty, -3) \cup (0, \infty)$ and $x \neq 2$

$\Rightarrow x \in (-\infty, -3) \cup (0, 2) \cup (2, \infty)$

4. To define $f(x)$, it must satisfy the following conditions:

$x^2 + 1 > 0$ and $2x - 3 \neq 0$

$\Rightarrow x \in \mathbb{R}$ and $x \neq \frac{3}{2}$

$\therefore$ **Domain:** $\mathbb{R} - \left\{\frac{3}{2}\right\}$

Self Assessment

Given, $f(x) = \ln(x^2 - 2x + 3)$

Base $= e > 0$ (not equal to 1); this means that the graph of this function will be similar to the graph of logarithmic function where $a > 1$

We know that the functions inside log must be greater than zero.

$\Rightarrow x^2 - 2x + 3 > 0$

$\Rightarrow (x - 1)^2 + 2 > 0$

$\Rightarrow x^2 - 2x + 3 \geq 2$ (As $(x - 1)^2 \geq 0 \ \forall \ x \in \mathbb{R}$, the domain is $x \in \mathbb{R}$)

On taking ln on both the sides, we get,

$\ln(x^2 - 2x + 3) \geq \ln 2$

(In an increasing function, the inequality sign does not change)

$f(x) \geq \ln 2$

Range: $f(x) \in [\ln 2, \infty)$.

BYJU'S Classes

MATHEMATICS

RELATIONS AND FUNCTIONS

TYPES OF FUNCTIONS

NOTES

What you already know

- Definition of a function
- Domain and range of a function

What you will learn

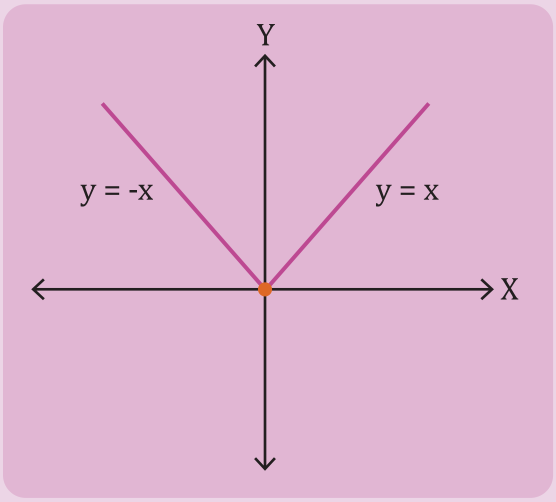
- Modulus function
- Greatest integer function
- Fractional part function
- Signum function
- Trigonometric function

Types of Functions

Modulus function

A modulus function is a function which gives the absolute value of a number or variable. It is also termed as an absolute value function. The outcome of this function is always non-negative, no matter what input has been given to the function.

Any value → Modulus function → non-negative



Mathematically, the definition of modulus function,

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain: $x \in \mathbb{R}$

Range: $y \in \mathbb{R}^{+} \cup \{0\}$ or $[0, \infty)$

Find the domain and the range of function $f(x) = \frac{\sqrt{x^2}}{|x|}$

Solution

Given, $f(x) = \frac{\sqrt{x^2}}{|x|}$

And

$|x| \neq 0$ (The denominator should not be equal to zero.)

$\Rightarrow x \neq 0$

Therefore,

Domain: $x \in \mathbb{R} - \{0\}$

$$\Rightarrow f(x) = \frac{|x|}{|x|} = 1$$

Range: $y \in \{1\}$

Greatest integer function (G.I.F)

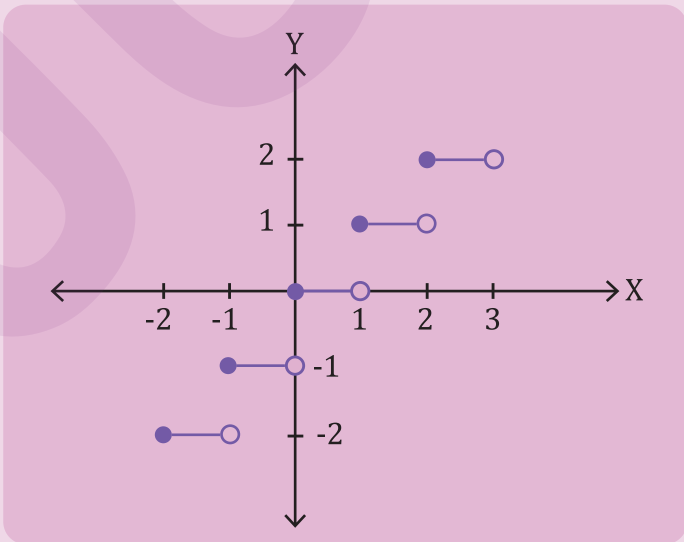
$y = [x]$ is the greatest integer less than or equal to x.

Examples:

$[2.3] = 2$ (Greatest integer less than or equal to 2.3 is 2.)

$[7.99] = 7$ (Greatest integer less than or equal to 7.99 is 7.)

$[-3.7] = -4$ (Greatest integer less than or equal to -3.7 is -4.)



$$y = [x] = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ -1, & -1 \le x < 0 \\ -2, & -2 \le x < -1 \\ -3, & -3 \le x < -2 \end{cases}$$

This is also referred as a step function.

Domain: $\mathbb{R}$

Range: $y \in \mathbb{I}$ or $\mathbb{Z}$ [$\mathbb{I}$, $\mathbb{Z}$ both represent the set of integers]

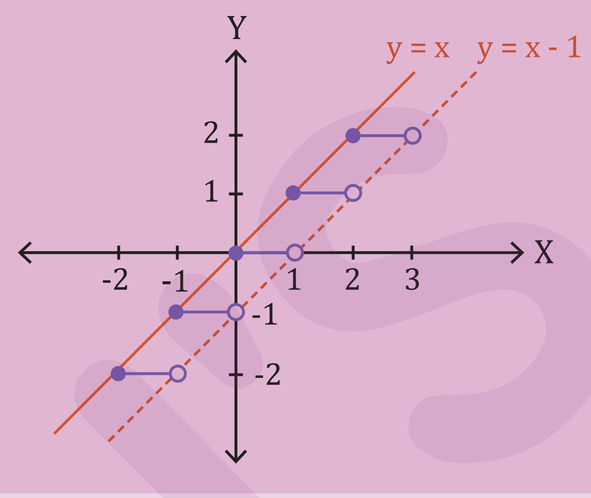
Properties

- $x - 1 < [x] \le x$

  Example:
  Let $x = 2.3$
  $2.3 - 1 < [2.3] \le 2.3$
- $[x + m] = [x] + m$; for $m \in \mathbb{Z}$

  Example:
  $[2.5 + 3] = [5.5] = 5$
  or $[2.5 + 3] = [2.5] + 3$
  $= 2 + 3 = 5$
- $[x]+[-x]=\begin{cases}0, & x \in \mathbb{I}\\ -1, & x \notin \mathbb{I}\end{cases}$

  Example:
  $[7] + [-7] = 7 - 7 = 0$
  $[7.1] + [-7.1] = 7 - 8 = -1$



Find the domain and the range of $f(x) = [x + 1] + 1$, where [.] denotes the G.I.F.

Solution

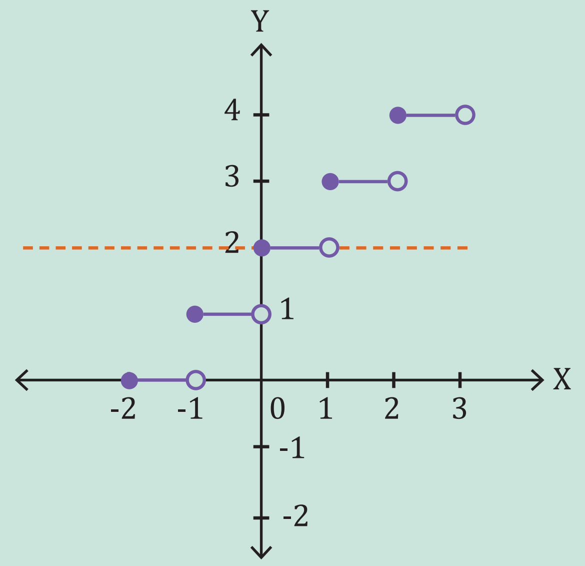
Method 1: Algebraic method

$f(x) = [x + 1] + 1 = [x] + 1 + 1$ (Using $\{[x + m] = [x] + m$, for $m \in \mathbb{Z}\}$)
$\Rightarrow f(x) = [x] + 2$
Domain: $x \in \mathbb{R}$, **Range:** $y \in \mathbb{Z}$

Method 2: Graphical method

$f(x) = [x + 1] + 1$
Using $\{[x + m] = [x] + m$, for $m \in \mathbb{Z}\}$, we get
$f(x) = [x + 1] + 1 = [x] + 2$
$\Rightarrow f(x) = [x] + 2$

Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{Z}$



Fractional part function

Any real number x can be written as $x = [x] + \{x\}$, where [.] represents the greatest integer function and {.} represents the fractional part function.

$\Rightarrow \{x\} = x - [x]$

Graph of fractional part function

We know that

$y = \{x\} = x - [x]$

$\Rightarrow 0 \le \{x\} < 1$

For plotting the graph of the fractional part function, let us break the domain into multiple parts.

Consider $0 \le x < 1$

$\Rightarrow [x] = 0 \Rightarrow y = \{x\} = x - 0$

$\Rightarrow y = x$

For $1 \le x < 2$,

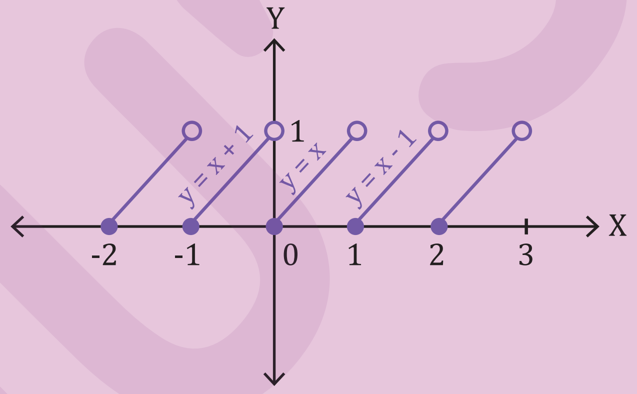
$[x] = 1 \Rightarrow y = \{x\} = x - 1$

$\Rightarrow y = x - 1$

For $-1 \le x < 0$,

$[x] = -1 \Rightarrow y = \{x\} = x + 1$

$\Rightarrow y = x + 1$



Domain of the fractional part function is $x \in \mathbb{R}$

Range of the fractional part function is $y \in [0, 1)$

Properties

- $\{x + n\} = \{x\}, n \in \mathbb{Z}$

  Example:

  Consider $x = 5.2$; then,

  $\{x + 3\} = \{8.2\} = 0.2 = \{x\}$

- $\{x\} + \{-x\} = 0$, if $x \in \mathbb{Z}$

  $\{x\} + \{-x\} = 1$, if $x \notin \mathbb{Z}$

  Example:

  Consider $x = 9 \in \mathbb{Z}$; then,

  $\{x\} + \{-x\} = \{9\} + \{-9\} = 0 + 0 = 0$

  Now, consider $x = 9.1 \notin \mathbb{Z}$

  $\{x\} + \{-x\} = \{9.1\} + \{-9.1\} = 0.1 + 0.9 = 1 \quad [\{-9.1\} = -9.1 + 10]$

Find the domain and the range of function $f(x) = 2\{x + 1\} + 3$, where $\{.\}$ denotes the fractional part function.

Solution

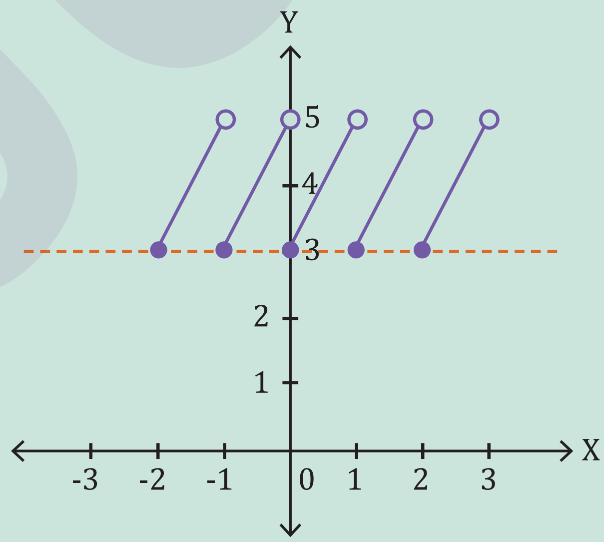
Method 1: Algebraic method

Given, $f(x) = 2\{x + 1\} + 3$, where $\{.\}$ denotes the fractional part function.
We have,
$f(x) = 2\{x + 1\} + 3$
$\Rightarrow f(x) = 2\{x\} + 3$ $\quad [\because \{X + n\} = \{X\}, n \in \mathbb{Z}]$
Now, we know that,
$0 \le \{x\} < 1$
$\Rightarrow 0 \le 2\{x\} < 2$ (Multiplying by 2 throughout the inequality)
$\Rightarrow 3 \le 2\{x\} + 3 < 5$ (Adding 3 throughout the inequality)
Therefore, the range of $f(x)$ is $[3, 5)$.
Domain: $x \in \mathbb{R}$

Method 2: Graphical method

Given, $f(x) = 2\{x + 1\} + 3$, where $\{.\}$ denotes the fractional part function.
After drawing the graph, we can see that,

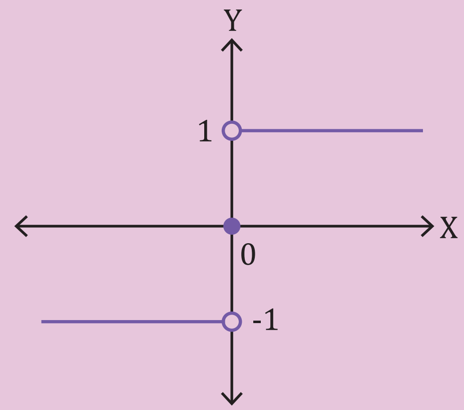
Domain: $x \in \mathbb{R}$
Range: $y \in [3, 5)$



Signum function

$$y = \operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$



The domain of the signum function is $\mathbb{R}$, and the range of the signum function has only three values, i.e., $y \in \{-1, 0, 1\}$. The graph of the signum function is shown in the above figure.

Note

$\text{sgn}(\text{sgn}(\text{sgn}(\ldots\ldots \text{sgn}(x)) = \text{sgn}(x)$

Explanation: Let us take $x = 5$. Then,

R.H.S $= \text{sgn}(5) = 1$

L.H.S $= \text{sgn}(\text{sgn}(\text{sgn}(\ldots\ldots \text{sgn}(5))$

$= \text{sgn}(\text{sgn}(\text{sgn}(\ldots\ldots \text{sgn}(1))$

$= \text{sgn}(\text{sgn}(\text{sgn}(1)) = \text{sgn}(\text{sgn}(1)) = \text{sgn}(1) = 1$

$\Rightarrow$ L.H.S = R.H.S

Find the domain and the range of function $f(x) = \text{sgn}\left(\dfrac{x^3 + x^2}{x+1}\right)$

Solution

We have,

$$f(x) = \text{sgn}\left(\frac{x^3 + x^2}{x+1}\right)$$

$\Rightarrow x + 1 \neq 0 \Rightarrow x \neq -1$ (∵ The denominator cannot be zero.)

Therefore, the domain is all real numbers except -1, i.e., $x \in \mathbb{R} - \{-1\}$

Now, let $y = f(x)$

$$\Rightarrow y = \text{sgn}\left(\frac{x^3 + x^2}{x+1}\right)$$

$y = \text{sgn}(x^2)$

For $x \neq 0$,

$\text{sgn}(x^2) = 1$ $[\because x^2 > 0]$

For $x = 0$,

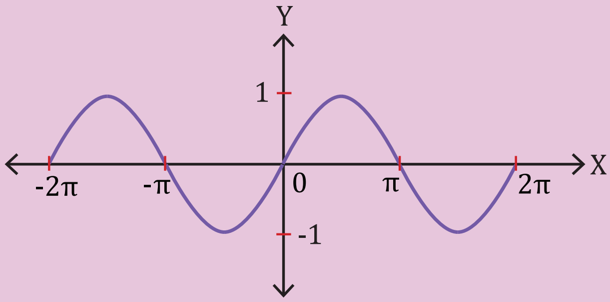
$\text{sgn}(x^2) = 0$

Therefore, the range of $f(x)$ is $\{0, 1\}$, i.e., $y \in \{0, 1\}$.

Trigonometric Function

f(x) = sin x

The graph of the sine function is

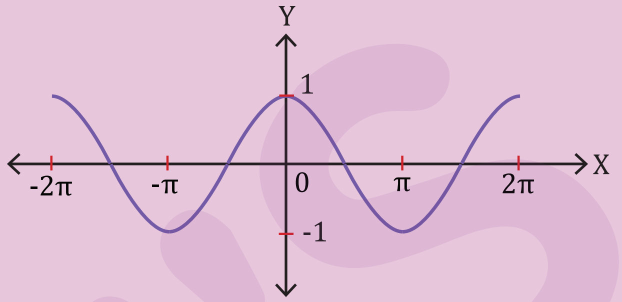


Domain: $x \in \mathbb{R}$

Range: $y \in [-1, 1]$

f(x) = cos x

The graph of the cosine function is

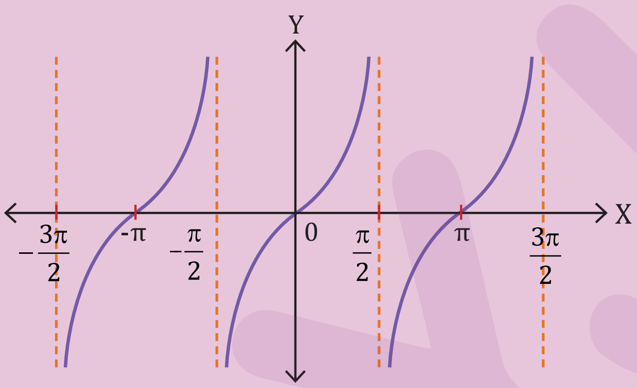


Domain: $x \in \mathbb{R}$

Range: $y \in [-1, 1]$

f(x) = tan x

The graph of the tangent function is

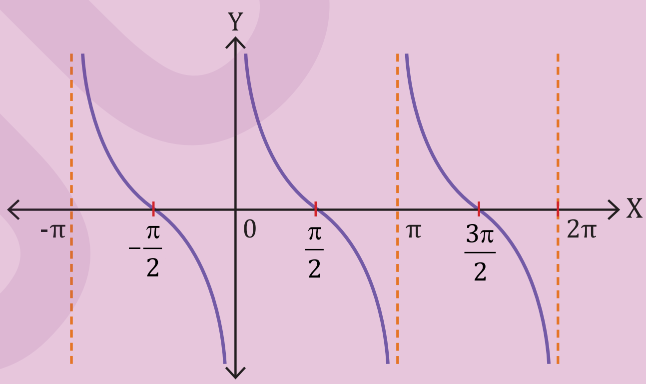


Domain: $x \in \mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$

Range: $y \in \mathbb{R}$

f(x) = cot x

The graph of the cotangent function is

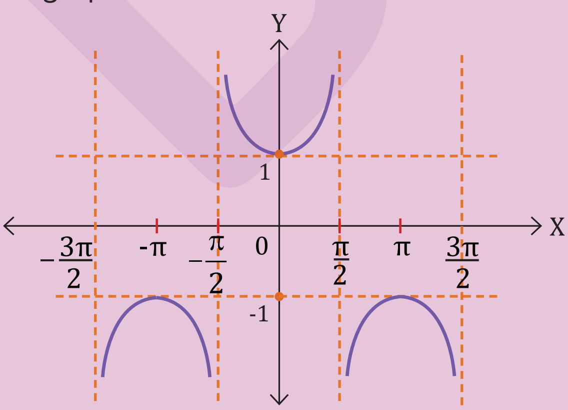


Domain: $x \in \mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$

Range: $y \in \mathbb{R}$

f(x) = sec x

The graph of the secant function is

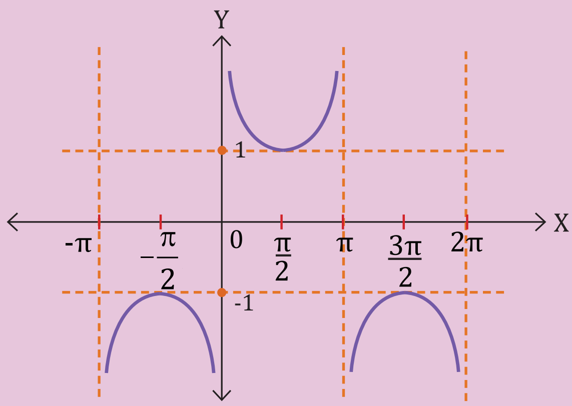


Domain: $x \in \mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$

Range: $y \in (-\infty, -1] \cup [1, \infty)$

f(x) = cosec x

The graph of the cosecant function is



Domain: $x \in \mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$

Range: $y \in (-\infty, -1] \cup [1, \infty)$

Solve $2[x] = x + 2\{x\}$

Solution

Given, $2[x] = x + 2\{x\}$
Substituting $x = \{x\} + [x]$ ------- (i),
We get, $2[x] = \{x\} + [x] + 2\{x\}$

$\Rightarrow [x] = 3\{x\} \Rightarrow \{x\} = \dfrac{[x]}{3}$ -------- (ii)

As, $0 \leq \{x\} < 1$

$\Rightarrow 0 \leq \dfrac{[x]}{3} < 1 \Rightarrow 0 \leq [x] < 3 \Rightarrow [x] = 0, 1, 2$

Now, we will solve the equation by taking different cases.

Case 1: When $[x] = 0$
Considering equations (i) and (ii), we get,
$x = 0 + 0 = 0$
$x = 0$

Case 2: When $[x] = 1$
Considering equations (i) and (ii), we get,
$x = \dfrac{1}{3} + 1 = \dfrac{4}{3}$
$\Rightarrow x = \dfrac{4}{3}$

Case 3: When $[x] = 2$
Considering equations (i) and (ii), we get,
$x = \dfrac{2}{3} + 2 = \dfrac{8}{3}$
$\Rightarrow x = \dfrac{8}{3}$

Thus, $x = \left\{0, \dfrac{4}{3}, \dfrac{8}{3}\right\}$

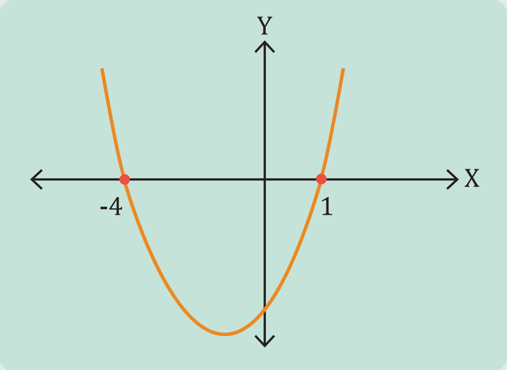
It is given that $y = \text{sgn}(x^2 + 3x - 4)$. find x for:

(a) $y = 1$ (b) $y = 0$ (c) $y = -1$

Solution

Given, $y = \text{sgn}(x^2 + 3x - 4)$
$\Rightarrow y = \text{sgn}[(x + 4)(x - 1)]$
By the definition of signum function,

$$y = \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$



When $y = 1$
$\Rightarrow x \in (-\infty, -4) \cup (1, \infty)$

When $y = 0$
$\Rightarrow x \in \{-4, -1\}$

When $y = -1$
$\Rightarrow x \in (-4, -1)$

Equal or Identical Function

Functions $f(x)$ and $g(x)$ are said to be equal or identical if,

- Domain of function f = Domain of function g
- Range of function f = Range of function g
- $f(x) = g(x)$ [graphs of both the functions overlap each other]

Example

$f(x) = x$, $g(x) = x^2 \times \frac{1}{x}$

Clearly, $D_f = \mathbb{R}$ and $D_g = \mathbb{R} - \{0\}$

Thus, functions f and g are not equal.

Find whether the functions $f(x) = 1$ and $g(x) = \sin^2 x + \cos^2 x$ are identical or not.

Solution

Given, $f(x) = 1$ and $g(x) = \sin^2 x + \cos^2 x$

Clearly, $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$

$R_f = 1 = R_g$

Thus, functions f and g are equal or identical.

Domain for Special Function

For ${}^nC_r$ and ${}^nP_r$ to be defined,

$n \in \mathbb{N},\ r \in \mathbb{W},\ n \ge r$

Find the range of function $f(x) = {}^{5-x}C_{3x-4}$

(a) $\{2\}$ (b) $\{3\}$ (c) $\{4\}$ (d) $\{5\}$

Solution

Given, $f(x) = {}^{5-x}C_{3x-4}$

As we know that,

For ${}^nC_r$ and ${}^nP_r$ to be defined,

$n \in \mathbb{N},\ r \in \mathbb{W},\ n \ge r$

$$\Rightarrow \begin{cases} (5-x) \in \mathbb{N} \ \text{--------(i)} \\ (3x-4) \in \mathbb{W} \ \text{---------(ii)} \\ (5-x) \ge (3x-4) \ \text{--------(iii)} \end{cases}$$

$\Rightarrow x = 2$

Now, substituting $x = 2$ in the original expression, we get,

${}^3C_2 = 3$

Hence, option (b) is the correct answer.

Concept Check

1. Find the domain of function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ **JEE MAIN 2011**

   (a) $(-\infty, \infty)$ (b) $(0, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, \infty) - \{0\}$

2. Find the range of function $f(x) = 1 - |x - 2|$

   (a) $(-\infty, 1]$ (b) $(-\infty, 2]$ (c) $(1, \infty)$ (d) $(-\infty, \frac{1}{2}]$

3. Find the domain of $f(x) = \sqrt{1 - [x]^2}$, where [.] denotes the G.I.F.

   (a) $(1, 2]$ (b) $[-1, 2)$ (c) $(1, 2)$ (d) $(-1, 0)$

4. Find the range of the following function: $f(x) = x^{[x]}$, $x \in [1, 3]$, where $[x]$ denotes the G.I.F.

5. Find the range of function $f(x) = \dfrac{\{x\}}{1 + \{x\}}$, where {.} denotes the fractional part function.

6. Find the domain and the range of function $f(x) = \text{sgn}(9 - 7 \sin x)$

   (a) $x \in \mathbb{R}, y = 1$ (b) $x \in \mathbb{R}, y = -1$ (c) $x \in \mathbb{R}, y = 0$ (d) $x \in \mathbb{R}, y \in \{1, -1\}$

7. Find the domain of function $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$.

8. Find the range of function $f(x) = \cos^2 x + 2 \cos x - 3$.

9. Find whether the following pairs of functions are identical or not.

   (a) $f(x) = \ln e^x$ and $g(x) = e^{\ln x}$

   (b) $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x^2}$

   (c) $f(x) = \tan x$ and $g(x) = \dfrac{1}{\cot x}$

Summary Sheet

Key Results

- Domain of modulus function: $x \in \mathbb{R}$
- Range of modulus function: $y \in \mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$
- Domain of greatest integer function: $\mathbb{R}$
- Range of greatest integer function: $y \in \mathbb{I}$ or $\mathbb{Z}$
- Domain of the fractional part function: $x \in \mathbb{R}$
- Range of the fractional part function: $y \in [0, 1)$

- The domain of the signum function is $\mathbb{R}$.
- The range of the signum function has only three values, i.e., $y \in \{-1, 0, 1\}$.

Mind Map

Types of functions
- Modulus function
- Greatest integer function
- Fractional part function
- Signum function
- Trigonometric functions

Self-Assessment

Find the domain and the range of function $f(x) = x^3 + 10$.

Answers

Concept Check

1.

Given, function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

Clearly, $|x| - x > 0$

Now, we will consider different cases:

Case 1: When $x \geq 0$

$\Rightarrow |x| - x = x - x = 0, \qquad [|x| = x, \text{ when } x \geq 0]$

Thus, $x \in \varphi$

Case 2: When $x < 0$

$\Rightarrow |x| - x = -x - x > 0, \qquad [|x| = -x, \text{ when } x < 0]$

$\Rightarrow x < 0$

Thus, $x \in (-\infty, 0)$

So, the domain is **(Case 1)** U **(Case 2)**

$\Rightarrow x \in (-\infty, 0)$

Hence, option (c) is the correct answer.

2.

Method 1: Algebraic method

Given, $f(x) = 1 - |x - 2|$

As we know that $|x| \geq 0, \forall x \in \mathbb{R}$

$\Rightarrow |x - 2| \geq 0$

Now, after multiplying by the -1 and then adding 1, we get,

$1 - |x - 2| \leq 1$

$\Rightarrow$ Range of function $f(x) = 1 - |x - 2|$ is $(-\infty, 1]$

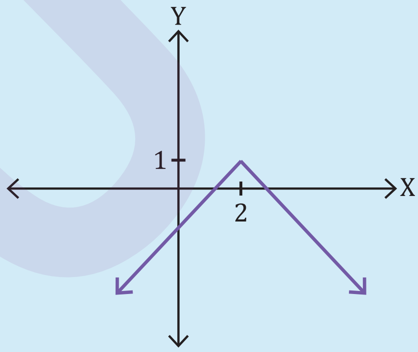
Hence, option (a) is the correct answer.

Method 2: Graphical method

Given, $f(x) = 1 - |x - 2|$

After drawing the figure, we can see that the span of graph on the vertical axis is $(-\infty, 1]$.

Hence, option (a) is the correct answer.



3.

Step 1:

Given, $f(x) = \sqrt{1 - [x]^2}$

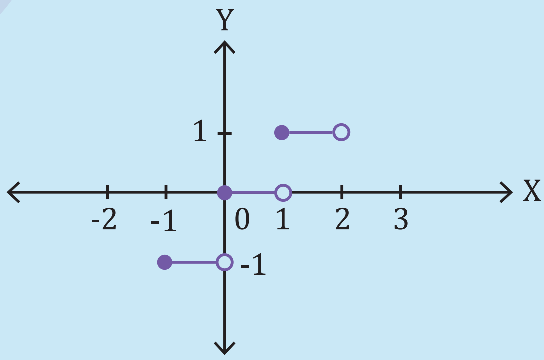
Where [.] denotes the G.I.F.

Now, $1 - [x]^2 \geq 0$

(The value inside the square root must be greater than equal to zero.)

$\Rightarrow [x]^2 - 1 \leq 0$ (Multiplying by -1 on both sides.)

$\Rightarrow [x]^2 \leq 1$ (Add 1 on both the sides.)



Step 2:

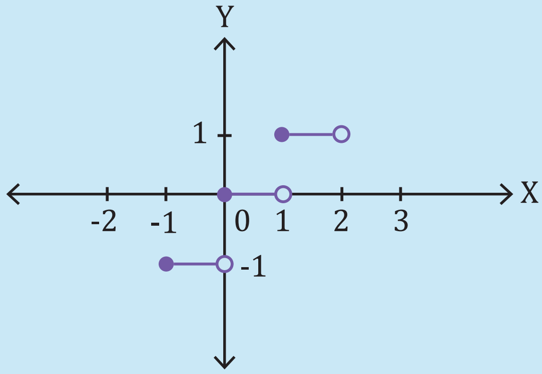
$\Rightarrow -1 \leq [x] \leq 1$

$\Rightarrow [x] = -1, 0, 1$

(Only integer values are to be considered.)

$\Rightarrow x \in [-1, 2)$

So, option (b) is the correct answer.



4.

Given, $f(x) = x^{[x]}, x \in [1, 3]$
Now, we will consider different cases to solve the problem.

Case 1: When $x \in [1, 2)$
$[x] = 1$
$\Rightarrow y = x^{[x]} = x^1$
If $x \in [1, 2)$
$\Rightarrow y \in [1, 2)$ ---- (i)

Case 2: When $x \in [2, 3)$
$[x] = 2$
$\Rightarrow y = x^{[x]} = x^2$
If $2 \le x < 3$
$\Rightarrow 4 \le x^2 < 9$
$\Rightarrow y \in [4, 9)$ ---- (ii)

Case 3: When $x = 3$
$[x] = 3$
$\Rightarrow y = x^{[x]} = 3^3 = 27$
$\Rightarrow y \in \{27\}$ ---- (iii)

The range of the given function is (i) U (ii) U (iii), i.e., $f(x) \in [1, 2) \cup [4, 9) \cup \{27\}$.

5.

Step 1:

We have,

$$f(x) = \frac{\{x\}}{1 + \{x\}}$$

The denominator $1 + \{x\} \neq 0$

Therefore, the domain of $f(x)$ is $\mathbb{R}$.

Now, in order to find the range, let $y = f(x)$

$$\Rightarrow y = \frac{\{x\}}{1 + \{x\}}$$

$$\Rightarrow y + y\{x\} = \{x\}$$

$$\Rightarrow \{x\} = \frac{y}{1 - y}$$

Step 2:

We know that $0 \le \{x\} < 1$

$$\Rightarrow 0 \le \{x\} < 1$$

$$\Rightarrow 0 \le \frac{y}{1 - y} < 1$$

$$\Rightarrow \frac{y}{1 - y} \ge 0 \text{ and } \frac{y}{1 - y} < 1$$

$$\Rightarrow \frac{y}{y - 1} \le 0 \text{ and } \frac{2y - 1}{1 - y} < 0$$

$$\Rightarrow \frac{y}{y - 1} \le 0 \text{ and } \frac{2y - 1}{y - 1} > 0$$

$$\Rightarrow y \in [0, 1) \text{ and } y \in (-\infty, \frac{1}{2}) \cup (1, \infty)$$

Taking the intersection of the two obtained sets, we get,

$y \in \left[0, \frac{1}{2}\right)$

Therefore, the range of $f(x)$ is $\left[0, \frac{1}{2}\right)$.

6.

Given, $f(x) = \text{sgn}(9 - 7\sin x)$
Since sin x is defined for all real numbers, domain $x \in \mathbb{R}$

As we know that $-1 \le \sin x \le 1$
$\Rightarrow -7 \le 7\sin x \le 7$
$\Rightarrow 7 \ge -7\sin x \ge -7$
$\Rightarrow 9 + 7 \ge 9 - 7\sin x \ge 9 - 7$
$\Rightarrow 16 \ge 9 - 7\sin x \ge 2$

By the definition of signum function, we get
$f(x) = \text{sgn}(9 - 7\sin x) = 1 \ \forall \ (9 - 7\sin x) \in [2, 16]$

Hence, option (a) is the correct answer.

7.

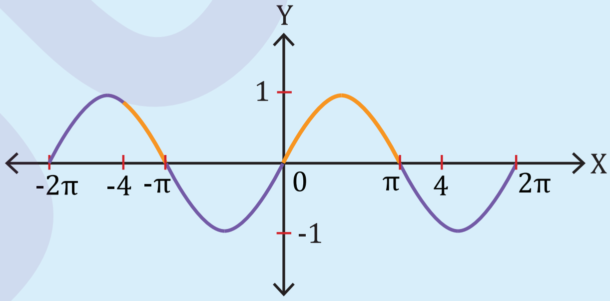
Given, $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

Clearly, $16 - x^2 \ge 0$

$\Rightarrow (x - 4)(x + 4) \le 0$
$\Rightarrow x \in [-4, 4]$

Now, after drawing the graph of sin x, we can see that,

Domain $x \in [-4, -\pi] \cup [0, \pi]$



8.

Given, $f(x) = \cos^2 x + 2\cos x - 3$

After re-writing in terms of perfect square, we get
$f(x) = (\cos x + 1)^2 - 4$

Now, as we know that $-1 \le \cos x \le 1$
$\Rightarrow 0 \le (\cos x + 1) \le 2$
$\Rightarrow 0 \le (\cos x + 1)^2 \le 4$
$\Rightarrow -4 \le (\cos x + 1)^2 - 4 \le 0$
$\Rightarrow f(x) \in [-4, 0]$

9.

(a) $f(x) = \ln e^x$ and $g(x) = e^{\ln x}$
Clearly, $D_f = \mathbb{R}$ and $D_g = \mathbb{R}^+$
Thus, functions f and g are not equal.

(b) $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$
Clearly, $D_f = \mathbb{R} - \{0\}$ and $D_g = \mathbb{R} - \{0\}$
$R_f = \mathbb{R} - \{0\} = R_g$
Thus, functions f and g are equal.

(c) $f(x) = \tan x$ and $g(x) = \dfrac{1}{\cot x} = \dfrac{1}{\dfrac{\cos x}{\sin x}}$

Clearly, $D_f = \mathbb{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$ and $D_g = \mathbb{R} - \{n\pi, (2n+1)\frac{\pi}{2}$, where $n \in \mathbb{Z}\}$

Thus, functions f and g are not equal.

Self-Assessment

Step 1:

Since $f(x)$ is a polynomial, its domain is $\mathbb{R}$.

$f(x) = x^3 + 10 = y$

$y = x^3 + 10$

$\Rightarrow$ Range = $\mathbb{R}$

BYJU'S Classes

MATHEMATICS

RELATIONS AND FUNCTIONS

ONE-ONE AND MANY-ONE FUNCTIONS

NOTES

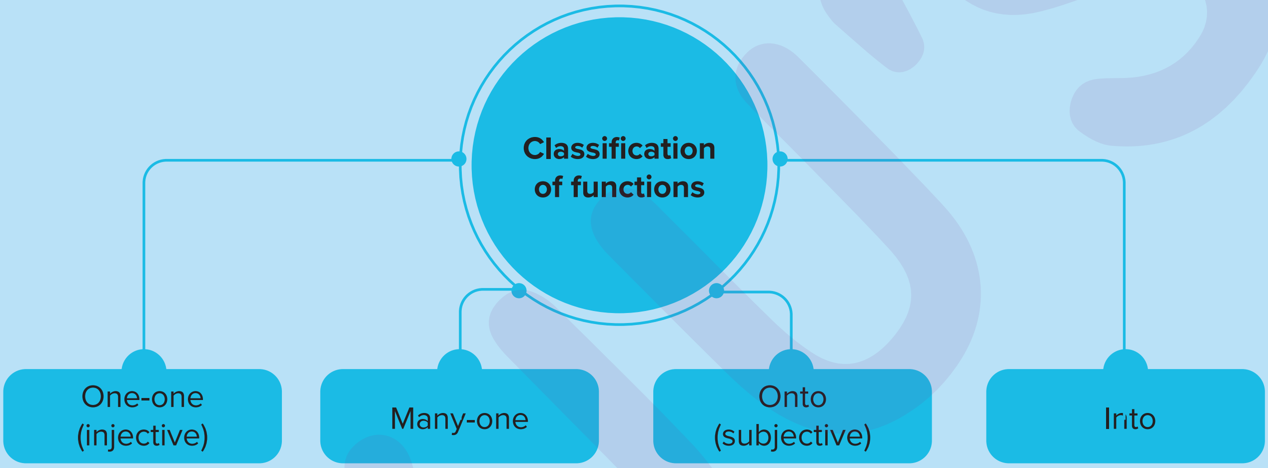
What you already know

- Relation
- Function
- Domain and range

What you will learn

- One-one function (Injective)
- Many-one function
- Number of one-one and many-one functions

Classification of Functions



One-one function (Injective function/injective mapping)

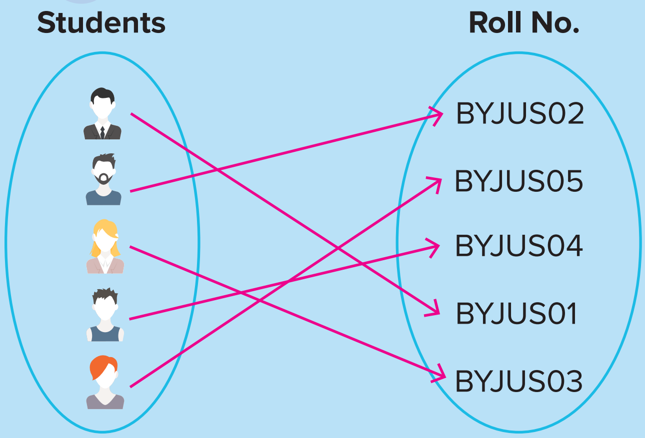
Name	Kishor
Roll no.	BYJUS01

Name	Roohi
Roll no.	BYJUS03

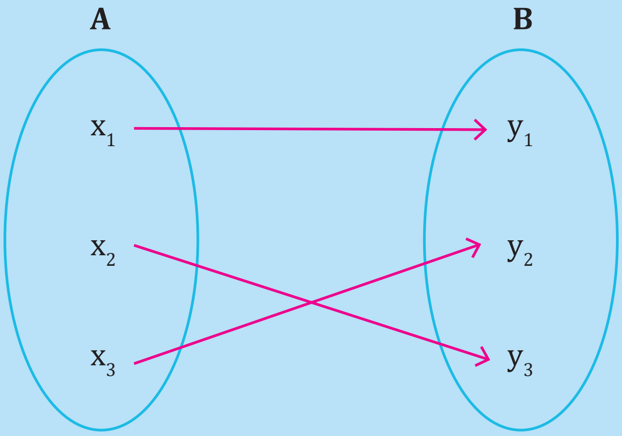
Name	Aryan
Roll no.	BYJUS04

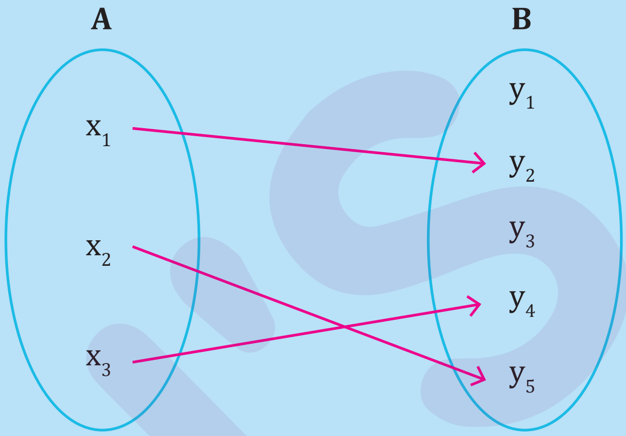
Name	Arya
Roll no.	BYJUS02

Name	Alia
Roll no.	BYJUS05



In the above mapping, two different roll numbers cannot be allotted to the same student, so it is a one-one mapping.
A function $f : A \rightarrow B$ is said to be a one-one function if different elements of set A have different images in set B.
Consider the following mappings:





In both the mappings, every element of set A is related to only one element of set B. No two elements in set A are related to the same element in set B. Therefore, both the mappings are injective or one-one.

Note

$f : A \rightarrow B$ is a one-one function iff same elements have the same images.

Methods to determine whether a function is one-one or not

Let $f : A \rightarrow B$ be a function, x_1 and x_2 be the two elements in set A, and the corresponding images of x_1 and x_2 in set B be $f(x_1)$ and $f(x_2)$, respectively. Now, for f to be a one-one function,
$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$

Check whether the given functions are injective or not.
(a) $f(x) = 3x + 5, f : \mathbb{R} \rightarrow \mathbb{R}$ (b) $f(x) = x^2, f : \mathbb{R} \rightarrow \mathbb{R}$ (c) $f(x) = x^2, f : \mathbb{N} \rightarrow \mathbb{N}$

Solution

(a)
Let $x_1, x_2 \in \mathbb{R}$
$f(x_1) = f(x_2)$
$\Rightarrow 3x_1 + 5 = 3x_2 + 5$
$\Rightarrow 3x_1 = 3x_2$
$\Rightarrow x_1 = x_2$
Therefore, the given function is a one-one function.

(b)
Let $x_1, x_2 \in \mathbb{R}$
$f(x_1) = f(x_2)$
$\Rightarrow x_1^{\ 2} = x_2^{\ 2}$
$\Rightarrow x_1^{\ 2} - x_2^{\ 2} = 0$
$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$
$x_1 = x_2$ or $x_1 = -x_2$
Here, we get the same output with two different inputs.
Therefore, it is not a one-one function.
$x_1 = -1 \Rightarrow f(-1) = 1$
$x_2 = 1 \Rightarrow f(1) = 1$

(c)
Let $x_1, x_2 \in \mathbb{N}$
$f(x_1) = f(x_2)$
$\Rightarrow x_1^2 = x_2^2$
$\Rightarrow x_1^2 - x_2^2 = 0$
$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$
$\Rightarrow x_1 = x_2$ or $x_1 \neq -x_2$ $\quad [\because x_1, x_2 \in \mathbb{N}]$
Therefore, the given function is a one-one function.

Many-one function

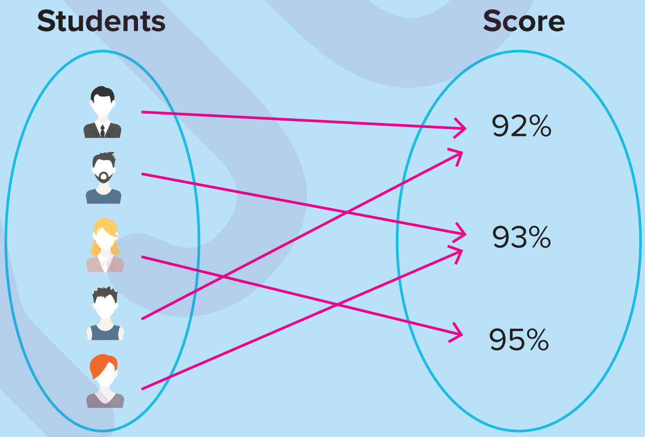
Name	Kishor
Roll no.	BYJUS01
Score	92%

Name	Roohi
Roll no.	BYJUS03
Score	95%

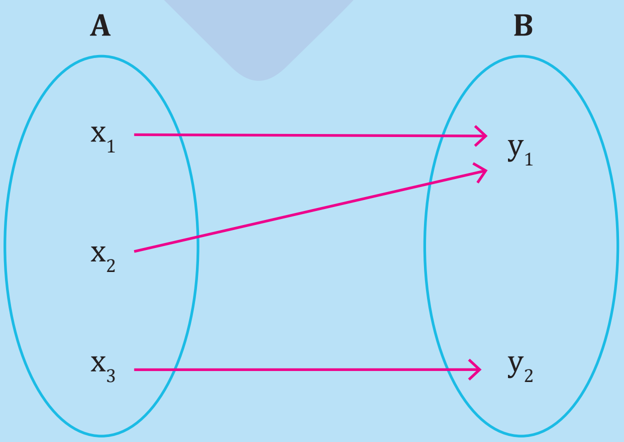
Name	Aryan
Roll no.	BYJUS04
Score	92%

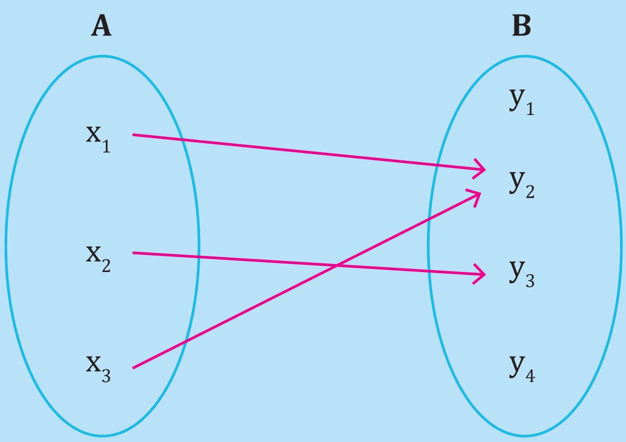
Name	Arya
Roll no.	BYJUS02
Score	93%

Name	Alia
Roll no.	BYJUS05
Score	93%



Two different students can get the same marks, therefore the above mapping is many-one.
A function $f : A \rightarrow B$ is said to be a many-one function if there exists at least two or more elements of set A that have the same image in set B.
Consider the following mappings:





In both the mappings, two elements of set A have the same image in set B. Therefore, both the mappings are many-one.

Methods to determine whether a function is one-one or many-one

For a function $f : A \rightarrow B$ to be many-one, there must be at least two elements, $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, $f(x_1), f(x_2) \in B$, but $x_1 = x_2$ will not be the only condition for that to happen.

Example: Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = x^2 + x + 2$

Let $f(x_1) = f(x_2)$, $x_1, x_2 \in \mathbb{R}$

$\Rightarrow x_1^{\,2} + x_1 + 2 = x_2^{\,2} + x_2 + 2$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow x_1 = x_2$ or $x_1 + x_2 + 1 = 0$

Here, $x_1 = x_2$ is not the only condition that we get. It implies that there are other x_1, x_2 conditions for which $f(x_1) = f(x_2)$ holds true. Therefore, the given function is a many-one function.

Note

If a function is one-one, then it cannot be many-one and vice versa.

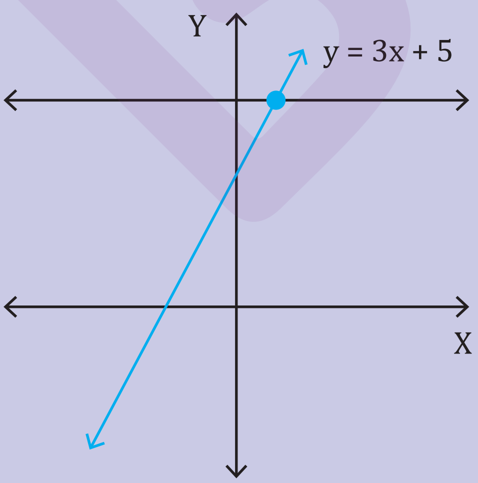
Graphical Test

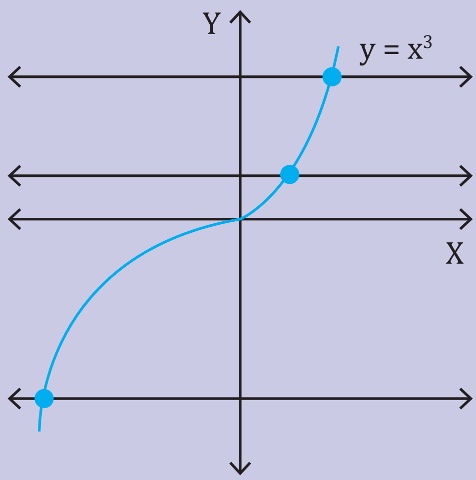
Horizontal line test

If we draw straight lines parallel to X-axis, and they cut the graph of the function at exactly one point, then the function is one-one.

Example: Consider the functions, $f(x) = 3x + 5$ and $f(x) = x^3$, $f : \mathbb{R} \rightarrow \mathbb{R}$

Both $f(x) = 3x + 5$ and $f(x) = x^3$ have only one point of intersection with the line parallel to the X-axis. Therefore, they are one-one functions.

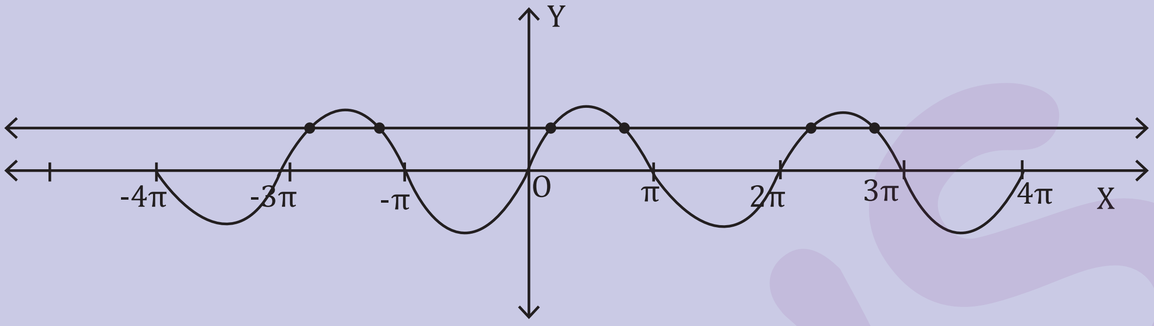




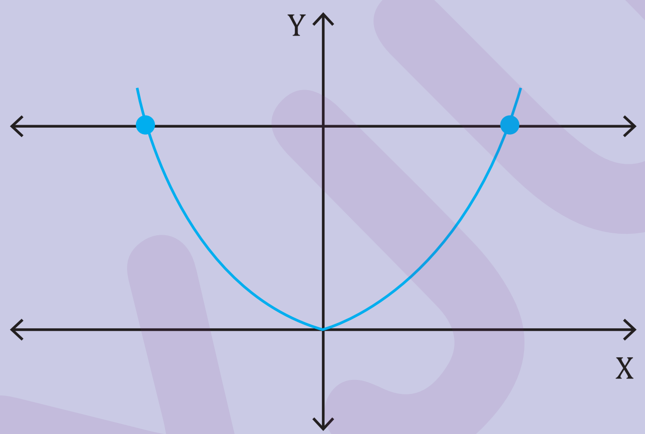
If there exists a straight line parallel to the X-axis, which cuts the graph of the function at two or more points, then the function is many-one.

Example:

- Consider $f(x) = \sin x$, $f : \mathbb{R} \to \mathbb{R}$
  The function $f(x) = \sin x$ has more than two points of intersection with the line parallel to the X-axis. Therefore, it is a many-one function.



- Consider $f(x) = x^2$, $f : \mathbb{R} \to \mathbb{R}$
  The function $f(x) = x^2$ has two points of intersection with the line parallel to the X-axis. Therefore, it is a many-one function.

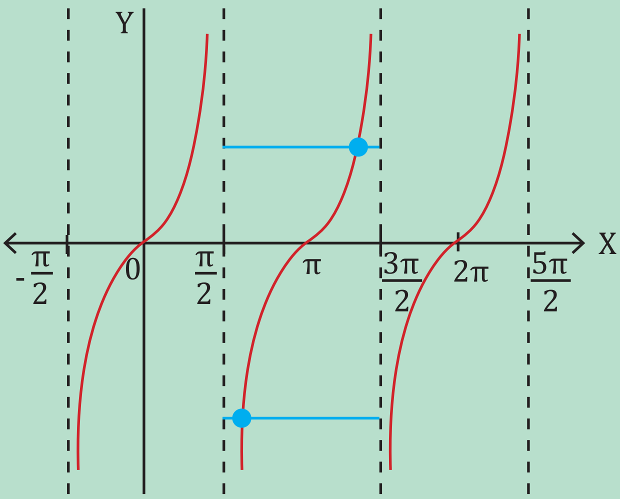


Identify the following function as one-one or many-one. $f(x) = 2\tan x;\ f : \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}$

Solution

Observe that the domain is restricted to $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

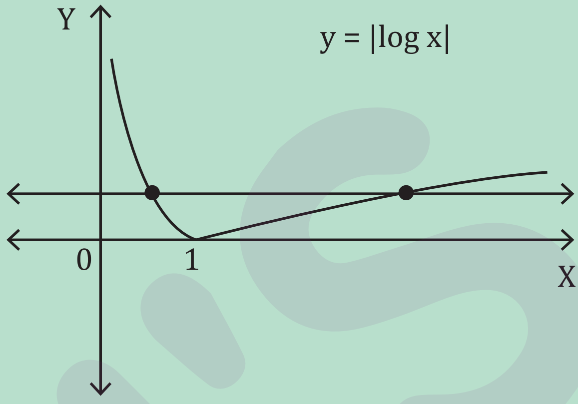
Clearly, the given function has only one point of intersection with the lines parallel to the X-axis. Therefore, it is a one-one function.



Identify the following function as one-one or many-one.
$f(x) = |\log x|$

Solution

Given, $f(x) = |\log x|$
Clearly, the given function has more than two points of intersection with the lines parallel to the X-axis. Therefore, it is a many-one function.



Number of Functions

Let a function be $f : A \rightarrow B$
$A = \{x_1, x_2, x_3, x_4\}$, $B = \{y_1, y_2, y_3, y_4, y_5\}$
$n(A) = 4$ and $n(B) = 5$.
Each element of A can choose one element out of the five elements from B.
Thus, total number of functions from A to B is $5 \times 5 \times 5 \times 5 = 5^4$

Generally, if $n(A) = m$ and $n(B) = n$,
Then, the total number of functions from A to B is $n \times n \times \ldots$ m times $= n^m$

Number of one-one functions

Let a function be $f : A \rightarrow B$
$A = \{x_1, x_2, x_3, x_4\}$, $B = \{y_1, y_2, y_3, y_4, y_5\}$
$n(A)= 4$ and $n(B) = 5$
Here, x_1 has to choose one out of five $\{y_1, y_2, y_3, y_4, y_5\}$.
Let us say x_1 chose y_5(5 ways).
Now, x_2 has to choose one out of four $\{y_1, y_2, y_3, y_4\}$.
Let us say x_2 chose y_4(4 ways).
Now, x_3 has to choose one out of three $\{y_1, y_2, y_3\}$.
Let us say x_3 chose y_2(3 ways).
Now, x_4 has to choose one out of two $\{y_1, y_3\}$. i.e 2 ways.
$\therefore$ Number of one-one mappings $= 5 \times 4 \times 3 \times 2 = 5(5 - 1)(5 - 2)(5 - 3) = {}^5P_4 = {}^5C_4 \times 4!$ (Selection of any four outputs and arranging them)
Generally, if $n(A) = m$ and $n(B) = n$, then the number of one-one mappings will be the following:
(a) If $n \geq m$, then ${}^nP_m$
(b) If $n < m$, then one-one function does not exist = 0

Consider a function, $f : A \rightarrow B$
$A = \{x_1, x_2, x_3\}$, $B = \{y_1, y_2\}$
$n(A)= 3$ and $n(B) = 2$
Here, x_1 has to choose one out of two $\{y_1, y_2\}$.

Let us say x_1 chose y_2.
Now, x_2 has only single option y_1.
Also, x_3 has no choice.
Number of one-one mappings = $2 \times 1 \times 0 = 0$

Note

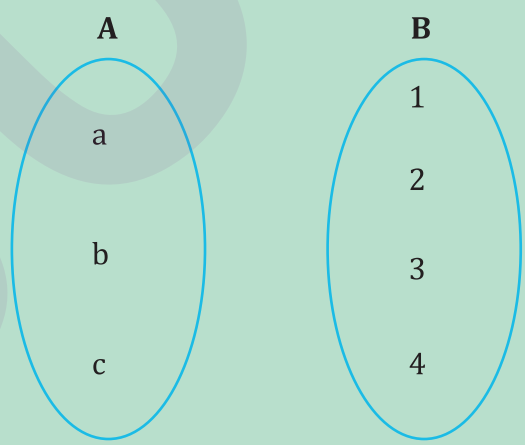
Number of many-one functions = (Total number of functions) - (Number of one-one functions)
In general, if $n(A) = m$ and $n(B) = n$, then the number of many-one mappings is $n^m - {}^nP_m$.

JEE MAIN-2020 (Sept)

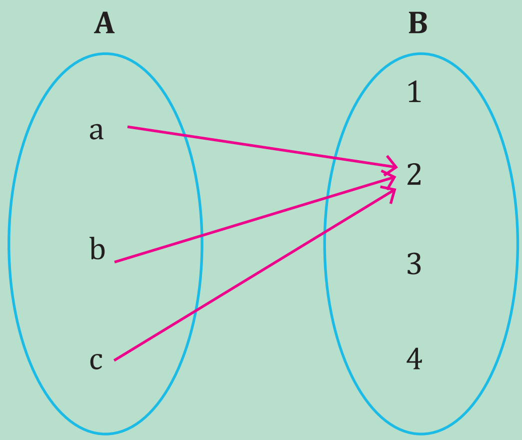
Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Find the number of elements in the set $C = \{f : A \to B|$ $2 \in f(A)$ and f is not one-one$\}$

Solution

Here, f is not one-one and $2 \in f(A)$.
We cannot consider three or more elements in set B as set A has three elements. 2 in B has to be considered always.
Hence, we have two cases.



Case 1:
f has only one image
Then, all the elements of A will map to 2 in B.
Hence, in this case, the number of elements in C will be 1.



Case 2:
f has only two images ('2' has to be there)
The number of elements in C will be ${}^3C_1\{2^3 - 2\}$.
${}^3C_1$ (To select one more image except the element '2' from B)
2^3 (Total number of functions from A to B)
2 (Two cases where all the elements of A map to either of two elements in B)

Subtracting 2 is necessary because of the following reasons:
Firstly, if all the elements of A are mapped to 2 in B, it becomes re-counting as this is exactly in case 1.
The other case is where all the elements of A are mapped to the other chosen element in B. This is not allowed as 2 has to be in the range of f.
Number of elements in C = ${}^3C_1\{2^3 - 2\} = 18$
Total number of elements in C = (Case 1) + (Case 2) = 1+ 18 = 19

Check if the function, f : ℝ → ℝ, is one-one or many-one, where f(x) = $\dfrac{x^2+4x+7}{x^2+x+1}$

Solution

Method 1

Given, f : ℝ → ℝ, where f(x) = $\dfrac{x^2+4x+7}{x^2+x+1}$

$\Rightarrow f(1) = \dfrac{1+4+7}{1+1+1} = 4$, and $f(-1) = \dfrac{1-4+7}{1-1+1} = 4$

Thus, the given function is a many-one function.

Method 2

Given, f : ℝ → ℝ, where f(x) = $\dfrac{x^2+4x+7}{x^2+x+1}$

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1^2+4x_1+7}{x_1^2+x_1+1} = \frac{x_2^2+4x_2+7}{x_2^2+x_2+1}$$

$$\Rightarrow x_1^2x_2^2 + x_1^2x_2 + x_1^2 + 4x_1x_2^2 + 4x_1x_2 + 4x_1 + 7x_2^2 + 7x_2 + 7$$
$$= x_1^2x_2^2 + 4x_1^2x_2 + 7x_1^2 + x_1x_2^2 + 4x_1x_2 + 7x_1 + x_2^2 + 4x_2 + 7$$
$$\Rightarrow 3x_1^2x_2 + 6x_1^2 - 3x_1x_2^2 + 3x_1 - 6x_2^2 - 3x_2 = 0$$
$$\Rightarrow 3x_1x_2(x_1 - x_2) + 6(x_1 - x_2)(x_1 + x_2) + 3(x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2)(3x_1x_2 + 6x_1 + 6x_2 + 3) = 0$$
$$\Rightarrow x_1 = x_2 \text{ or } x_1 = \frac{-(2x_2+1)}{x_2+2}$$

Thus, the given function is a many - one function.

Concept Check

1. Check whether the given functions are one-one or many-one functions.
   (a) $y = \log_2 x, x > 0$ (b) $y = \{x\}, x \in \mathbb{R}$ (c) $y = a^x, x \in \mathbb{R}$ (d) $y = \text{sgn}(x), x \in \mathbb{R}$

2. Identify the following function as one-one or many-one: $f(x)=\sqrt{\left(1-e^{\left(\frac{1}{x}-1\right)}\right)};(x\neq 0)$ in its domain.

3. If $A = \{1, 2, 3, 4\}$, then find the number of functions on set A that are not one-one. **JEE MAIN-2020 (Jan)**
   (a) 240 (b) 248 (c) 232 (d) 256

4. Let x denote the total number of one-one functions from set A with 3 elements to set B with 5 elements, and let y denote the total number of one-one functions from set A to set $A \times B$. Which of the following is correct?
   (a) $y = 273x$ (b) $2y = 91x$ (c) $y = 91x$ (d) $2y = 273x$

Summary Sheet

Key Takeaways

Horizontal line test

» If we draw a straight line parallel to the X-axis, and when it intersects the graph of the function at **exactly one point**, then the function is **one-one**.

» If there exists a straight line parallel to the X-axis, which cuts the graph of the function **at two or more points**, then the function is **many-one**.

- If $n(A) = m$ and $n(B) = n$, then
  Total number of functions from A to B is $n \times n \times \ldots\ m$ times $= n^m$
- If $n(A) = m$ and $n(B) = n$ then the number of one-one mappings is
  » ${}^{n}P_{m}$; where $n \geq m$
  » 0, (one-one function does not exist) ; where $n < m$
- The number of many-one mappings is $n^m - {}^{n}P_{m}$.

Mind Map

- Functions
  - One - one function
  - Many-one function
  - Number of functions

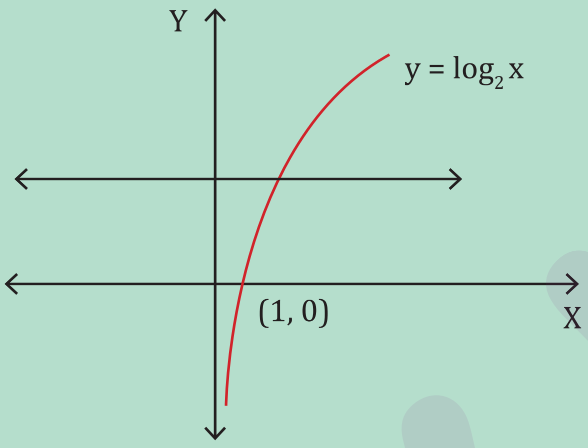
Self-Assessment

Identify $f(x) = |x|x^2, x \in \mathbb{R}$ as a one-one or a many-one function.

Answers

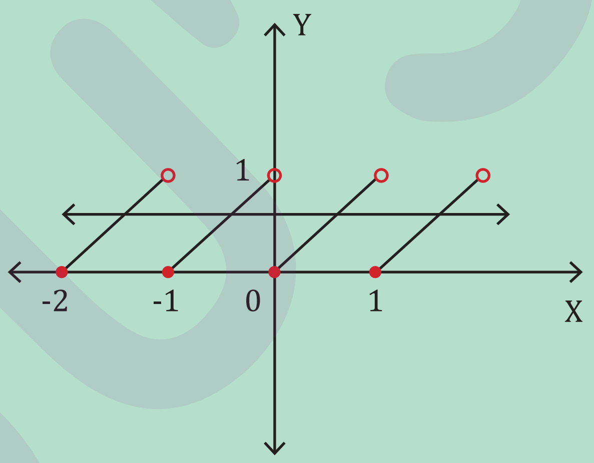
Concept Check

1. (a) Given, $y = \log_2 x, x > 0$



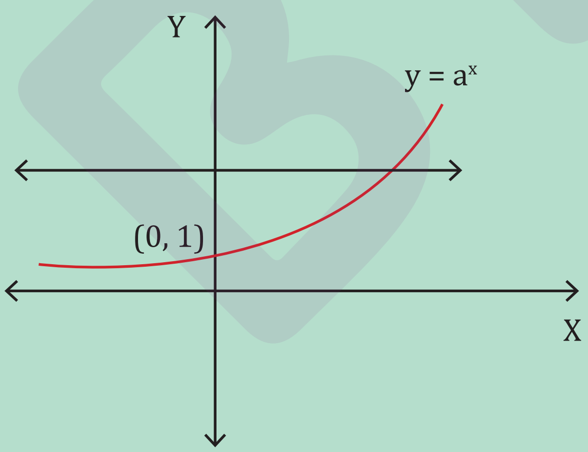
Clearly, the given function has only one point of intersection with the line parallel to the X-axis. Therefore, it is a one-one function.

(b) Given, $y = \{x\}, x \in \mathbb{R}$



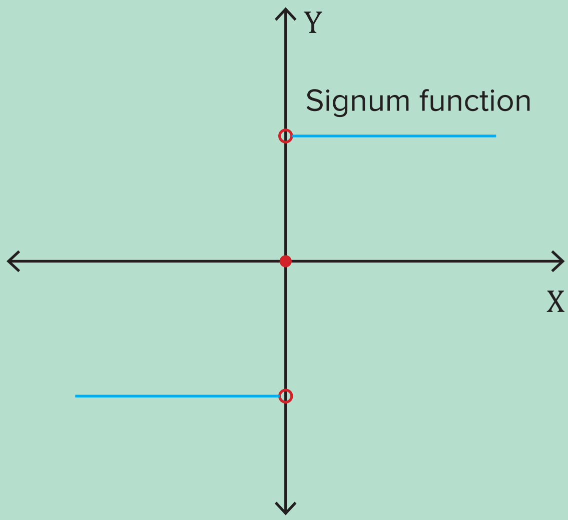
Clearly, the given function has more than one points of intersection with the line parallel to the X-axis. Therefore, it is a many-one function.

(c) Given, $y = a^x, x \in \mathbb{R}$



Clearly, the given function has only one point of intersection with the line parallel to the X-axis. Therefore, it is a one-one function.

(d) Given, $y = \text{sgn}(x), x \in \mathbb{R}$



Clearly, the given function has more than one points of intersection with the line parallel to the X-axis. Therefore, it is a many-one function.

2.

Consider

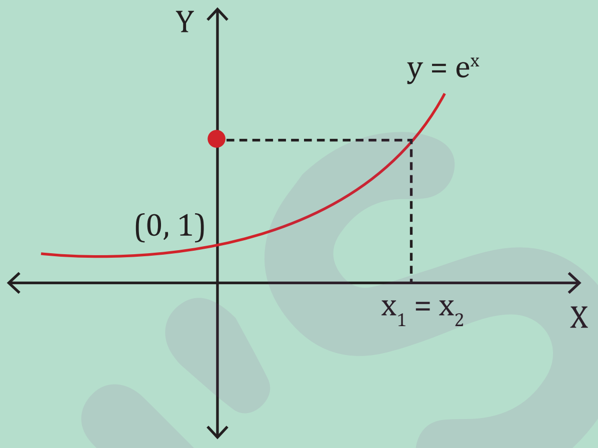
$f(x_1) = f(x_2); x_1, x_2$ in the domain

$$\Rightarrow \sqrt{\left(1 - e^{\left(\frac{1}{x_1} - 1\right)}\right)} = \sqrt{\left(1 - e^{\left(\frac{1}{x_2} - 1\right)}\right)}$$

On squaring both the sides, we get,

$$\Rightarrow \left(1 - e^{\left(\frac{1}{x_1} - 1\right)}\right) = \left(1 - e^{\left(\frac{1}{x_2} - 1\right)}\right)$$

$$e^{\frac{1}{x_1}} = e^{\frac{1}{x_2}}$$



From the graph, we can observe the following:

$$\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$$

Hence, the given function is a one - one function.

3.

Number of not one-one functions = (Total number of functions) - (Number of one-one functions)

$= n^m - {}^nP_m$

Here, $n = m = 4$

Number of non-one-one functions $= 4^4 - {}^4P_4 = 256 - 24 = 232$

Hence, option (c) is the correct answer.

4.

Given,

$n(A) = 3$

$n(B) = 5$

$n(A \times B) = 15$

Here,

x = Total number of one-one functions from set A to set B $= 5 \times 4 \times 3$ ---(i)

y =Total number of one-one functions from set A to set A × B $= 15 \times 14 \times 13$ ------(ii)

$\Rightarrow 2y = 91x$

Hence, option (b) is the correct answer.

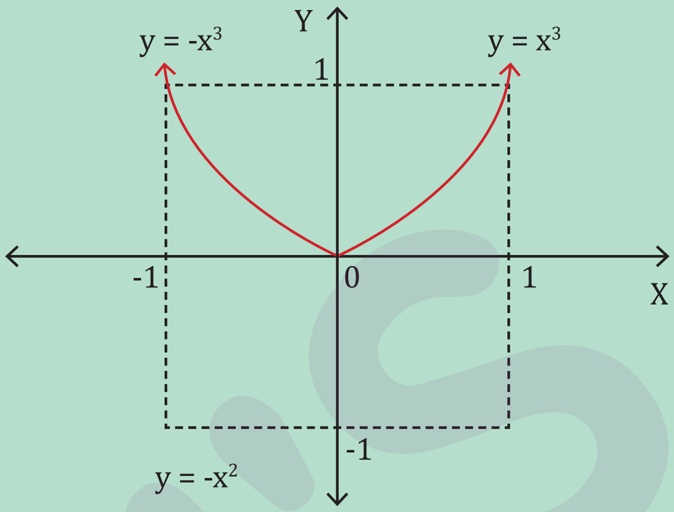
Self-Assessment

Method 1

Given, $f(x) = |x|x^2$

$$y = |x|x^2 = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$$

Clearly, a line drawn parallel to the X-axis intersects the graph at two distinct points. Therefore, the given function will not be a one-one function.



Method 2

f(1) = 1, f(-1) = 1

Hence, not a one-one function.

MATHEMATICS

RELATIONS AND FUNCTIONS

ONTO AND INTO FUNCTIONS

NOTES

What you already know

- Definition of a function
- Domain and range of a function
- Different types of functions

What you will learn

- Classification of functions
- Principle of inclusion and exclusion
- Number of onto functions

Classification of Functions

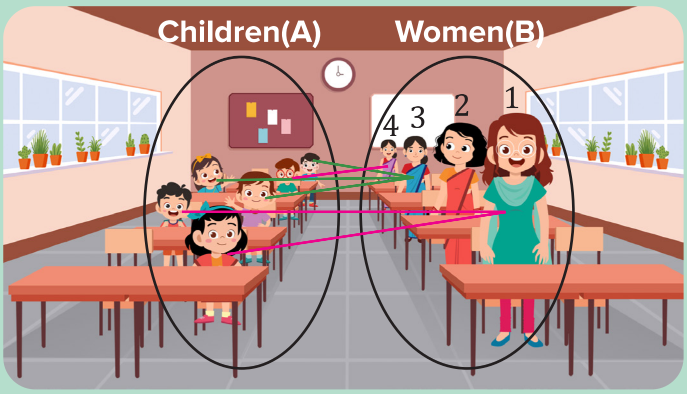
Classification of Functions

- One-One (Injective)
- Many-one
- Onto (Surjective)
- Into

Let us consider two sets, A and B,
Where A is a set of children and B is a set of women.
The association from A to B refers to ‘is the child of’.
Hence, all the elements of set A have to be connected to set B uniquely, as there cannot be a possibility of a child having more than one mother.
Now, let us discuss the two situations.

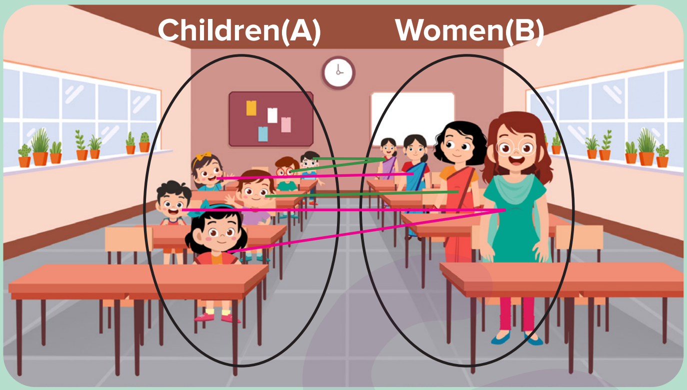
Situation 1:

Here, we can see that woman 2 is not connected to any child of the set A.
Hence,
Range $= \{1, 3, 4\}$
Codomain $= \{1, 2, 3, 4\}$
$\Rightarrow$ Range $\subset$ Codomain
Such a function is known as an into function.



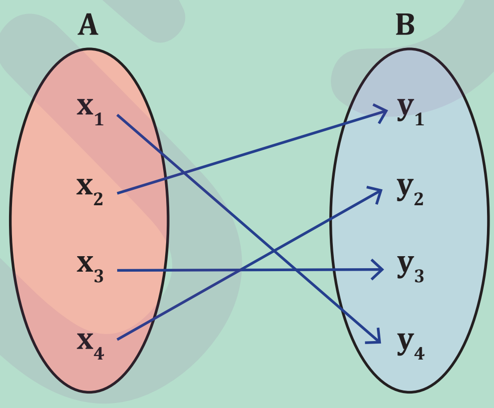
Situation 2:

In this case, all the women (elements of set B) are connected to at least one element of set A.
Hence,
Range = $\{1, 2, 3, 4\}$
Codomain = $\{1, 2, 3, 4\}$
$\Rightarrow$ Range = Codomain
Such a function is known as an onto function or surjective function.



Onto function (Surjective mapping)

If the function $f : A \to B$ is such that each element in the B (codomain) has at least one pre-image in A, then we say that the function $f : A \to B$ is an onto function.



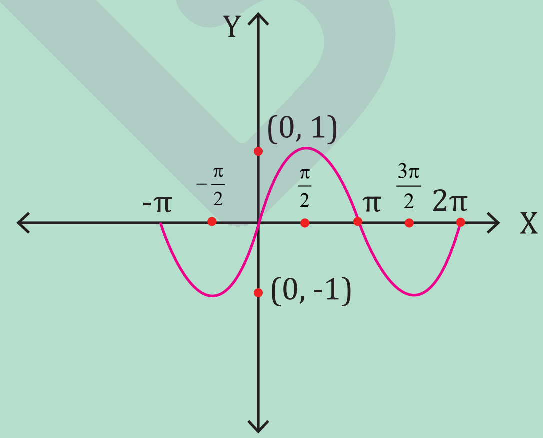
$f : A \to B$ is surjective iff $\forall\, b \in B$, there exists some $a \in A$ such that $f(a) = b$

Note:

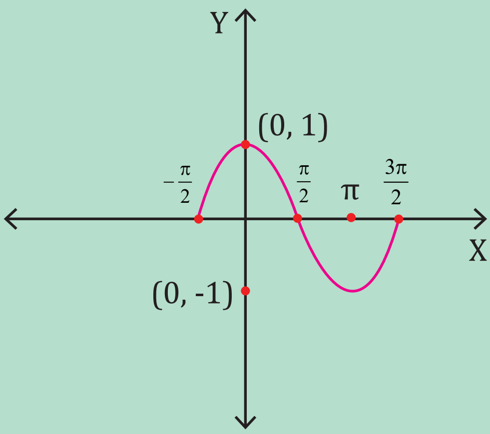
- If the range of f = codomain of f, then the function f is onto function
- If codomain of a function is not given then it is taken as $\mathbb{R}$.

Examples:

Let us consider a function, $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$.
Here, Range = Codomain
$\Rightarrow$ Function $f(x)$ is onto



Let us consider another function, $f : \mathbb{R} \to [-1, 2]$, $f(x) = \cos x$;
We know that the range of $f(x)$ is $[-1, 1]$
Here,
Range $\subset$ Codomain
$\Rightarrow$ Function $f(x)$ is not onto

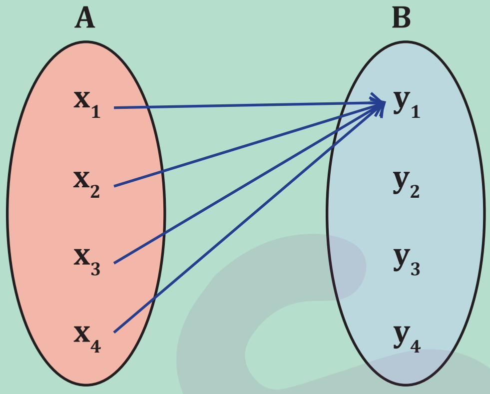


Into function

If the function $f : A \to B$ is such that there exists at least one element in B (codomain), which has no preimage in the set A, then $f : A \to B$ is called an into function.

Note:

- If the range of $f \neq$ codomain of f, then the function f is into function
- If a function is onto, it cannot be into and vice versa.

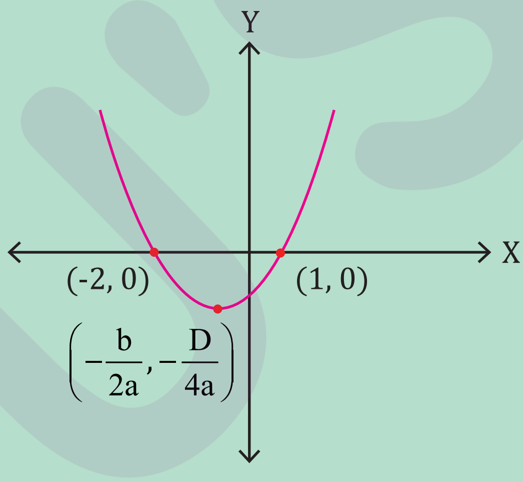


Examples:

(i) Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2 + x - 2$

$\Rightarrow f(x) = (x + 2)(x - 1)$

Here f(x) is a quadratic whose minimum point on the curve is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \equiv \left(-\frac{1}{2}, -\frac{9}{4}\right)$



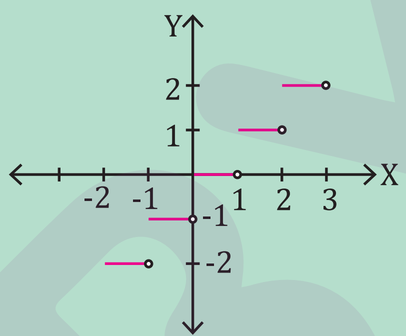
Hence, the range of the function f(x) is $\left[-\frac{9}{4}, \infty\right)$

But given, Codomain $= \mathbb{R}$

$\Rightarrow$ Range $\neq$ Codomain

$\Rightarrow$ The function is an into function

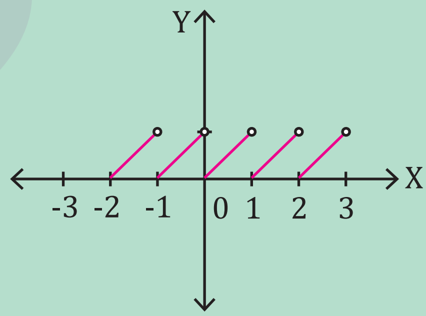
(ii) $f : \mathbb{R} \to \mathbb{R}, f(x) = [x]$



Range $= \mathbb{Z} \subset \mathbb{R}$

$\Rightarrow$ f(x) is an into function

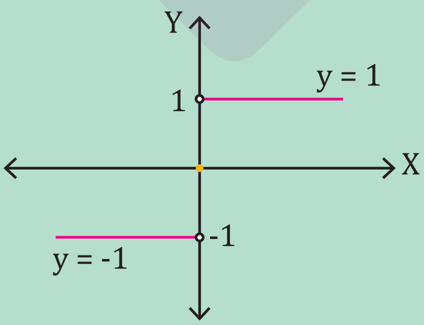
(iii) $f : \mathbb{R} \to \mathbb{R}, f(x) = \{x\}$



Range $= [0, 1) \subset \mathbb{R}$

$\Rightarrow$ f(x) is an into function

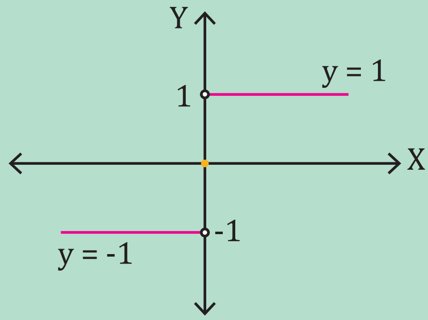
(iv) $f : \mathbb{R} \to \mathbb{R}, f(x) = \operatorname{sgn}(x)$



Range $= \{-1, 0, 1\} \subset \mathbb{R}$

$\Rightarrow$ f(x) is an into function

(v) $f : \mathbb{R} \to \{-1, 0, 1\}, f(x) = \operatorname{sgn}(x)$



Range $= \{-1, 0, 1\} =$ Codomain

$\Rightarrow$ f(x) is an onto function

Note

- When a function is both one-one and onto, then the function is known as a bijective function.
- A function can be of one of the following four types:
  (i) One-one, onto (injective and surjective); also known as bijective functions
  (ii) One-one, into (injective but not surjective)
  (iii) Many-one, onto (surjective but not injective)
  (iv) Many-one, into (neither surjective nor injective)

If the function $f : \mathbb{R} - \{-1, 1\} \rightarrow A$, defined by $f(x) = \dfrac{x^2}{1 - x^2}$ is surjective, then what is A equal to?

(a) $\mathbb{R} - [-1, 0)$ (b) $\mathbb{R} - \{-1\}$ (c) $\mathbb{R} - (-1, 0)$ (d) $[0, \infty)$

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Solution

Step 1:

Let $f(x) = y = \dfrac{x^2}{1 - x^2}$

$\Rightarrow x^2 = y - yx^2$

$\Rightarrow x^2(1 + y) = y$

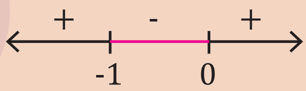
$\Rightarrow x^2 = \dfrac{y}{(1 + y)}$

$\Rightarrow x^2 = \dfrac{y(1 + y)}{(1 + y)^2}$

$\Rightarrow y(y + 1) \geq 0; y \neq -1$

Step 2:

By using the wavy curve method, we get,



$y \in (-\infty, -1) \cup [0, \infty)$

$\Rightarrow y \in \mathbb{R} - [-1, 0)$

As the given function is surjective,

$\Rightarrow$Range of f = Codomain of f

$\Rightarrow A \in \mathbb{R} - [-1, 0)$

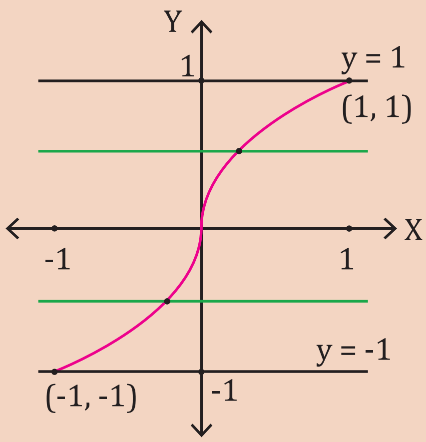
Option (a) is the correct answer.

$f(x) = \sin\left(\dfrac{\pi x}{2}\right) : [-1, 1] \rightarrow [-1, 1]$ is ____

(a) One-one, onto (b) Many-one, onto (c) One-one, into (d) Many-one, into

Solution

The function $f(x) = \sin\left(\dfrac{\pi x}{2}\right)$ can be plotted as shown in the diagram.



From the diagram, we can see

Range of the function = $[-1, 1]$

Hence, Range = Codomain

$\Rightarrow$ The function is onto

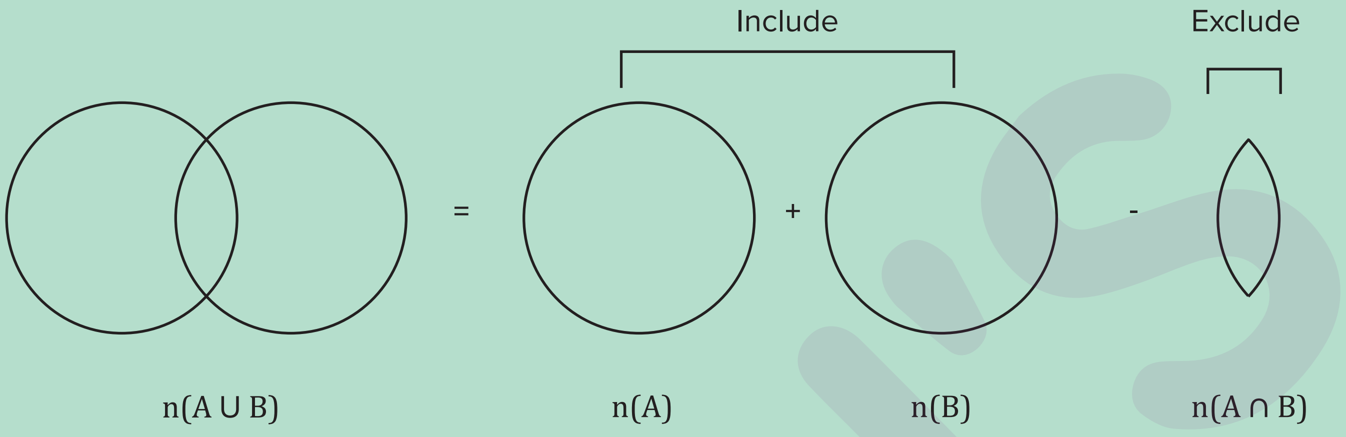
By the horizontal line test, $f(x)$ is one-one

$\Rightarrow$ The given function $f(x)$ is one-one and onto

$\therefore$ **Option (a) is the correct answer**

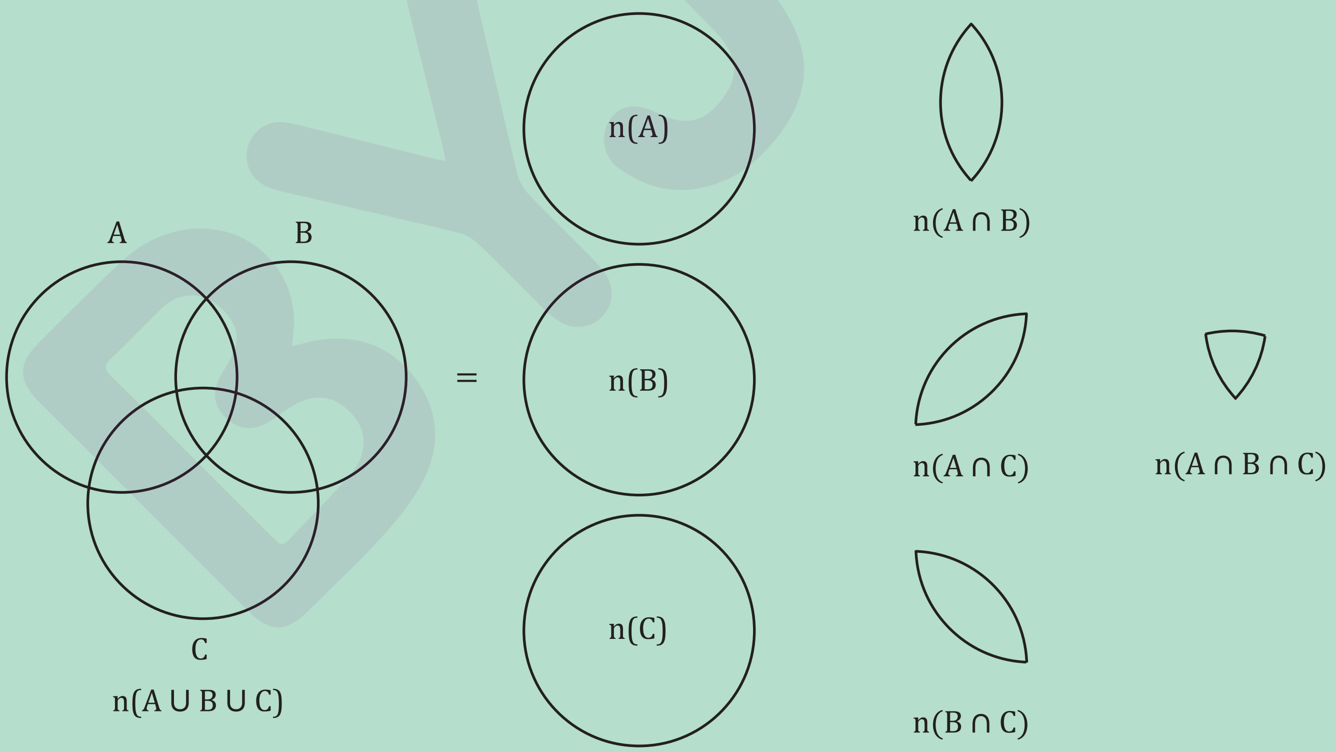
Principle of inclusion and exclusion

- Let us consider two sets, A and B.



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
As $n(A \cap B)$ is included in both $n(A)$ and $n(B)$, it is excluded once to get $n(A \cup B)$
Let $n(A) + n(B) = S_1$
And $n(A \cap B) = S_2$
Then, $n(A \cup B) = S_1 - S_2$

Similarly, for the three sets: A, B, and C.



$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
$\Rightarrow n(A \cup B \cup C) = S_1 - S_2 + S_3$
The obtained result can be generalised as follows:
$n(A_1 \cup A_2 \cup \ldots \cup A_n) = S_1 - S_2 + S_3 - \ldots + (-1)^{n+1}S_n$

Number of onto functions

Let the function be $f : A \to B$,
Where $n(A) = m$ and $n(B) = n$
Case 1: When $m > n$

- $n(A_i)$ = Total number of functions when y_i (from set B) is excluded = ${}^nC_1(n - 1)^m$
- $n(A_i \cap A_j)$ = Total number of functions when y_i and y_j are excluded = ${}^nC_2(n - 2)^m$
- Total number of functions where at least one element is excluded

$= n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$
$= \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \ldots$
$= {}^nC_1(n - 1)^m - {}^nC_2(n - 2)^m + \ldots$

Number of onto functions $= n^m - n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$
$= n^m - ({}^nC_1(n - 1)^m - {}^nC_2(n - 2)^m + ..)$

Case 2: When $m = n$, number of onto functions $= n!$
Case 3: When $n > m$, no onto functions exist.

Number of onto functions $\begin{cases} 0, (m < n) \\ n!, (m = n) \\ n^m - ({}^nC_1(n - 1)^m - {}^nC_2(n - 2)^m + ..), (m > n) \end{cases}$

Note

Number of into functions = (Total number of functions) - (Number of onto functions).

Find the number of ways of distributing 5 balls into three distinct boxes considering the following conditions:
(a) Any number of balls can go to any number of boxes
(b) Each box has at least one ball

Solution

5 balls represent set A.
3 boxes represent set B.
Mapping has to happen from A to B for the given conditions.
(a) Since each ball has 3 choices,
Number of ways = Total number of functions $= 3^5$
(b) Here, we want the number of onto functions as none of the boxes is empty in this case.
Therefore, Total number of onto functions $= 3^5 - [{}^3C_1(2)^5 - {}^3C_2(1)^5] = 150$

Which of the following defines the function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ as $f(x) = \frac{x}{1+x^2}$?

(a) Both injective and surjective
(b) Injective but not surjective
(c) Surjective but not injective
(d) Neither injective nor surjective

Solution

Step 1:

$f(x) = f(y)$

$\Rightarrow \frac{x}{1+x^2} = \frac{y}{1+y^2}$

$\Rightarrow x + xy^2 = y + yx^2$

$\Rightarrow x - y + xy^2 - yx^2 = 0$

$\Rightarrow (x - y) + xy(y - x) = 0$

$\Rightarrow (x - y)(1 - xy) = 0$

$\Rightarrow x = y;\ xy \neq 1$

$\Rightarrow$ f(x) is a one-one (injective) function.

Step 2:

Let $f(x) = y = \frac{x}{1+x^2}$

$\Rightarrow x = y + yx^2$

$\Rightarrow x^2y - x + y = 0$

For $x \in \mathbb{R}$,

$D \geq 0$

$\Rightarrow 1 - 4y^2 \geq 0$

$\Rightarrow (2y - 1)(2y + 1) \leq 0$

$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (By wavy curve method)

Hence, Range = Codomain

f(x) is an onto (surjective) function.

$\therefore$ **Option (a) is the correct answer.**

Concept Check

1. If $f : \mathbb{R} \to [a, b]$, $f(x) = 2\sin x - 2\sqrt{3}\cos x + 1$ is onto, then find the value of b - a.
2. If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x|x|$, then f(x) is
   (a) One-one, onto (b) Many-one, onto (c) One-one, into (d) Many-one, into
3. Let $A = \{x \in \mathbb{R};$ x is not a positive integer$\}$. Define a function $f : A \to \mathbb{R}$ as $f(x) = \frac{2x}{x-1}$, then which of the following represents f ?
   (a) Not injective
   (b) Injective but not surjective
   (c) Surjective but not injective
   (d) Neither injective nor surjective

Summary Sheet

Key Takeaways

- Function $f : A \to B$ is known as an onto function iff $\forall\ b \in B$, there exists some $a \in A$ such that $f(a) = b$.
- A function that is both one-one and onto is known as a bijective function.

- If the function is $f : A \rightarrow B$, where $n(A) = m$ and $n(B) = n$, then
  Case 1: When $m > n$,
  Number of onto functions = $n^m - ({}^nC_1(n-1)^m - {}^nC_2(n-2)^m + ..)$
  Case 2: When $m = n$,
  Number of onto functions = $n!$
  Case 3: When $n > m$,
  No onto functions exist.
- Number of into functions = Total number of functions - Number of onto functions

Mind Map

Classification of functions ← Priciples of exclusion and inclusion → Number of onto functions

Self-Assessment

Determine whether the following functions are into or onto functions.
(a) $f(x) = |x|$, where $x \in \mathbb{R}^+$
(b) $f(x) = c$, where c is a constant real number

Answers

Concept Check

1.

$f(x) = 2\sin x - 2\sqrt{3}\cos x + 1$

We know, $\alpha\cos\theta + \beta\sin\theta \in \left[-\sqrt{\alpha^2+\beta^2}, \sqrt{\alpha^2+\beta^2}\right]$

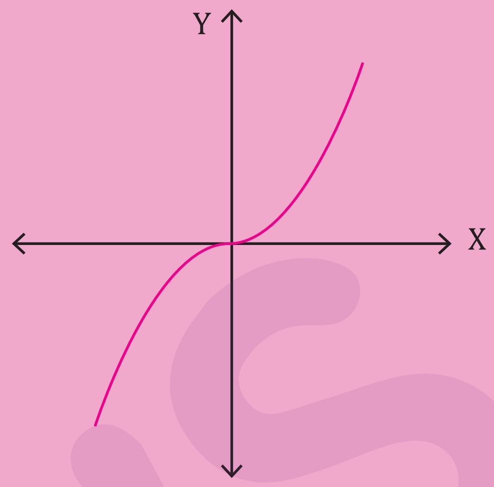
$\Rightarrow 2\sin x - 2\sqrt{3}\cos x \in \left[-\sqrt{2^2+\left(2\sqrt{3}\right)^2}, \sqrt{2^2+\left(2\sqrt{3}\right)^2}\right]$

$\Rightarrow 2\sin x - 2\sqrt{3}\cos x \in [-4, 4]$
$2\sin x - 2\sqrt{3}\cos x + 1 \in [-4+1, 4+1]$
$\Rightarrow f(x) \in [-3, 5]$
As the given function is onto, we get,
$\Rightarrow [a, b] = [-3, 5]$
$\Rightarrow b - a = 5 - (-3) = 8$

2.

$$f(x) = x|x| \begin{cases} -x^2, x \le 0 \Rightarrow f(x) \in (-\infty, 0] \\ x^2, x > 0 \Rightarrow f(x) \in (0, \infty) \end{cases}$$

The given function can be plotted as shown in the figure.
The range of the function is $\mathbb{R}$.
Here,
Range = Codomain
$\Rightarrow$The given function is an onto function.
By horizontal line test, $f(x)$ is a one-one function.
$\Rightarrow$The given function $f(x)$ is one-one and onto.
$\therefore$ **Option (a) is the correct answer.**



3.

Step 1:

$f(x) = f(y)$

$\Rightarrow \dfrac{2x}{x-1} = \dfrac{2y}{y-1}$

$\Rightarrow xy - x = xy - y$

$\Rightarrow x = y$

$\Rightarrow$ The function $f(x)$ is one-one (injective).

Step 2:

Let $f(x) = y = \dfrac{2x}{x-1}$

$\Rightarrow 2x = xy - y$

$\Rightarrow y = x(y - 2)$

$\Rightarrow x = \dfrac{y}{y-2}$

$\Rightarrow x = \dfrac{y(y-2)}{(y-2)^2}$

$\Rightarrow y(y - 2) \le 0 \quad (\because x \le 0)$

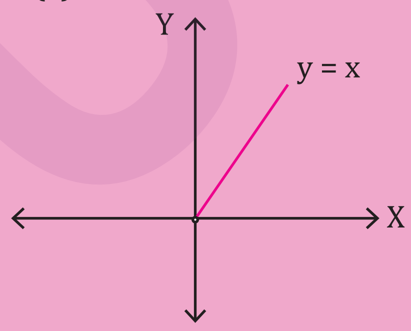
(y cannot be 2 as $y - 2$ is in denominator)

$\Rightarrow y \in [0, 2)$ (By wavy curve method)

Here,
Range $\neq$ Codomain
The function $f(x)$ is not onto (surjective).
$\therefore$ **Option (b) is the correct answer.**

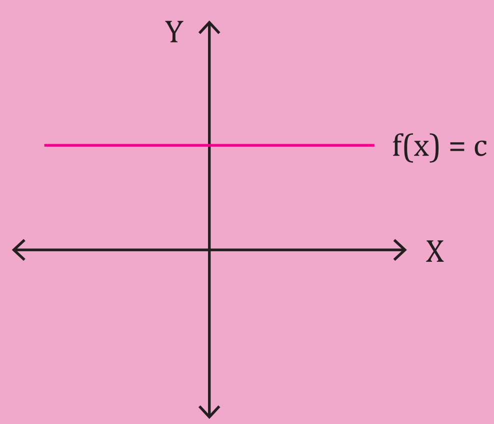
Self-Assessment

(a) $f(x) = |x|$, where $x \in \mathbb{R}^+$
For $x \in \mathbb{R}^+$, $f(x) = x$



For the domain, $x \in \mathbb{R}^+$, the range of the function is $y \in \mathbb{R}^+$.
As the co-domain is not given, it is taken as $\mathbb{R}$.
Since the range and the co-domain of the given function are not equal, the given function is an into function.

(b) $f(x) = c$, where c is a constant real number



The range of the function is $\{c\}$, while the co-domain is $\mathbb{R}$. Therefore, the given function is an into function.

RELATIONS AND FUNCTIONS

ODD AND EVEN FUNCTIONS

What you already know

- One-one and many-one functions
- Methods to identify one-one and many-one functions
- Onto functions and into functions

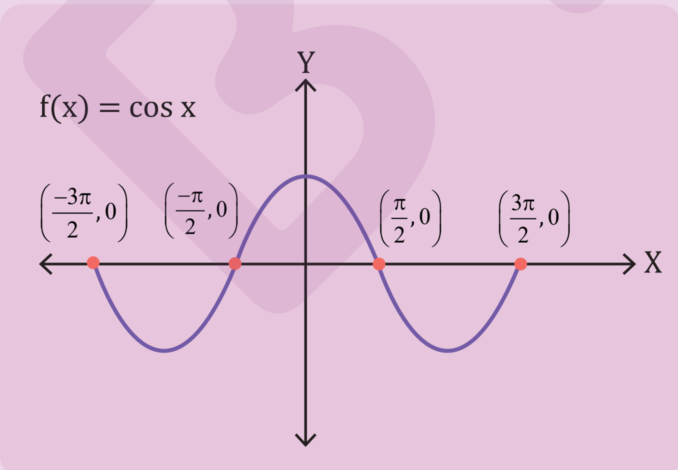
What you will learn

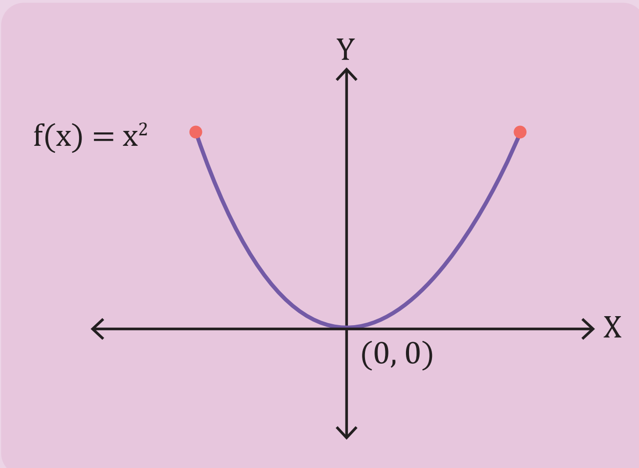
- Even and odd functions
- Properties of even and odd functions
- Composite function

Even Function

If $f(-x) = f(x)\ \forall\ x$ in the domain of f, then f is said to be an even function.
Some examples of even functions are, $f(x) = \cos x$, $g(x) = |x|$, and $h(x) = x^2 + 3$
The graph of every even function is symmetric about the Y-axis.

Example





In both the cases graph is symmetric about the Y-axis.

Identify whether function $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$ is even or not.

Solution

Step 1

Given, $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$

$$\Rightarrow f(-x)=\frac{-x}{e^{-x}-1}+\frac{-x}{2}+1$$

$$\Rightarrow f(-x)=\frac{-x}{\frac{1-e^x}{e^x}}-\frac{x}{2}+1$$

$$\Rightarrow f(-x)=\frac{-xe^x}{1-e^x}-\frac{x}{2}+1$$

$$\Rightarrow f(-x)=\frac{-xe^x+x-x}{1-e^x}-\frac{x}{2}+1$$

$$\Rightarrow f(-x)=\frac{x(e^x-1)+x}{e^x-1}-\frac{x}{2}+1$$

Step 2

$$\Rightarrow f(-x)=\frac{x}{e^x-1}+x-\frac{x}{2}+1$$

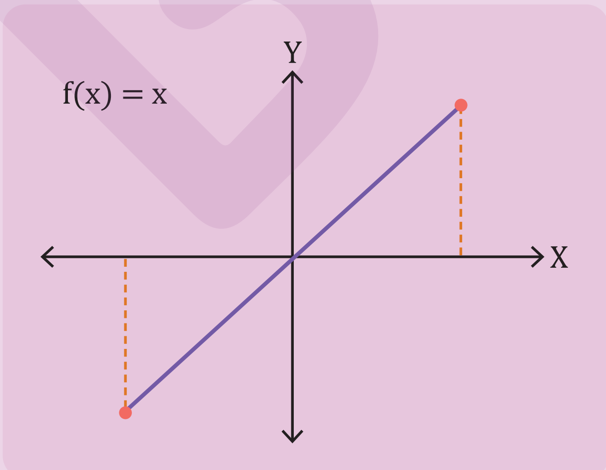
$$\Rightarrow f(-x)=\frac{x}{e^x-1}+\frac{x}{2}+1=f(x)$$

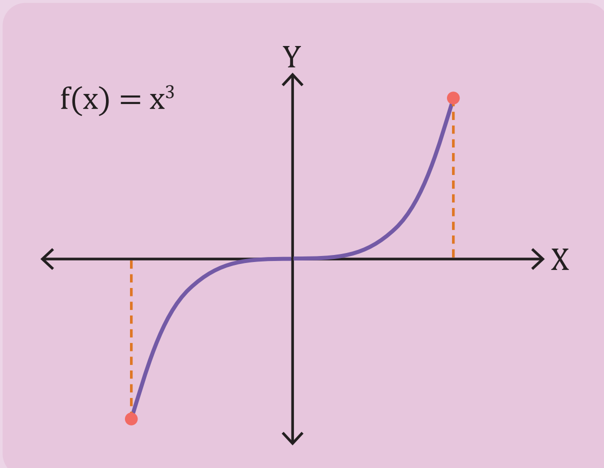
Hence, function f(x) is an even function.

Odd Function

If f(-x) = -f(x) ∀ x in the domain of f, then f is said to be an odd function.
Some examples of odd functions are, $f(x) = x$, $g(x) = \sin x$ and $h(x) = \tan x$
The graph of an odd function is symmetric about the origin.

Example





In both the cases, the graph is symmetric about the origin.

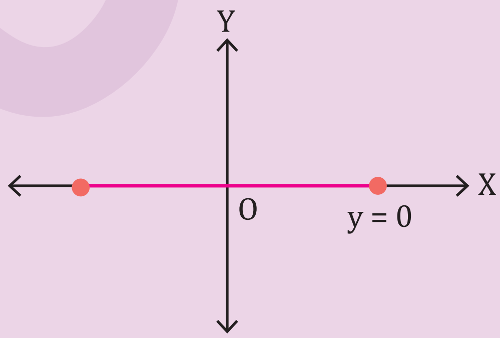
Note

- For an odd function, $f(-x) = -f(x)\ \forall\ x$ in the domain of f.
  Now, substituting $x = 0$, we get
  $f(0) = -f(0)$
  $\Rightarrow f(0) = 0$
  If an odd function is defined at $x = 0$, then $f(0) = 0$
- Some functions may neither be even nor odd (NENO)
  For example, $f(x) = 3x + 2$
  $f(-x) = -3x + 2$
  Clearly, for $f(x)$, neither $f(-x) = f(x)$ nor $f(-x) = -f(x)$
  Therefore, $f(x)$ is neither even nor odd.

Properties of Even and Odd Functions

- The only function that is defined on the entire number line and is even as well as odd is $f(x) = 0$

For function $f(x) = 0$,
$f(-x) = f(x) = 0 \Rightarrow$ Even function
$f(-x) = -f(x) = 0 \Rightarrow$ Odd function



- All functions (whose domain is symmetric about origin) can be expressed as the sum of an even and an odd function.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\textbf{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\textbf{Odd}}$$

Proof

Let's consider

$h(x) = \dfrac{f(x)+f(-x)}{2}$ and $g(x) = \dfrac{f(x)-f(-x)}{2}$

Now, $h(-x) = \dfrac{f(-x)+f(x)}{2}$ and $g(-x) = \dfrac{f(-x)-f(x)}{2}$

$\Rightarrow h(-x) = h(x)$ and $g(-x) = -g(x)$

Hence, function $h(x)$ is even and $g(x)$ is odd.

Express function f(x) = x + e^x as the sum of an even and an odd function.

Solution

Given, $f(x) = x + e^x$

As we know that,

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\textbf{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\textbf{Odd}}$$

$$f(x)=\frac{(x+e^x)+(-x+e^{-x})}{2}+\frac{(x+e^x)-(-x+e^{-x})}{2}$$

$$\Rightarrow f(x)=\frac{e^x+e^{-x}}{2}+\frac{(2x+e^x-e^{-x})}{2}$$

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Let $f(x) = a^x$; $(a > 0)$ be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function, then find the value of $f_1(x + y) + f_1(x - y)$.

(a) $2f_1(x+y)\, f_1(x-y)$

(b) $2f_1(x)\, f_1(y)$

(c) $2f_1(x)\, f_2(y)$

(d) $2f_1(x+y)\, f_2(x-y)$

Solution

Given, $f(x) = a^x$; $(a > 0)$ and $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$, is an odd function.

$$\Rightarrow f_1(x) = \frac{f(x)+f(-x)}{2}=\frac{a^x+a^{-x}}{2}$$

$$\Rightarrow f_1(x+y)+f_1(x-y)=\frac{a^{(x+y)}+a^{-(x+y)}}{2}+\frac{a^{(x-y)}+a^{-(x-y)}}{2}$$

$$\Rightarrow f_1(x+y)+f_1(x-y)=\frac{a^xa^y+a^{-x}a^{-y}+a^xa^{-y}+a^{-x}a^y}{2}$$

$$=\frac{a^x(a^y+a^{-y})+a^{-x}(a^y+a^{-y})}{2}$$

$$=\frac{(a^x+a^{-x})(a^y+a^{-y})}{2}$$

$\Rightarrow f_1(x + y) + f_1(x - y) = 2f_1(x)\, f_1(y)$

Hence, option (b) is the correct answer.

- If $f(x) = x^2$ = Even and $g(x) = |x|$ = Even, then,

  $f \pm g = x^2 \pm |x|$ = Even

  $f.g = x^2 \times |x|$ = Even

  $\frac{f}{g} = \frac{x^2}{|x|}$ = Even $x \neq 0$

f	g	$f \pm g$	$f \times g$	$\frac{f}{g}, g \neq 0$
Even	Even	Even	Even	Even

- If $f(x) = x$ = Odd and $g(x) = \sin x$ = Odd, then,

f	g	$f \pm g$	$f \times g$	$\frac{f}{g}, g \neq 0$
Odd	Odd	Odd	Even	Even

- If $f(x) = x^2$ = Even and $g(x) = x$ = Odd, then,

f	g	$f \pm g$	$f \times g$	$\frac{f}{g}, g \neq 0$
Even	Odd	NENO	Odd	Odd

Note

f	g	$f \pm g$	$f . g$	$\frac{f}{g}, g \neq 0$
Even	Even	Even	Even	Even
Odd	Odd	Odd	Even	Even
Even	Odd	NENO	Odd	Odd

Find whether the following functions are even, odd, or none.

(a) $f(x) = x \times \tan x + \cos^2 x$ (b) $f(x) = \ln\left(\frac{1+x}{1-x}\right), |x| < 1$

Solution

(a) Given,

$f(x) = x \times \tan x + \cos^2 x$

$\Rightarrow f(-x) = -x \times \tan(-x) + \cos^2(-x)$

$= x \times \tan x + \cos^2 x$

$\Rightarrow f(-x) = f(x) =$ Even function

(b) Given,

$f(x) = \ln\left(\frac{1+x}{1-x}\right), |x| < 1$

$\Rightarrow f(-x) = \ln\left(\frac{1-x}{1+x}\right)$

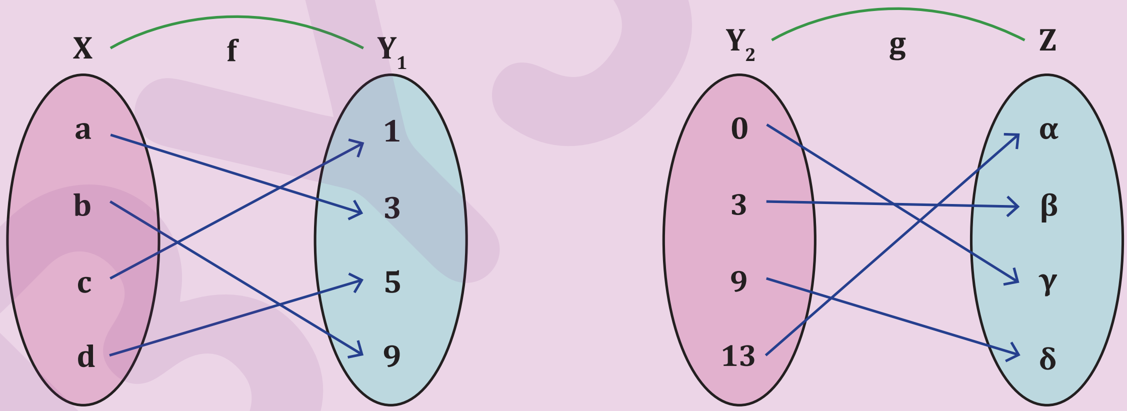
$= -\ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow f(-x) = -f(x) =$ Odd function

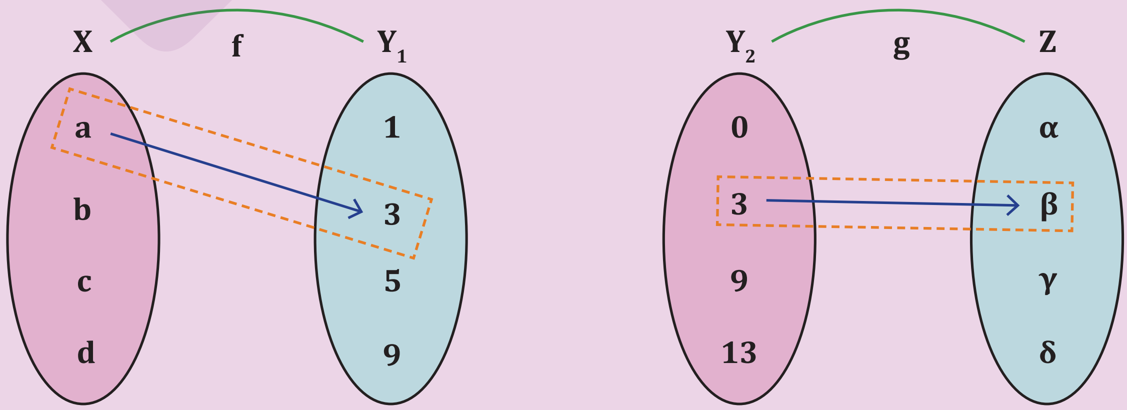
Composite Function

The composition of a function is an operation where two functions, say f and g, generate a new function, say h, in such a way that $h(x) = g(f(x))$. In this operation, the function g is applied to the result of applying the function f to x.

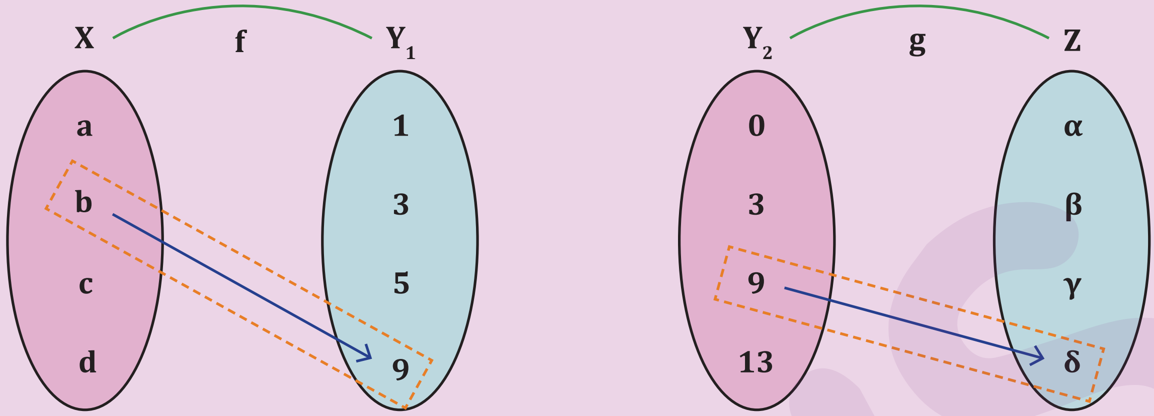
Let us consider functions f: $X \to Y_1$ and g: $Y_2 \to Z$



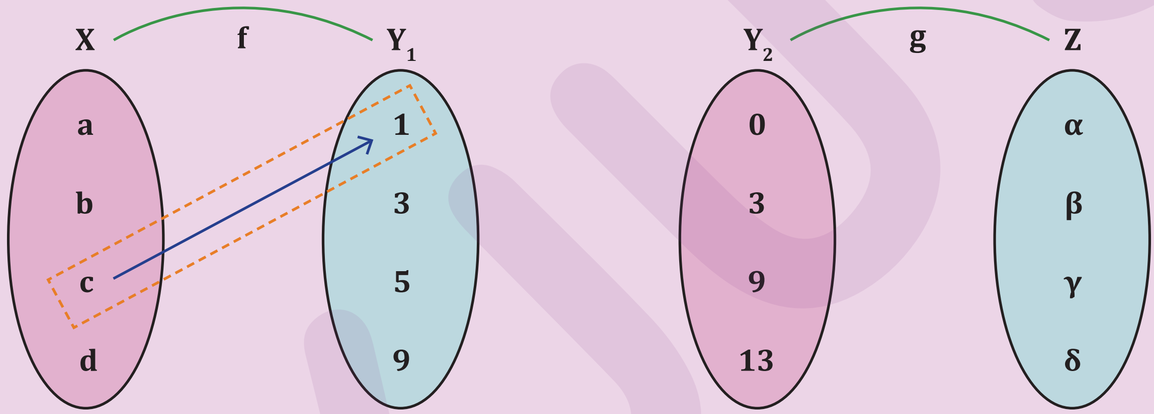
Here, $g(f(a)) = g(3) = \beta$



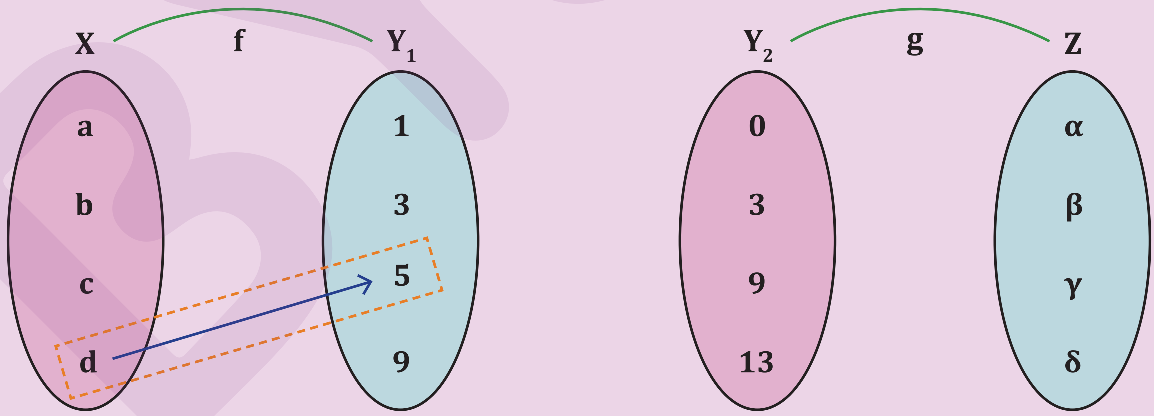
Here, $g(f(b)) = g(9) = \delta$

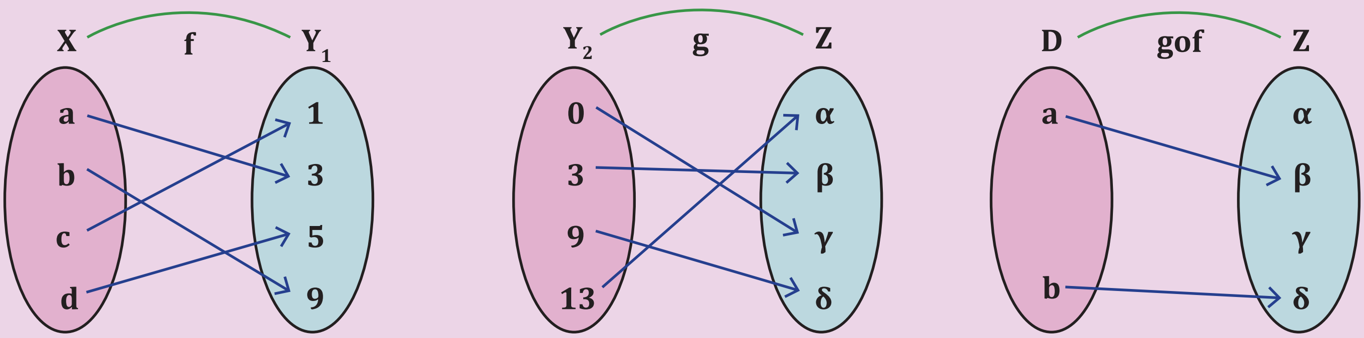


Here, $g(f(c)) = g(1) =$ Not defined



Here, $g(f(d)) = g(5) =$ Not defined



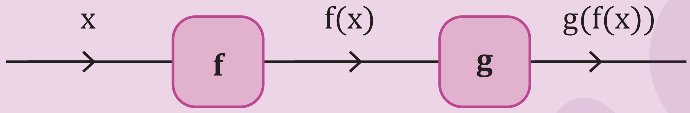


So, $g(f(x))$ is defined for only those values of x for which the range of f is a subset of the domain of g.

$\therefore f : X \rightarrow Y_1$ and $g: Y_2 \rightarrow Z$ are two functions and D is set of x such that if $x \in X$, then $f(x) \in Y_2$

If domain, $D \neq \varphi$, then function h defined by $h(x) = g(f(x))$ is known as the composite function of g and f. A composite function is denoted by $gof(x)$ or $g(f(x))$. It is also known as a function of a function.

Pictorially, $gof(x)$ can be viewed as follows:



Note

The domain of $gof(x)$ is D which is a subset of X (the domain of f). The range of $gof(x)$ is a subset of the range of g. If $D = X$, then $f(x) \subseteq Y_2$

If two functions f and g are defined from $\mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = x + 1$ and $g(x) = x + 2$, then, find the values of $g(f(x))$, $f(g(x))$.

Solution

Given, $f(x) = x + 1$, $g(x) = x + 2$ and f and g are defined from $\mathbb{R} \rightarrow \mathbb{R}$

$g(f(x)) = f(x) + 2$

$\quad = x + 1 + 2 = x + 3$

$\Rightarrow g(f(x)) = x + 3$

$f(g(x)) = g(x) + 1$

$\quad = x + 2 + 1 = x + 3$

$\Rightarrow f(g(x)) = x + 3$

Here, $g(f(x)) = f(g(x))$

Note

In general, $g(f(x)) \neq f(g(x))$

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If $f(x) = \log_e\left(\frac{1-x}{1+x}\right)$, $|x| < 1$, then find $f\left(\frac{2x}{1+x^2}\right)$.

(a) $2f(x)$ **(b)** $[f(x)]^2$ **(c)** $2f(x^2)$ **(d)** $-2f(x)$

Solution

Given, $f(x) = \log_e\left(\frac{1-x}{1+x}\right)$, $|x| < 1$

Let us consider $g(x) = \frac{2x}{1+x^2}$, then,

$$f\left(\frac{2x}{1+x^2}\right) = f(g(x)) = \ln\left(\frac{1-g(x)}{1+g(x)}\right)$$

$$\Rightarrow f(g(x)) = \ln\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right) = \ln\left(\frac{1+x^2-2x}{1+x^2+2x}\right) = \ln\left(\frac{1-x}{1+x}\right)^2$$

$$\Rightarrow f(g(x)) = 2\ln\left(\frac{1-x}{1+x}\right) = 2f(x)$$

Hence, option (a) is the correct answer.

Properties of Composite functions

Associative property of composite functions

Composite functions are associative, i.e., if three functions $f : A \to B$, $g : B \to C$ and $h : C \to D$ are such that fo(goh) and (fog)oh are defined, then fo(goh) = (fog)oh

Example

If $f(x) = x$, $g(x) = \sin x$, $h(x) = e^x$, the domain of f, g, h is $\mathbb{R}$. Prove that fo(goh) = (fog)oh

Solution

As the domain of the three given functions f, g, h is a real number, the composite functions fo(goh) and (fog)oh are defined.

fo(goh) = fo(g(h(x)) = fo(g(e^x))

$\Rightarrow$ fo(goh) = f(sin e^x) = sin e^x

(fog)oh =(fog)(h(x)) and (fog)(x) = f(g(x)) = f(sin x) = sin x

$\Rightarrow$ (fog)(h(x)) = sin(h(x)) = sin e^x

Therefore, fo(goh) = (fog)oh

For $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$, find the domain and range of gof(x).

Solution

Step 1

Domain of f(x) : [-3, ∞)
Range of f(x) : [0, ∞)
Domain of g(x) : ℝ
Range of g(x) : [1, ∞)

Step 2

For gof(x), since the range of f is a subset of domain of g, the domain of gof(x) is [-3, ∞).

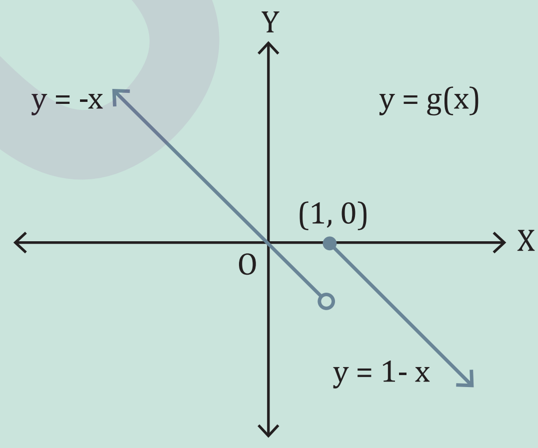
Step 3

$gof(x) = g(f(x)) = 1 + (f(x))^2 = 1 + x + 3 = x + 4$
The range of gof(x) is [1, ∞).

$f(x) = \begin{cases} 1-x, & x \le 0 \\ x^2, & x > 0 \end{cases}$ $\quad g(x) = \begin{cases} -x, & x < 1 \\ 1-x, & x \ge 1 \end{cases}$ **Define $f(g(x))$.**

Solution

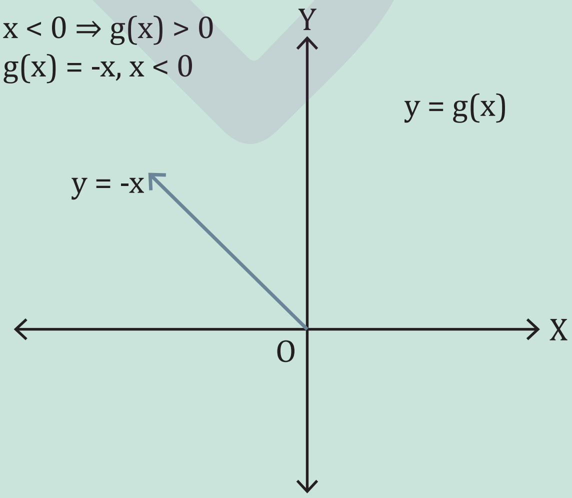
For f(g(x)) to be defined, $R_g \subseteq D_f$
As we can observe that the domain of the function f is a real number, i.e., $D_f = \mathbb{R}$
For the range of the function g, let us plot the graph of this linear function.
The range of function g is also a real number.
Therefore, the composite function f(g(x)) is valid as range of g is domain of f.
Replace x with g(x) in the given definition of f(x).



$$\Rightarrow f(g(x)) = \begin{cases} 1 - g(x), & g(x) \le 0 \\ (g(x))^2, & g(x) > 0 \end{cases}$$

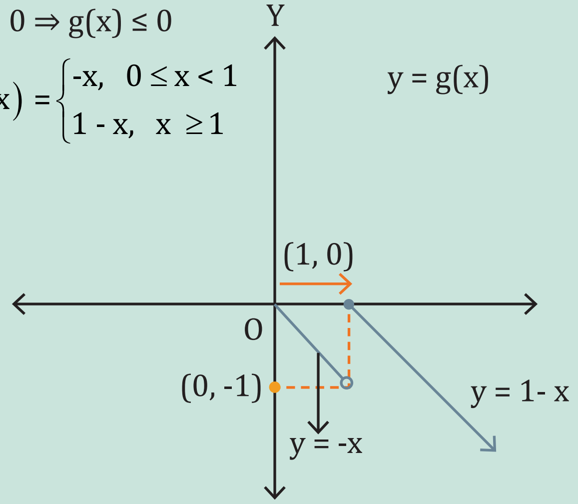
Identify the portions of the function g, where g(x) > 0 and g(x) ≤ 0 along with the corresponding values of x.

$x < 0 \Rightarrow g(x) > 0$
$g(x) = -x, x < 0$



$x \ge 0 \Rightarrow g(x) \le 0$

$g(x) = \begin{cases} -x, & 0 \le x < 1 \\ 1-x, & x \ge 1 \end{cases}$



Therefore,

$$f(g(x)) = \begin{cases} 1-(-x),\ g(x) \leq 0,\ 0 \leq x < 1 \\ 1-(1-x),\ g(x) \leq 0,\ x \geq 1 \\ (-x)^2,\ g(x) > 0,\ x < 0 \end{cases}$$

$$\Rightarrow\ f(g(x)) = \begin{cases} 1+x,\ 0 \leq x < 1 \\ x,\ x \geq 1 \\ x^2,\ x < 0 \end{cases}$$

Concept Check

1. Identify whether the function $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$ is even or odd.
2. Find whether the function $f(x)=10x^3-4x^2+3x-8$ even/odd or none.
3. Two functions f and g are defined from $\mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$, $g(x) = x + 1$. Find $g(f(x))$ and $f(g(x))$.
4. If $g(x) = x^2 + x - 1$ and $g(f(x)) = 4x^2 - 10x + 5$, then what is the value of $f\left(\frac{5}{4}\right)$?

   (a) $-\frac{3}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

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Summary Sheet

Key Takeaways

- If $f(-x) = f(x)\ \forall\ x$ in the domain of f, then f is said to be an even function.
- If $f(-x) = -f(x)\ \forall\ x$ in the domain of f, then f is said to be an odd function.
- For an odd function, $f(-x)= -f(x)\ \forall\ x$ in the domain of f, $f(0) = 0$
- All the functions (whose domain is symmetric about origin) can be expressed as the sum of an even and an odd function.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\textbf{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\textbf{Odd}}$$

- The composition of a function is an operation where two functions, say f and g, generate a new function, say h, in such a way that $h(x) = g(f(x))$. It means that function g is applied to the function of x. Basically, a function is applied to the result of another function.
- The domain of gof is D which is a subset of X (the domain of f). The range of gof is a subset of the range of g. If $D = X$, then $f(x) \subseteq Y_2$.
- Composite functions are associative, i.e., if three functions $f : A \to B$, $g : B \to C$, and $h : C \to D$ are such that $fo(goh)$ and $(fog)oh$ are defined, then $fo(goh) = (fog)oh$

Mind Map

Relations and Functions

- Even function
- Odd function
- Composite function

Self-Assessment

Find if the given functions in their respective domains are even, odd, or neno.

$$f(x) = \sqrt{1+x+x^2} + \sqrt{1-x+x^2}$$

Answers

Concept Check

1.

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right)$$

$$\Rightarrow f(-x) = \ln\left(-x + \sqrt{1+x^2}\right) = \ln\left(\frac{\left(-x+\sqrt{1+x^2}\right)\left(x+\sqrt{1+x^2}\right)}{\left(x+\sqrt{1+x^2}\right)}\right)$$

$$= \ln\left(\frac{1}{\left(x+\sqrt{1+x^2}\right)}\right)$$

$$= -\ln\left(x+\sqrt{1+x^2}\right)$$

$$\Rightarrow f(-x) = -f(x)$$

Hence, f(x) is an odd function.

2.

$$f(x) = 10x^3 - 4x^2 + 3x - 8$$

$$\Rightarrow f(-x) = 10(-x)^3 - 4(-x)^2 + 3(-x) - 8$$

$$= -10x^3 - 4x^2 - 3x - 8$$

$$= -\left(10x^3 + 4x^2 + 3x + 8\right)$$

f(x) is neither even nor odd.

3.

Given, $f(x) = x^2$, $g(x) = x + 1$ and f and g are defined from $\mathbb{R} \rightarrow \mathbb{R}$

$\Rightarrow g(f(x)) = f(x) + 1$

$= x^2 + 1$

$\Rightarrow g(f(x)) = x^2 + 1$

$\Rightarrow f(g(x)) = [g(x)]^2$

$= (x + 1)^2 = x^2 + 2x + 1$

$\Rightarrow f(g(x)) = x^2 + 2x + 1$

Here, $g(f(x)) \neq f(g(x))$

4.

Given, $g(x) = x^2 + x - 1$ and $g(f(x)) = 4x^2 - 10x + 5$

By substituting $x = \frac{5}{4}$ in g(f(x)), we get,

$$\Rightarrow g\left(f\left(\frac{5}{4}\right)\right) = 4\left(\frac{5}{4}\right)^2 - 10\left(\frac{5}{4}\right) + 5 = \frac{25}{4} - \frac{25}{2} + 5$$

$$\Rightarrow g\left(f\left(\frac{5}{4}\right)\right) = -\frac{5}{4}$$

Let $f\left(\frac{5}{4}\right) = a$

$$\Rightarrow g(a) = -\frac{5}{4}$$

As the definition of function g is known, the input a for which the output $-\frac{5}{4}$ is generated, can be easily found.

Therefore, $g(a) = a^2 + a - 1 = -\frac{5}{4}$

$$\Rightarrow a^2 + a - 1 + \frac{5}{4} = 0$$

$$\Rightarrow a^2 + a + \frac{1}{4} = 0$$

$$\Rightarrow \left(a + \frac{1}{2}\right)^2 = 0$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow f\left(\frac{5}{4}\right) = -\frac{1}{2}$$

Self-Assessment

Given, $f(x) = \sqrt{1+x+x^2} + \sqrt{1-x+x^2}$

$$\Rightarrow f(-x) = \sqrt{1-x+(-x)^2} + \sqrt{1+x+(-x)^2}$$

$$= \sqrt{1-x+x^2} + \sqrt{1+x+x^2} = \sqrt{1+x+x^2} + \sqrt{1-x+x^2}$$

$\Rightarrow f(-x) = f(x)$

Hence, function f(x) is an even function

MATHEMATICS

RELATIONS AND FUNCTIONS

PERIODIC FUNCTIONS

What you already know

- Types of functions
- Composite functions

What you will learn

- Properties of composite functions
- Periodic functions
- Properties of periodic functions

Properties of a Composite Function

Property 1

If the functions f and g are one-one functions, then gof, if defined, will be a one-one function.

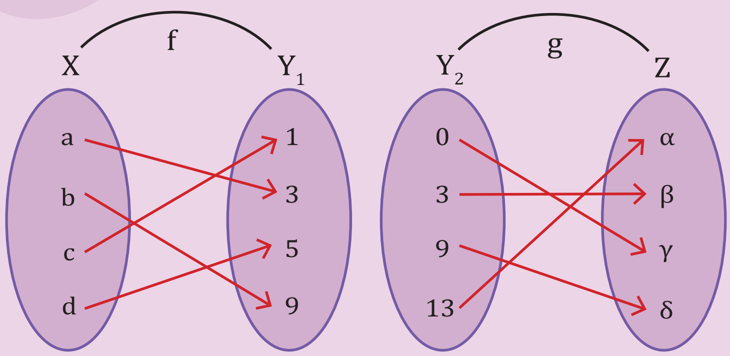
Example

Let the functions be f and g that are shown in the arrow diagram.

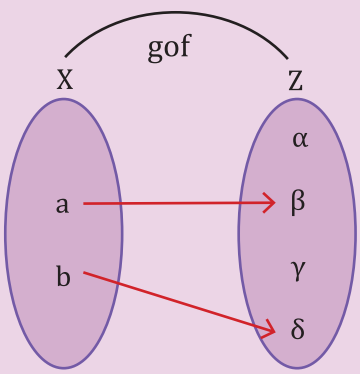
Function gof is defined because R_f and D_g have common elements.

$R_f = \{1, 3, 5, 9\}$ and $D_g = \{0, 3, 9, 13\}$

$\Rightarrow R_f \cap D_g = \{3, 9\}$



As we can observe that f and g are one-one functions, they result in a valid composite function gof, which is also a one-one function

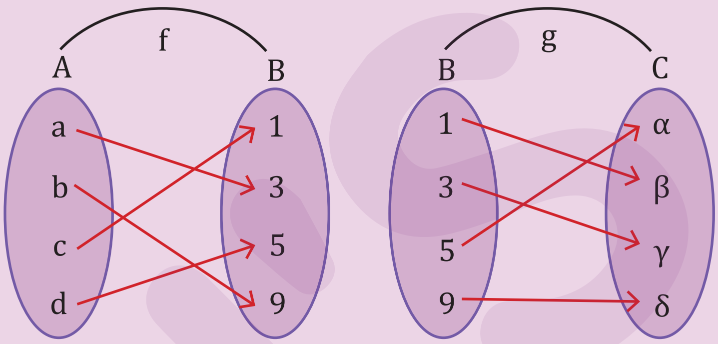


Property 2

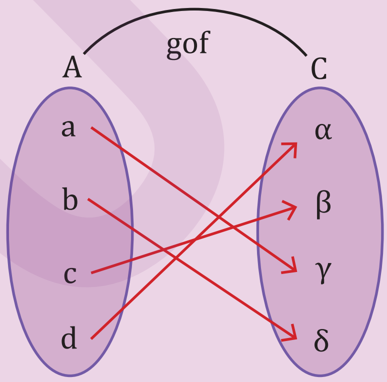
If the functions f and g are bijective functions and gof is defined, then gof will be a bijection iff range of f is equal to the domain of g.

Example

It is given that the two functions are bijections, i.e., functions f and g, are both one-one and onto.
$f : A \to B$ and $g : B \to C$



As $R_f = D_g$, and thus, gof is defined.
The composite function $gof : A \to C$ is as shown.
Therefore, $gof : A \to C$ is a bijection from its definition.



Property 3

The composition of functions (which are either even or odd) is even if at least one of them is even.

Example

Let $f(x) = x$, $g(x) = \sin x$, $h(x) = \cos x$. Check whether $f(g(h(x)))$ is an even function or not?

Solution

Step 1

$h(x) = \cos x$ is an even function.
$f(g(h(x))) = f(g(\cos x)) = \sin(\cos x)$ ... (i)
$f(g(h(-x))) = \sin(\cos(-x))$
$= \sin(\cos x)$ ... (ii)

Step 2

From (i) and (ii), we can say,
$f(g(h(x))) = f(g(h(-x)))$
So, $f(g(h(x)))$ is an even function. ($\because F(x) = F(-x) \Rightarrow F(x)$ is an even function)
As $h(x) = \cos x$ is an even function, composition $f(g(h(x)))$ is an even function.

Periodic Functions

A periodic function is a function that repeats its values after every particular interval.

Mathematically, a function $f(x)$ is said to be a periodic function if there exists a positive real number T, such that $f(x + T) = f(x), \forall x \in D_f ; T > 0$

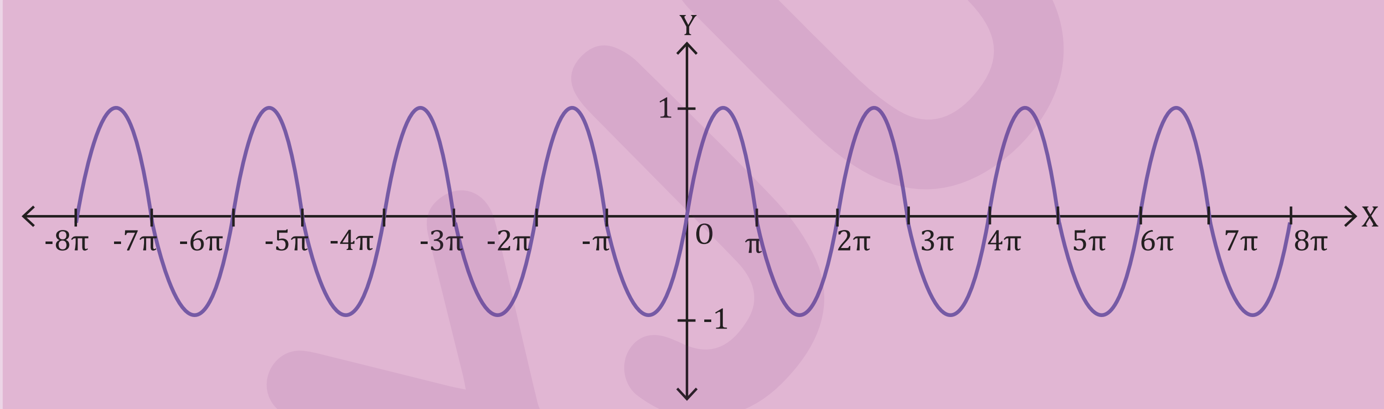
Here, T is known as a period of function f and the smallest value of T is known as the fundamental period.

Note

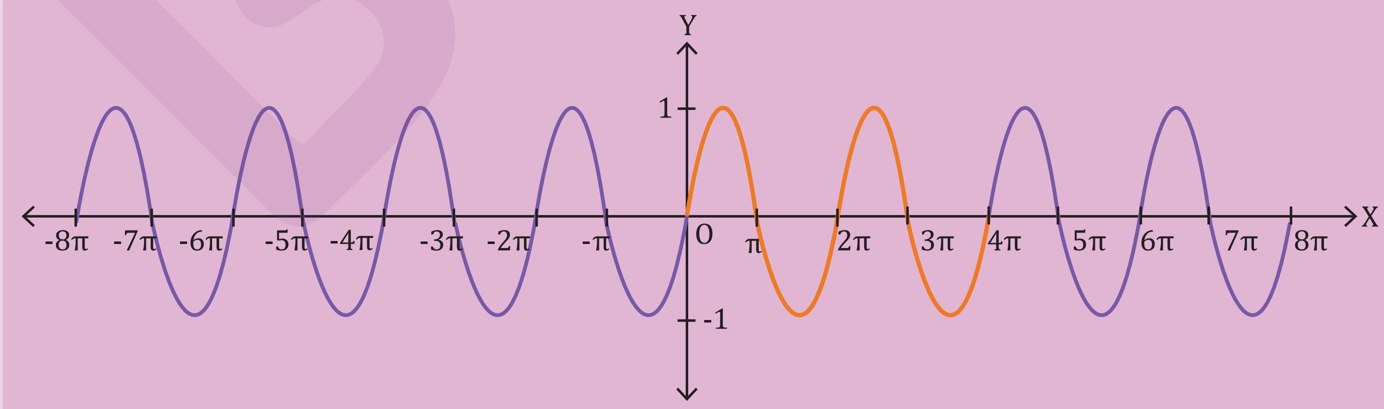
The domain of periodic function should not be restricted (bounded).

Examples of periodic function:

1. $f(x) = \sin x$



We can observe that $f(x) = \sin x$ repeats its value after every 2π interval.
Hence, $f(x) = \sin x$ is a periodic function with a period of 2π.
As the graph repeats for every 2π interval length, it will also repeat for 4π or 6π and so on.



Mathematical justification

If the period of a function is T, then $f(x + aT) = f(x), a \in \mathbb{N}$

Let $f(x) = \sin x$ with a period T

$\Rightarrow \sin(x + T) = \sin x$

$\Rightarrow \sin(x + T) - \sin x = 0$

$\Rightarrow 2\cos\left(\frac{2x + T}{2}\right) \sin\left(\frac{T}{2}\right) = 0$

$\cos\left(\frac{2x + T}{2}\right) = 0$ is not considered because $\cos\left(\frac{2x + T}{2}\right) = \cos(2n + 1)\frac{\pi}{2}$ does not give the value of T.

$\Rightarrow \sin\left(\frac{T}{2}\right) = 0$ for $\forall\, x \in \mathbb{R}$

$\Rightarrow \frac{T}{2} = n\pi$

$\Rightarrow T = 2n\pi$

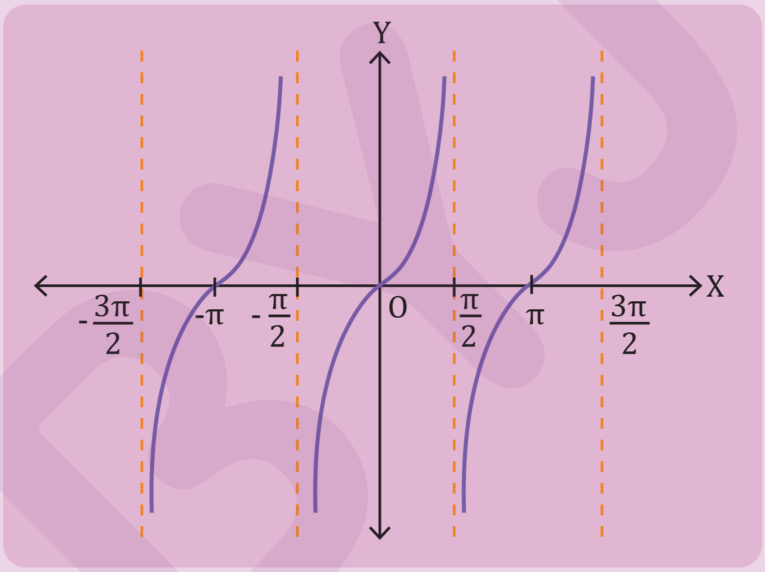
$\Rightarrow \sin(x + 2n\pi) = \sin x$

$\Rightarrow f(x + 2n\pi) = f(x)$

As $T > 0$, the possible values of the period T are $2\pi, 4\pi, 6\pi$, and so on.

The smallest amongst these values is 2π. This is known as the fundamental period of the function.

2. $f(x) = \tan x$



The function $\tan x$ is periodic with fundamental period as π.

$\Rightarrow \tan(x + \pi) = \tan x$

$\Rightarrow \tan(x + a\pi) = \tan x, a \in \mathbb{N}$

Also, we can observe that the discontinuity of function $\tan x$ also repeats for every π interval.

Note

If a function is discontinuous and periodic with period T, then its discontinuity also repeats after every T interval length.

Properties of Periodic Functions

Property 1

If a function f(x) has period T, then $\frac{1}{f(x)}, (f(x))^n\ (n \in \mathbb{N}), |f(x)|, \sqrt{f(x)}$ also have period T. (T may or may not be a fundamental period.)

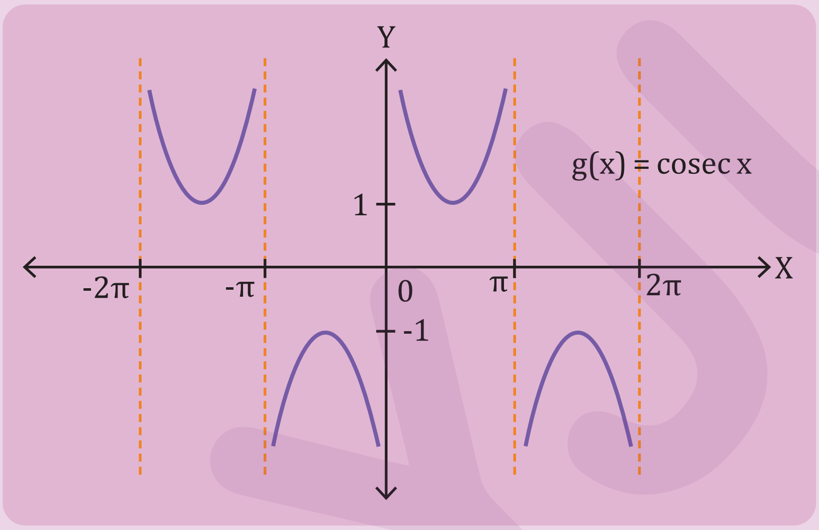
(i) $g(x) = \operatorname{cosec} x$

$$\Rightarrow g(x) = \frac{1}{\sin x}, f(x) = \sin x$$

$$\Rightarrow g(x) = \frac{1}{f(x)}$$

The period of function cosec x remains same as of the function sin x, which is 2π. So, cosec x has the fundamental period as 2π.

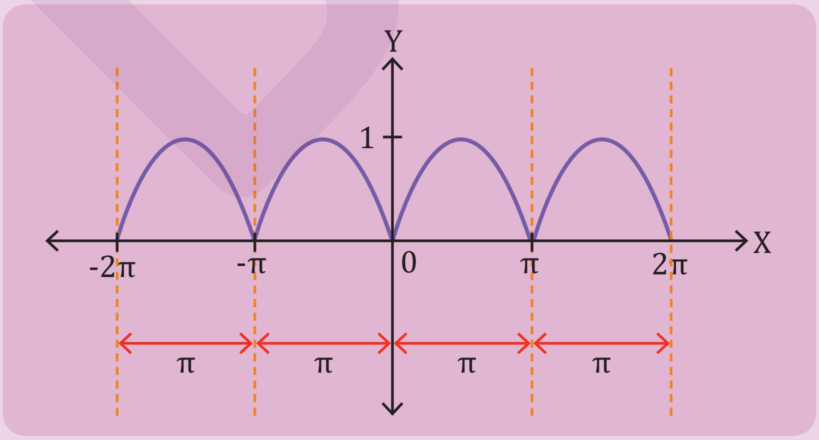
Let us verify it using the graph of $g(x) = \operatorname{cosec} x$



We can observe that $g(x) = \operatorname{cosec} x$ repeats its value after every 2π interval.

(ii) $g(x) = |\sin x|$

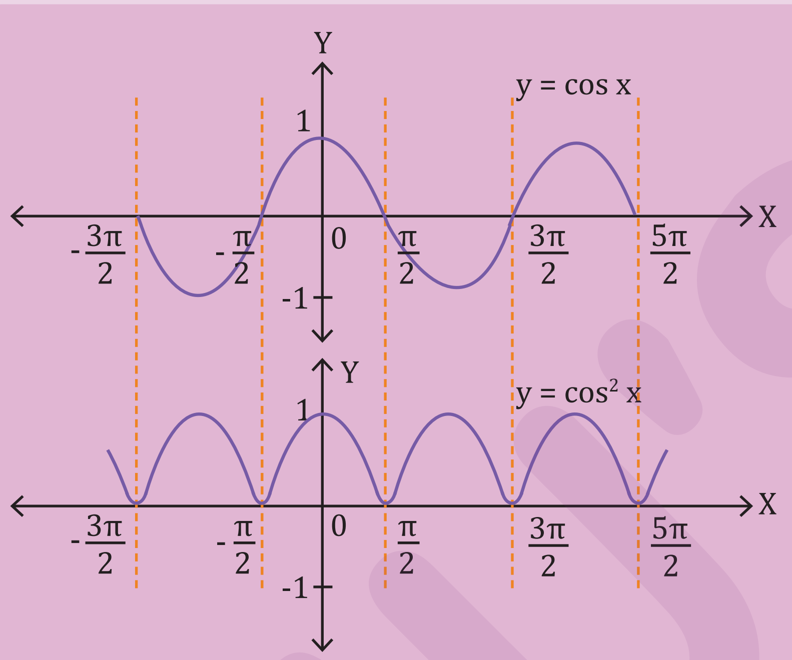
Let $f(x) = \sin x$, $g(x) = |f(x)|$



Here, the fundamental period of |sin x| is π unlike sin x whose fundamental period 2π. However, one can say that the period of |sin x| is 2π.

(iii) $g(x) = \cos^2 x$

Let $f(x) = \cos x$, $g(x) = (f(x))^2$

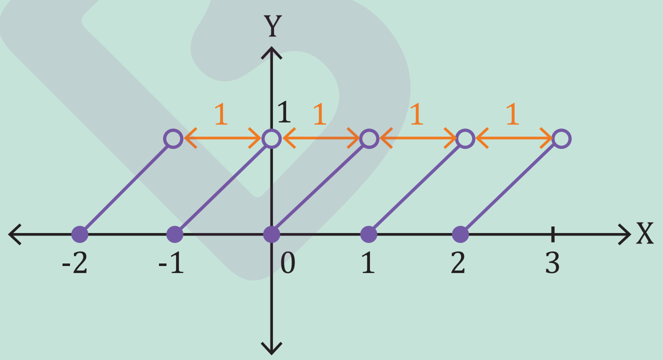


Here, the fundamental period of $\cos^2 x$ is π unlike $\cos x$ with fundamental period 2π. However, one can say that the period of $\cos^2 x$ is 2π.

Find the period of the function $f(x) = \{x\}$, if possible, where $\{.\}$ represents the fractional part function.

Solution

$f(x) = \{x\}$, $\{.\}$ is the fractional part function.



From the graph, it is clear that the $\{x\}$ repeats its value after an interval of 1. So, the period of $f(x) = \{x\}$ is 1.

Property 2

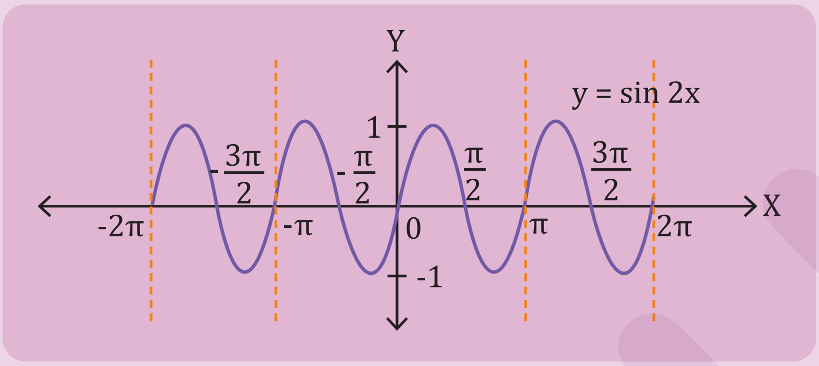
If a function f(x) has period T, then f(ax + b) has period $\frac{T}{|a|}$

Examples:

(i) $y = \sin 2x$

By the property of periodic functions, sin 2x will have a period of $\frac{2\pi}{2} = \pi$

Let us verify it using the graph of $y = \sin 2x$



We can observe that $y = \sin 2x$ repeats its value after every π interval.

(ii) $y = \left\{\frac{x}{3}\right\}$ where {.} denotes fractional function

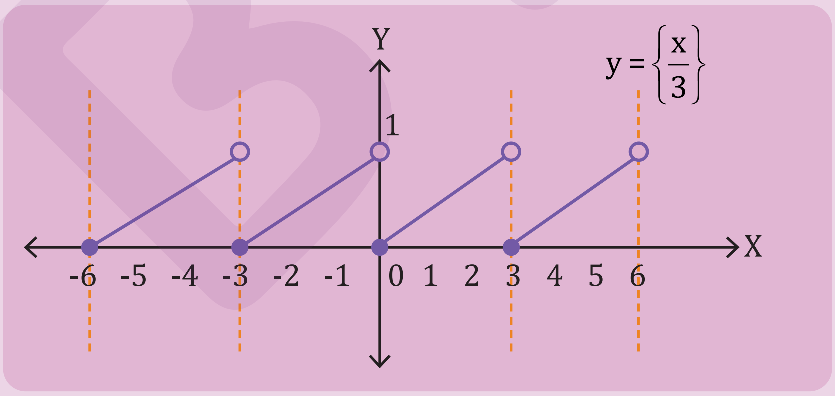
Let $f(x) = \{x\}$, $y = f\left(\frac{x}{3}\right)$

Period of function f(x), T = 1

As the coefficient of x in the given function y is $\frac{1}{3}$, its period is given by $\frac{T}{\left(\frac{1}{3}\right)} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

⇒ The period of $\left\{\frac{x}{3}\right\}$ is 3.

Let us verify it using the graph of $y = \left\{\frac{x}{3}\right\}$



We can observe that $y = \left\{\frac{x}{3}\right\}$ repeats its value after every interval of 3.

Property 3

Every constant function defined for an unbounded domain is always periodic with no fundamental period.

Examples:

(i) $f(x) = \sin^2x + \cos^2x$, domain is $\mathbb{R}$

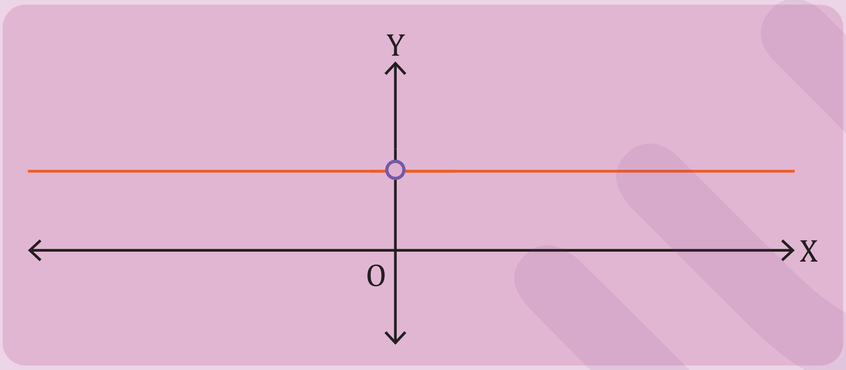
$\Rightarrow f(x) = 1$

This constant function is periodic but with no fundamental period.

(ii) $f(x) = x.\frac{1}{x}$

$\Rightarrow f(x) = 1$

Domain: $x \in \mathbb{R} - \{0\}$



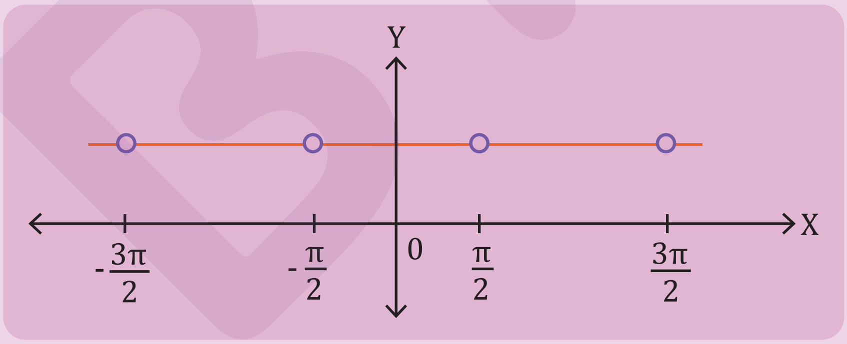
This is a constant function but not a periodic function as the graph does not repeat due to a single discontinuity. This discontinuity does not repeat/occur in regular intervals. Thus, the function can be concluded as non-periodic.

(iii) $f(x) = \cos x.\sec x$

$\Rightarrow f(x) = 1$

Domain: $x \in \mathbb{R} - \{(2n + 1)\frac{\pi}{2}, n \in \mathbb{Z}\}$

Let us have a look at the graph.



We can say that the values are repeated after every π interval. So, the period of the function is π.

Property 4

If $f(x)$ has a period T_1 and $g(x)$ has a period T_2, then $f(x) \pm g(x)$, $f(x).g(x)$, and $\frac{f(x)}{g(x)}$ are also periodic with period L.C.M (T_1, T_2), provided that the L.C.M of T_1, T_2 exists.

However, the L.C.M need not be a fundamental period.

Note

- $\text{L.C.M}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{L.C.M}(a, c)}{\text{H.C.F}(b, d)}$ (a, b, c, and d are rational)
- If the L.C.M of periods of f(x) and g(x) does not exist, then f(x) ± g(x), f(x).g(x), $\frac{f(x)}{g(x)}$ are non-periodic.

Find the period of the following functions:

(i) $f(x) = \sin\left(\frac{3x}{2}\right) + \cos\left(\frac{9x}{4}\right)$

(ii) $f(x) = |\sin x| + |\cos x|$

Solution

(i) We know that if $f(x)$ has period T, then $f(ax+b)$ has period $\frac{T}{|a|}$.

$$\therefore T_1 = \text{Period of } \sin\left(\frac{3x}{2}\right) = \frac{2\times 2\pi}{3} = \frac{4\pi}{3}$$

$$T_2 = \text{Period of } \cos\left(\frac{9x}{4}\right) = \frac{4\times 2\pi}{9} = \frac{8\pi}{9}$$

$$\text{L.C.M}\left(\frac{4\pi}{3}, \frac{8\pi}{9}\right) = \frac{\text{L.C.M}\,(4\pi, 8\pi)}{\text{H.C.F}\,(3, 9)} = \frac{8\pi}{3}$$

$$\Rightarrow \text{Period of } f(x) = \frac{8\pi}{3}$$

(ii) We know that the period of g(x) ± h(x) is the L.C.M of (T_1, T_2), where T_1 and T_2 are the periods of g(x) and h(x), respectively.

Period of |sin x| = π
Period of |cos x| = π
L.C.M (π, π) = π
⇒ Period of f(x) = π
π may not be the fundamental period.

Let us check for $\frac{\pi}{2}$.

$$f\left(x+\frac{\pi}{2}\right) = \left|\sin\left(x+\frac{\pi}{2}\right)\right| + \left|\cos\left(x+\frac{\pi}{2}\right)\right|$$

$$= |\cos x| + |-\sin x|$$

$$= |\cos x| + |\sin x| = f(x)$$

⇒ $\frac{\pi}{2}$ is also a period of f(x), which is a fundamental period.

Property 5

If g is a function such that gof is defined on the domain of f and f is periodic with period T, then gof is also periodic with T as one of its periods.

Example:

Let $f(x) = \cos x$, $g(x) = \{x\}$

Here, $f(x)$ has period 2π and $g(x)$ has period 1.

$gof(x) = g(f(x)) = \{f(x)\} = \{\cos x\}$

It is also a periodic function with period 2π.

$fog(x) = f(g(x)) = \cos(g(x)) = \cos\{x\}$

It is also a periodic function with period 1.

Note

If g is a function such that gof is defined on the domain of f and f is non-periodic, then gof may or may not be periodic.

Example:

Let $f(x) = x + \sin x$, $g(x) = \cos x$

$f(x)$ is not periodic, while $g(x)$ is periodic.

Let $h(x) = g(f(x))$

$\Rightarrow h(x) = \cos(x + \sin x)$

Now, $h(x + 2\pi) = \cos(x + 2\pi + \sin(x + 2\pi))$

$= \cos(x + \sin x)$

$\Rightarrow h(x)$, i.e., $g(f(x))$ is periodic with the period 2π.

Property 6

The sum or the difference of two periodic functions can be non-periodic, while the sum or the difference of two non-periodic functions can be periodic.

Examples:

(i) $f(x) = \{x\} + \sin x$

We know that the period of $\{x\}$ is 1 and the period of $\sin x$ is 2π.

And L.C.M $(1, 2\pi)$ does not exist.

So, $\{x\} + \sin x$ is not periodic.

(ii) $f(x) = x - [x]$ (Where [.] represents the G.I.F.)

Both x and $[x]$ are non-periodic functions.

However, $x - [x] = \{x\}$ is periodic with the fundamental period as 1.

Find the fundamental period of the function $f(x) = \cos(\sin x) + \cos(\cos x)$

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

Solution

$f(x + 2\pi) = \cos(\sin(x + 2\pi)) + \cos(\cos(x + 2\pi)) = \cos(\sin x) + \cos(\cos x)$

$f(x + \pi) = \cos(\sin(x + \pi)) + \cos(\cos(x + \pi)) = \cos(\sin x) + \cos(\cos x)$

$f(x + \frac{\pi}{2}) = \cos(\sin(x + \frac{\pi}{2})) + \cos(\cos(x + \frac{\pi}{2})) = \cos(\sin x) + \cos(\cos x)$

∴ The fundamental period of $f(x)$ is $\frac{\pi}{2}$.

So, option (c) is the correct answer.

If fundamental period of $f(x) = \sin([a]x)$ is π, where [.] represents greatest integer function, then what is the range of value of a?

(a) $a \in [3, \infty)$ **(b)** $a \in (1, 3)$ **(c)** $a \in [2, 3)$ **(d)** $a \in (-2, 0)$

Solution

Given, fundamental period of $f(x) = \sin([a]x)$ is π.

$\Rightarrow \pi = \frac{2\pi}{[a]}$

$\Rightarrow [a] = 2$

$\Rightarrow a \in [2, 3)$

∴ **Option (c) is the correct answer.**

Find the period of $f(x) + f(x + 1) = 0, \ \forall x \in \mathbb{R}$.

Solution

Given, $f(x) + f(x + 1) = 0$ ...(i)

On replacing x by $x + 1$, we get,

$f(x + 1) + f(x + 2) = 0$ ...(ii)

On subtracting equation (i) from equation (ii), we get,

$f(x + 2) - f(x) = 0$

$\Rightarrow f(x + 2) = f(x)$

∴ $f(x)$ is periodic with period 2.

Concept Check

1. Find the period of the function $f(x) = [x]$, if possible, where [.] represents the G.I.F.
2. Find the period of the function $f(x) = \sqrt{\sec(3x - 1)}$

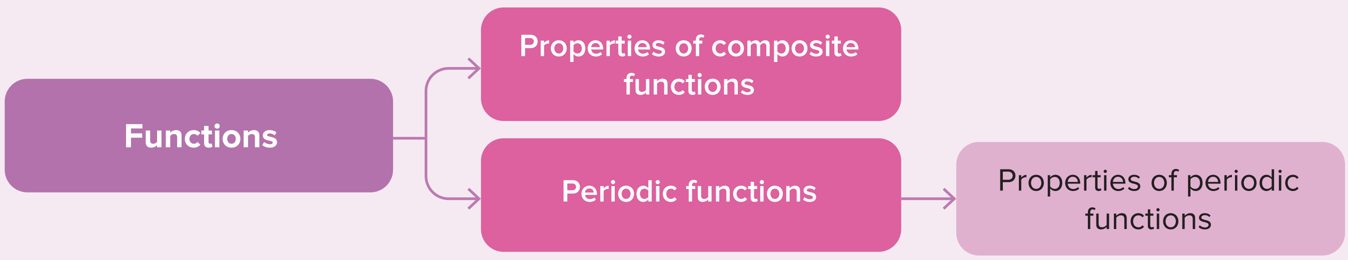
Summary Sheet

Key Takeaways

- If the functions f and g are one-one functions, then gof, if defined, will be a one-one function.
- If the functions f and g are bijective functions and gof is defined, then gof will be a bijection iff range of f is equal to the domain of g.
- The composition of functions (which are either even or odd) will be even, if at least one of them is even.
- A function $f(x)$ is said to be a periodic function if there exists a positive real number T, such that $f(x + T) = f(x), \forall\, x \in D_f\,;\, T > 0$
- If a function $f(x)$ has period T, then $\frac{1}{f(x)}, \left(f(x)\right)^n (n \in \mathbb{N}), |f(x)|, \sqrt{f(x)}$ also have period T.

  (T may or may not be a fundamental period)
- If a function $f(x)$ has period T, then $f(ax + b)$ has period $\frac{T}{|a|}$.
- Every constant function defined for an unbounded domain is always periodic with no fundamental period.
- If $f(x)$ has a period T_1 and $g(x)$ has a period T_2, then $f(x) \pm g(x)$, $f(x) . g(x)$, and $\frac{f(x)}{g(x)}$ are also periodic with period L.C.M (T_1, T_2), provided that the L.C.M of T_1, T_2 exists.
- If g is a function such that gof is defined on the domain of f and f is periodic with period T, then gof is also periodic with T as one of its periods.
- If g is a function such that gof defined on the domain of f and f is non-periodic, then gof may or may not be periodic.
- The sum or the difference of two periodic functions can be non-periodic, while the sum or the difference of two non-periodic functions can be periodic.

Mind Map



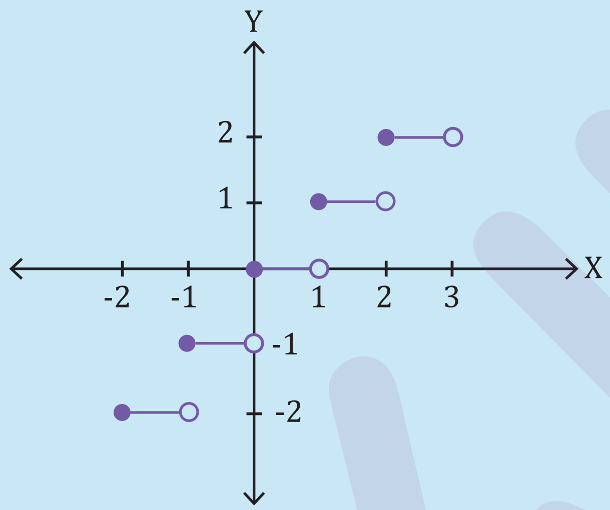
Self-Assessment

Find the period of the function $\frac{\sin x+\cos x}{\sqrt{2}}$.

Answers

Concept Check

1. Let us see the graph of the G.I.F. y = [x].



The values does not repeat after any interval of x. So, f(x) = [x] is not periodic.

2. If a function f(x) has period T, then $\frac{1}{f(x)}$, $(f(x))^n$ $(n \in \mathbb{N})$, $|f(x)|$, $\sqrt{f(x)}$ also have period T (T may or may not be a fundamental period). If a function f(x) has period T, then f(ax + b) has period $\frac{T}{|a|}$.

$\Rightarrow$ Period of $f(x)=\sqrt{\sec(3x-1)}=\frac{2\pi}{3}$

Self-Assessment

Given, $\frac{\sin x+\cos x}{\sqrt{2}}$

Let $f(x)=\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)$

$\Rightarrow f(x)=\sin\left(x+\frac{\pi}{4}\right)$

Therefore, period is same as the sin x function, which is 2π.

BYJU'S Classes

MATHEMATICS

RELATIONS AND FUNCTIONS

INVERSE FUNCTIONS

NOTES

What you already know

- Periodic functions
- Properties of periodic functions

What you will learn

- Inverse function
- Properties of inverse function

Inverse Function

Let $y = f(x): A \to B$ be a one-one and onto function, i.e., a bijection. Then, there exists a function $x = g(y) : B \to A$ such that if $(\alpha, \beta) \in f \Rightarrow (\beta, \alpha) \in g$

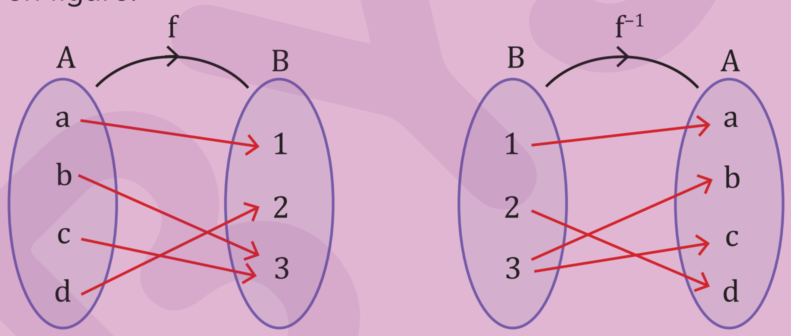
Then f and g are said to be invertible functions of each other.

$g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$

Why does a function have to be bijective to be invertible?

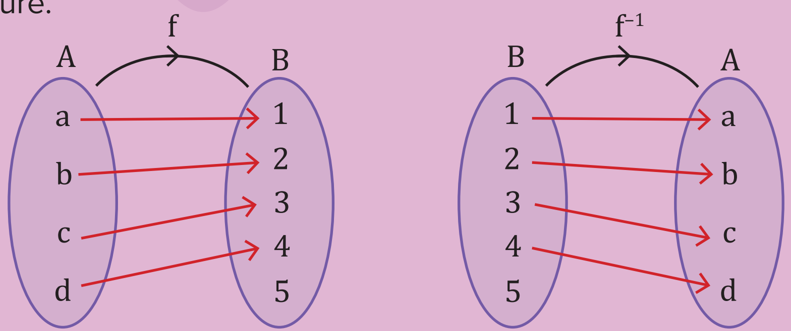
Case 1: Many-one function

Let us consider a many-one function f. Then, its inverse f^{-1} is not a function as shown in the given figure.



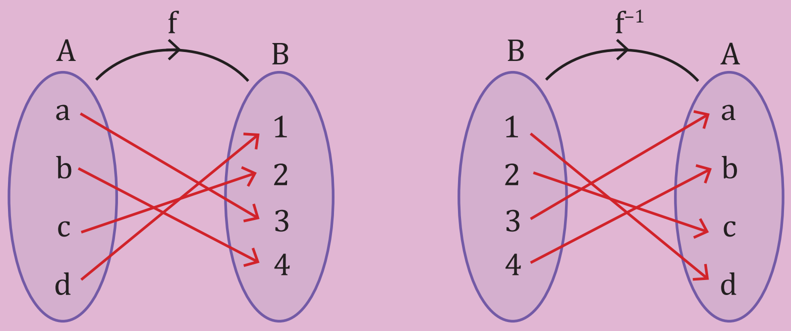
Case 2: Into function

Let us consider a into function f. Then, its inverse f^{-1} is not a function as shown in the given figure.



Case 3: Both one-one and onto function

Let us consider a one-one and onto (bijective) function f. Then, its inverse f^{-1} is also a one-one and onto function as shown in the given figure.



Here, f^{-1} is also a bijective function.
Hence, the inverse of a bijection is unique and also a bijection.

To find the inverse:

Step 1: Let $y = f(x)$
Step 2: Express x in terms of y, $x = g(y)$
Step 3: Now, we get the inverse, i.e., $f^{-1}(x)$.

Note

To get the inverse in step 3, y is replaced by x. Therefore, the domain and the range of $f(x)$ become the range and the domain of $f^{-1}(x)$, respectively.

Example

If $f(x) = \dfrac{2x+3}{4}$, $f : \mathbb{R} \to \mathbb{R}$, then find its inverse.

Solution

Since $f(x)$ is a linear function, it is a bijective function.
Hence, the inverse of function $f(x)$ exists.
Given,

$$y = f(x) = \frac{2x+3}{4}$$

$$\Rightarrow 4y = 2x + 3$$

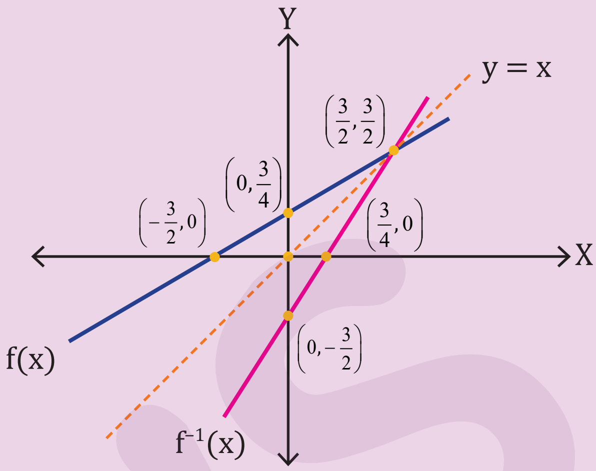
$$\Rightarrow x = \frac{4y-3}{2}$$

$$g(x) = f^{-1}(x) = \frac{4x-3}{2}; g : \mathbb{R} \to \mathbb{R}$$

Observation:

Let us plot a graph of $f(x)$ and $f^{-1}(x)$.

From the figure, we can see that $f(x)$ and $f^{-1}(x)$ are the mirror images of each other about the line $y = x$



$f(x) = x^2 + x + 1$; $f : [0, \infty) \to [1, \infty)$. Find its inverse.

Solution

Step 1:

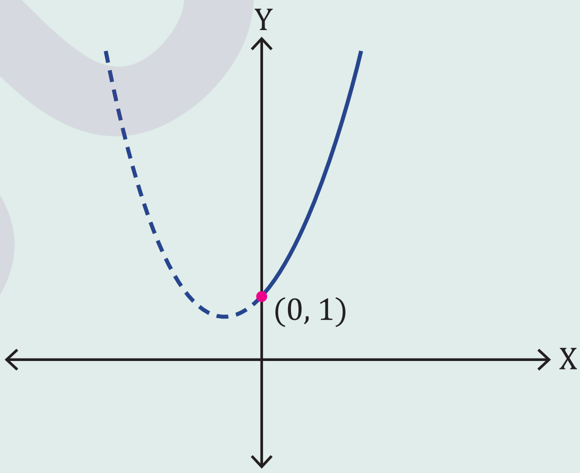
The given quadratic function $y = f(x)$ can be plotted as shown in the figure.

For $x \in [0, \infty)$, $f(x)$ is a one-one function (by horizontal line test)

For $x \in [0, \infty)$, $y \in [1, \infty)$

Hence, Range of $f(x)$ = Codomain of $f(x)$
$\Rightarrow f(x)$ is an onto function.

Hence, function $f(x)$ is bijective and its inverse exists.



Step 2:

Given, $y = x^2 + x + 1$
$\Rightarrow x^2 + x + (1 - y) = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1 - y)}}{2}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{4y - 3}}{2}$$

We know that the domain of a function is the range of its inverse.
$\Rightarrow$ The range of $f^{-1}(x)$ is $[0, \infty)$.

So, $y = \dfrac{-1 - \sqrt{4x - 3}}{2}$ cannot be inverse of the function.

Hence, the inverse of the function is given as follows:

$$f^{-1}(x) = y = \frac{-1 + \sqrt{4x - 3}}{2}$$

Properties of Inverse Function

Property 1

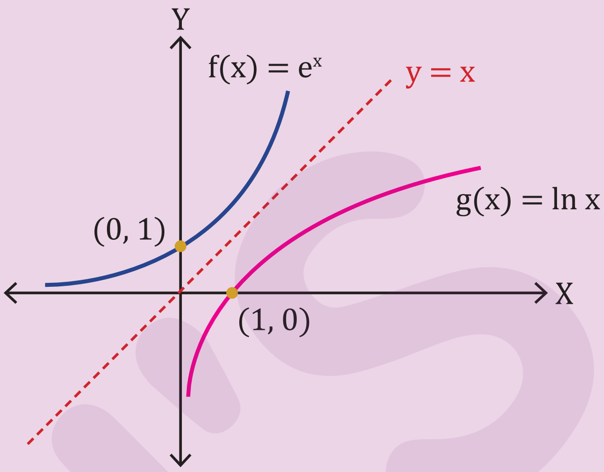
The graphs of function f and its inverse g are the mirror images of each other about the line $y = x$

Example:

$f(x) = e^x$, $g(x) = f^{-1}(x) = \ln x$

So, they can be plotted as shown in the figure.

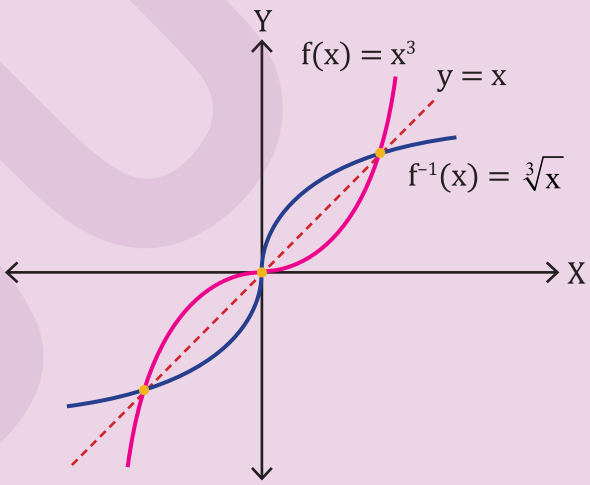
Hence, the function and its inverse are the mirror images of each other about the line $y = x$



Property 2

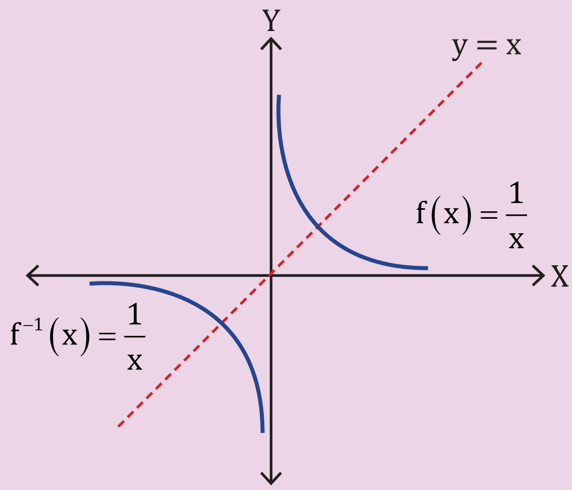
If functions f and f^{-1} intersect, then at least one point of intersection lie on the line $y = x$

Let us consider a function $f(x) = x^3$, then the inverse of function $f^{-1}(x)$ intersects $y = x$ as shown in the figure.



Let us consider another function $f(x) = \frac{1}{x}$

Then, its inverse function $f^{-1}(x) = \frac{1}{x}$ is an example of self-inverse.



Find the solution of the given equation.

$$x^2 - 3x = \frac{3-\sqrt{9+4x}}{2}, x \in \left[\frac{-9}{4}, 1\right]$$

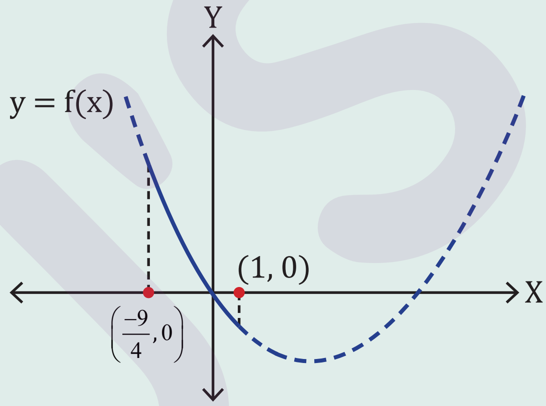
Solution

Step 1:

Given, $x^2 - 3x = \dfrac{3-\sqrt{9+4x}}{2}$, $x \in \left[\dfrac{-9}{4}, 1\right]$

Let $y = f(x) = x^2 - 3x$

For $x \in \left[\dfrac{-9}{4}, 1\right]$, $f(x)$ is bijective



Hence, its inverse exists.

$y = x^2 - 3x$

$x^2 - 3x - y = 0$

$$x = \frac{3 \pm \sqrt{9+4y}}{2}$$

By interchanging x and y, we get,

$$y = \frac{3 \pm \sqrt{9+4x}}{2}$$

Step 2:

We know that the domain of a function is the range of its inverse.

$\Rightarrow$ The range of $f^{-1}(x)$ is $\left[\dfrac{-9}{4}, 1\right]$.

So, $y = \dfrac{3+\sqrt{9+4x}}{2}$ cannot be the inverse of the function.

Hence, the inverse of the function is given as follows:

$$f^{-1}(x) = \frac{3-\sqrt{9+4x}}{2}$$

Step 3:

Hence, the equation is of form $f(x) = f^{-1}(x)$

We know that if functions f and f^{-1} intersect, then at least one point of intersection lie on the line $y = x$

$\Rightarrow f(x) = x^2 - 3x = x$

$\Rightarrow x^2 - 4x = 0$

$\Rightarrow x(x - 4) = 0$

$\Rightarrow x = 0, 4$

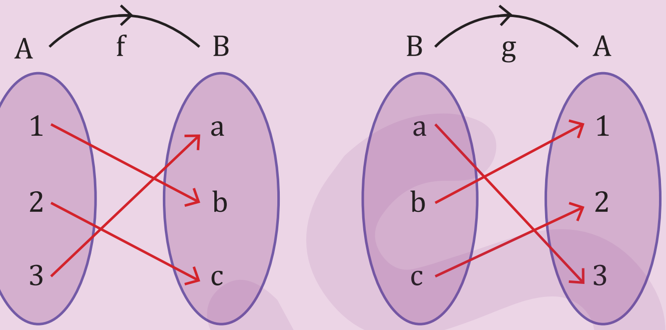
However, $x \in \left[\dfrac{-9}{4}, 1\right]$

$\Rightarrow x = 0$

Property 3

If f and g are the inverse of each other, then fog = gof = x

Let us consider the mapping shown in the figure.



Here, we can see that,
$fog(a) = f(g(a)) = f(3) = a$
$fog(b) = f(g(b)) = f(1) = b$
$fog(c) = f(g(c)) = f(2) = c$

$gof(1) = g(f(1)) = g(b) = 1$
$gof(2) = g(f(2)) = g(c) = 2$
$gof(3) = g(f(3)) = g(a) = 3$

Can we conclude if $fog = gof \Rightarrow$ f and g are inverses of each other?

No, fog and gof can be equal even if f and g are not the inverse of each other. However, in that case, $fog = gof \neq x$

Let us consider two functions, $f(x) = x + 2$ and $g(x) = x + 1$
$fog(x) = f(g(x)) = g(x) + 2 = x + 3$
$gof(x) = g(f(x)) = f(x) + 1 = x + 3$

Hence, $fog = gof \neq x$
Thus, fog = gof, but f and g are not inverse of each other.

Property 4

If $f : A \to B$ and $g : B \to C$ are two bijections, then inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$

Example:

$f(x) = x + 1,\ g(x) = 2x - 1$

$gof(x) = g(f(x)) = g(x+1) = 2(x+1) - 1 = 2x + 1$

$2x + 1 = y \Rightarrow x = \dfrac{y-1}{2}$

$\Rightarrow (gof)^{-1}(x) = \dfrac{x-1}{2}$

$y = x + 1 \Rightarrow x = y - 1$

$\Rightarrow f^{-1}(x) = x - 1$

and, $2x - 1 = y \Rightarrow x = \dfrac{y+1}{2}$

$\Rightarrow g^{-1}(x) = \dfrac{x+1}{2}$

$f^{-1}og^{-1}(x) = f^{-1}(g^{-1}(x)) = f^{-1}\left(\dfrac{x+1}{2}\right) = \dfrac{x-1}{2}$

Hence $(gof)^{-1} = f^{-1}og^{-1}$

If $f(x) = ax + b$ and the equation $f(x) = f^{-1}(x)$ can be satisfied by every real value of x, then

(a) $a = 2, b = -1$ **(b)** $a = -1, b \in \mathbb{R}$ **(c)** $a = 1, b \in \mathbb{R}$ **(d)** $a = 1, b = -1$

Solution

$$f(x) = y = ax + b \Rightarrow x = \frac{y}{a} - \frac{b}{a}$$

$$\Rightarrow f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$$

Given, $f(x) = f^{-1}(x)\ \forall\ x \in \mathbb{R}$

$$\Rightarrow ax + b = \frac{x}{a} - \frac{b}{a}$$

$$\Rightarrow \frac{1}{a} = a,\ b = -\frac{b}{a}$$

$$\Rightarrow a^2 = 1 \Rightarrow a = \pm 1 \qquad (a \neq 0)$$

If $a = 1 \Rightarrow b = -b \Rightarrow b = 0$

If $a = -1 \Rightarrow b = b \Rightarrow b \in \mathbb{R}$

Hence, option (b) is the correct answer.

Concept Check

1. Find the inverse function of $f(x) = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}},\ x \in (-1, 1)$ **JEE MAIN 2020**

(a) $\frac{1}{4}\log_8 e \times \log_e\left(\frac{1-x}{1+x}\right)$ (b) $\frac{1}{4}\log_8 e \times \log_e\left(\frac{1+x}{1-x}\right)$

(c) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

2. Find the inverse of the following function (assuming one-one and onto):

$$y = \log_a\left(x + \sqrt{x^2+1}\right),\ (a > 1)$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible function defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$. Find its inverse.

Summary Sheet

Key Takeaways

- Let $y = f(x): A \rightarrow B$ be a one-one and onto function, i.e., a bijection. Then, there exists a function $x = g(y) : B \rightarrow A$ such that if $(\alpha, \beta) \in f \Rightarrow (\beta, \alpha) \in g$
  Then f and g are said to be invertible functions of each other.
- The inverse of a bijection is unique and also a bijection.
- The graphs of function f and its inverse g are the mirror images of each other about the line $y = x$
- If functions f and f^{-1} intersect, then at least one point of intersection lies on the line $y = x$
- If f and g are the inverse of each other, then $fog = gof = x$
- If $f : A \rightarrow B$, and $g : B \rightarrow C$ are two bijections, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$

Mind Map

Inverse function → Properties of inverse function

Self-Assessment

If $f(x) = 3x - 2$ and $(gof)^{-1}(x) = x - 2$, then find the function $g(x)$.

Answers

Concept Check

1. Let $f(x) = y = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}} \Rightarrow y = \dfrac{8^{4x} - 1}{8^{4x} + 1}$

$$\Rightarrow y \times 8^{4x} + y = 8^{4x} - 1 \Rightarrow 8^{4x} = \frac{y+1}{1-y}$$

$$\Rightarrow 4x = \log_8\left(\frac{y+1}{1-y}\right) \Rightarrow x = \frac{1}{4}\log_8\left(\frac{y+1}{1-y}\right)$$

$$\Rightarrow x = \frac{1}{4}\log_8 e \times \log_e\left(\frac{y+1}{1-y}\right) = g(y) \qquad \{\log_b a \,.\, \log_c b = \log_c a\}$$

$$\therefore g(x) = f^{-1}(x) = \frac{1}{4}\log_8 e \times \log_e\left(\frac{x+1}{1-x}\right)$$

Hence, option (b) is the correct answer.

2. $y = \log_a\left(x + \sqrt{x^2+1}\right), (a > 1)$

$a^y = x + \sqrt{x^2+1} \quad ...(1)$

$\Rightarrow a^{-y} = \dfrac{1}{x+\sqrt{x^2+1}} \Rightarrow a^{-y} = \dfrac{1}{x+\sqrt{x^2+1}} \cdot \dfrac{-x+\sqrt{x^2+1}}{-x+\sqrt{x^2+1}}$

$\Rightarrow a^{-y} = -x + \sqrt{x^2+1} \quad ...(2)$

By subtracting (2) from (1), we get,

$a^y - a^{-y} = 2x \Rightarrow x = \dfrac{1}{2}\left(a^y - a^{-y}\right)$

$f^{-1}(x) = \dfrac{1}{2}\left(a^x - a^{-x}\right)$

3. $f(x) = \dfrac{e^x - e^{-x}}{2} = y$

$\Rightarrow e^x - \dfrac{1}{e^x} = 2y \Rightarrow e^{2x} - 2ye^x - 1 = 0$

After substituting $e^x = t$, we get,

$t^2 - 2yt - 1 = 0 \Rightarrow t = \dfrac{2y \pm \sqrt{4y^2+4}}{2}$

$\Rightarrow e^x = y \pm \sqrt{y^2+1}$

Since $e^x < 0$ is not possible,

$\Rightarrow e^x = y + \sqrt{y^2+1} \quad \Rightarrow x = \log\left(y + \sqrt{y^2+1}\right)$

$\Rightarrow f^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

Self-Assessment

$f(x) = y = 3x - 2 \Rightarrow x = \dfrac{y+2}{3}$

$f^{-1}(x) = \dfrac{x+2}{3}$

Given, $(gof)^{-1}(x) = x - 2$

$\Rightarrow f^{-1}og^{-1}(x) = x - 2 \Rightarrow f^{-1}\left(g^{-1}(x)\right) = x - 2$

$\Rightarrow \dfrac{g^{-1}(x)+2}{3} = x - 2 \Rightarrow g^{-1}(x) = 3x - 8$

$\Rightarrow g(x) = \dfrac{x+8}{3}$

MATHEMATICS

RELATIONS AND FUNCTIONS

TRANSFORMATION OF GRAPHS

NOTES

What you already know

- Types of functions
- Composite functions
- Periodic functions
- Inverse functions

What you will learn

- Functional equation
- Transformation of graphs

Functional equation

If x and y are independent real variables, then,

(i) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx;\ k \in \mathbb{R}$

Justification:

$f(x) = kx$

$f(x+y) = k(x+y) = kx + ky = f(x) + f(y)$

(ii) $f(x+y) = f(x)f(y) \Rightarrow f(x) = a^{kx};\ k \in \mathbb{R}$ and $a > 0$

Justification:

$f(x) = a^{kx}$

$f(x+y) = a^{k(x+y)} = a^{kx} \times a^{ky} = f(x)f(y)$

(iii) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x;\ k \in \mathbb{R}, x > 0$

Justification:

$f(x) = k \ln x$

$f(xy) = k \ln (xy) = k(\ln x + \ln y) = k \ln x + k \ln y = f(x) + f(y)$

(iv) $f(xy) = f(x)f(y) \Rightarrow f(x) = x^n;\ n \in \mathbb{R}$

Justification:

$f(x) = x^n$

$f(xy) = (xy)^n = x^n y^n = f(x)f(y)$

(v) If f(x) is a polynomial of degree n such that $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n;\ n \in \mathbb{R}$

Justification:

$f(x) = 1 \pm x^n$

$$f(x)f\left(\frac{1}{x}\right) = (1 \pm x^n)\left(1 \pm \left(\frac{1}{x}\right)^n\right) = 1 \pm \left(\frac{1}{x}\right)^n \pm x^n + 1 = f(x) + f\left(\frac{1}{x}\right)$$

Condition	Function
$f(x+y) = f(x) + f(y)$	$f(x) = kx;\ k \in \mathbb{R}$
$f(x+y) = f(x)f(y)$	$f(x) = a^{kx};\ k \in \mathbb{R}$ and $a > 0$
$f(xy) = f(x) + f(y)$	$f(x) = k \ln x;\ k \in \mathbb{R}, x > 0$
$f(xy) = f(x)f(y)$	$f(x) = x^n;\ n \in \mathbb{R}$
If $f(x)$ is a polynomial of degree n such that $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$	$f(x) = 1 \pm x^n;\ n \in \mathbb{R}$

If a function $f(x)$ satisfies the relation $f(x+y) = f(x) + f(y)$, where $x, y \in \mathbb{R}$ and $f(1) = 4$, then find the value of $\sum_{r=1}^{10} f(r)$.

(a) 100 **(b) 220** **(c) 160** **(d) 300**

Solution

Step 1:
Given,

$f(x+y) = f(x) + f(y)$

$\Rightarrow f(x) = kx$

It is also given that,

$f(1) = 4$

$\Rightarrow k(1) = 4$

$\Rightarrow k = 4$

$\therefore$ The function is $f(x) = 4x$

Step 2:
Now,

$$\sum_{r=1}^{10} f(r) = f(1) + f(2) + \ldots + f(10)$$

$$= 4 + 4(2) + 4(3) + \ldots + 4(10)$$

$$= 4(1 + 2 + 3 + \ldots + 10)$$

$$= 4 \times \frac{10}{2} \times (10+1) \qquad \left(\sum n = \frac{n(n+1)}{2}\right)$$

$$= 220$$

$\therefore$ **Option (b) is the correct answer.**

If $f(x)$ is a polynomial function such that $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(3) = -26$, then find $f(4)$.

(a) 64 (b) -65 (c) -63 (d) 65

Solution

Step 1:

Given,

$$f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n$$

Also given,

$$f(3) = -26 \Rightarrow 1 \pm 3^n = -26$$

Step 2:

Now, we have two cases,

Case I:

$$1 + 3^n = -26$$

$$\Rightarrow 3^n = -27$$

It is not possible.

Case II:

$$1 - 3^n = -26$$

$$\Rightarrow 3^n = 27 \quad \Rightarrow n = 3$$

Thus, we have the function $f(x) = 1 - x^3$

$\therefore f(4) = 1 - 4^3 = -63$

$\therefore$ **Option (c) is the correct answer.**

Let the function $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \frac{4^x}{4^x + 2}$, then find the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \ldots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$$

Solution

Step 1:

$$f(x) + f(1 - x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2}$$

$$= \frac{4^x}{4^x + 2} + \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2}$$

$$= \frac{4^x}{4^x + 2} + \frac{2}{2 + 4^x}$$

$$= \frac{4^x + 2}{4^x + 2} = 1$$

$$\therefore f(x) + f(1 - x) = 1$$

Step 2:

$$f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+\ldots+f\left(\frac{19}{40}\right)+f\left(\frac{20}{40}\right)+\ldots+f\left(\frac{38}{40}\right)+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$$

$$=\left\{f\left(\frac{1}{40}\right)+f\left(\frac{39}{40}\right)\right\}+\left\{f\left(\frac{2}{40}\right)+f\left(\frac{38}{40}\right)\right\}+\ldots+f\left(\frac{20}{40}\right)-f\left(\frac{1}{2}\right)$$

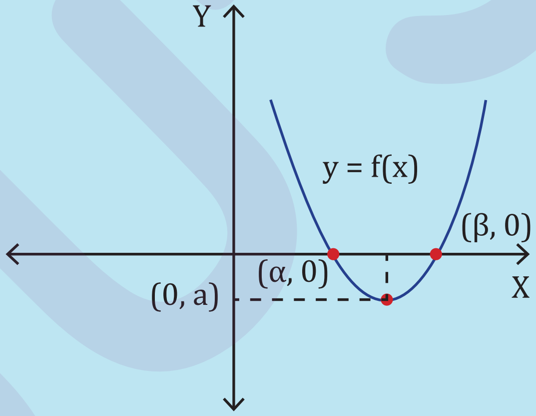
$= 1 + 1 + 1 + \ldots$ 19 terms

$= 19$

Transformation of Graphs

Horizontal shift

Let us consider a function $y = f(x)$ intersecting the X-axis at $(\alpha, 0)$ and $(\beta, 0)$ as shown in the figure.

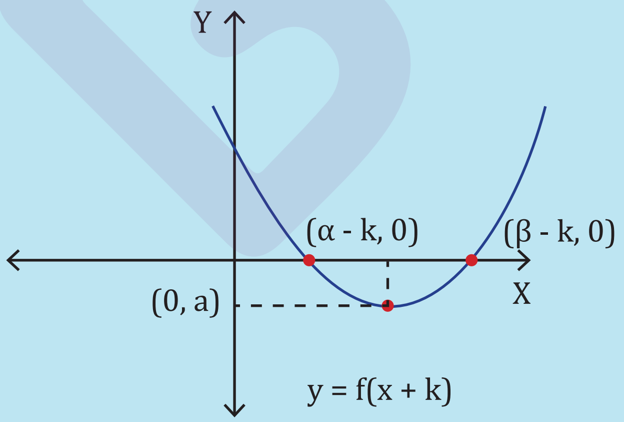


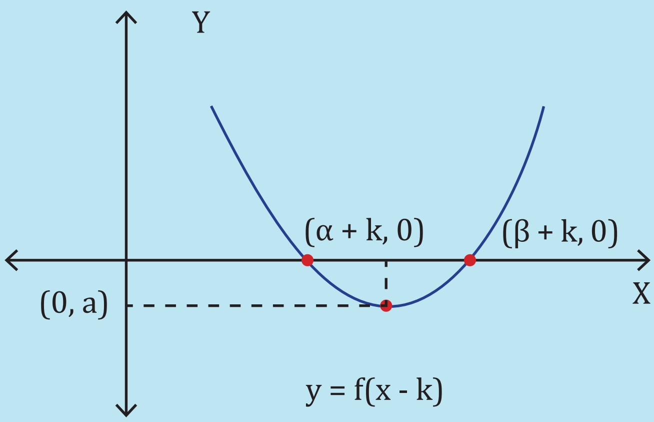
When the given graph of f(x) is shifted by k units toward the left, it gives the graph of the function $y = f(x + k)$, $(k > 0)$

In this case, the graph cuts the X-axis at the points $(\alpha - k, 0)$ and $(\beta - k, 0)$ as shown in the figure.

When the given graph of f(x) is shifted by k units toward the right, it gives the graph of the function $y = f(x - k)$, $(k > 0)$

In this case, the graph cuts the X-axis at the points $(\alpha + k, 0)$ and $(\beta + k, 0)$ as shown in the figure.

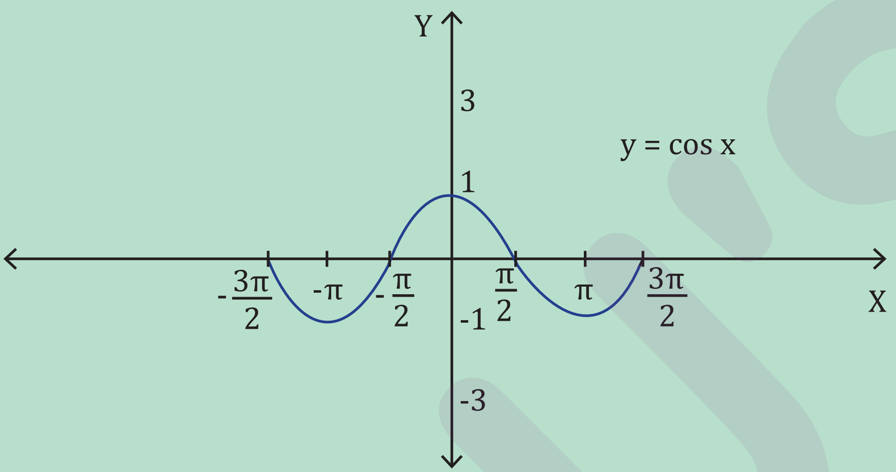




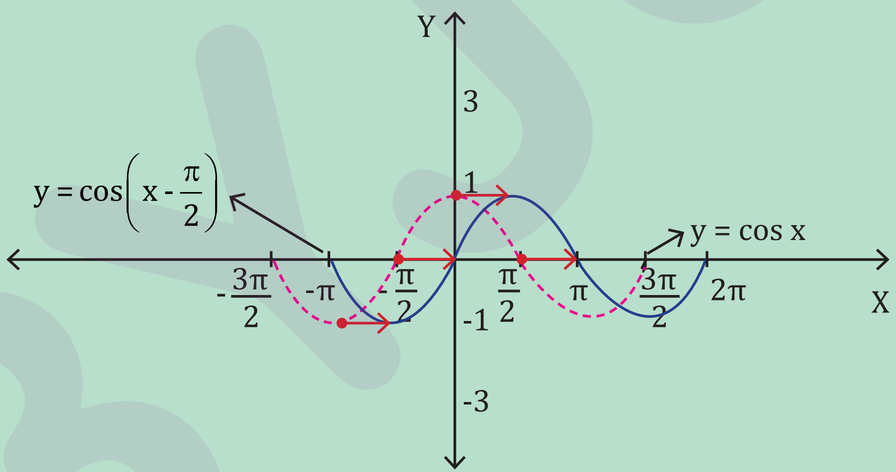
Plot graph of the function $y = \cos\left(x - \frac{\pi}{2}\right)$

Solution

Step 1: Plot the graph of the function $y = \cos x$

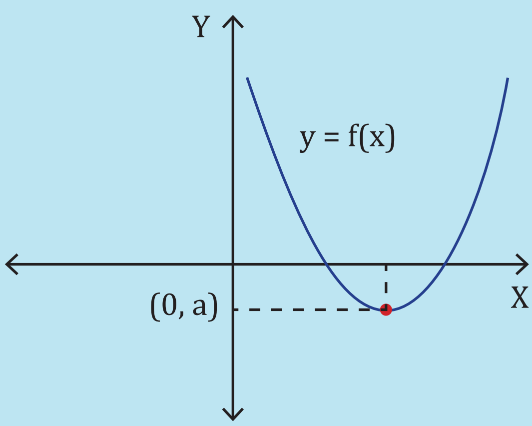


Step 2: Shift the graph of $y = \cos x$ in the direction of positive X-axis (by $\frac{\pi}{2}$ units) to get the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$



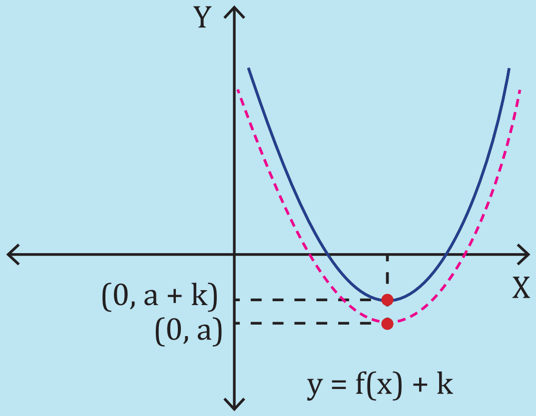
Vertical shift

Let us consider a function $y = f(x)$ having minimum value a as shown in the figure.



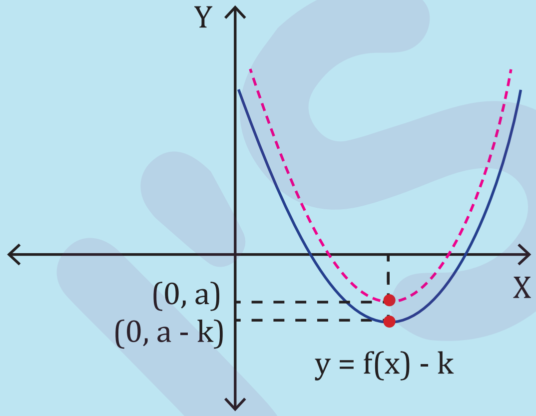
When the given graph of f(x) is shifted upwards by k units, it gives the graph of the function $y = f(x) + k; k > 0$

In this case, the minimum value of the function $y = f(x) + k$ is $a + k$



When the given graph of f(x) is shifted downwards by k units, it gives the graph of the function $y = f(x) - k; k > 0$

In this case, the minimum value of the function $y = f(x) - k$ is $a - k$.

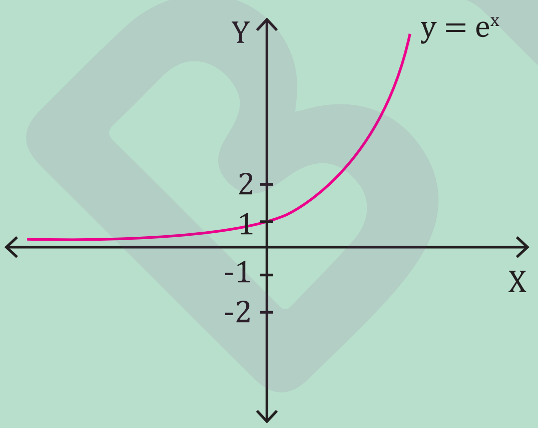


Plot graph of the function $y = e^x - 2$.

Solution

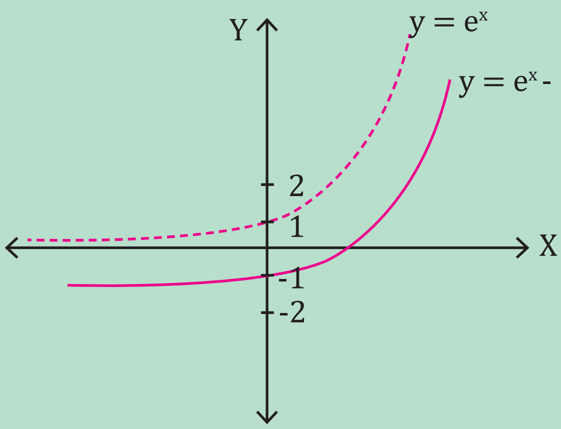
Step 1:

Plot the graph of the function $y = e^x$



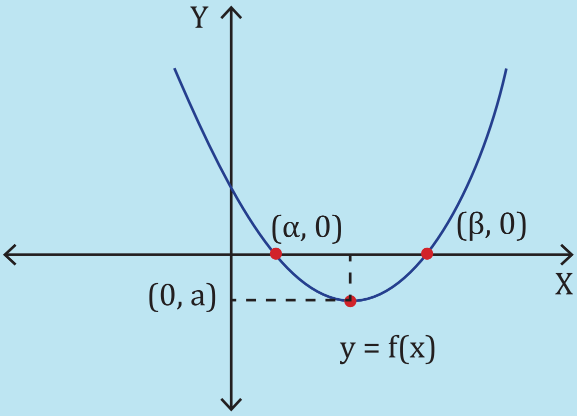
Step 2:

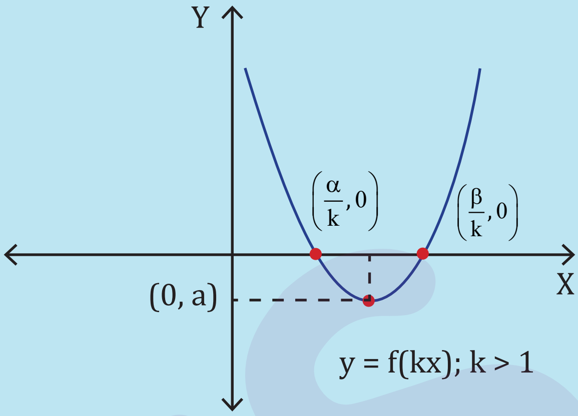
Shift the graph of $y = e^x$ by 2 units downwards to plot the graph of $y = e^x - 2$



Horizontal stretch

Let us consider a function $y = f(x)$. When the points on the X-axis of the graph of the given function f(x) are divided by k units, it gives the graph of $y = f(kx); k > 1$



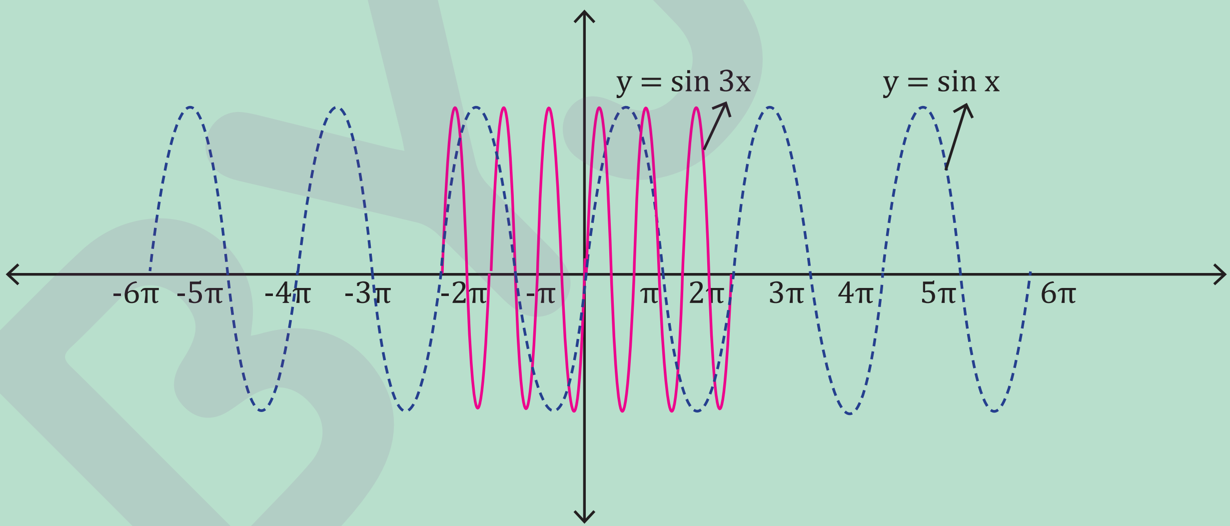


Plot graph of the function $y = \sin 3x$.

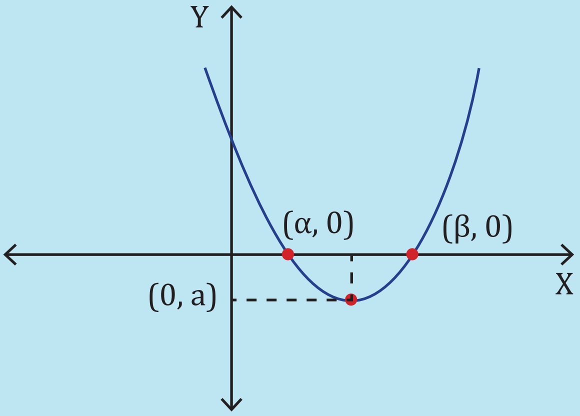
Solution

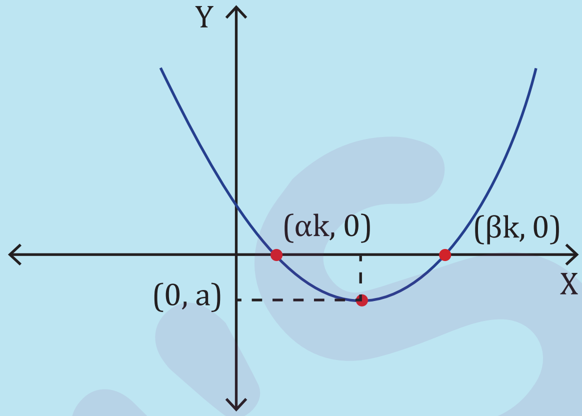
To get the graph of $y = \sin 3x$, the points on the X-axis of the graph of the given function $f(x)$ are divided by 3 units.

Here, the period of $\sin 3x$ is $\frac{2\pi}{3}$.



Let us consider a function $y = f(x)$. When the points on the X-axis of the graph of the given function $f(x)$ are multiplied by k units, it gives the graph of $y = f\left(\frac{x}{k}\right); k > 1$

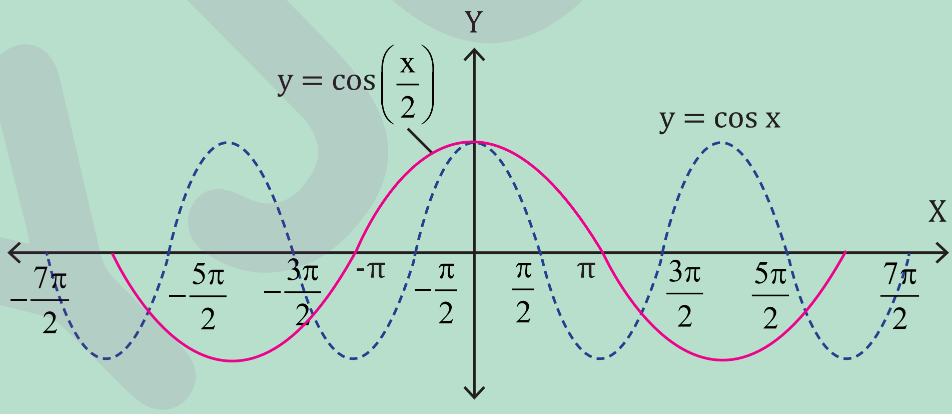




Plot graph of the function $y = \cos\left(\frac{x}{2}\right)$.

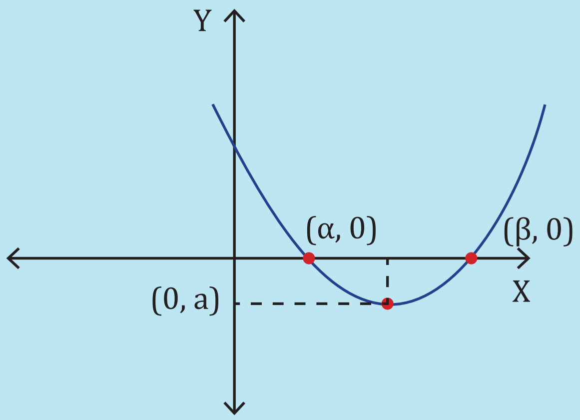
Solution

To get the graph of $y = \cos\left(\frac{x}{2}\right)$, the points on the X-axis of the graph of the given function f(x) are multiplied by 2. Here, the period of $\cos\left(\frac{x}{2}\right)$ is 4π.

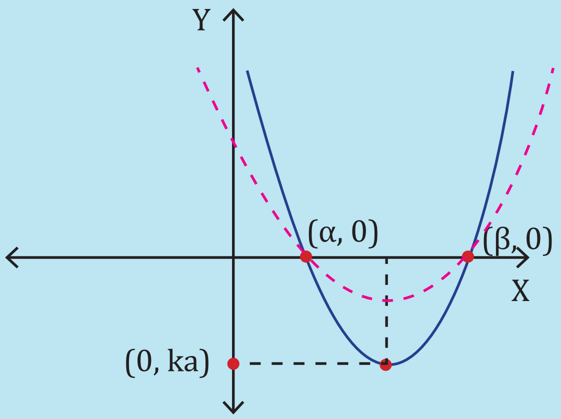


Vertical stretch

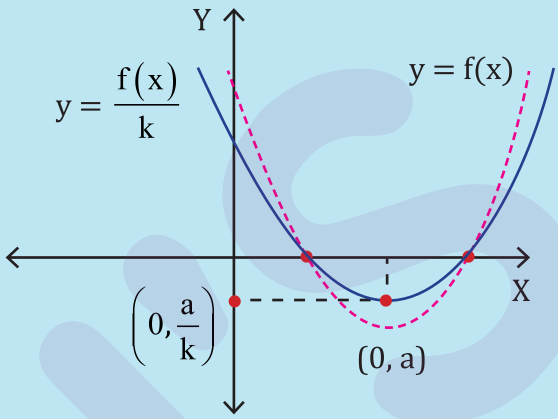
Let us consider a function y = f(x)



When the points on the Y-axis of the graph of the given function $f(x)$ are multiplied by k units, it gives the graph of $y = k\,f(x)$; $k > 1$



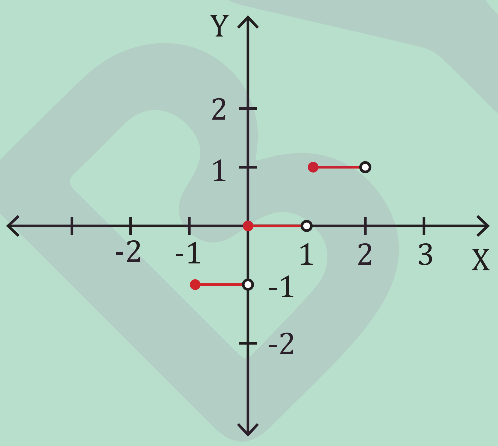
When the points on the Y-axis of the graph of the given function $f(x)$ are divided by k units, it gives the graph of $y = \frac{f(x)}{k}$; $k > 1$



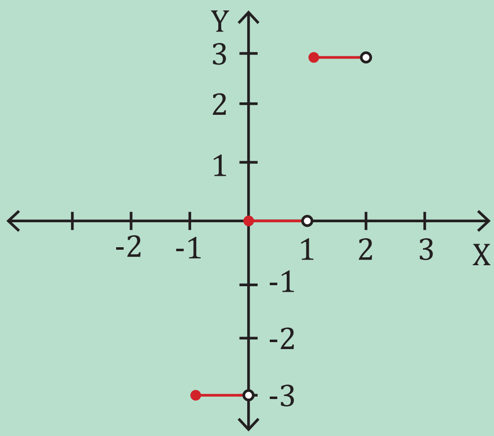
Plot graph of the function $y = 3[x]$; where [.] denotes G.I.F.

Solution

Step 1:
Plot the graph of the function $y = [x]$



Step 2:
Multiply the points on the Y-axis of the graph by 3 units to get the graph of $y = 3[x]$

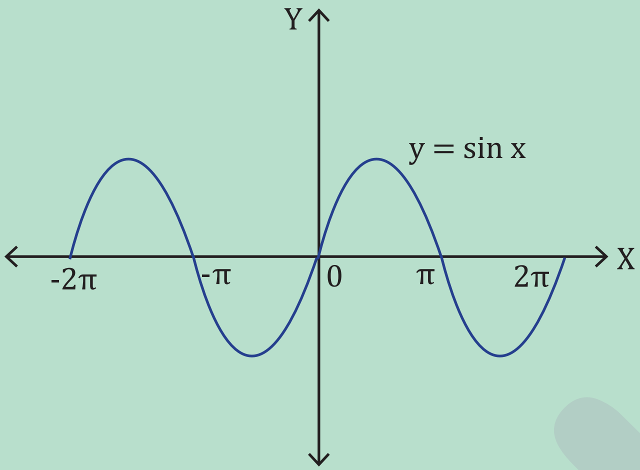


Plot graph of the function $y = \left|\sin x + \frac{1}{2}\right|$.

Solution

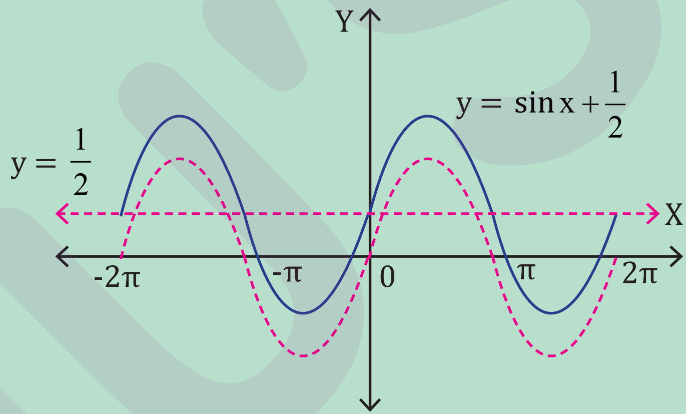
Step 1:

Plot the graph of the function $y = \sin x$



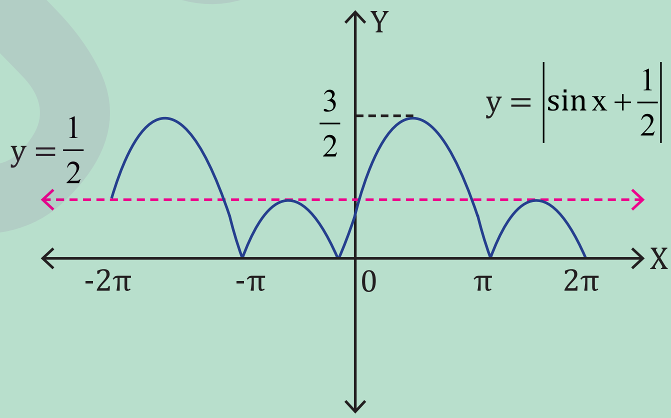
Step 2:

Shift the graph of $y = \sin x$ by $\frac{1}{2}$ units upwards to plot the graph of $y = \sin x + \frac{1}{2}$



Step 3:

To plot $y = \left|\sin x + \frac{1}{2}\right|$, take the mirror image of $y = \sin x + \frac{1}{2}$ about the X-axis of that part of the graph which lies below the X-axis.



Find the number of solutions of the equation $|x| = \cos x$.

Solution

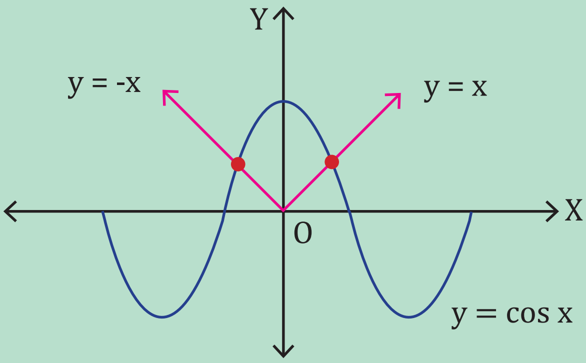
Step 1:

Given, $|x| = \cos x$

Number of real solutions = Number of points of intersection of the curves

Let us plot $y = |x|$ and $y = \cos x$

Step 2:



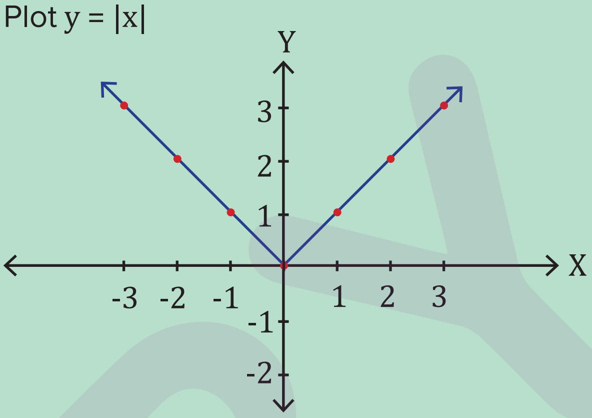
Clearly, both the graphs intersect at two points. Therefore, there are two solutions for $|x| = \cos x$

Plot the graph of the function $y = |2 - |x - 1||$

Solution

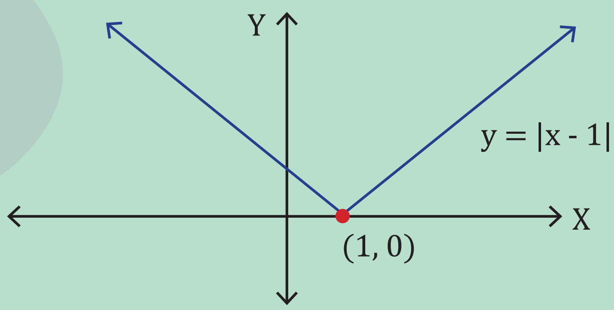
Step 1:

Plot $y = |x|$



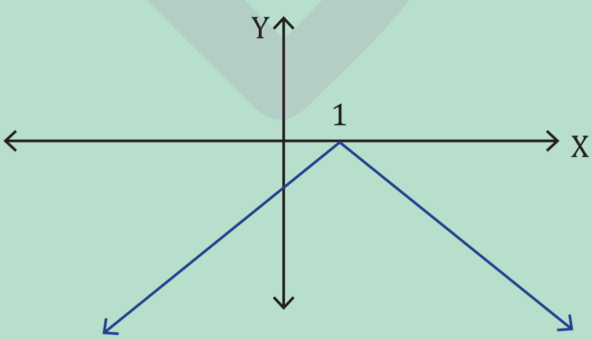
Step 2:

Shift the graph of $y = |x|$ towards the right by 1 unit to obtain the graph of $y = |x - 1|$



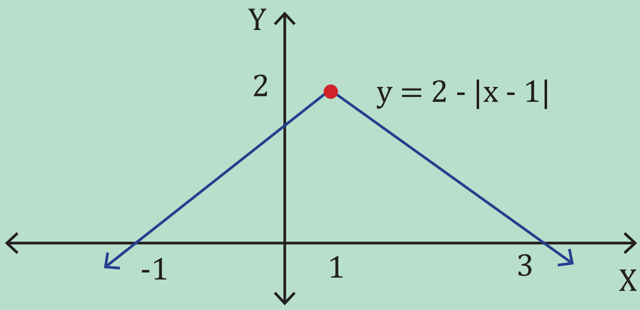
Step 3:

To plot $y = -|x - 1|$, take the reflection of $y = |x - 1|$ about the X-axis.



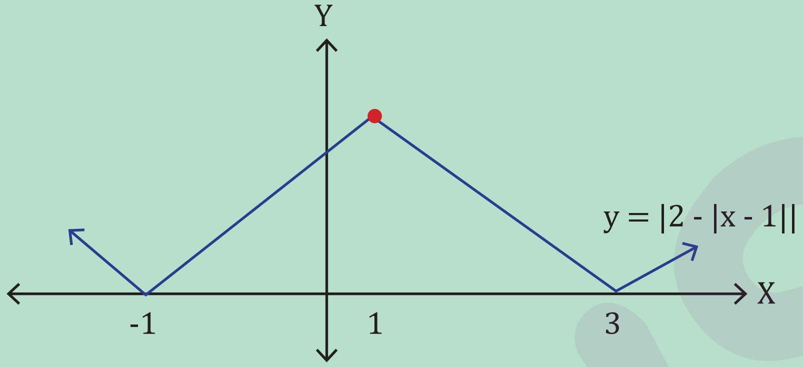
Step 4:

To plot $y = 2 - |x - 1|$, shift the graph of $y = -|x - 1|$ by 2 units in the upward direction.



Step 5:

To plot $y = |2 - |x - 1||$, take the mirror image of $y = 2 - |x - 1|$ about the X-axis of that part of the graph which lies below the X-axis.



Concept Check

1. For $x \in \mathbb{R} - \{0\}$, the function $f(x)$ satisfies $f(x) + 2f(1 - x) = \frac{1}{x}$. Find the value of $f(2)$.
   (a) $-\frac{5}{6}$ (b) $\frac{1}{2}$ (c) -2 (d) $\frac{3}{4}$

2. A function $f(x)$ is given by $f(x) = \frac{5^x}{5^x + 5}$.
   Find the sum of the series $f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{3}{20}\right) + \ldots\ldots\ldots + f\left(\frac{39}{20}\right)$.
   (a) $\frac{19}{2}$ (b) $\frac{49}{2}$ (c) $\frac{39}{2}$ (d) $\frac{29}{2}$

3. Plot the graph of of $y = \ln(x + 2)$
4. Plot the graph of the function $y = \sin x + 3$
5. Plot the graph of the function $y = \frac{|x|}{5}$
6. Plot the graph of the function $y = 2 \sin 2x$

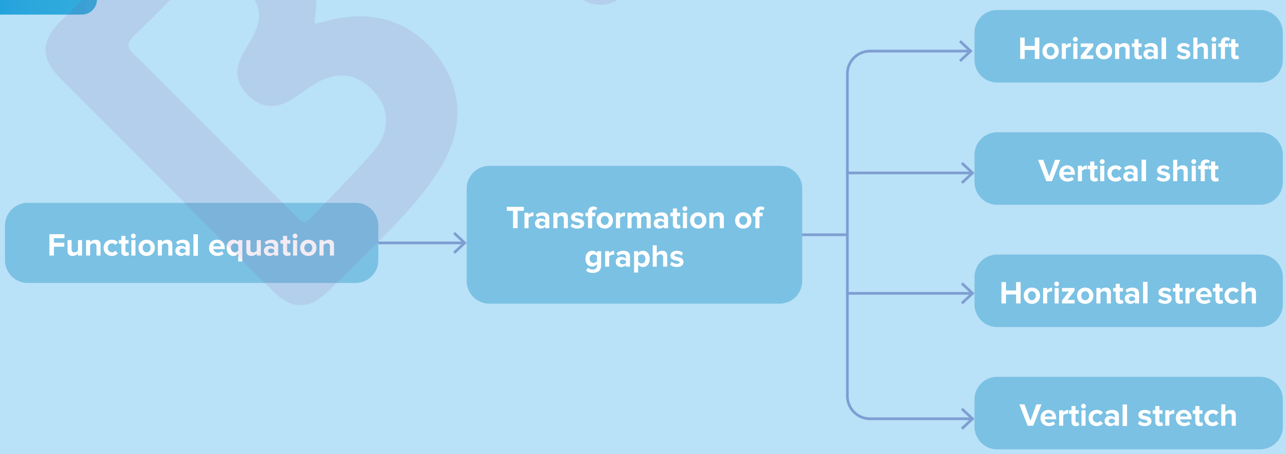
Summary Sheet

Key Takeaways

- $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx;\ k \in \mathbb{R}$
- $f(x + y) = f(x)\, f(y) \Rightarrow f(x) = a^{kx};\ k \in \mathbb{R}$ and $a > 0$
- $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \log_a x;\ k \in \mathbb{R}$, $a > 0$, and $a \neq 1$
- $f(xy) = f(x)\, f(y) \Rightarrow f(x) = x^n;\ n \in \mathbb{R}$
- If $f(x)$ is a polynomial of degree n such that, $f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n;\ n \in \mathbb{R}$

Condition	Function
• $f(x)$ is shifted by k units towards the left	$y = f(x + k)$
• $f(x)$ is shifted by k units towards the right	$y = f(x - k)$
• $f(x)$ is shifted by k units upwards	$y = f(x) + k;\ k > 0$
• $f(x)$ is shifted by k units downwards	$y = f(x) - k;\ k > 0$
• When the points on the X-axis of the graph of the given function $f(x)$ are divided by k units	$y = f(kx);\ k > 1$
• When the points on the Y-axis of the graph of the given function $f(x)$ are divided by k units	$y = f\left(\frac{x}{k}\right);\ k > 1$
• When the points on the Y-axis of the graph of the given function $f(x)$ are multiplied by k units	$y = k\,f(x);\ k > 0$

Mind Map



Self-Assessment

Find the number of solutions for the following function: $\sin x = x^2 + x + 1, x \in \mathbb{R}$

Answers

Concept Check

1.

Given,

$$f(x) + 2f(1 - x) = \frac{1}{x} \ldots(i)$$

By substituting $x = 2$ in equation (i), we get,

$$f(2) + 2f(-1) = \frac{1}{2} \ldots(ii)$$

By substituting $x = -1$ in equation (i), we get,

$$f(-1) + 2f(2) = -1 \ldots(iii)$$

Here, we have two unknowns, f(2) and f(-1), and two equations (ii) and (iii).

By solving equations (ii) and (iii), we get,

$$f(2) = -\frac{5}{6}$$

∴ **Option (a) is the correct answer.**

2.

Step 1:

$$f(x) = \frac{5^x}{5^x + 5} \text{ ....(i)}$$

$$f(2-x) = \frac{5^{2-x}}{5^{2-x} + 5} = \frac{5}{5^x + 5} \text{ ....(ii)}$$

Step 2:

Adding (i) and (ii), we get,

$$f(x) + f(2-x) = \frac{5^x}{5^x + 5} + \frac{5}{5^x + 5} = 1$$

$$f\left(\frac{1}{20}\right) + f\left(\frac{39}{20}\right) = 1$$

$$f\left(\frac{2}{20}\right) + f\left(\frac{38}{20}\right) = 1$$

.

.

.

$$f\left(\frac{19}{20}\right) + f\left(\frac{21}{20}\right) = 1$$

Step 3:

$$f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{3}{20}\right) + \text{.........} + f\left(\frac{39}{20}\right)$$

$$= f\left(\frac{1}{20}\right) + f\left(\frac{39}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{38}{20}\right) + \text{.......} + f\left(\frac{19}{20}\right)$$

$$+ f\left(\frac{21}{20}\right) + f\left(\frac{20}{20}\right)$$

$$= 1 + 1 + \text{.........} (19 \text{ times}) + f(1)$$

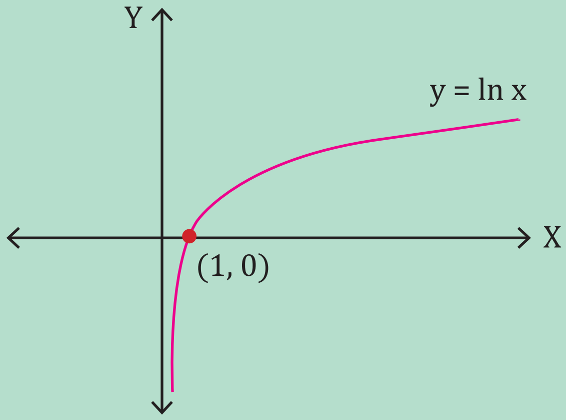
$$= 19 + \frac{1}{2} \qquad \left(\because f(x) = \frac{5^x}{5^x + 5}\right)$$

$$= \frac{39}{2}$$

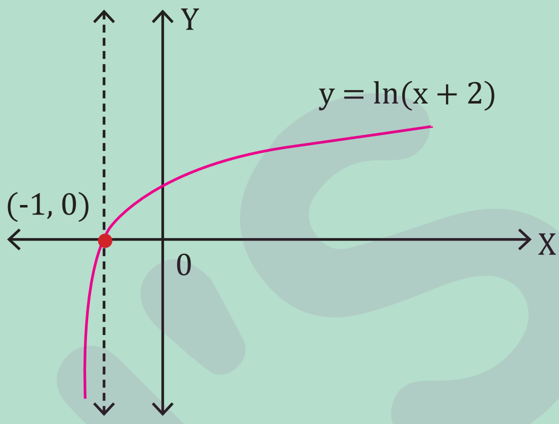
So, option (c) is the correct answer.

3.

Let us plot the graph of $y = \ln x$

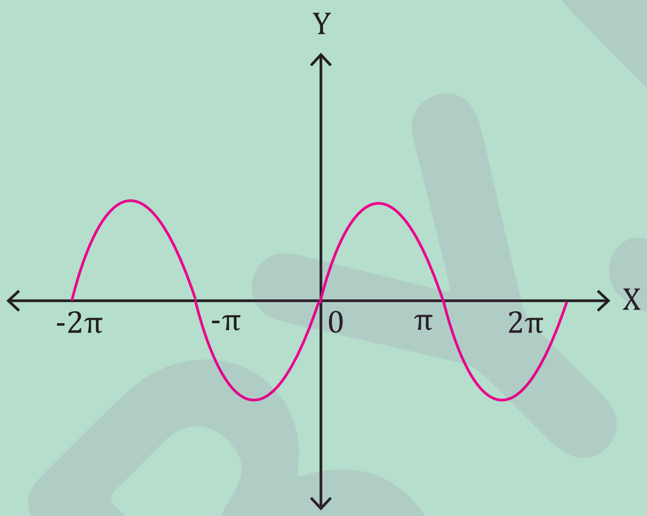


To plot $y = \ln(x + 2)$, shift the graph of $y = \ln x$ leftwards by 2 units.

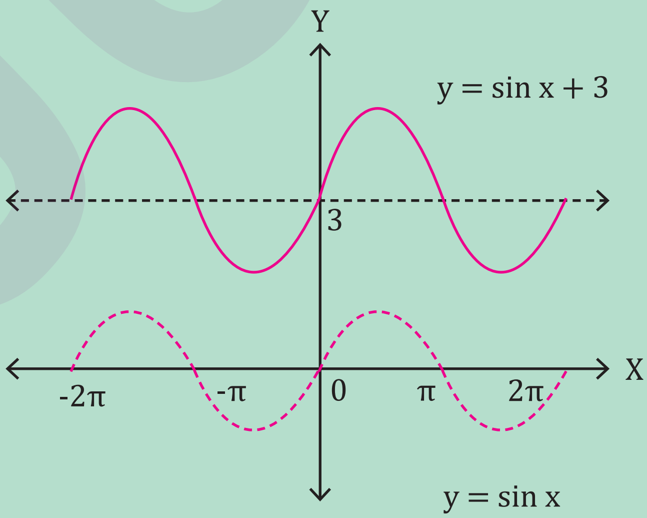


4.

Let us plot the graph of $y = \sin x$

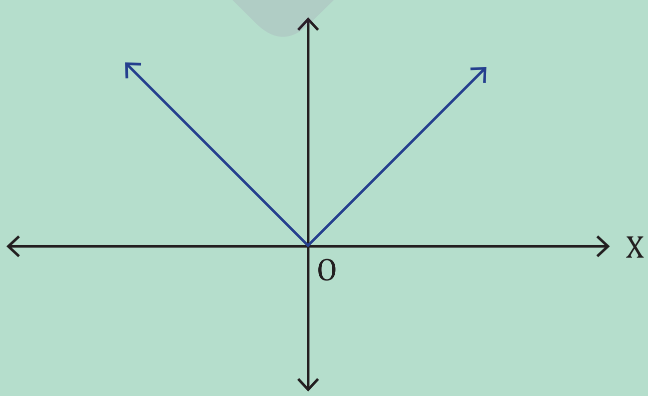


The whole graph of $y = \sin x$ is shifted upwards by 3 units to get the graph of $y = \sin x + 3$

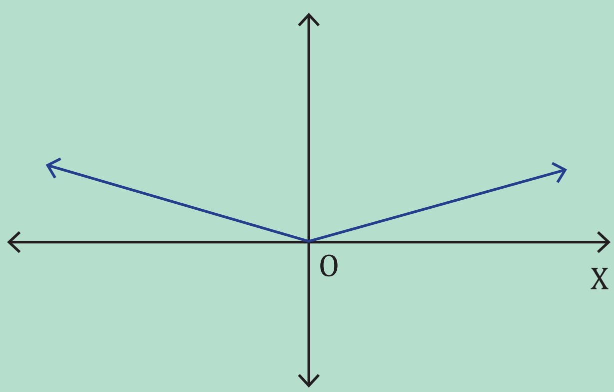


5.

Let us plot the graph of $y = |x|$

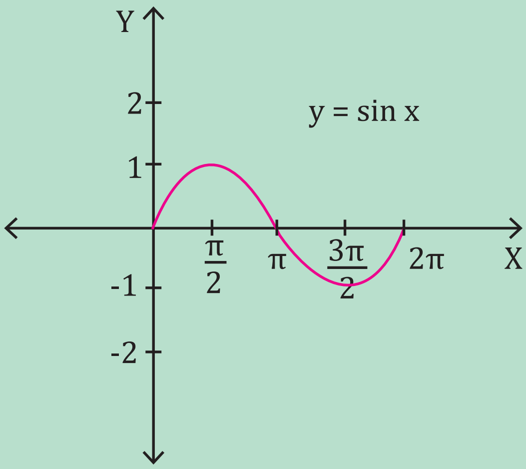


Divide the points on the Y-axis of the graph by 5 to get the graph of $y = \frac{|x|}{5}$

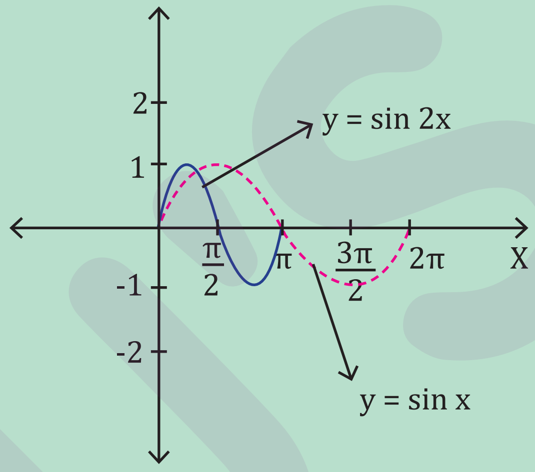


6.

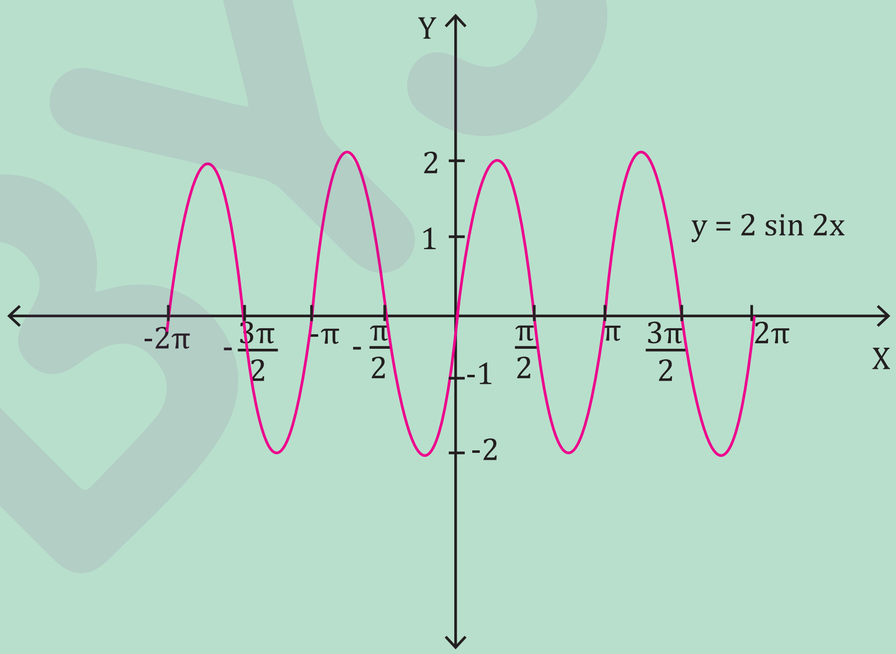
Step 1:
Plot the graph of the function
$y = \sin x$



Step 2:
Divide the points on the X-axis of the graph by 2 to get the graph of
$y = \sin 2x$

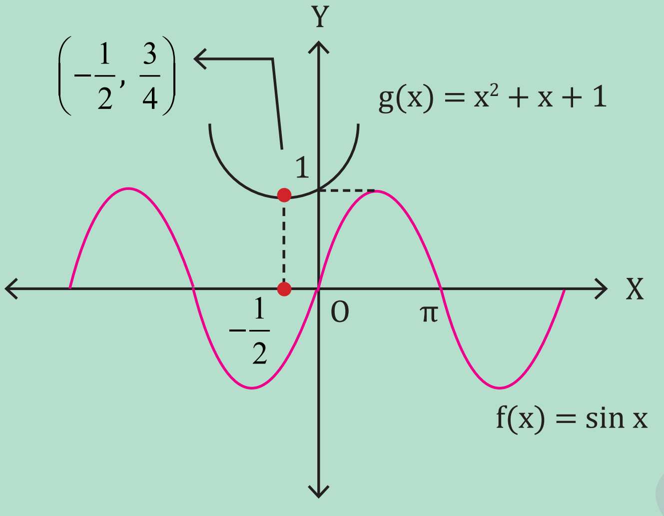


Step 3:
Multiply the points on the Y-axis of the graph by 2 to get the graph of $y = 2 \sin 2x$



Self-Assessment

Plot the graph of $y = f(x) = \sin x$, $y = g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$



From the graph, we can observe that the number of points of intersection is zero.
Hence, $\sin x = x^2 + x + 1$ has no solution.

MATHEMATICS

RELATIONS AND FUNCTIONS

MORE ON TRANSFORMATION OF GRAPHS

NOTES

What you already know

- Types of functions
- Graphs of functions
- Domain and range of a function

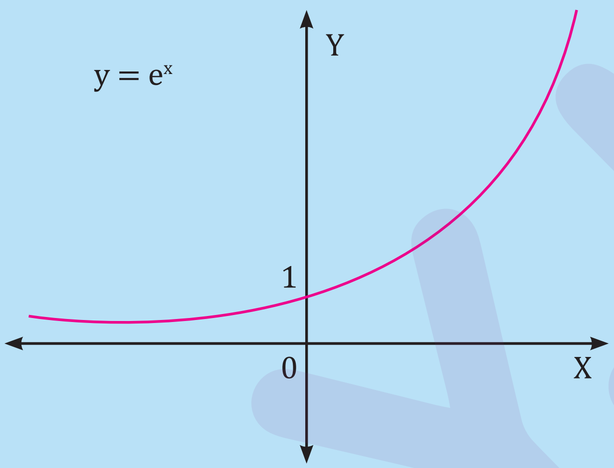
What you will learn

- Different transformations of graphs
- Number of solutions of an equation using graphs

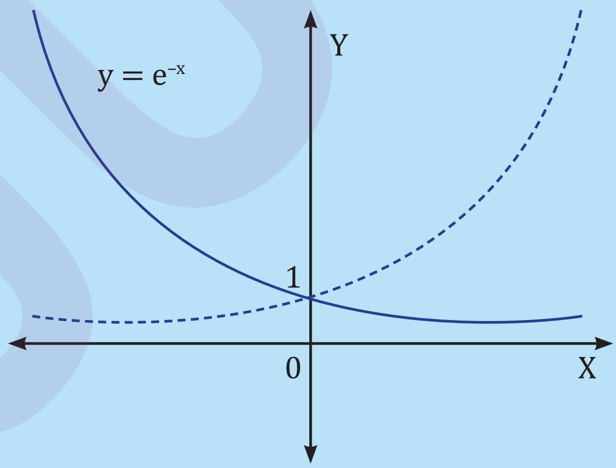
Transformation of Graphs

$f(x) \to f(-x)$ (Mirror image about the Y-axis)

Let us consider the example of the following function: $f(x) = e^x$

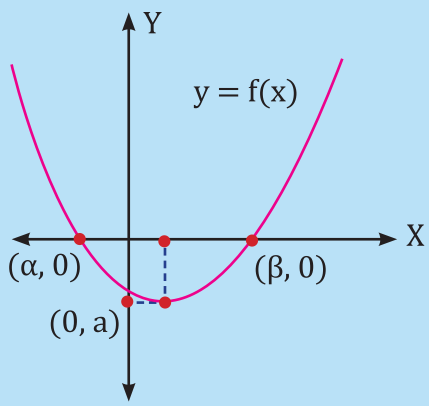


We can obtain the graph of $f(-x) = e^{-x}$ by taking the mirror image about the Y-axis.

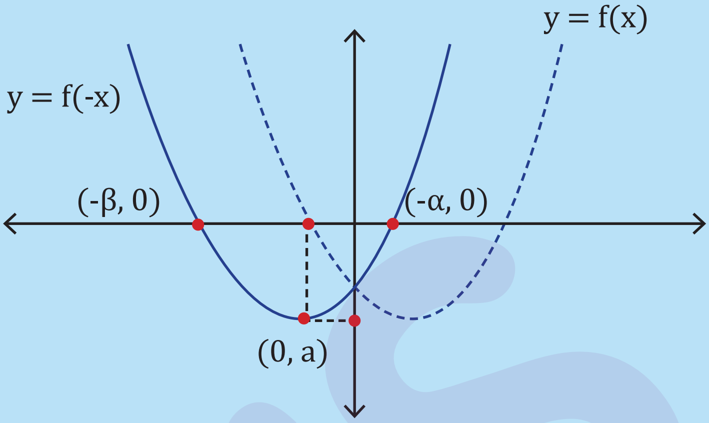


From the given example, we can deduce that for $f(x) = a^x, a > 0$, the mirror image about the Y-axis becomes $f(-x) = a^{-x}, \forall\, x$ in the domain.

Now, let us consider any random quadratic function graph of $y = f(x)$ like the following:



Similarly, the graph of $y = f(-x)$ can be obtained by taking the mirror image about the Y-axis.

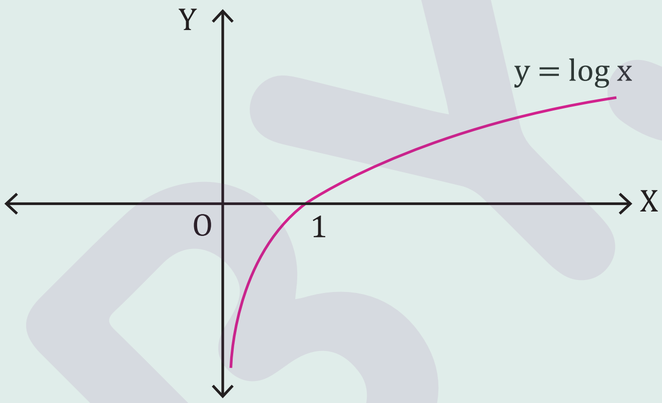


From the given example, we can deduce that for any random quadratic equation $f(x)$, the graph of $f(-x)$ is obtained by taking the reflection about the Y-axis.

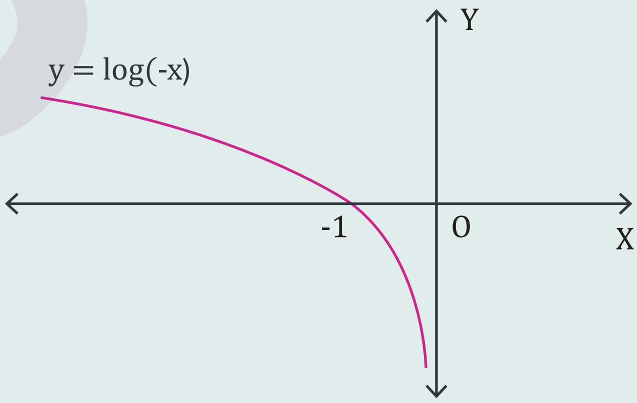
Plot the graph of $y = \log(-x)$ when the graph of $y = \log x$ is given.

Solution

Given: Graph of $y = \log x$

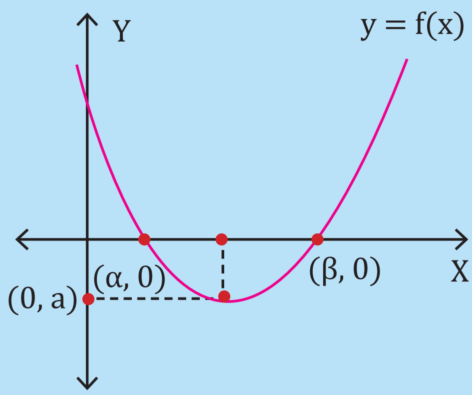


The graph of $y = \log(-x)$ is obtained by taking the reflection about Y-axis

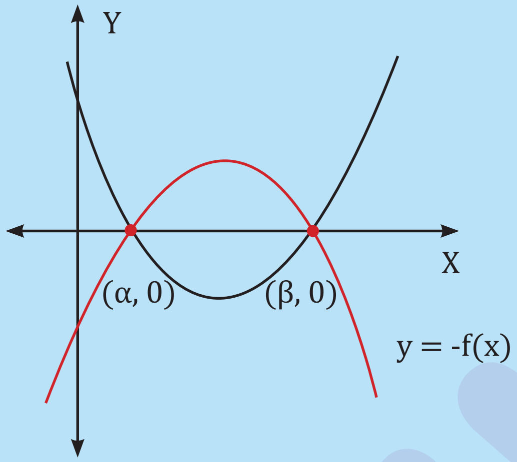


$f(x) \rightarrow -f(x)$ (Mirror image about the X-axis)

Consider the following graph for $y = f(x)$:



We can obtain the graph of $y = -f(x)$ by taking the mirror image about the X-axis.

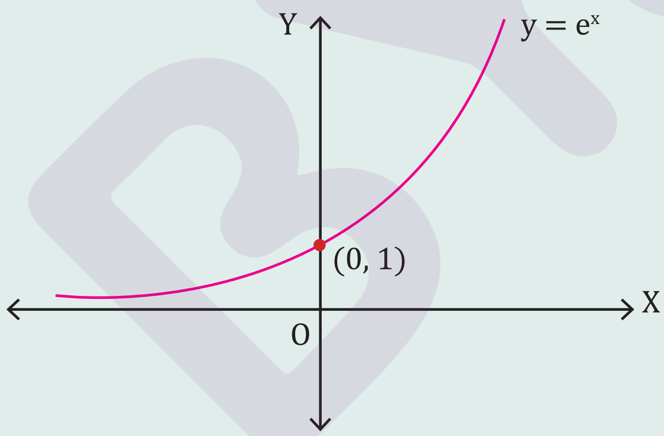


From the given example, we can deduce that the points where the graph intersects the X-axis do not change (invariant points). The graph of $y = -f(x)$ can be obtained by multiplying the values of $f(x)$ by -1 $\forall$ x in the domain.

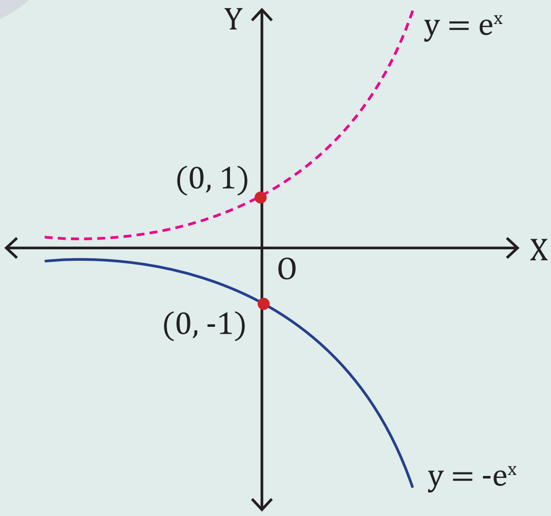
Plot the graph of $y = -e^x$ when the graph of $y = e^x$ is given.

Solution

Given: Graph of $y = e^x$

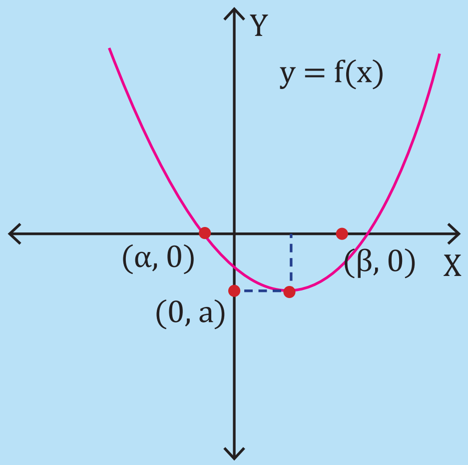


The graph of $y = -e^x$ is obtained by taking the reflection about X-axis



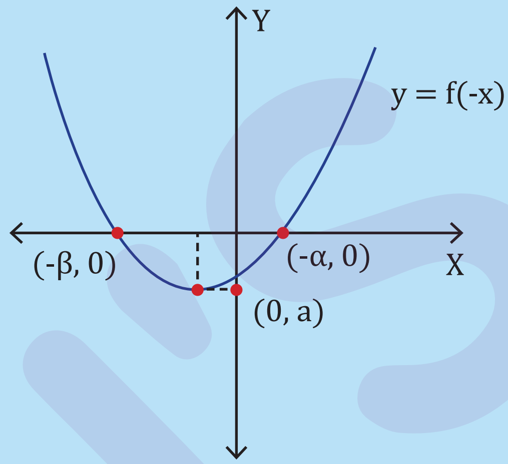
$f(x) \rightarrow -f(-x)$

Consider the following graph for $f(x)$:

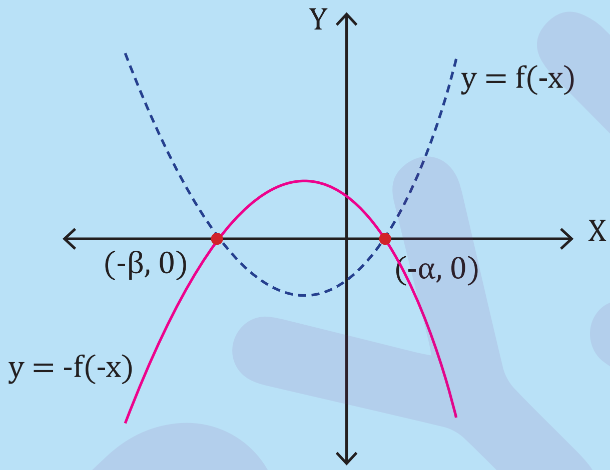


We can obtain the graph of $-f(-x)$ in the following two steps:

(1) $f(x) \rightarrow f(-x)$



(2) $f(-x) \rightarrow -f(-x)$

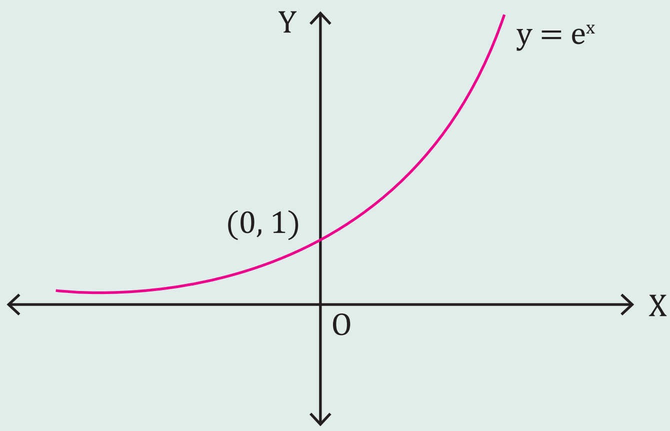


So, it means that take the mirror image of the graph about the Y-axis and then about the X-axis.

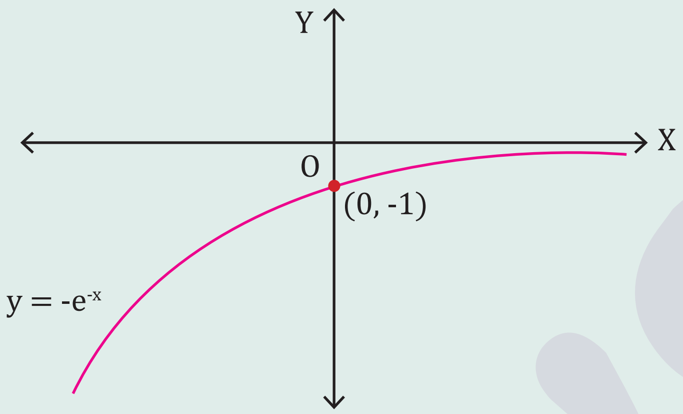
Plot the graph of $y = -e^{-x}$ when the graph of $y = e^{x}$ is given.

Solution

Given: Graph of $y = e^{x}$

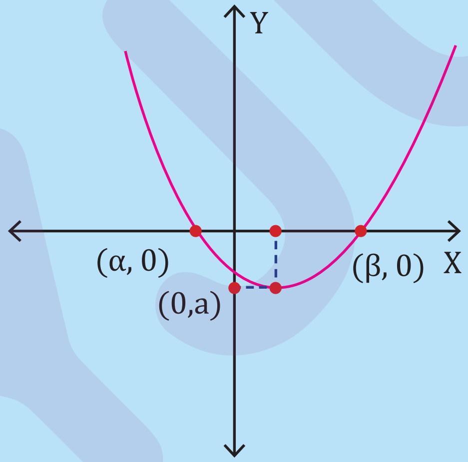


The required graph can be obtained by taking the mirror image about the Y-axis and then about the X-axis

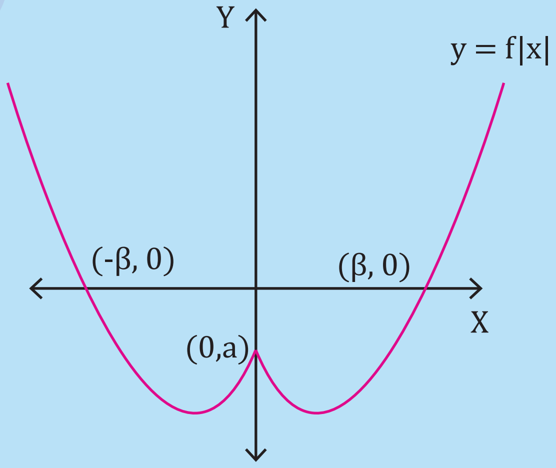


f(x) → f(|x|) (Horizontal flip)

Consider the following graph for f(x):



We can obtain the graph of f(|x|) by the following ways:
1. Retaining the graph corresponding to only the non-negative values of x
2. Taking the mirror image of the retained graph about the Y-axis

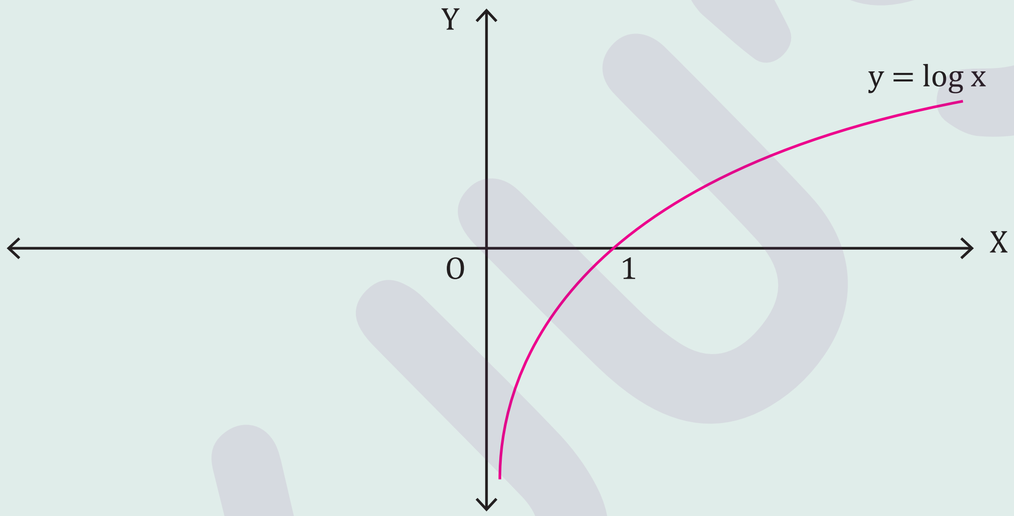


Note

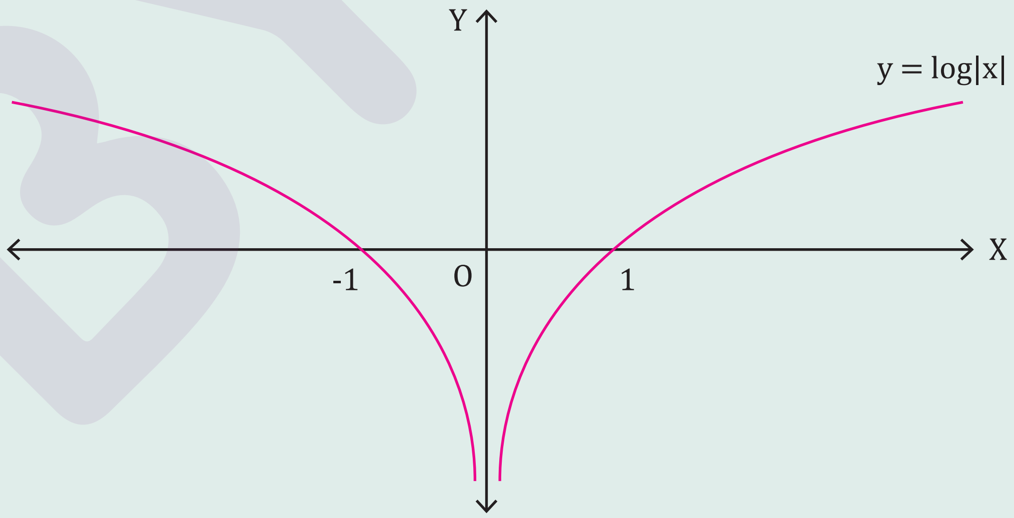
$f(|x|)$ is an even function as $f(x) = f(-x)$, i.e., the graph of $f(|x|)$ is symmetric about the Y-axis.

Plot the graph of $y = \log |x|$ when the graph of $y = \log x$ is given.

Solution

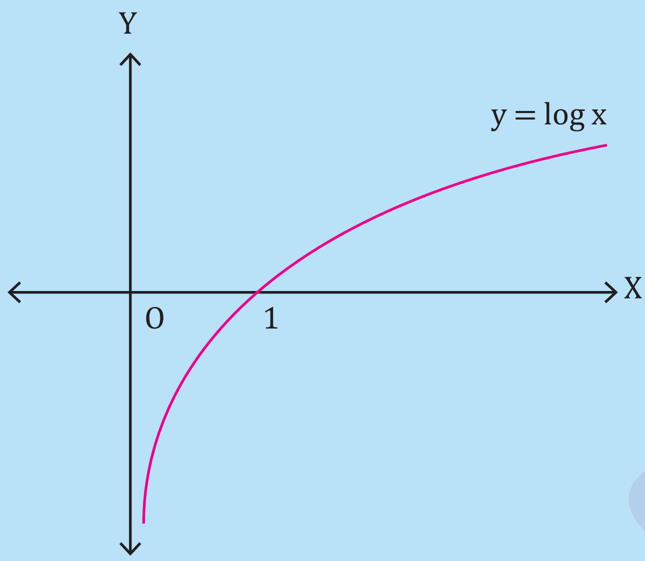


Since there is no portion in the graph of $y = \log x$ which corresponds to negative value of x, hence the entire graph is retained and the graph of $y = \log x$ is obtained by simply taking mirror image about the Y-axis



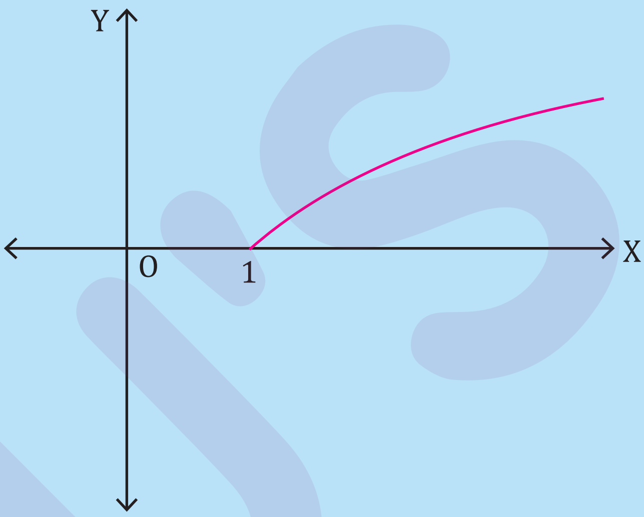
$f(x)\rightarrow|f(x)|$ (Vertical flip)

Consider the following graph for $f(x)$:

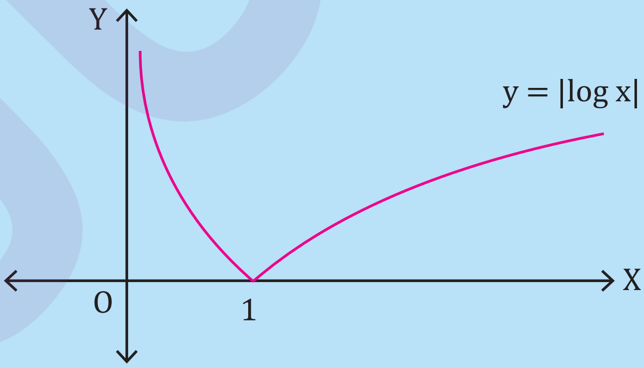


We can obtain the graph of $|f(x)|$ by the following ways:

1. Retaining the graph that is above the X-axis

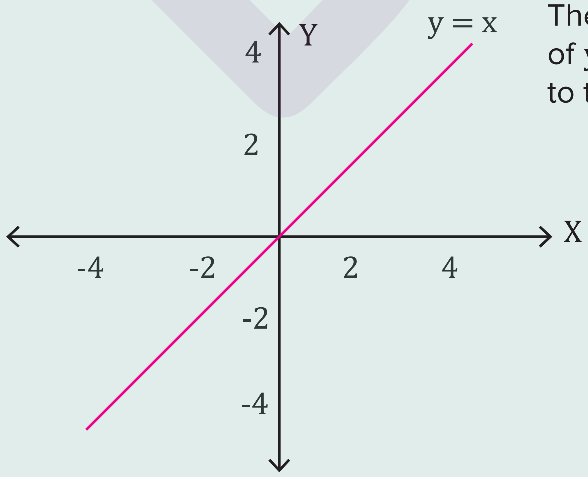


2. Flipping the graph corresponding to the negative values of y about the X-axis

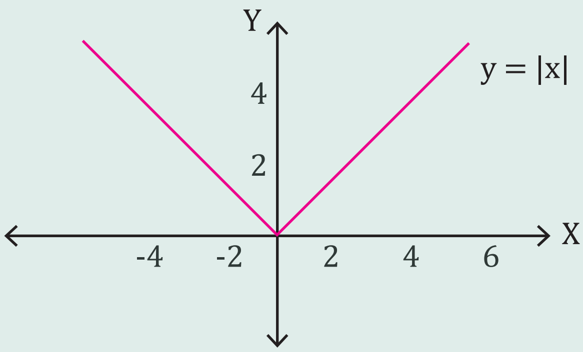


Plot the graph of $y = |x|$ when the graph of $y = x$ is given.

Solution

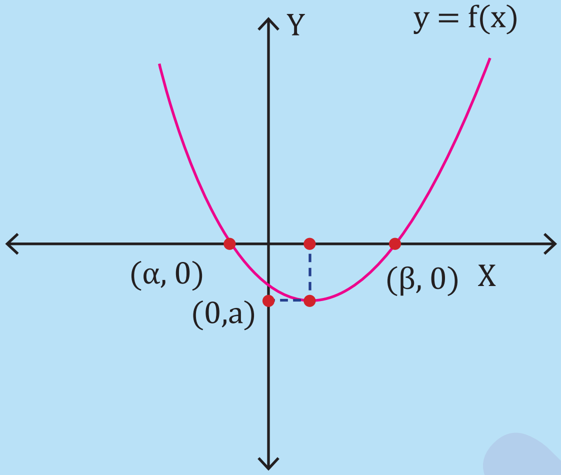


The graph of $y = x$ can be obtained by retaining the graph of $y = x$ above the X-axis and flip the graph corresponding to the negative values of y about the X-axis.



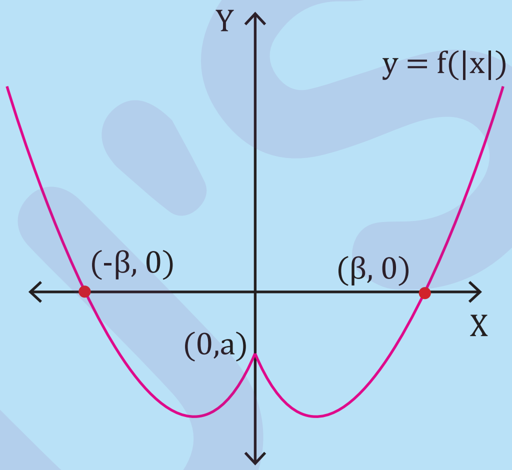
$f(x) \to |f(|x|)|$

Consider the following graph for $f(x)$:

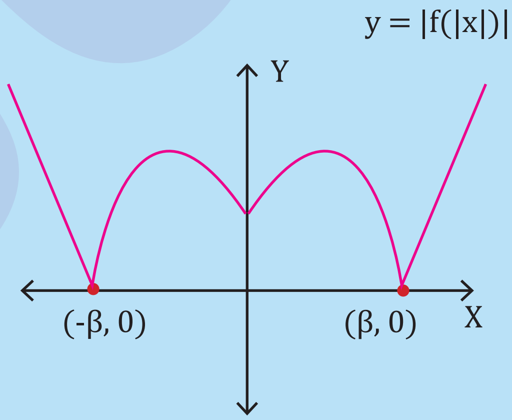


We can obtain the graph of $|f(|x|)|$ in the following steps:

(1) The first step is to carry out the transformation $f(x) \to f(|x|)$ with reference to the graph of $y = f(x)$



2. Now, carry out the transformation $f(|x|) \to |f(|x|)|$ with reference to the graph of $y = f(|x|)$ obtained in the above step

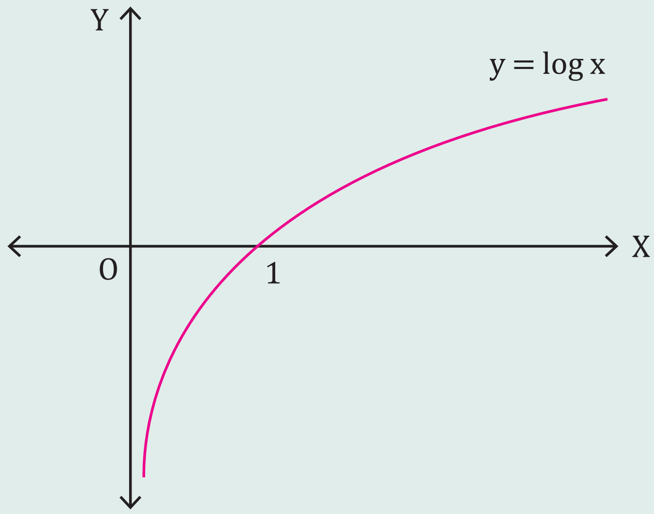


Note

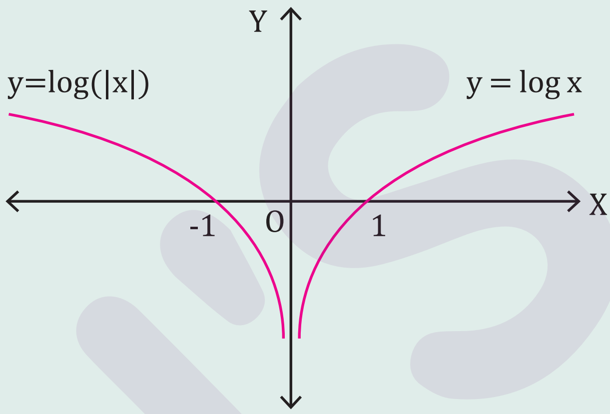
1. $|f(|x|)|$ is an even function as the graph of $|f(|x|)|$ is symmetric about the Y-axis.
2. Carrying out the transformations $f(x) \to |f(x)|$ and $|f(x)| \to |f(|x|)|$ will give the same result.

Plot the graph of $y = |\log(|x|)|$ when the graph of $y = \log x$ is given.

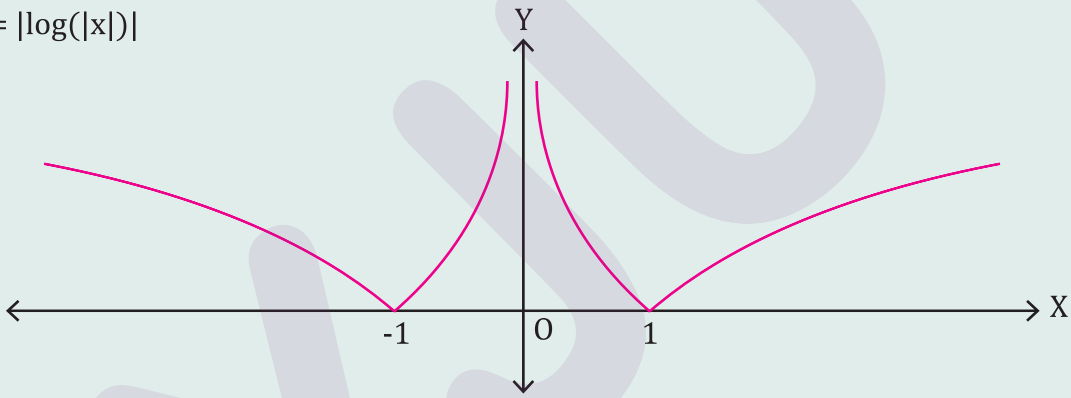
Solution



Step 1:
With respect to the graph of y = log x carry out the transformation $\log x \rightarrow \log(|x|)$

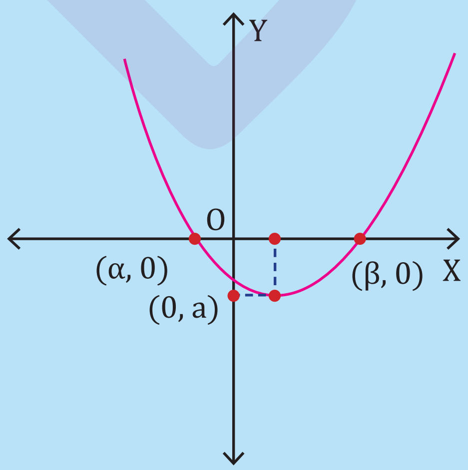


Step 2: $y = |\log(|x|)|$

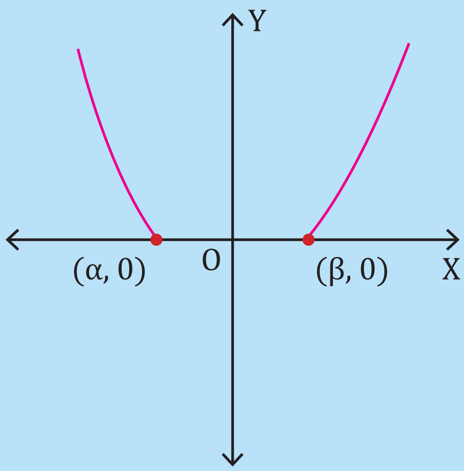


|y| = f(x)

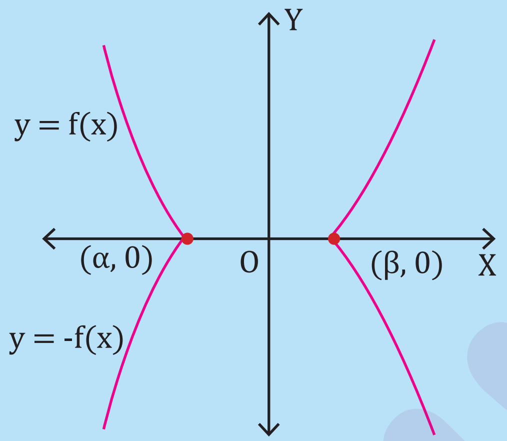
Consider the following graph for y= f(x):



1. For |y| = f(x), the region below the X-axis cannot be considered as |y| gives the non-negative values.



2. As $|y| = f(x) \Rightarrow y = f(x)$ and $-y = f(x)$, we also consider the mirror image of the given graph about the X-axis.



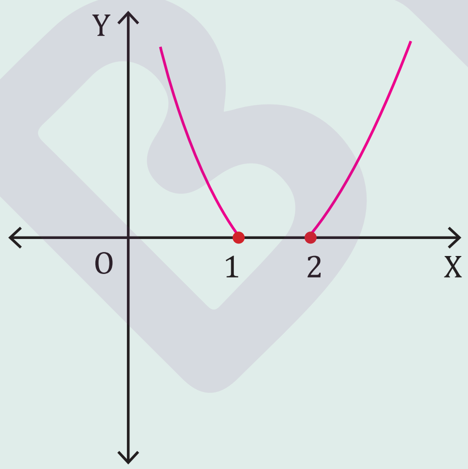
Hence, graph of $|y| = f(x)$ is obtained.

Draw the graph of $|y| = (x - 1)(x - 2)$

Solution

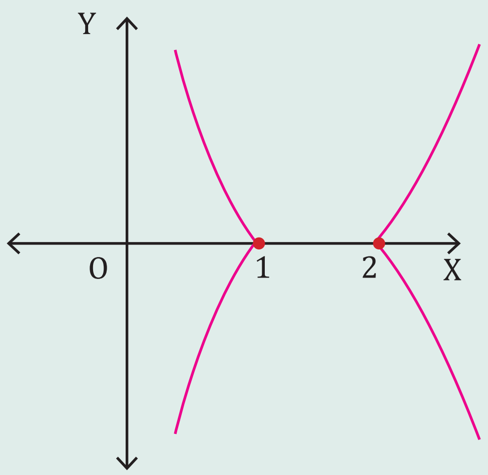
Step 1:

The region below the y axis of $|y| = (x - 1)(x - 2)$ is not to be considered as $|y|$ gives non-negative values.



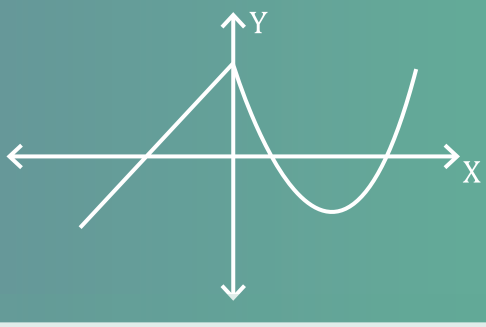
Step 2:

Now take the mirror image of the above graph about the X-axis

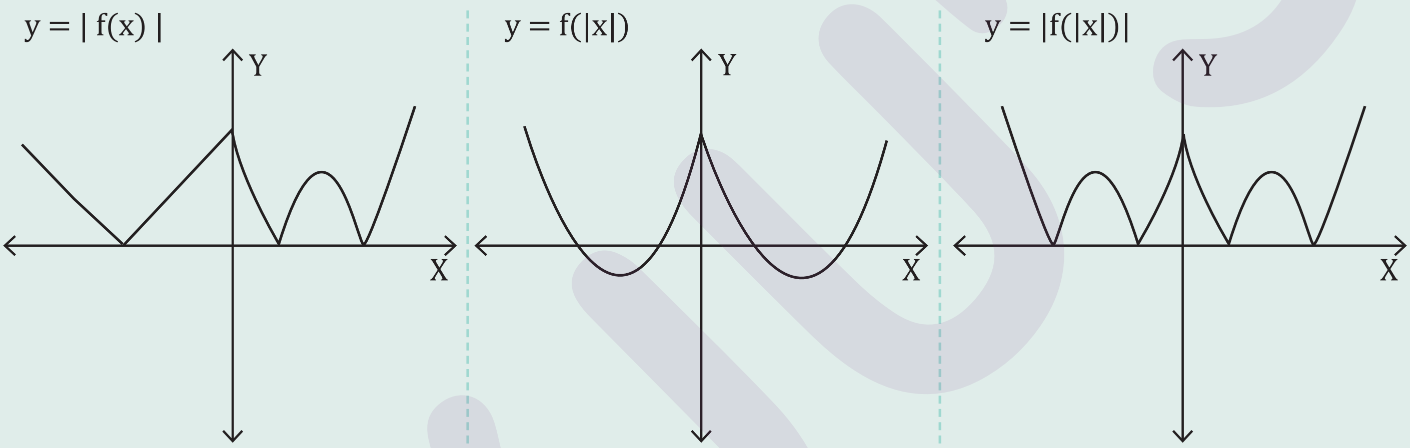


Hence, the graph of $|y| = (x - 1)(x - 2)$ is obtained as above.

Draw the graphs of $|f(x)|$, $f(|x|)$, $|f(|x|)|$ for the given function $f(x)$.



Solution



Finding the number of solutions

To find the number of solutions for $f(x) = g(x)$, try to plot the graphs of both the functions and look for the number of points of intersection, i.e., plot $y = f(x), y = g(x)$ in the same XY–plane within their respective domains, and identify the points where these functions intersect.

Find the number of solutions for the following function:

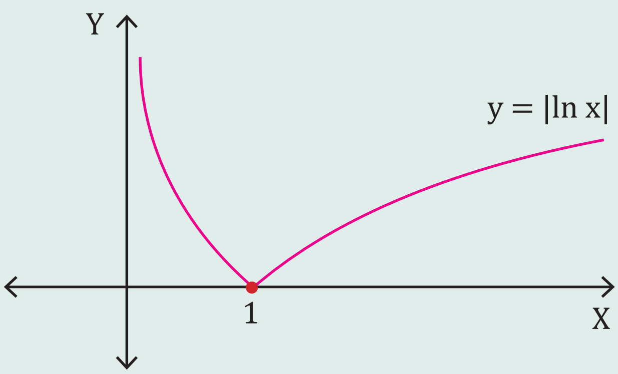
$|\ln x| = 2^{-x}$

Solution

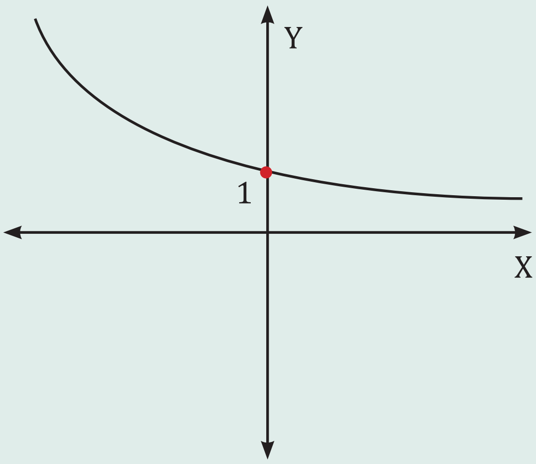
Step 1:

Plot the graph of the function:

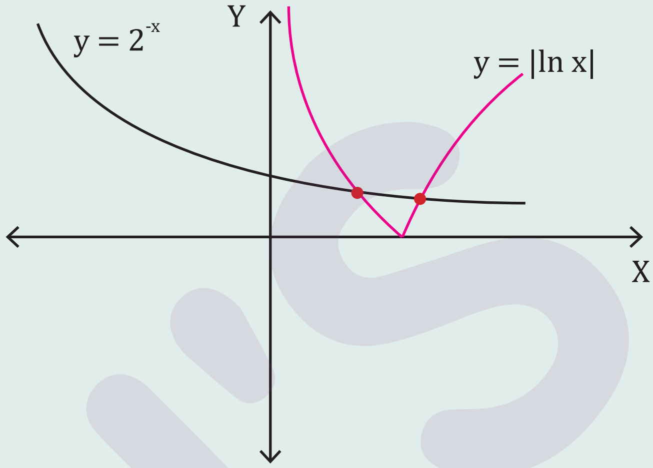
$y = | \ln x |$



Step 2:
Plot the graph of the function:
$y = 2^{-x}$



Step 3:
Combine both the graphs.



From the graph, we can observe that the number of solutions for $|\ln x| = 2^{-x}$ are two.

$f(x) \to [f(x)]$, [.] denotes the GIF function

Consider $y = [\sin x], x \in [0, 2\pi]$

Explanation:

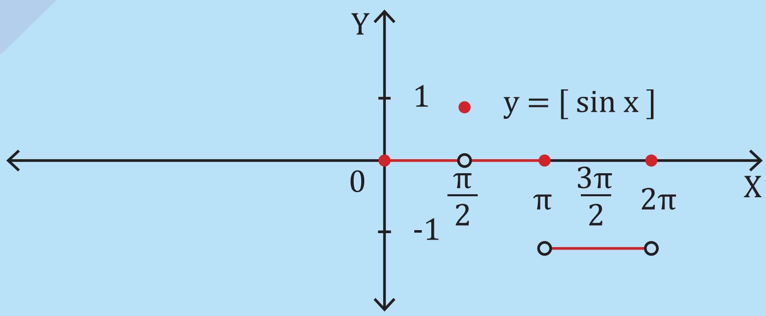
We know that $[x] = x$ for $x \in \mathbb{Z}$
Hence,
$[\sin x] = 0$ for $x\ [0, \frac{\pi}{2})$

$[\sin x] = 1$ for $x = \frac{\pi}{2}$

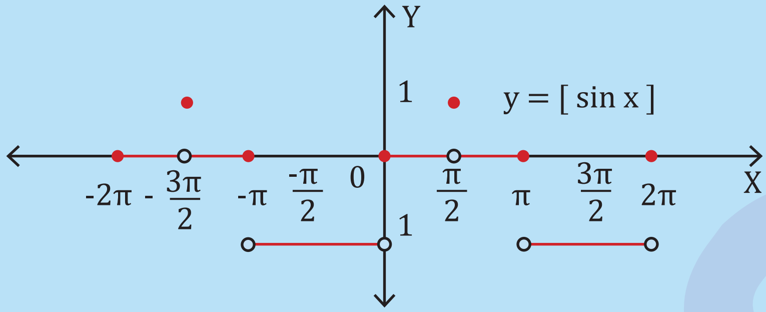
$[\sin x] = 0$ for $x \in (\frac{\pi}{2}, \pi\,]$

$[\sin x] = -1$ for $x \in (\pi, 2\pi)$
$[\sin x] = 0$ for $x = 2\pi$

Hence, the graph of $y = [\sin x]$ for $x \in [0, 2\pi]$ is as follows:



As $\sin x$ has a period of 2π, the period of $[\sin x]$ remains 2π. Hence, the graph would repeat after every 2π interval.

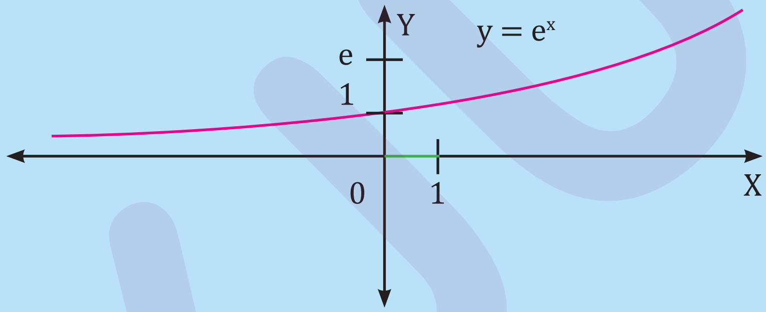


$f(x) \rightarrow (f\{x\})$

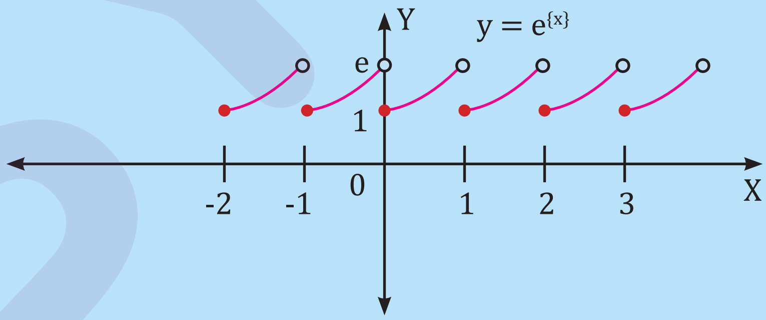
Consider $y = e^{\{x\}}$, $x \in (-\infty, \infty)$

Explanation:

For $y = e^x$, the graph is given as follows:



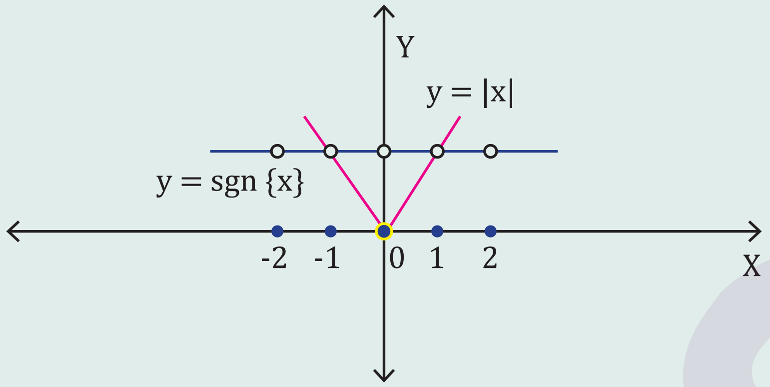
We know the following:
$\{x\} \in [0, 1) \Rightarrow e^{\{x\}} \in [1, e)$
As $\{x\}$ is a periodic function with period 1, $e^{\{x\}}$ is also a periodic function with the same period.



Find the number of solutions for $|x| = \text{sgn}\{x\}$, where $\{.\}$ denotes a fractional part function.

Solution

Plot the graphs of $y = |x|$ and $y = \text{sgn}\{x\}$ simultaneously.

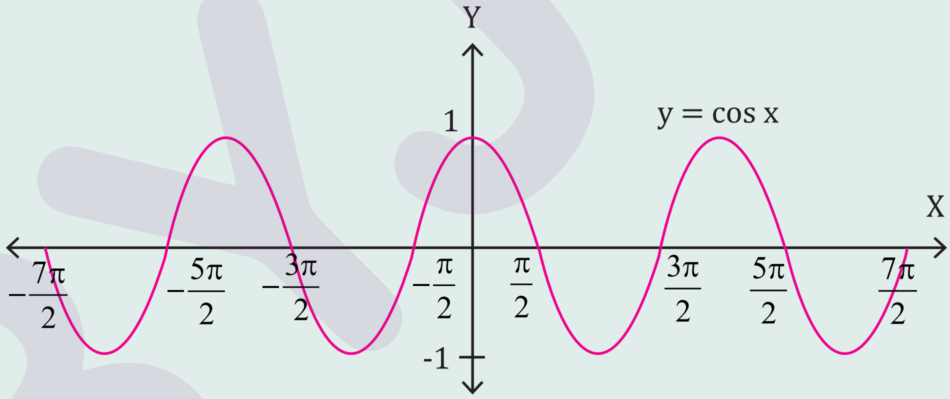


From the graph, we can observe that both the functions intersect only at origin. Hence, the number of required solutions is one.

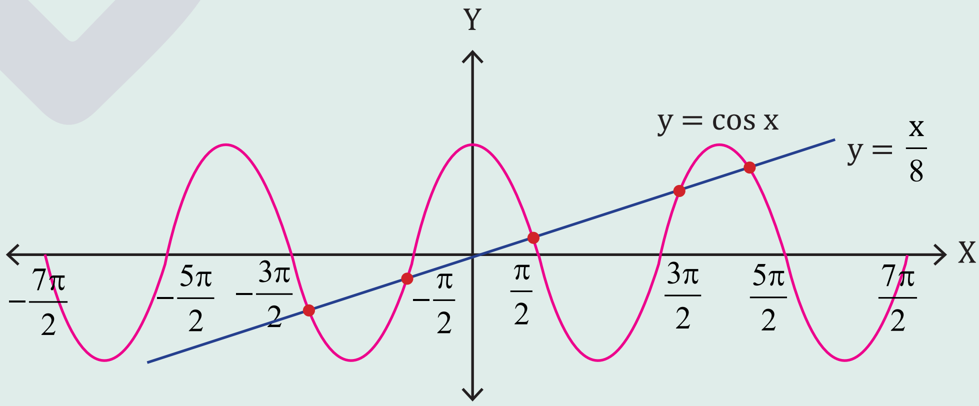
Find the number of solutions of the equation $\cos x = \frac{x}{8}$

Solution

Step 1: Plot $y = \cos x$



Step 2: Plot $y = \frac{x}{8}$ in the same graph and hence obtain the points of intersection



From the graph, we can observe that the number of solutions is five

Concept Check

1. Find the number of solutions for the following: $|y| = \cos x$ and $y = -x^2$
2. Find the number of solutions for the following: $|[x]| = |x - 1|$

Summary Sheet

Key Takeaways

- $f(x) \to f(-x)$ is obtained by taking a mirror image of the graph about the Y-axis.
- $f(x) \to -f(x)$ is obtained by taking a mirror image of the graph about the X-axis.
- $f(x) \to -f(-x)$ can be obtained by taking a mirror image of the graph about the Y-axis and then about the X-axis or vice versa.
- $f(x) \to f(|x|)$ can be obtained by retaining the graph corresponding to the non-negative values of x and taking the mirror image of it about the Y-axis.
- $f(x) \to |f(x)|$ can be obtained by retaining the graph above the X-axis and flip the graph corresponding to the negative values of y about the X-axis.
- $f(x) \to |f(|x|)|$ can be obtained in the following two steps:

  (1) $f(x) \to |f(x)|$

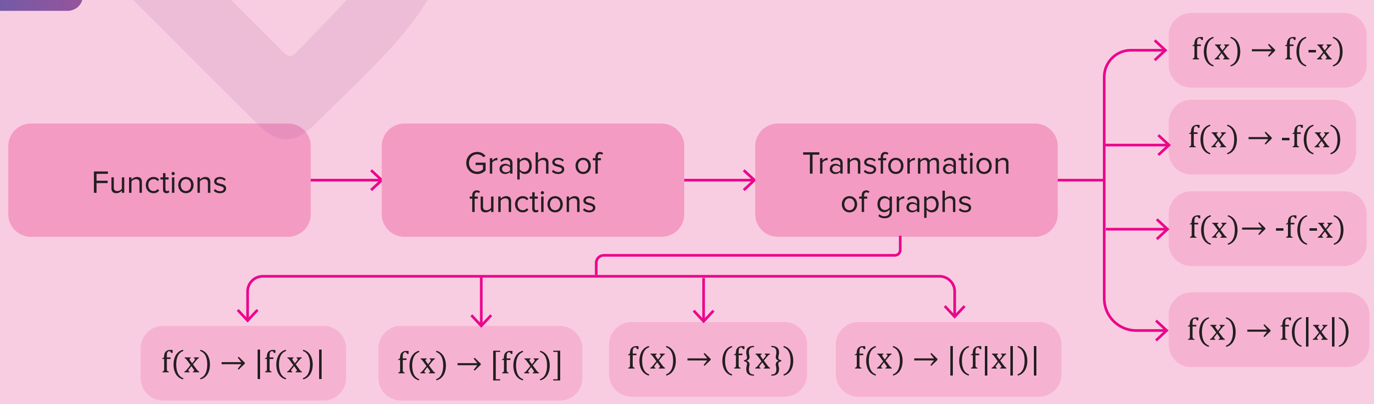
  (2) $|f(x)| \to |f(|x|)|$

  It can also be obtained as follows:

  (1) $f(x) \to f(|x|)$

  (2) $f(|x|) \to |f(|x|)|$.
- To find the number of solutions for $f(x) = g(x)$, plot the graphs of both the functions and look for the number of points of intersection.

Mind Map



Self-Assessment

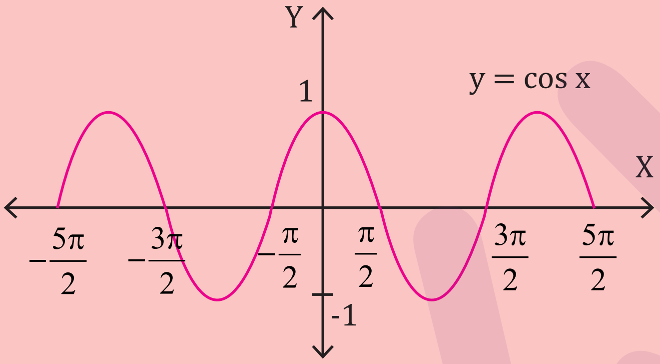
Find the number of solutions for the following function: $\frac{-3}{4}x^2 + 3x + 1 = 2^x$

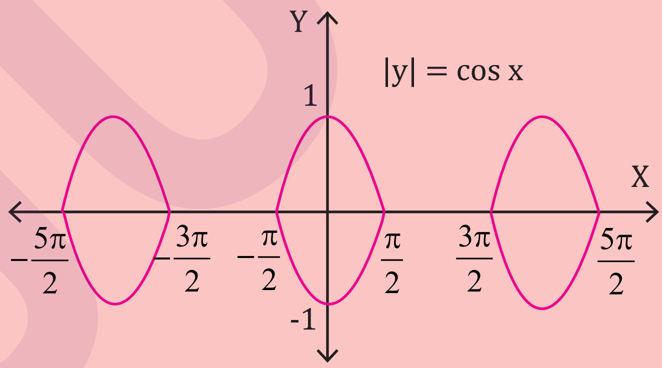
Answers

Concept Check

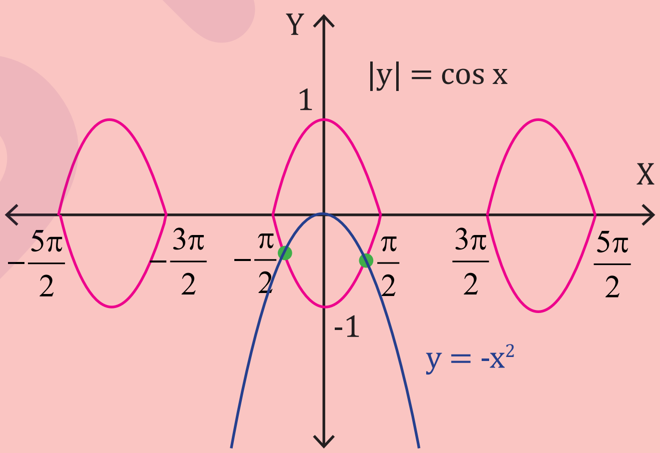
1.

Step 1:
Plot the graph of the function:
$|y| = \cos x$





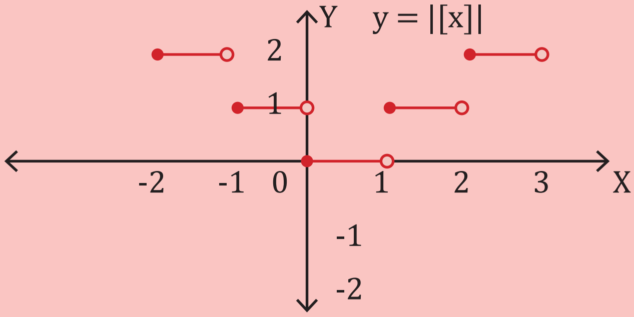
Step 2:
Plotting the graph of $y = -x^2$ and obtaining the number of solutions by counting the number of points of intersection



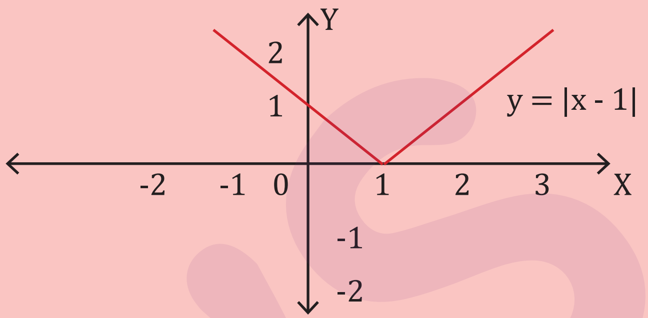
From the graph, we can observe that the number of solutions is two.

2.

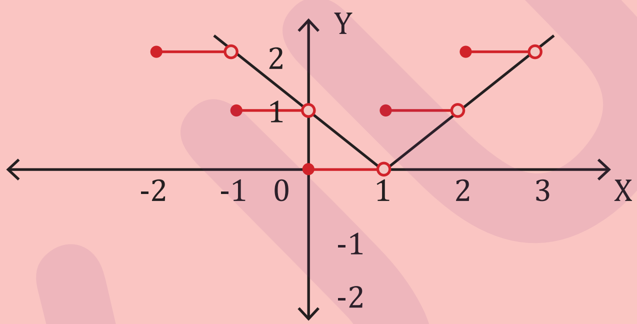
Step 1:
Plotting the graph of the function $y = |[x]|$



Step 2:
Plot the graph of the function:
$y = | x - 1 |$



Step 3:
Plotting both the graphs.

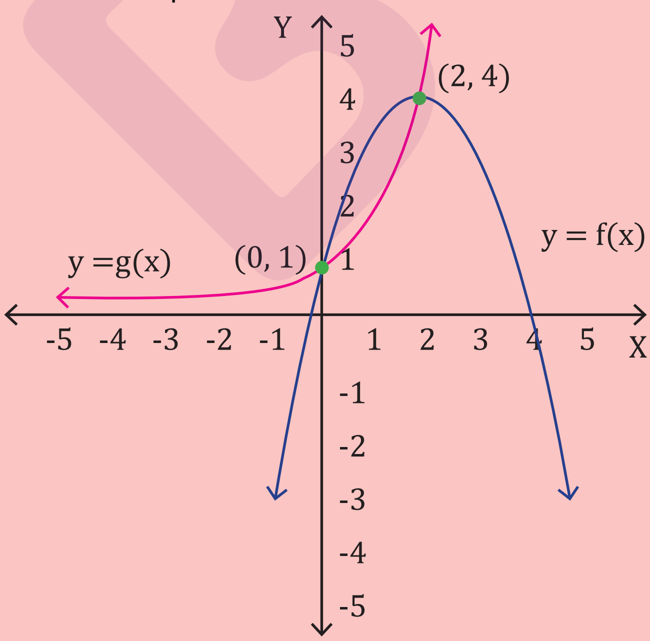


From the graph, we can observe that the number of solutions for $|[x]| = |x - 1|$ is zero.

Self-Assessment

Plot the graph for the following:

$y = f(x) = \frac{-3}{4}x^2 + 3x + 1$ and $y = g(x) = 2^x$



We can observe that the number of points of intersection for both the graphs is two.

Hence, $\frac{-3}{4}x^2 + 3x + 1 = 2^x$ has two solutions.