BYJU'S Classes

Application of Derivatives

Introduction to Application of Derivatives

Road Map

Approximation

Derivative as Rate of Change

Application of Derivatives

Application of Derivatives

Derivative is defined as the rate of change of one quantity with respect to another.

Here, it is used to calculate

- Rate measure
- Approximation
- Tangent and Normal to a curve
- Increasing/Decreasing functions
- Maxima and Minima of a function

Derivative as Rate of Change

If y is a function of x, then small change in x has a small change in y.

Rate of change of $'y'$ w.r.t $'x' = \dfrac{\Delta y}{\Delta x}$,

where $\Delta x \rightarrow$ change in $'x'$ and $\Delta y \rightarrow$ change in $'y'$.

As $\Delta x \rightarrow 0$, rate of change becomes instantaneous.

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}}$$

Illustration

If the radius of a circle is increasing at a uniform rate of $2\ cm/s$, find the rate of increase of area of circle, at the instant when the radius is $20\ cm$.

Solution:

Let us consider A to be the area of circle having radius r.

Given, $\frac{dr}{dt} = 2\ cm/s$ and for $r = 20\ cm$; $\frac{dA}{dt} = ?$

As A is the area of circle $\Rightarrow A = \pi r^2$

Differentiating w.r.t time $'t'$, we get,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \times 20 \times 2 \Rightarrow \frac{dA}{dt} = 80\pi\ cm^2/s$$

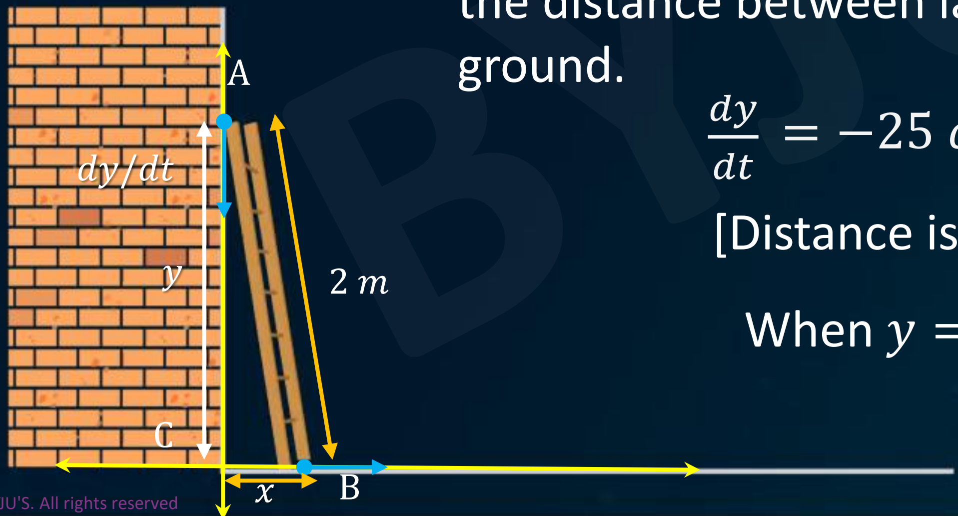
Illustration

JEE Main - 2019 (April)

A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate of 25 $cm/s.$, then the rate (in cm/s) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is :

a. $25/\sqrt{3}$ b. $25\sqrt{3}$ c. 25 d. $25/3$

Solution:



Let the ladder touch the wall y $meter$ above the ground and the distance between ladder and wall be x $meter$ on the ground.

$$\frac{dy}{dt} = -25\ cm/s = -0.25\ m/s$$

[Distance is decreasing with time]

When $y = 1m$, $\frac{dx}{dt} = ?$

In triangle ABC, At any instant , $x^2 + y^2 = 4$, when $y = 1, x = \sqrt{3}$.

Differentiating w.r.t time $'t'$, we get,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

Substituting $\frac{\mathrm{dy}}{\mathrm{dt}} = -0.25$, $y = 1, x = \sqrt{3}$, we get,

$$\frac{dx}{dt} = \frac{0.25}{\sqrt{3}}m/s = \frac{25}{\sqrt{3}}cm/s$$

Rate at which bottom of ladder slides is $\frac{25}{\sqrt{3}}cm/s$.

Hence, option (a) is the correct answer.

Illustration

On the curve $x^3 = 12\,y$, abscissa changes at a faster rate than the ordinate, then x belongs to __________.

a. $(2, \infty)$ b. $(-2,2)$ c. $(-2,2) - \{0\}$ d. $(0, \infty)$

Solution:

$$x^3 = 12y$$

$$3x^2 \frac{dx}{dt} = 12 \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{3x^2}{12} \frac{dx}{dt}$$ [Differentiating w.r.t 't']

$$\left|\frac{dx}{dt}\right| > \left|\frac{dy}{dt}\right| \Rightarrow \frac{\left|\frac{dy}{dt}\right|}{\left|\frac{dx}{dt}\right|} < 1$$

$$\Rightarrow \left|\frac{3x^2}{12}\right| < 1 \Rightarrow \frac{x^2}{4} < 1 \Rightarrow (x-2)(x+2) < 0.$$ Hence, $\boxed{x \in (-2,2)}$

Hence, option (b) is the correct answer.

Illustration

A water tank has the shape of an inverted right circular cone , whose semi – vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 m^3/min. Then the rate (in m/min), at which the level of water is rising at the instant when the depth of water in tank is $10\ m$ is:

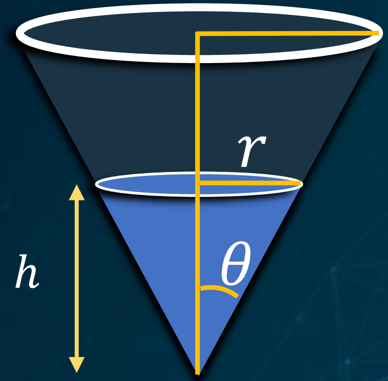
a. $\frac{1}{5\pi}$ b. $\frac{1}{15\pi}$ c. $\frac{2}{\pi}$ d. $\frac{1}{10\pi}$

Solution:

Let volume of water inside the inverted right circular cone be V.

$$\theta = \tan^{-1}\left(\frac{1}{2}\right), \frac{dV}{dt} = 5m^3/min$$

When $h = 10m, \frac{dh}{dt} = ?$



We know, $V = \frac{1}{3}\pi r^2 h$

$$\tan\theta = \frac{r}{h} \Rightarrow \frac{1}{2} = \frac{r}{h} \Rightarrow r = \frac{h}{2}$$

$$\therefore V = \frac{1}{3}\pi\frac{h^3}{4}$$

Differentiating w.r.t. $'t'$, we get,

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

$$\Rightarrow 5 = \frac{100\pi}{4}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{5\pi}\ m/min$$

Hence, option (a) is the correct answer.

Illustration

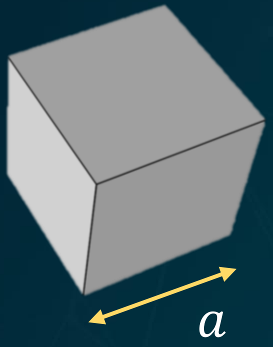
JEE Main - 2020 (Sept)

If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape, then the rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10 cm, is :

$a.$ 18 $b.$ 20 $c.$ 10 $d.$ 9

Solution:

Let the surface area of the cube is S and Volume be V.



Given, $\frac{dS}{dt} = 3.6\ cm^2/sec$

When $a = 10\ cm$, then $\frac{dV}{dt} = ?$

$S = 6a^2$

Differentiating both the sides w.r.t time $'t'$, we get,

$$\frac{dS}{dt} = 12a \times \frac{da}{dt}$$

$$3.6 = 12 \times 10 \times \frac{da}{dt} \Rightarrow \frac{da}{dt} = 0.03\ cm/sec$$

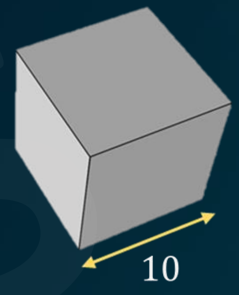
$V = a^3$

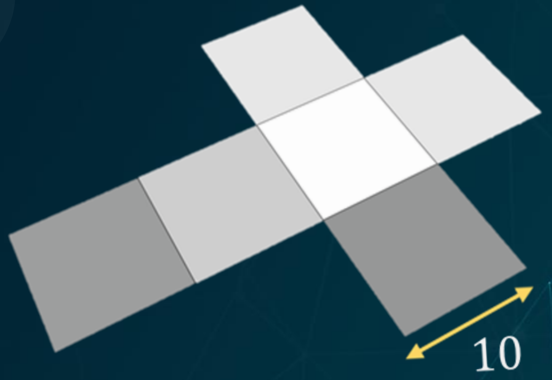
Differentiating both the sides w.r.t time $'t'$, we get,

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}$$

$$\frac{dV}{dt} = 3 \times (10)^2 \times 0.03 \Rightarrow \frac{dV}{dt} = 9\ cm^3/sec$$

Hence, option (d) is the correct answer.





Illustration

JEE Main - 2019 (April)

A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate (in cm/min.) at which the thickness of ice decreases, is:

a. $\frac{1}{18\pi}$

b. $\frac{5}{6\pi}$

c. $\frac{1}{36\pi}$

d. $\frac{1}{54\pi}$

Solution:

Let volume and uniform thickness of ice be V and r, respectively.

Given, melting rate $\frac{dV}{dt} = 50\ cm^3/min$

The radius of the spherical ball is 10 cm.

Now, volume of ice $\mathrm{V} = \frac{4}{3}\pi(10 + r)^3 - \frac{4}{3}\pi(10)^3$

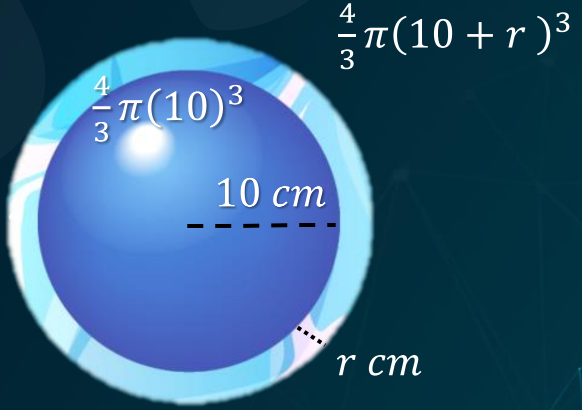
Differentiating w.r.t time ‘t’, we get,

$$\frac{dV}{dt} = 4\pi(10 + r)^2 \frac{dr}{dt}$$

$$50 = 4\pi(15)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{50}{4\pi(15)(15)}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{18\pi} cm/min}$$



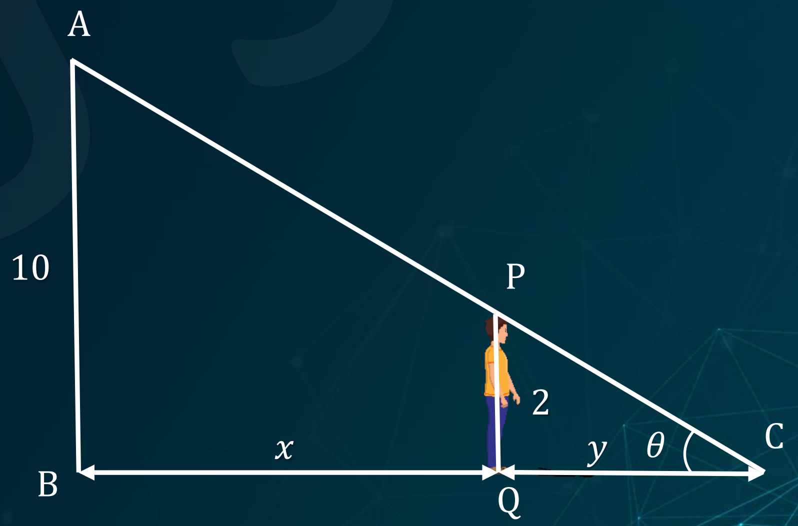
Hence, option (a) is the correct answer.

Illustration

The rate of increase of length of the shadow of a man $2m$ height, due to a lamp at $10m$ height, when he is moving away from it at the rate of $2\frac{m}{sec}$ is ______.

Solution:

Let "x" be the distance between pole and the man, "y" be the length of the shadow and θ is the angle made by shadow as shown in figure.



$$\frac{dx}{dt} = 2\frac{m}{sec} \text{ (given)}$$

$$\frac{10}{x+y} = \frac{2}{y}$$ [Clearly, both triangles ABC and PQC are similar with AAA rule
$\Rightarrow$ Corresponding sides of these triangle will be proportional]

$$\Rightarrow 8y = 2x$$

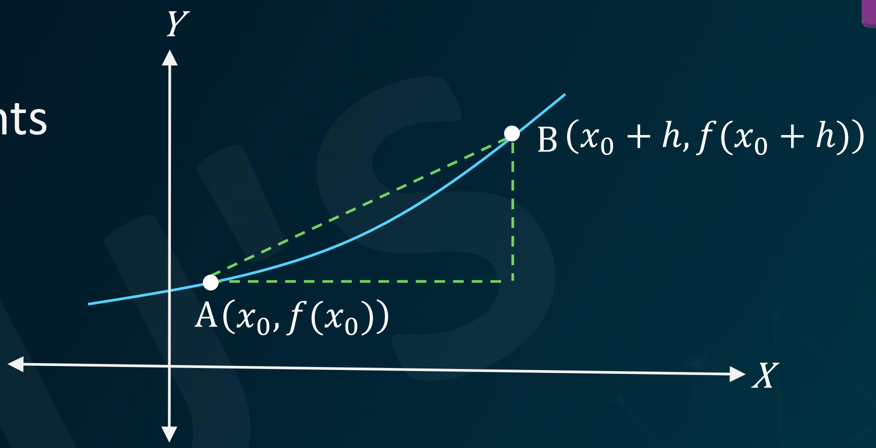
$$\Rightarrow 8\frac{dy}{dt} = 2\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{4}(2)$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\frac{m}{sec}$$

Approximation

Consider a curve $y = f(x)$ with two points A, B as shown in the figure.



As $h \to 0$, $\frac{dy}{dx} = f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Approximate change in value of $'f'$

$$f(x_0 + h) \approx f(x_0) + h.f'(x_0)$$

Illustration

Use differential to find approximate value of $\sqrt{25.2}$.

$a.$ 5.02 $b.$ 5.01 $c.$ 5.03 $d.$ 5.04

Solution:

Let $f(x) = \sqrt{x}, x_0 = 25, h = 0.2$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(25.2) = f(25 + 0.2) \approx f(25) + 0.2 \cdot f'(25)$$

$$\boxed{f(x_0 + h) \approx f(x_0) + h \cdot f'(x_0)}$$

$$\Rightarrow \sqrt{25.2} \approx 5 + 0.2\left(\frac{1}{10}\right)$$

$$\sqrt{25.2} \approx 5.02$$

Hence, option (a) is the correct answer.

Illustration

The approximate value of $\tan 44°$ is (take $\pi = 22/7$) .

a. 0.782 b. 0.965 c. 0.873 d. 0.999

Solution:

Let $f(x) = \tan x\,, x_0 = 45°, h = -1°$

$f'(x) = \sec^2 x$

$$\boxed{f(x_0 + h) \approx f(x_0) + h.\, f'(x_0)}$$

$\Rightarrow \tan 44° \approx \tan 45° + (-1°)\,.\sec^2 45°$

$$\approx 1 + \left(\frac{-1}{180} \times \frac{22}{7}\right).\,2$$

$\tan 44° \approx 0.965$

$180° = \pi$ radians

$\Rightarrow h = \dfrac{-1\times\pi}{180}$ radians

Hence, option (b) is the correct answer.

Summary Sheet

- Derivative is defined as the rate of change of one quantity with respect to another.

- If y is a function of x, then small change in x has a small change in y.
  Rate of change of $'y'$ w.r.t $'x' = \frac{\Delta y}{\Delta x}$.

- Approximate change in value of 'f' is $f(x_0 + h) \approx f(x_0) + h.f'(x_0)$.

BYJU'S Classes

Application of Derivatives

Tangents and Normal using Derivative

Road Map

- Tangent and Normal
- Miscellaneous Problems
- Approximation

Tangent and Normal

Let $y = f(x)$ be a curve and there is a point $P(x_1, y_1)$ on it.

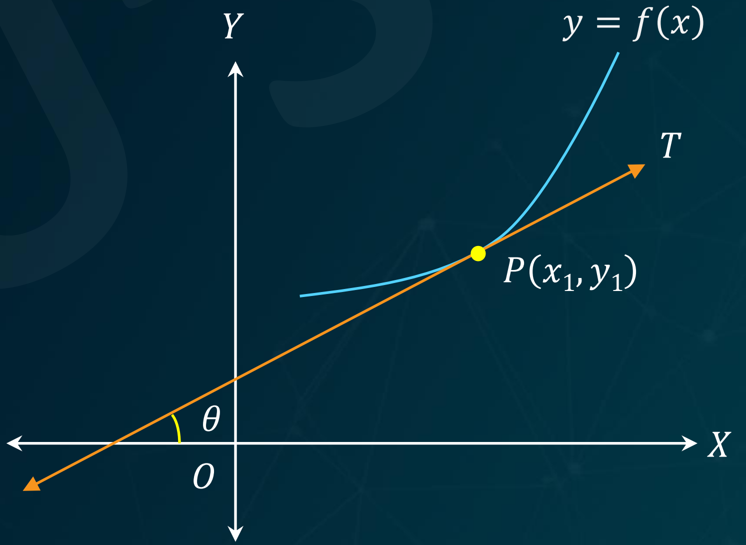
Slope of Tangent at point P

$$= \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = m = \tan\theta = f'(x)$$

$\therefore$ Equation of Tangent T :

(By point-slope form)

$$\boxed{(y - y_1) = m(x - x_1)}$$



Tangent and Normal

Let $y = f(x)$ be a curve and there is a point $P(x_1, y_1)$ on it.

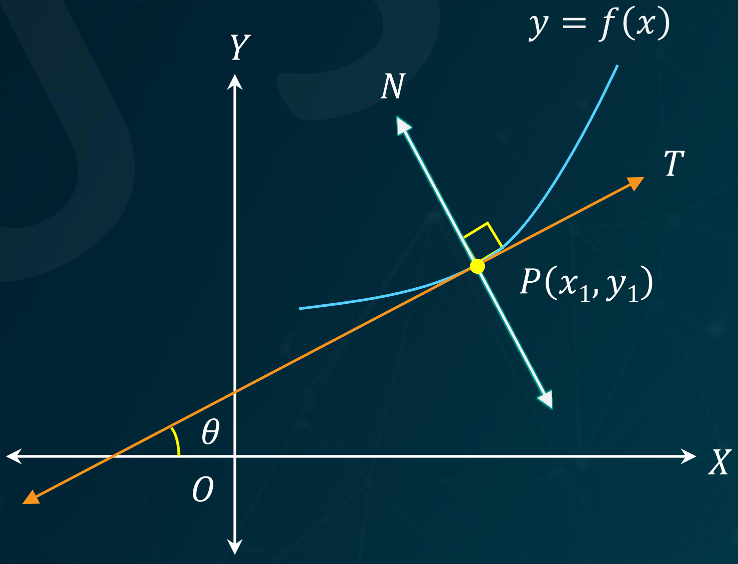
Slope of Normal at point P

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = -\frac{1}{m}$$

∴ Equation of Normal N :

(By point-slope form)

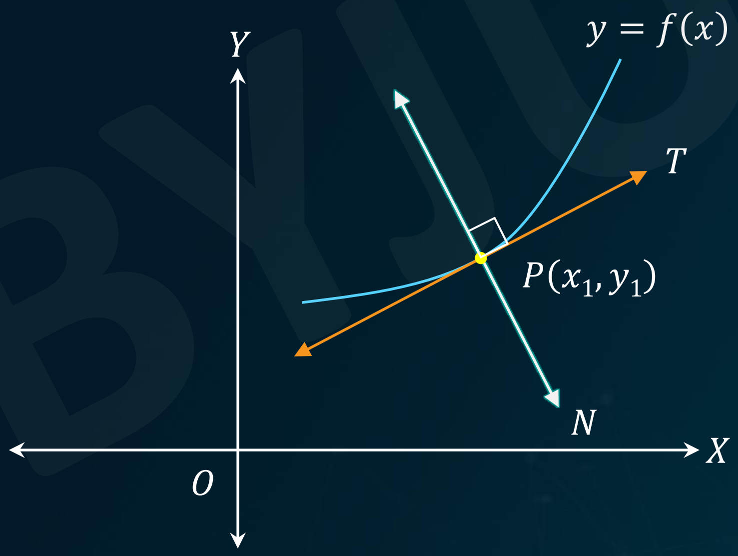
$$\boxed{(y - y_1) = -\frac{1}{m}(x - x_1)}$$



Tangent and Normal

Note:

The point $P(x_1, y_1)$ will satisfy the equation of the curve, the equation of the tangent and the equation of the normal.

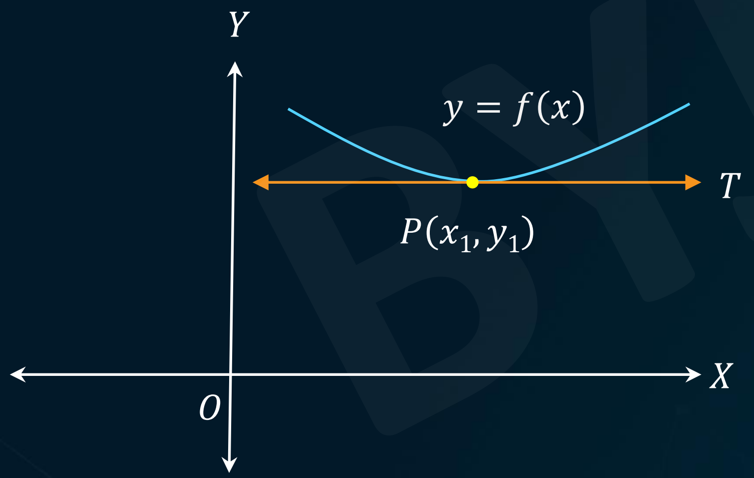


Tangent and Normal

Note:

If the tangent at any point P on the curve is parallel to the X - axis,

then $\frac{dy}{dx} = 0$ at point P.



Here, tangent T to the curve is parallel to the X – axis.

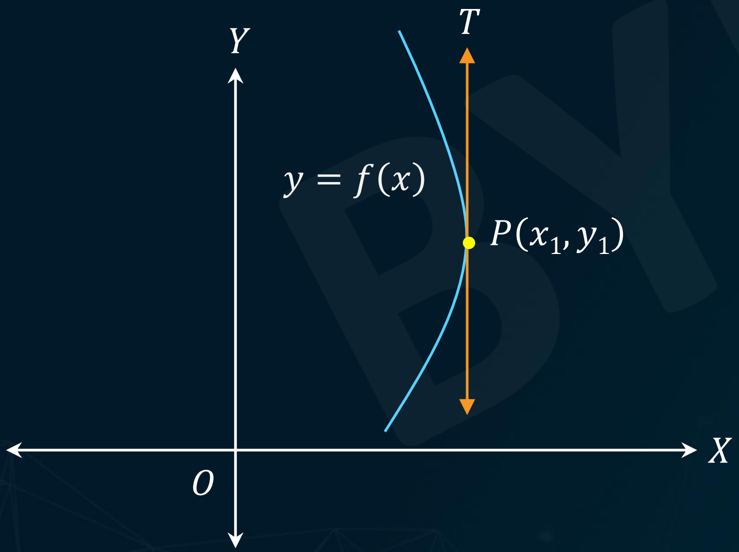
Hence, Slope = $\frac{dy}{dx} = \tan 0^0 = 0$

Tangent and Normal

Note:

If the tangent at any point P on the curve is parallel to the Y – axis,

then $\frac{dy}{dx} \to \infty$ or $\frac{dx}{dy} = 0$ at point P.



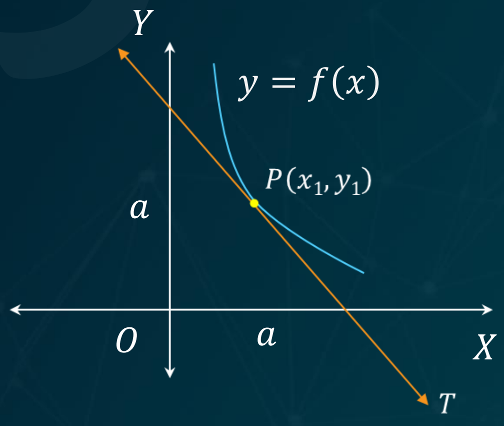
Here, tangent T to the curve is perpendicular to the X – axis.

Hence, Slope = $\frac{dy}{dx} = \tan 90^0 \to \infty$

Tangent and Normal

Note: If the tangent at any point P on the curve makes equal and positive intercepts on the coordinate axes, then $\frac{dy}{dx} = -1$ at the point P.

Proof: Let the tangent to a curve make equal intercepts a on positive coordinate axes.



Equation of tangent is:

$$\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a$$

Now, Slope of the tangent $(m) = -\frac{Coefficient\ of\ x}{Coefficient\ of\ y}$

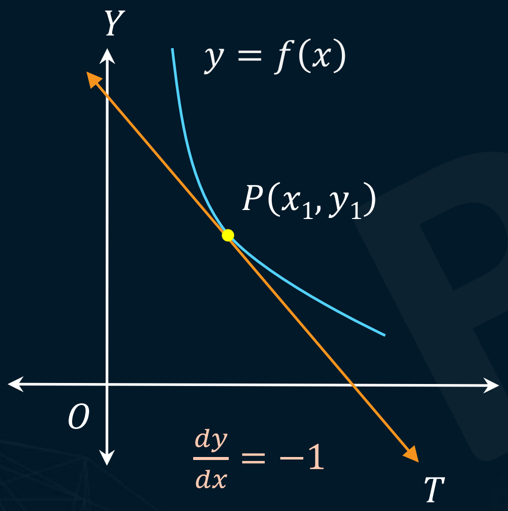
$$= -1$$

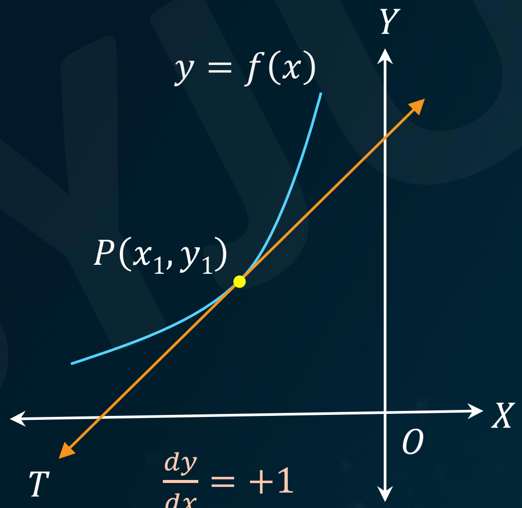
$$\Rightarrow \frac{dy}{dx} = -1 \qquad \left(\because \text{Slope} = \frac{dy}{dx}\right)$$

Tangent and Normal

Note:

If the tangent at any point P on the curve is equally inclined to both the axes, then $\frac{dy}{dx} = \pm 1$ at the point P.





For the tangent , intercepting the negative X-axis and negative Y-axis, $\frac{dy}{dx} = -1$.

And for the tangent , intercepting the positive X-axis and negative Y-axis, $\frac{dy}{dx} = 1$

Illustration

Find slope of tangent & normal to the curve $y = x^3 - 3x$, at the point $(2, 2)$.

Solution:

Given curve, $y = x^3 - 3x$

Differentiating w.r.t x at point $(2,2)$, we get,

$$\left(\frac{dy}{dx}\right)_{(2,\,2)} = (3x^2 - 3)_{(2,\,2)} = 9$$

We know that, slope of tangent at point $(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

$\Rightarrow$ Slope of tangent at point $(2,2) = \left(\frac{dy}{dx}\right)_{(2,\,2)} = 9$

And slope of normal at point $(2,2) = -\dfrac{1}{\left(\frac{dy}{dx}\right)_{(2,2)}} = -\dfrac{1}{9}$

Illustration

Find equation of tangent and normal to the curve $y = \frac{x^3 - x}{1 + x^2}$ at the point $x = 1$.

Solution:

Given curve, $y = \frac{x^3 - x}{1 + x^2}$

At $x = 1$; $y = \frac{1^3 - 1}{1 + 1} = 0$

$$y(1 + x^2) = x^3 - x$$

Differentiating w.r.t x, we get,

$$\frac{dy}{dx}(1 + x^2) + y(2x) = 3x^2 - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 1 - 2xy}{1 + x^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,0)} = \frac{2}{2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1$$

$\therefore$ Slope of the tangent is 1 and slope of normal is -1.

Now, by using point slope form, we get,

Equation of tangent : $y = x - 1$

Equation of normal : $y = -(x - 1)$

Illustration

Which of the following points lie on the tangent to the curve $x^4e^y + 2\sqrt{y+1} = 3$, at the point $(1,0)$?

$a.\ (2,2)$ $b.\ (-2,6)$ $c.\ (-2,4)$ $d.\ (2,6)$

Solution:

Given curve, $x^4e^y + 2\sqrt{y+1} = 3$

Differentiating w.r.t x, we get,

$$4x^3e^y + x^4e^y y' + \frac{1}{\sqrt{y+1}}.y' = 0$$

$$\Rightarrow y' = \frac{-4x^3e^y}{\left(x^4e^y + \frac{1}{\sqrt{y+1}}\right)}$$

On substituting point $(1,0)$, we get $y' = \dfrac{-4 \times 1^3 \times e^0}{\left(1^4 e^0 + \dfrac{1}{\sqrt{0+1}}\right)} = -2$

$\Rightarrow$ Slope of the tangent to the given curve at the point $(1,0)$ is -2

$\therefore$ Equation of tangent by point slope form is: $(y-0) = -2(x-1)$

$$\Rightarrow 2x + y = 2$$

Among the given points only $(-2, 6)$ satisfies the above equation.

So, option (b) is the correct answer.

Illustration

If the tangent to the curve $y = x + \sin y$, at a point (a, b) is parallel to the line joining points $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$, then :

a. $b = a$

b. $b = \frac{\pi}{2} + a$

c. $|b - a| = 1$

d. $|a + b| = 1$

Solution:

Given curve, $y = x + \sin y$

Slope of line joining points $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right) = \frac{2 - \frac{3}{2}}{\frac{1}{2} - 0} = 1 \quad \cdots (i)$

On differentiating the curve $y = x + \sin y$, we get,

$$\frac{dy}{dx} = 1 + \cos y \,.\, \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,b)} = 1 + \cos b \,.\, \left(\frac{dy}{dx}\right)_{(a,b)} \quad \cdots (ii)$$

From (i) and (ii)

$$1 = 1 + \cos b \quad (1)$$

$$\Rightarrow \cos b = 0 \Rightarrow b = \pm\frac{\pi}{2} \Rightarrow \sin b = \pm 1$$

Also , $b = a + \sin b$

$$\Rightarrow |b - a| = 1$$

So, option (c) is the correct answer.

Illustration

The length of the perpendicular from the origin on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at the point $(2,2)$ is :

$a.$ 2 $b.$ $\sqrt{2}$ $c.$ $2\sqrt{2}$ $d.$ $4\sqrt{2}$

Solution:

Given curve, $x^2 + 2xy - 3y^2 = 0$

Differentiating w.r.t x, we get,

$2x + 2y + 2xy' - 6yy' = 0$,

$\Rightarrow y'_{(2,2)} = 1$

$\Rightarrow$ Slope of normal at $(2, 2) = -1$

Equation of normal : $y - 2 = -(x - 2)$

$\Rightarrow x + y - 4 = 0$

We know,

Length of the perpendicular from origin on $ax + by + c = 0$ is $\left|\frac{c}{\sqrt{a^2+b^2}}\right|$.

$\therefore$ Length of the perpendicular from origin on $x + y - 4 = 0$ is $\left|\frac{-4}{\sqrt{1^2+1^2}}\right| = 2\sqrt{2}$

So, option (c) is the correct answer.

Illustration

Sum of intercepts of the tangent at any point to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, on the coordinate axes is :

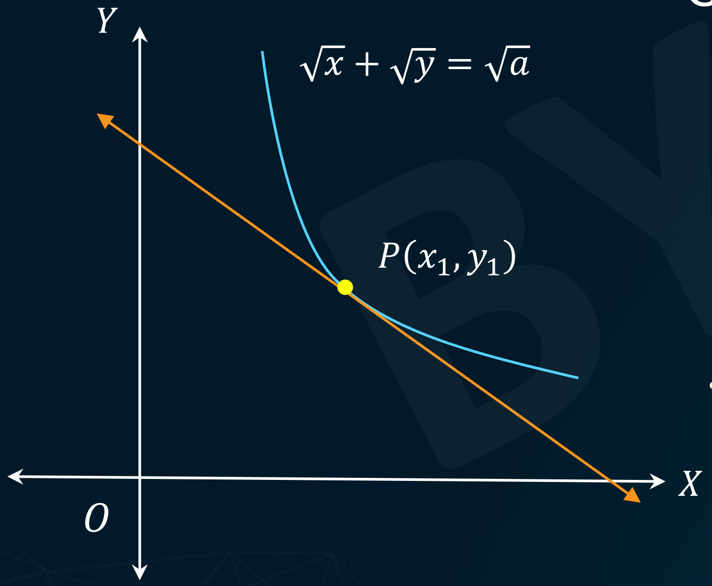
a. a

b. $2\sqrt{a}$

c. $\sqrt{a}$

d. $2a$

Solution:



Given curve, $\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}=0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

Hence, Slope of the tangent at the point $P(x_1, y_1)$ is $-\dfrac{\sqrt{y_1}}{\sqrt{x_1}}$

$\therefore$ Equation of tangent :

$$y-y_1=-\frac{\sqrt{y_1}}{\sqrt{x_1}}(x-x_1)\ldots(i)$$

On substituting $y = 0$ in the equation (i) , we get,

$$x = x_1 + \sqrt{x_1}\sqrt{y_1}$$

$\therefore X$- intercept, $X_{int.} = x_1 + \sqrt{x_1}\sqrt{y_1} = \sqrt{x_1}(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{x_1}\sqrt{a}$

On substituting $x = 0$ in the equation (i) , we get,

$$y = y_1 + \sqrt{x_1}\sqrt{y_1}$$

$\therefore Y$- intercept, $Y_{int.} = y_1 + \sqrt{x_1}\sqrt{y_1} = \sqrt{y_1}(\sqrt{y_1} + \sqrt{x_1}) = \sqrt{y_1}\sqrt{a}$

$$X_{int.} + Y_{int.} = \sqrt{a}\left(\sqrt{x_1} + \sqrt{y_1}\right) = \sqrt{a} \times \sqrt{a} = a$$

So, option(a) is the correct answer.

Illustration

Find the equation of tangent and normal to the curve $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$ at the point for which $t = \frac{1}{2}$.

Solution:

$$x = \frac{2at^2}{1+t^2}$$

Differentiating w.r.t. t, we get,

$$\frac{dx}{dt} = 2a\left[\frac{(1+t^2)2t - t^2(2t)}{(1+t^2)^2}\right]$$

$$= \frac{4at}{(1+t^2)^2}$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{1}{2}} = \frac{32a}{25}$$

$$y = \frac{2at^3}{1+t^2}$$

Differentiating w.r.t. t, we get,

$$\frac{dy}{dt} = 2a\left[\frac{(1+t^2)(3t^2) - t^3(2t)}{(1+t^2)^2}\right]$$

$$= \frac{2at^2(3+t^2)}{(1+t^2)^2}$$

$$\left.\frac{dy}{dt}\right|_{t=\frac{1}{2}} = \frac{26a}{25}$$

At $t = \frac{1}{2}$, $\frac{dy}{dx} = \frac{\frac{26a}{25}}{\frac{32a}{25}} = \frac{13}{16}$

$$x_1 = \frac{2a\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2} = \frac{2a}{5} \text{ and } y_1 = \frac{2a\left(\frac{1}{2}\right)^3}{1+\left(\frac{1}{2}\right)^2} = \frac{a}{5}$$

∴ Equation of tangent: $y - \frac{a}{5} = \frac{13}{16}\left(x - \frac{2a}{5}\right)$

And equation of normal: $y - \frac{a}{5} = -\frac{16}{13}\left(x - \frac{2a}{5}\right)$

Summary Sheet

- Equation of tangent at the point $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$(y - y_1) = m(x - x_1)$$

Where m is slope of the tangent.

Equation of normal at the point $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$(y - y_1) = -\frac{1}{m}(x - x_1)$$

- If the tangent at any point P on the curve is parallel to the X - axis, then $\frac{dy}{dx} = 0$ at the point P.
- If the tangent at any point P on the curve is parallel to the Y – axis, then $\frac{dy}{dx} \to \infty$ or $\frac{dx}{dy} = 0$ at the point P.
- If the tangent at any point P on the curve makes equal and positive intercepts on the coordinate axes, then $\frac{dy}{dx} = -1$ at the point P.

BYJU'S Classes

Application of Derivatives

Length of Tangents and Normals using Derivative

Road Map

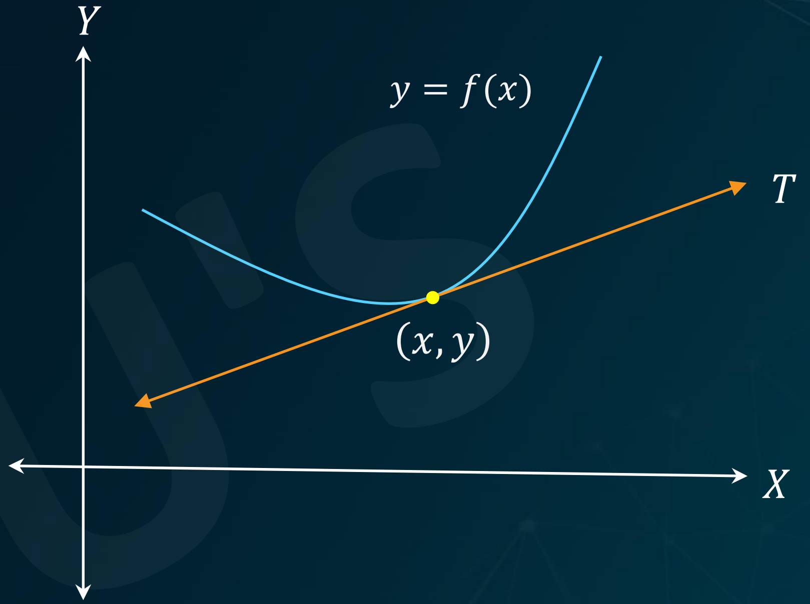
Illustrations based on Tangents and Normals

Length of Tangent, Normal,
Sub-Tangent & Sub-Normal

Tangent and Normal

Tangent and Normal

Let's say Line T is tangent to a curve $y = f(x)$



On solving tangent line with curve, we will get an equation which has repeated roots.

Note:

When the curve is a 2^{nd} degree polynomial, then the resulting equation (obtained by solving the tangent line with curve) will have two equal roots i.e., the discriminant will be zero.

Illustration

If tangent at point $(2,8)$ on the curve $y = x^3$ meets the curve again at Q. Then the co-ordinates of point Q is _____.

$a.\ (-4, -64)$ $\quad$ $b.\ (0,0)$ $\quad$ $c.\ (1,1)$ $\quad$ $d.\ (4,64)$

Solution:

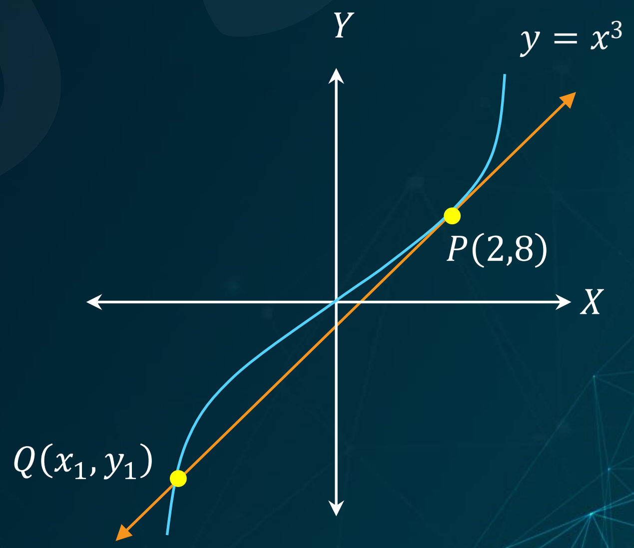
Given curve, $y = x^3$

By differentiating w.r.t to x, we get,

$$y' = 3x^2$$

$$y'_{(2,8)} = 12$$

Equation of tangent : $(y - 8) = 12(x - 2)$



Equation of tangent : $12x - y = 16 \cdots (i)$

Solving with the curve we get,

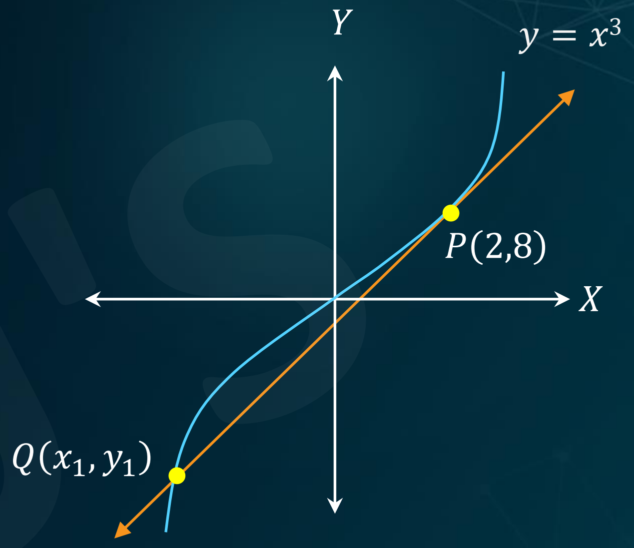
$$12x - 16 = x^3$$

$$\Rightarrow x^3 - 12x + 16 = 0$$

2

2

x_1



We get repeated roots at $P(2,8)$ as it is tangent point.

By theory of equation, $x_1 + x_2 + x_3 = -\frac{b}{a} \Rightarrow x_1 + 2 + 2 = 0 \Rightarrow x_1 = -4$

Subsituting in (i), gives $y_1 = -64$ $\boxed{\Rightarrow Q : (-4, -64)}$

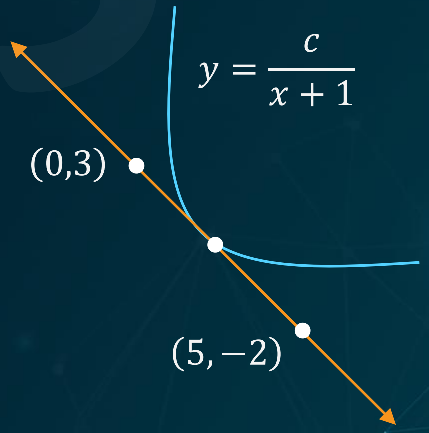
Hence, option (a) is the correct answer.

Illustration

If line joining points $(0,3)$ & $(5,-2)$ is a tangent to the curve $y = \frac{c}{x+1}$. Then the value of c is :

$a.\ -2$ $\quad$ $b.\ 5$ $\quad$ $c.\ -4$ $\quad$ $d.\ 4$

Solution:



Equation of line joining points $(0,3), (5,-2)$ is given by

$$(y-3) = \frac{3-(-2)}{0-5}(x-0) \Rightarrow x+y=3$$

Solving with the curve, $y = \frac{c}{x+1}$

$$\Rightarrow 3 - x = \frac{c}{x+1}$$

$$\Rightarrow x^2 - 2x + c - 3 = 0$$

Since the curve has repeated roots, $D = 0$

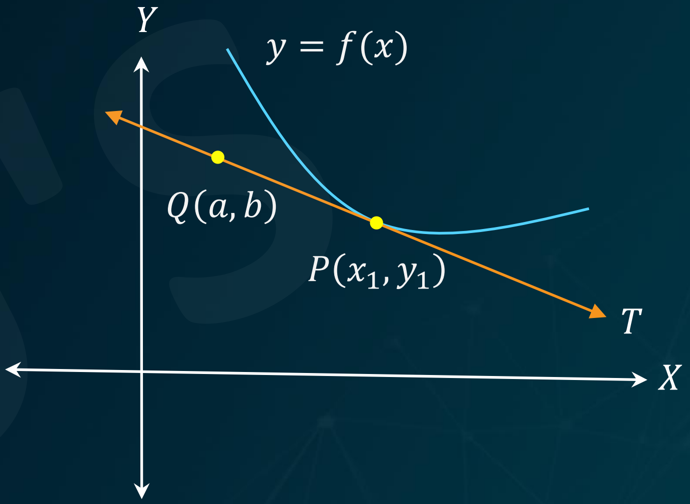
$$\Rightarrow 4 - 4(c-3) = 0 \Rightarrow \boxed{c = 4}$$

Hence, option (d) is the correct answer.

Equation of Tangent and Normal From External Point

Tangent is drawn from the point $Q(a, b)$

to the curve $y = f(x)$.



Let $P(x_1, y_1)$ be the point of contact.

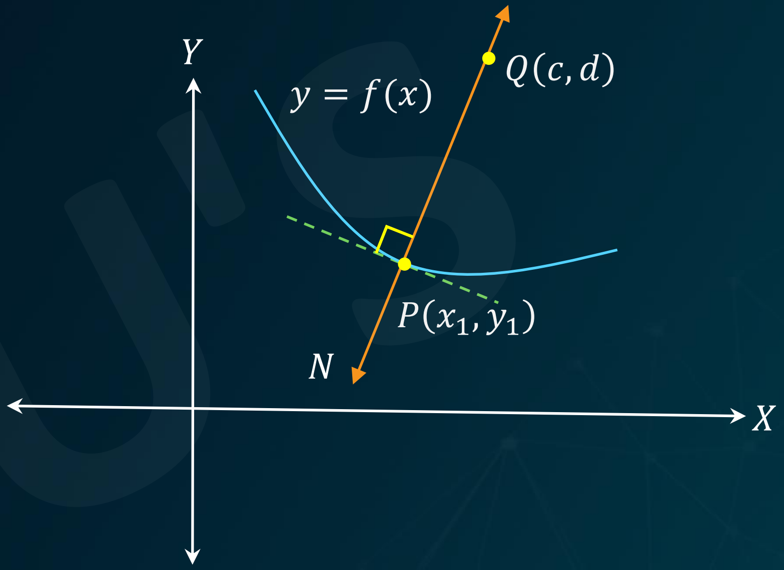
Then, slope of tangent at point $P = m_{PQ} = m_T$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \underbrace{\frac{y_1 - b}{x_1 - a}}$$

Slope of tangent (m_T)

Equation of tangent will be: $\boxed{(y - y_1) = m_T\,(x - x_1)}$

Equation of Tangent and Normal From External Point



Slope of normal m_N is given by,

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = \left(\frac{y_1 - d}{x_1 - c}\right)$$

Equation of normal will be: $\boxed{(y - y_1) = m_N (x - x_1)}$

Illustration

The equation of tangent drawn to the curve $xy = 4$ from point $(0, 1)$ is :

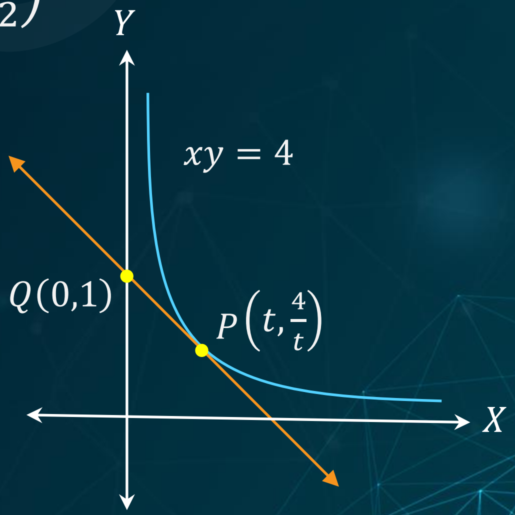
a. $y - \frac{1}{2} = -\frac{1}{16}(x + 8)$

b. $y - \frac{1}{2} = -\frac{1}{16}(x - 8)$

c. $y + \frac{1}{2} = -\frac{1}{16}(x - 8)$

d. $y - 8 = -\frac{1}{16}\left(x - \frac{1}{2}\right)$

Solution:



Let $P\left(t, \frac{4}{t}\right)$ be a point on curve $xy = 4$

Differentiating $xy = 4$, we get ,

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(t, \frac{4}{t}\right)} = -\frac{4}{t^2}$$

Slope of $PQ = \dfrac{\frac{4}{t}-1}{t}$

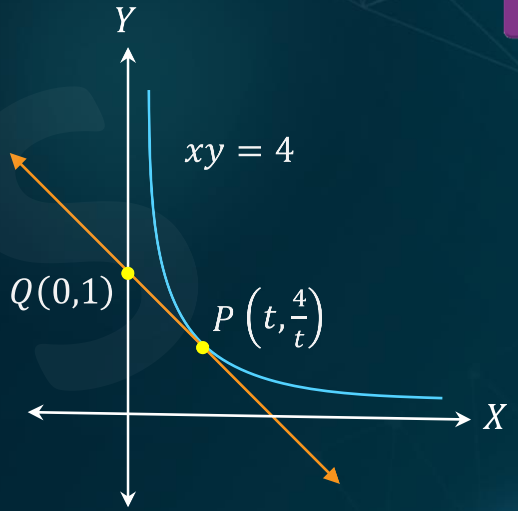
$$\Rightarrow \quad -\frac{4}{t^2} = \frac{\frac{4}{t}-1}{t}$$

$$\Rightarrow t = 8 \Rightarrow m_T = -\frac{1}{16}$$

Point $P(t, \frac{4}{t})$ will be $\left(8, \frac{1}{2}\right)$.

So, equation of tangent is : $\boxed{y - \frac{1}{2} = -\frac{1}{16}(x-8)}$

Hence, option (b) is the correct answer.



Illustration

JEE Main – 2020 (Jan)

Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the Y – axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to ____.

Solution:

Let $P \equiv (x_1, y_1)$

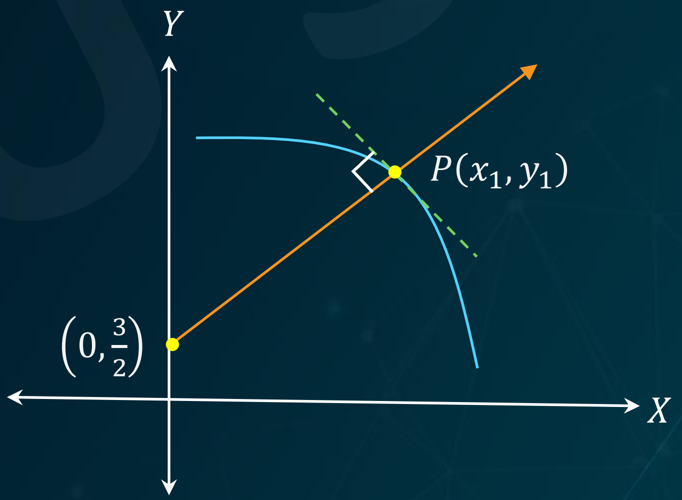
$$y^2 - 3x^2 + y + 10 = 0 \ldots (1)$$

Differentiating w.r.t. x, we get,

$$2yy' - 6x + y' = 0$$

$$\Rightarrow y'_{(x_1, y_1)} = \frac{6x_1}{1 + 2y_1}$$

So, $\dfrac{\frac{3}{2} - y_1}{-x_1} = -\dfrac{1+2y_1}{6x_1}$



$$\Rightarrow 9 - 6y_1 = 1 + 2y_1 \Rightarrow y_1 = 1$$

By Substituting the value of $y_1 = 1$ in equation (1),

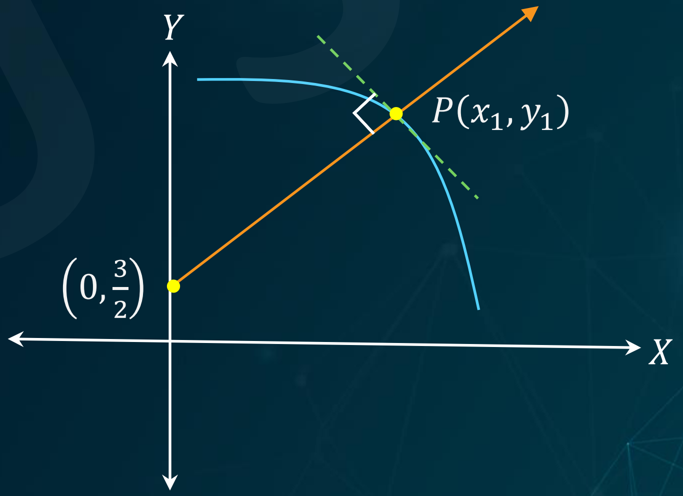
$$1^2 - 3x_1^2 + 1 + 10 = 0$$

$$\Rightarrow 3x_1^2 = 12 \Rightarrow x_1 = \pm 2$$

$$y'_{(x_1,y_1)} = \frac{6x_1}{1+2y_1},\ y_1 = 1\,, x_1 = \pm 2$$

So , $y'_{(x_1,y_1)} = \pm 4 = m$

$$\boxed{|m| = 4}$$



Illustration

Find equation(s) of tangent to the curve $y = (x+1)^3$, drawn from the origin.

a. $y = x$

b. $y = 0$

c. $y = 3x$

d. $y = \frac{27}{4}x$

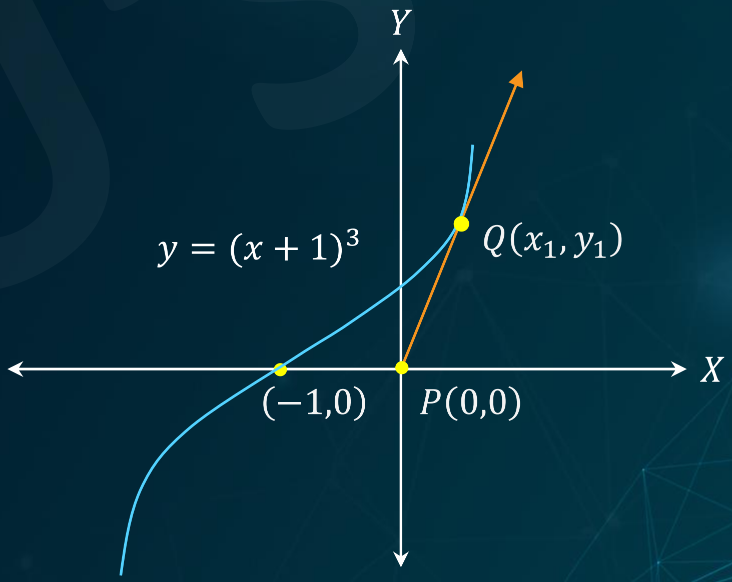
Solution:

$$y = (x+1)^3$$

Differentiating w.r.t x, we get,

$$y'_{(x_1,y_1)} = 3(x_1+1)^2 \ldots (i)$$

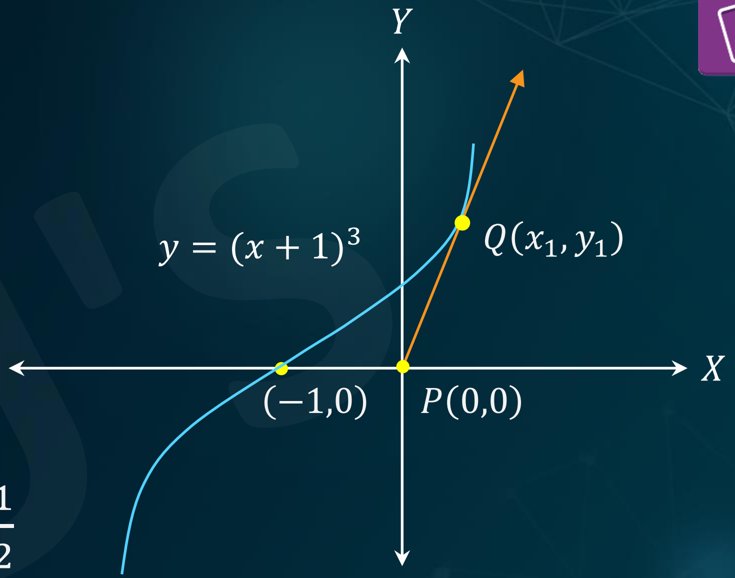
$$y'_{(x_1,y_1)} = m_{PQ}$$



$$\Rightarrow 3(x_1+1)^2 = \frac{y_1}{x_1}$$

$$\Rightarrow 3(x_1+1)^2 = \frac{(x_1+1)^3}{x_1} \quad \{\text{As } y=(x+1)^3\}$$

$$\Rightarrow x_1 = -1, \frac{1}{2}$$



$x_1 = -1$

$y_1 = 0$

$y'_{(-1,0)} = 0$

Equation of tangent : $\boxed{y = 0}$

$x_1 = \frac{1}{2}$

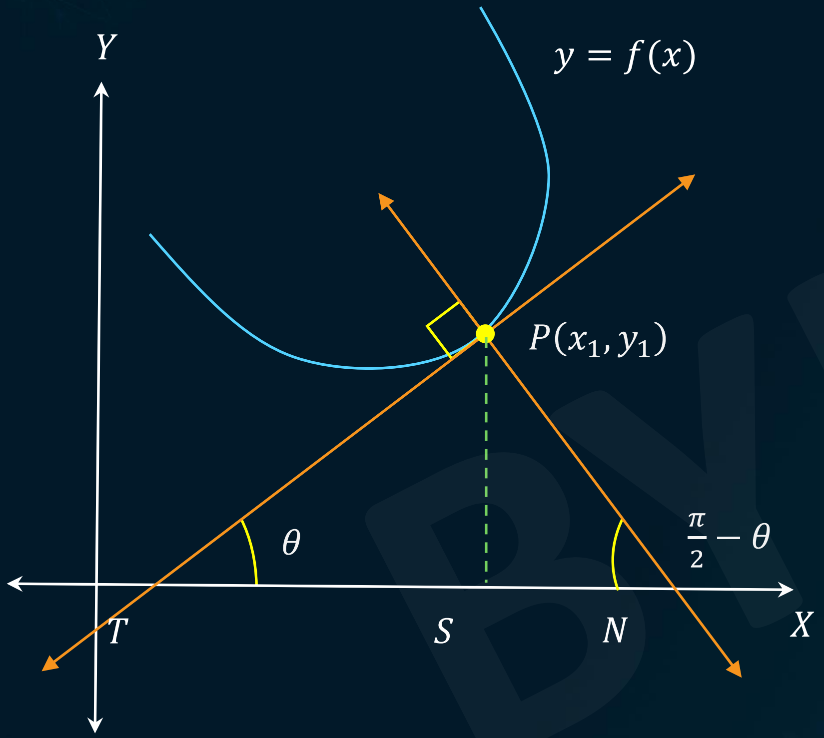
$y_1 = \frac{27}{8}$

$y'_{\left(\frac{1}{2}, \frac{27}{8}\right)} = \frac{27}{4}$

Equation of tangent : $\boxed{y = \frac{27}{4}x}$

Hence, option(b) and (d) are the correct answers.

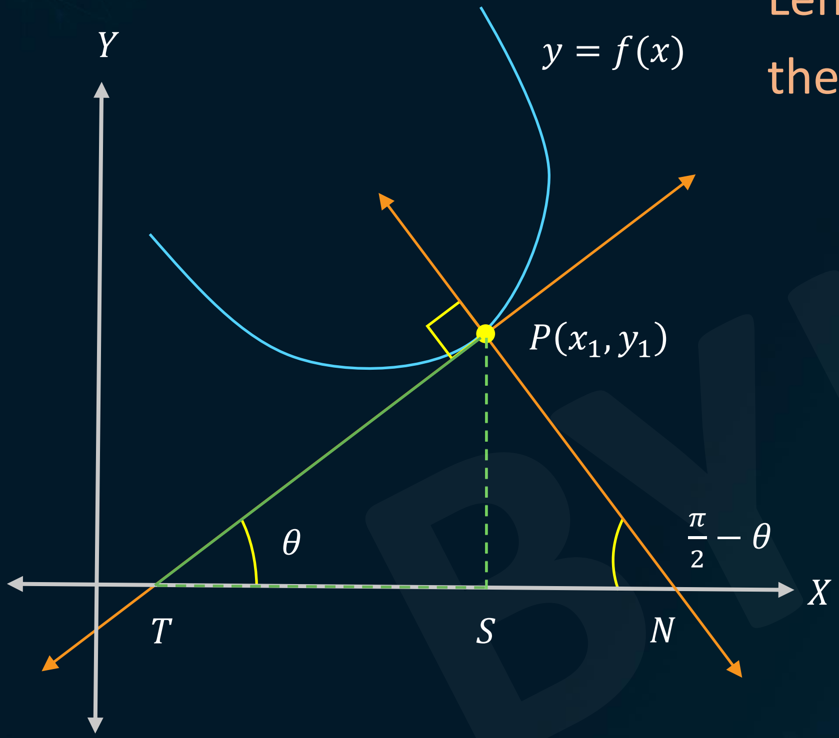
Length of Tangent, Normal,Sub-Tangent & Sub-Normal



Let the curve be $y = f(x)$ and at a point P on the curve, tangent and normal are drawn.

$$\& \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = m = \tan\theta$$

Length of Tangent



Length of tangent is the length of segment PT of the tangent between point of contact and X-axis.

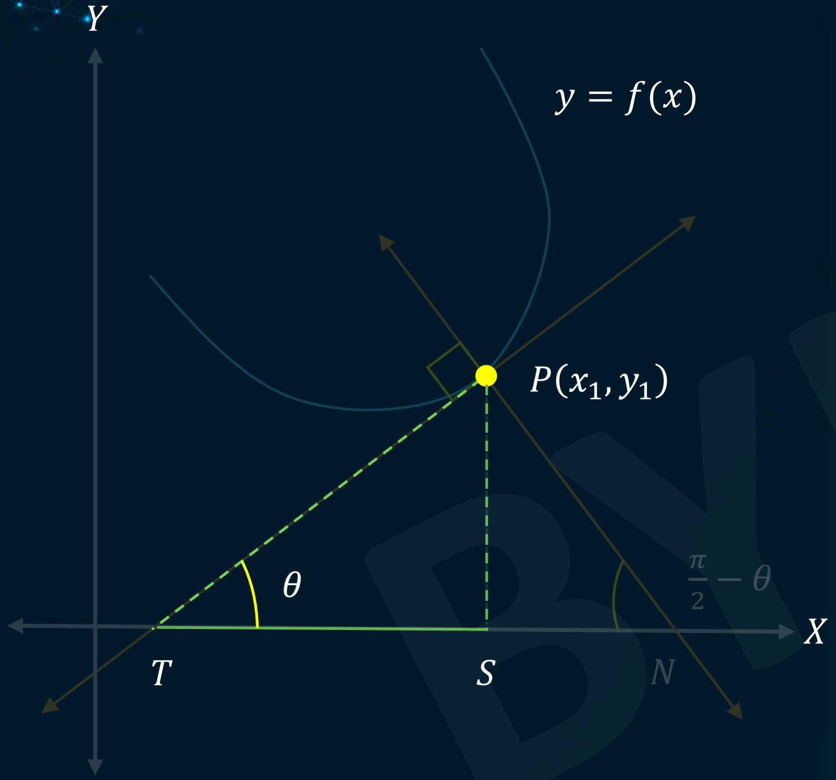
In ΔPST , $\sin\theta = \dfrac{PS}{PT}$

$$\Rightarrow PT = |y_1| \operatorname{cosec}\theta$$

$$\Rightarrow P = |y_1| \sqrt{1 + \frac{1}{\tan^2\theta}}$$

$$\boxed{L_T = |y_1| \sqrt{1 + \frac{1}{m^2}}}$$

Length of Sub-Tangent



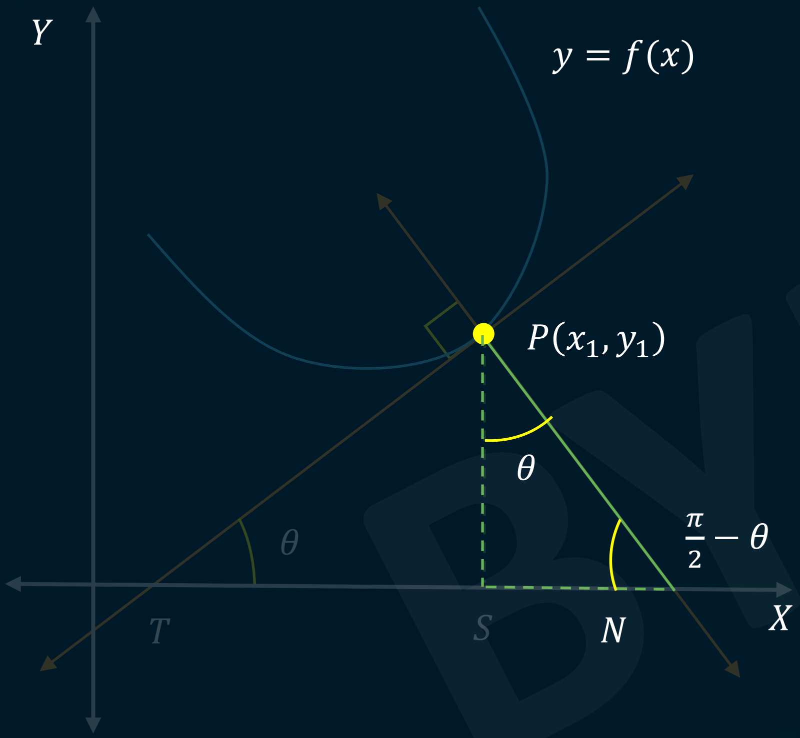
Length of sub-tangent is the projection of segment PT along X-axis (ST).

$$\text{In } \Delta PST\,,\ \tan\theta = \frac{PS}{ST}$$

$$\Rightarrow ST = \frac{|y_1|}{\tan\theta}$$

$$\boxed{L_{ST} = \left|\frac{y_1}{m}\right|}$$

Length of Normal



Length of normal is the length of segment PN of the normal intercepted between point of contact & X-axis.

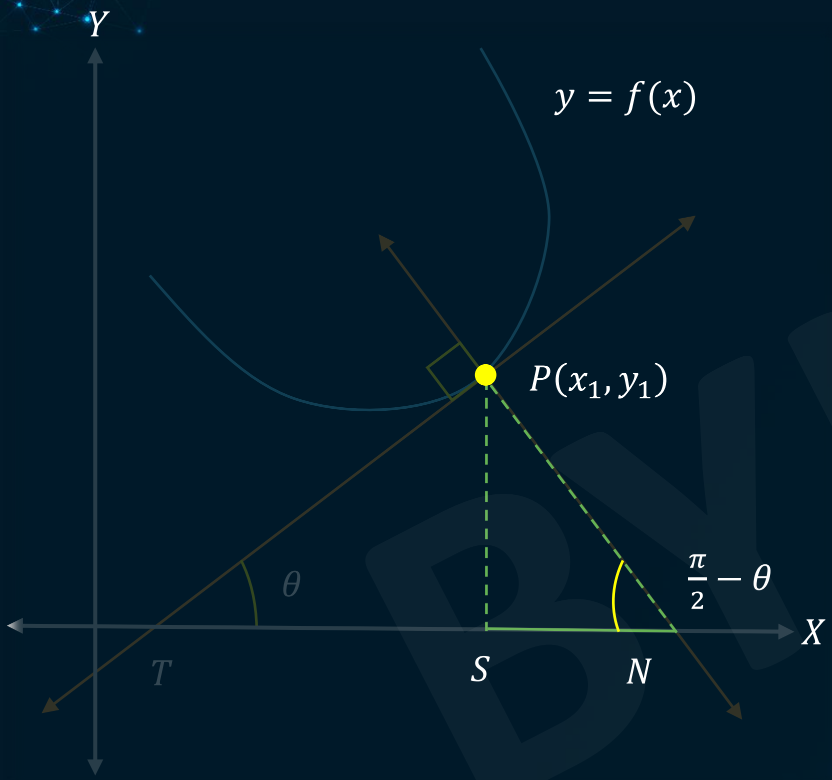
$$\text{In } \Delta PSN \text{ , } \cos\theta = \frac{PS}{PN}$$

$$\Rightarrow PN = |y_1| \sec\theta$$

$$\Rightarrow PN = |y_1|\sqrt{1 + \tan^2\theta}$$

$$\boxed{L_N = |y_1|\sqrt{1 + m^2}}$$

Length of Sub-Normal



Length of sub-normal is the projection of segment PN along X-axis (SN).

In ΔPSN , $\tan\theta = \frac{SN}{PS} \Rightarrow SN = |y_1| \tan\theta$

$$\boxed{L_{SN} = |y_1.m|}$$

Illustration

Find length of sub-tangent to curve $y = x^3 - 3x^2 + x$ at $x = -1$

Solution:

$$y = x^3 - 3x^2 + x$$

Differentiating w.r.t. x, we get,

$$\Rightarrow y' = 3x^2 - 6x + 1$$

$$\Rightarrow y'\Big|_{x=-1} = m = 3(-1)^2 + 6 + 1 = 10$$

$$x_1 = -1\,, y_1 = -5$$

$$L_{ST} = \left|\frac{y_1}{m}\right| \boxed{= \frac{1}{2}}$$

Illustration

For the curve $y = be^{\frac{x}{a}}$, length of sub-normal at the point (x_1, y_1) is :

$a.\ |ay_1|$ $\quad b.\ \left|\frac{y_1}{a}\right|$ $\quad c.\ \frac{(y_1)^2}{|a|}$ $\quad d.\ \left|\frac{b}{a}(y_1)^3\right|$

Solution:

$$y = be^{\frac{x}{a}}$$

Differentiating w.r.t. x, we get,

$$\Rightarrow y'|_{x=x_1} = m = \frac{b}{a}e^{\frac{x_1}{a}} = \frac{y_1}{a}$$

$$L_{SN} = |y_1.m|$$

$$\boxed{= \frac{(y_1)^2}{|a|}}$$

Hence, option (c) is the correct answer.

Summary Sheet

- Length of Tangent $= L_T = |y_1|\sqrt{1+\frac{1}{m^2}}$

- Length of Sub-tangent $= L_{ST} = \left|\frac{y_1}{m}\right|$

- Length of Normal $= L_N = |y_1|\sqrt{1+m^2}$

- Length of Sub-normal $= L_{SN} = |y_1.m|$

BYJU'S Classes

Application of Derivatives

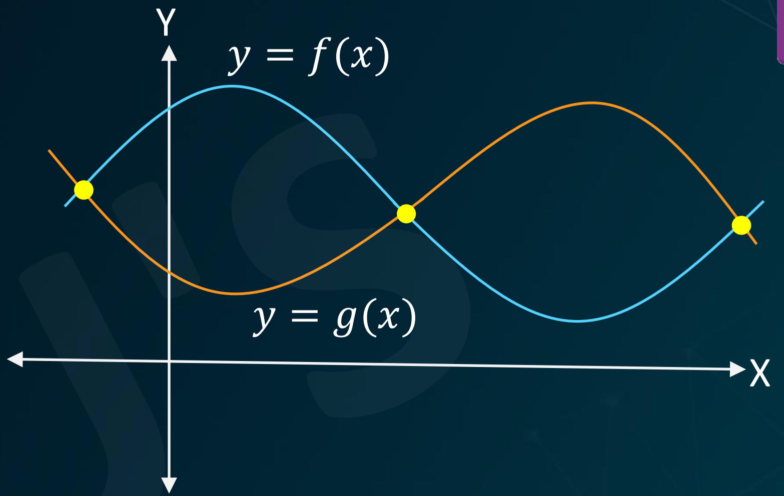
Angle between two curves

Road Map

- Illustrations
- Shortest Distance Between Two Curves
- Orthogonal Curves
- Angle Between Two Curves

Angle Between Two Curves

Let $y = f(x)$ and $y = g(x)$ be two curves, then angle between them is defined as angle between their tangents at their point of intersection.

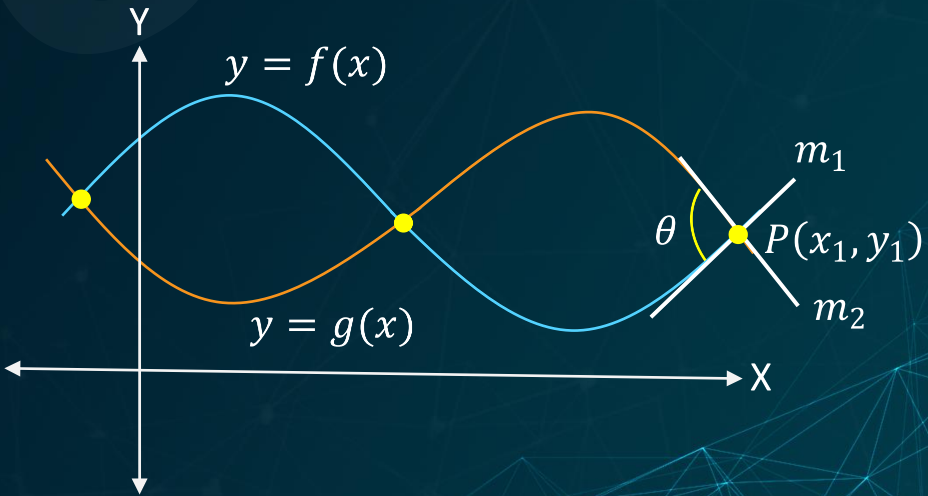


At point $P(x_1, y_1)$,

Slope of tangent for $f(x)$: $m_1 = \frac{d}{dx}(f(x))$

Slope of tangent for $g(x)$: $m_2 = \frac{d}{dx}(g(x))$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$ (For acute angle)



Illustration

Curves $y = \sin x$ & $y = \cos x$, intersect at infinite points. Find angle between them at one such point of intersection.

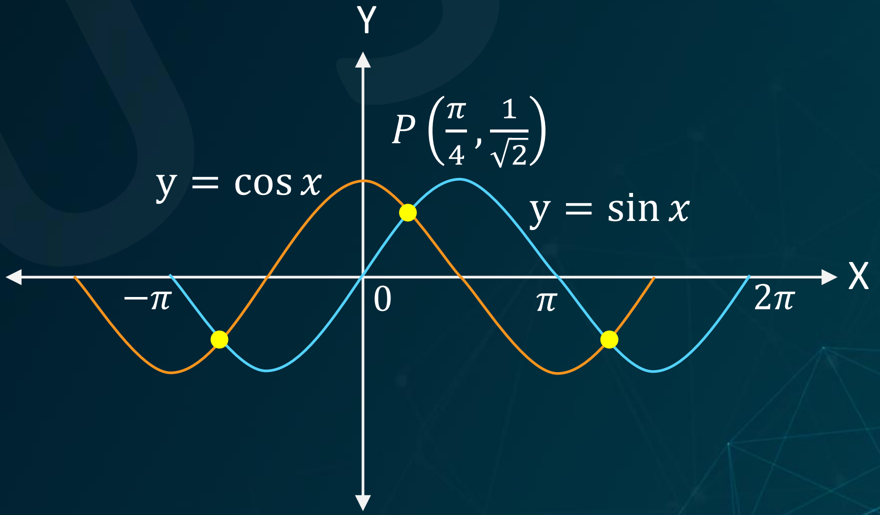
Solution:

Let one such point be $P\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

Let the slope of the tangent at point P to the curve $y = \sin x$ be m_1.

$$\frac{dy}{dx} = \cos x$$

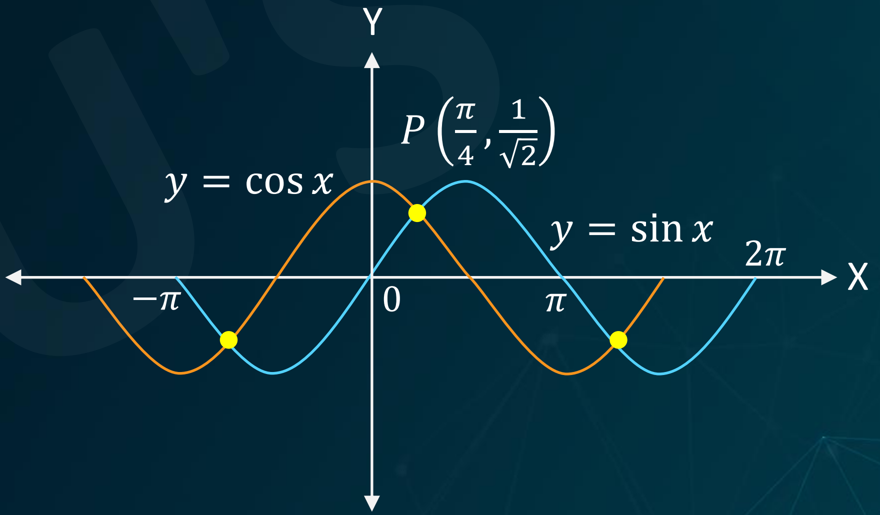
At $x = \frac{\pi}{4} \Rightarrow m_1 = \frac{1}{\sqrt{2}}$



Let the slope of tangent at point P to the curve $y = \cos x$ be m_2.

$$\frac{dy}{dx} = -\sin x$$

At $x = \frac{\pi}{4} \Rightarrow m_2 = -\frac{1}{\sqrt{2}}$



Now, angle between both the curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \tan\theta = \left|\frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| \Rightarrow \tan\theta = 2\sqrt{2}$$

$$\boxed{\Rightarrow \theta = \tan^{-1}(2\sqrt{2})}$$

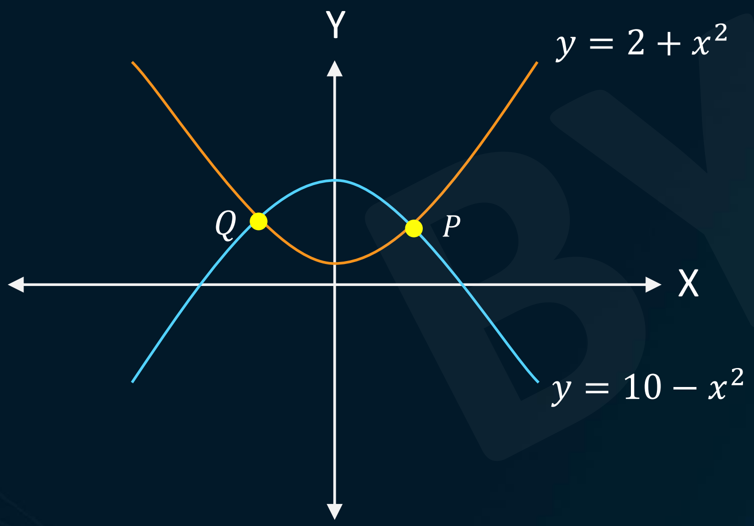
Illustration

JEE Main-2019 (Jan)

If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to:

a. $\frac{8}{17}$ b. $\frac{8}{15}$ c. $\frac{4}{9}$ d. $\frac{7}{17}$

Solution:



Let point of intersection of both the curves to be (x, y)

Now, for point of intersection, we have,

$$10 - x^2 = 2 + x^2$$

$$\Rightarrow x = \pm 2$$

Let $P: (2,6), Q: (-2,6)$

At point $P(2,6)$

Slope of tangent for $y = 2 + x^2$ is

$$\frac{dy}{dx} = 2x \Rightarrow m_1 = +4$$

At point $P(2,6)$

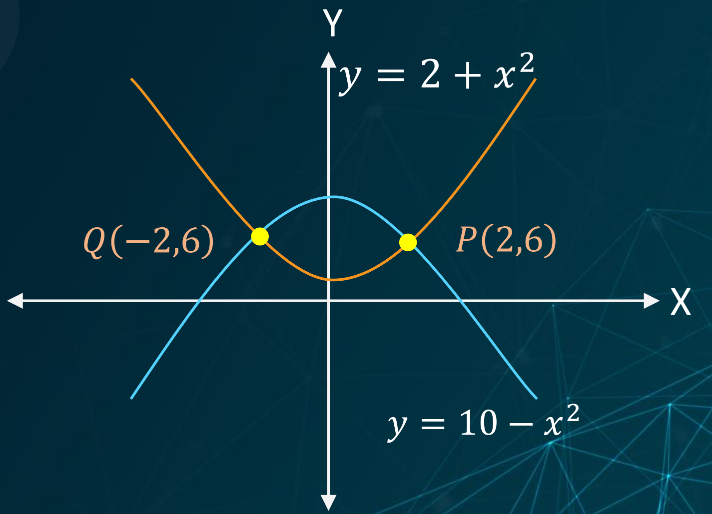
Slope of tangent for $y = 10 - x^2$ is

$$\frac{dy}{dx} = -2x \Rightarrow m_2 = -4$$

Given, the acute angle between two given curves is θ: $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \tan\theta = \left|\frac{4-(-4)}{1-16}\right| \quad \Rightarrow \tan\theta = \left|\frac{8}{15}\right|$$

$$|\tan\theta| = \frac{8}{15}$$



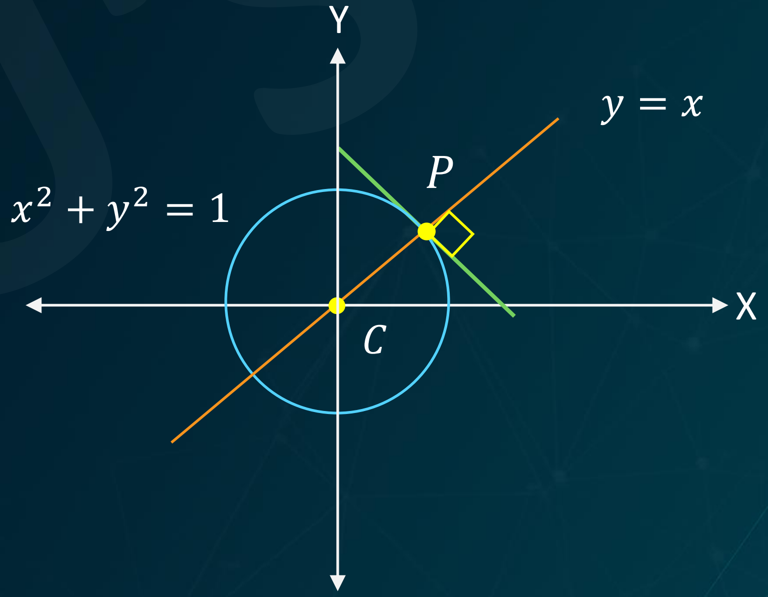
From symmetry, the angle between the curves at $Q(-2, 6)$ is also the same.

Hence, option (b) is the correct answer

Orthogonal Curves

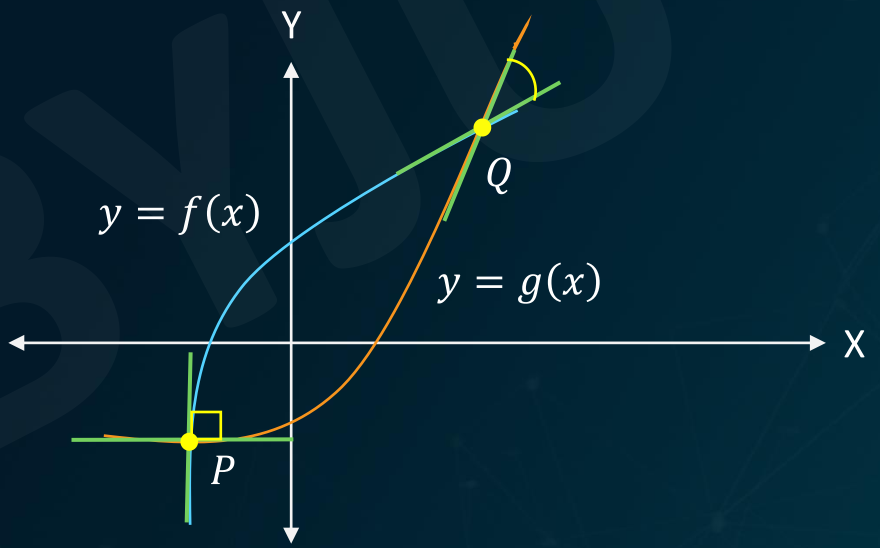
Two curves are said to be orthogonal to each other, if angle between them is 90° at their point of intersection.

For a circle centered at the origin $(0,0)$, all the lines passing through the origin will act as normal.



Note:

At the points of intersection of the curves $y = f(x)$ & $y = g(x)$, we observe that angle between the tangents is not 90^0. Hence these are not orthogonal curves. For orthogonality, the angle between the curves is 90^0 at every point of intersection.



Illustration

JEE Main-2018 (Jan)

If the curves $y^2 = 6x$ & $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

$a.$ 4 $\qquad b.$ $\frac{9}{2}$ $\qquad c.$ 6 $\qquad d.$ $\frac{7}{2}$

Solution:

Let point of intersection of both the curves to be: $P(x_1, y_1)$

$y_1{}^2 = 6x_1 \cdots (i) \qquad 9x_1{}^2 + by_1{}^2 = 16 \cdots (ii)$

Now, slope of tangent to the curve $y^2 = 6x$ is $\frac{dy}{dx} = \frac{3}{y} \Rightarrow m_1 = \frac{3}{y_1}$

Slope of tangent to the curve $9x^2 + by^2 = 16$ is $\frac{dy}{dx} = -\frac{9x}{b\,y} \Rightarrow m_2 = -\frac{9\,x_1}{b\,y_1}$

Since, curves are orthogonal $\Rightarrow \frac{3}{y_1} \cdot -\frac{9\,x_1}{b\,y_1} = -1$

$$27x_1 = by_1{}^2 \ldots (iii)$$

After substituting $y_1^2 = 6x_1$ in equation (iii), we get

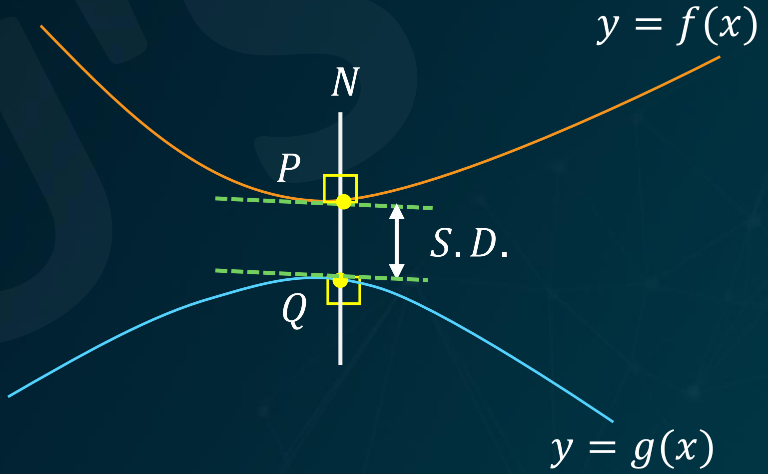
$$27x_1 = 6bx_1$$

$$\boxed{b = \frac{27}{6} = \frac{9}{2}}$$

Hence, option (b) is the correct answer

Shortest Distance Between Two Curves

Shortest distance between two continuous, differentiable & non – intersecting curves occurs along the common normal.



Steps to find shortest distance:

1. Find the slopes (m_P, m_Q) of normals at points P & Q.

2. Apply the condition $m_P = m_Q = m_{PQ}$ to get points P and Q

3. Find the shortest distance PQ using distance formula

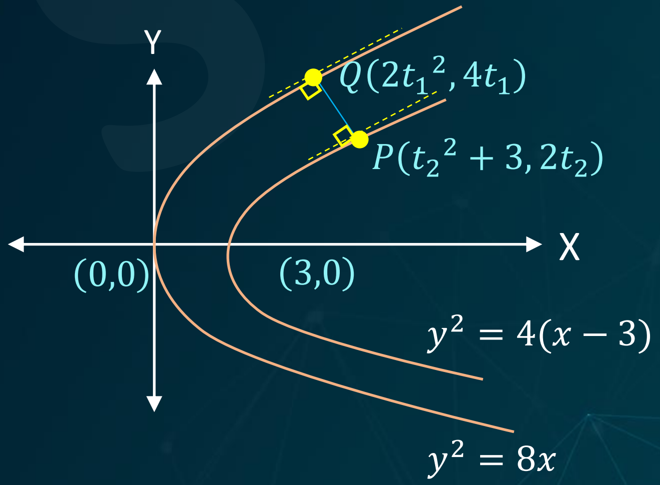
Illustration

The shortest distance between curves $y^2 = 8x$ and $y^2 = 4(x-3)$ is

$a. \sqrt{2}$ $\quad$ $b.\ 2\sqrt{2}$ $\quad$ $c.\ 3\sqrt{2}$ $\quad$ $d.\ 4\sqrt{2}$

Solution:

Any point on the curve $(y-\beta)^2 = 4a(x-\alpha)$ in the parametric form will be $(at^2+\alpha, 2at+\beta)$



$\therefore$ Parametric coordinates on curve P and Q will be $(t_2{}^2+3, 2t_2)$ and $(2t_1{}^2, 4t_1)$ respectively

Here, we get slope m_{PQ} using two point form

$$m_{PQ} = \frac{2t_2 - 4t_1}{t_2^2 + 3 - 2t_1^2}$$

Let Slope of normal at point Q to be m_Q on curve $y^2 = 8x$

$$\Rightarrow m_Q = -\frac{1}{\left(\frac{dy}{dx}\right)_Q} = -\frac{-2\times 4t_1}{8} = -t_1$$

Slope of normal at P to be m_P on curve $y^2 = 4(x-3)$

$$\Rightarrow m_P = -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\frac{-2\times 2t_2}{4} = -t_2$$

$$\Rightarrow -t_1 = \frac{-2t_1}{3-t_1{}^2}$$

Using condition, $m_P = m_Q = m_{PQ}$

We get, $-t_1 = -t_2 = \frac{2t_2-4t_1}{t_2{}^2+3-2t_1{}^2}$

$$\Rightarrow 3t_1 - t_1{}^3 = 2t_1 \Rightarrow t_1{}^3 - t_1 = 0 \Rightarrow t_1(t_1-1)(t_1+1) = 0 \Rightarrow t_1 = 0, 1, -1$$

For $t_1 = 0; P \equiv (3,0),\ Q \equiv (0,0) \Rightarrow PQ = 3$

For $t_1 = 1; P \equiv (4,2),\ Q \equiv (2,4) \Rightarrow PQ = 2\sqrt{2}$

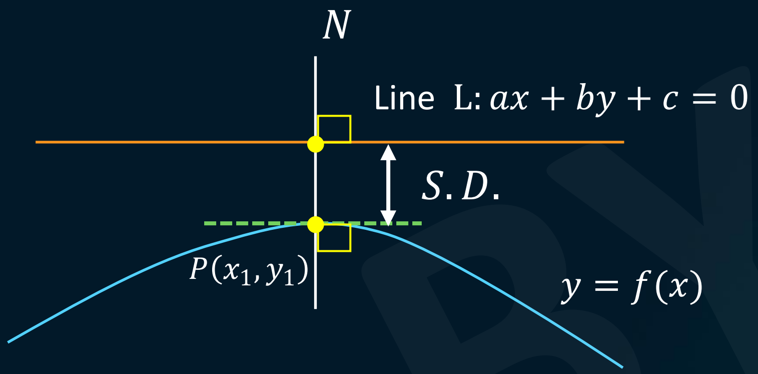
For $t_1 = -1; P \equiv (4,-2),\ Q \equiv (2,-4) \Rightarrow PQ = 2\sqrt{2}$

Shortest distance $= 2\sqrt{2}$

Hence, option (b) is the correct answer

Shortest Distance Between Two Curves if one Curve is a Line

$$m_P = m_L = -\frac{a}{b}$$



Steps to find shortest distance:

1. Find the slope of tangent (m_P) at point P

2. Apply the condition $m_p = m_L = -\frac{a}{b}$ to get the point P

3. Find shortest distance which is distance of point P from the line.

$$S.D. = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

Illustration

The shortest distance between the line $y = x$ and the curve $y^2 = x - 2$ is :

$a.\ \frac{7}{4\sqrt{2}}$ $b.\ 2$ $c.\ \frac{7}{8}$ $d.\ \frac{11}{4\sqrt{2}}$

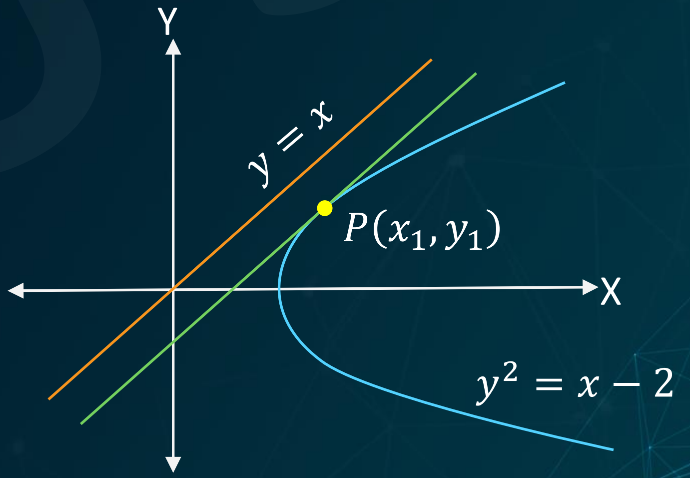
Solution:

Given, the line $y = x$ and the curve $y^2 = x - 2$

Let us consider the point $P(x_1, y_1)$ lies on the curve $y^2 = x - 2$

$\Rightarrow y_1^2 = x_1 - 2 \cdots (i)$

Slope at point $P = \left(\frac{dy}{dx}\right)_P = \frac{1}{2y_1}$

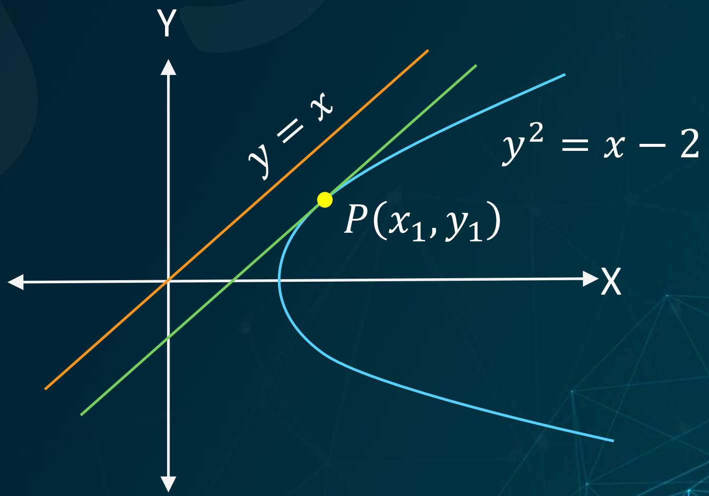


Slope of tangent at point P = Slope of line $y - x = 1$

$\frac{1}{2y_1} = 1 \Rightarrow y_1 = \frac{1}{2} \Rightarrow x_1 = \frac{9}{4}$ So , point $P : \left(\frac{9}{4}, \frac{1}{2}\right)$

Now, shortest distance between point P and line $y = x$, using following formula

$$S.D. = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$



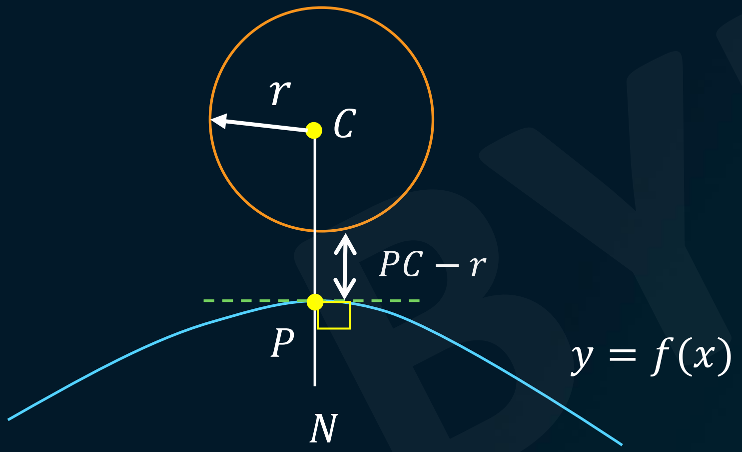
We get, $S.D. = \left|\frac{\frac{9}{4} - \frac{1}{2}}{\sqrt{1^2 + 1^2}}\right|$

$$\text{Shortest distance} = \frac{7}{4\sqrt{2}}$$

Hence, option (a) is the correct answer

Shortest Distance Between Two Curves if one Curve is a Circle

Let us consider a normal at point P on the curve $y = f(x)$, the normal will pass through the center C of the circle.



Steps to find shortest distance:

1. Find point P using $m_p = m_{CP}$; where $m_p =$ slope of normal at P

2. Find Shortest distance = $|PC - r|$

Illustration

Find shortest distance between the curves $y^2 = x^3$ and $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{1}{4}$

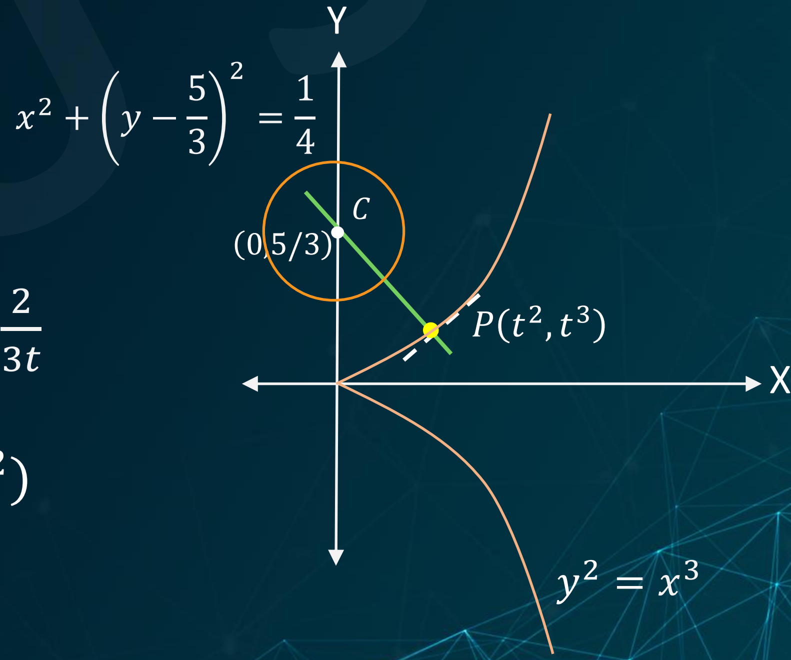
$a.$ 1 $b.$ $\frac{1}{2}$ $c.$ $\frac{\sqrt{13}}{3}$ $d.$ $\frac{\sqrt{13}}{3} - \frac{1}{2}$

Solution:

For the curve $y^2 = x^3$, let us consider parametric coordinates of point P be (t^2, t^3)

Slope of normal at point $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$

$$= -\frac{1}{\frac{3x^2}{2y}} = -\frac{1}{\frac{3t^4}{2t^3}} = -\frac{2}{3t}$$



Equation of normal PC is : $y - t^3 = -\frac{2}{3t}(x - t^2)$

Normal passes through $\left(0, \frac{5}{3}\right) \Rightarrow \frac{5}{3} - t^3 = \frac{2t}{3}$

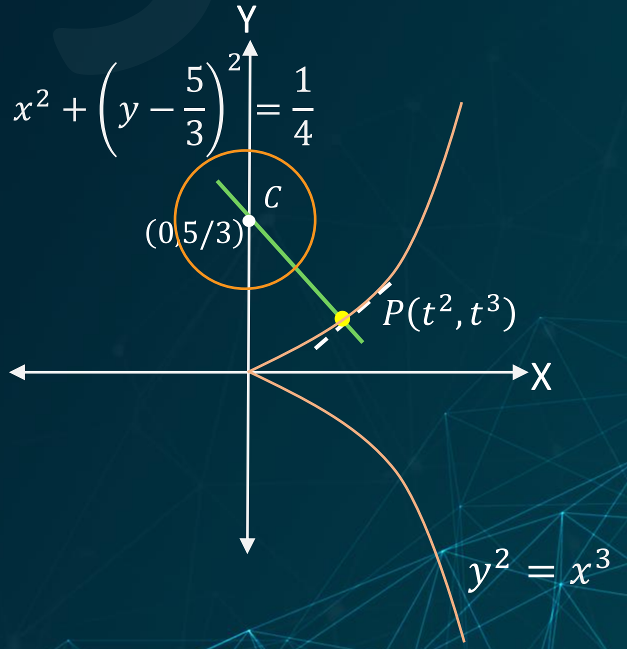
$$3t^3 + 2t - 5 = 0 \Rightarrow (t-1)(3t^2 + 3t + 5) = 0$$

$$\Rightarrow (t-1) = 0, (3t^2 + 3t + 5) = 0$$

$3t^2 + 3t + 5$ has no real roots as the discriminant is less than zero

For $t = 1$, $P \equiv (1,1)$

$$\Rightarrow PC = \sqrt{(1-0)^2 + \left(1 - \frac{5}{3}\right)^2} = \frac{\sqrt{13}}{3}$$



Shortest distance between point P and circle $= \left|PC - \frac{1}{2}\right|$

$$= \frac{\sqrt{13}}{3} - \frac{1}{2}$$

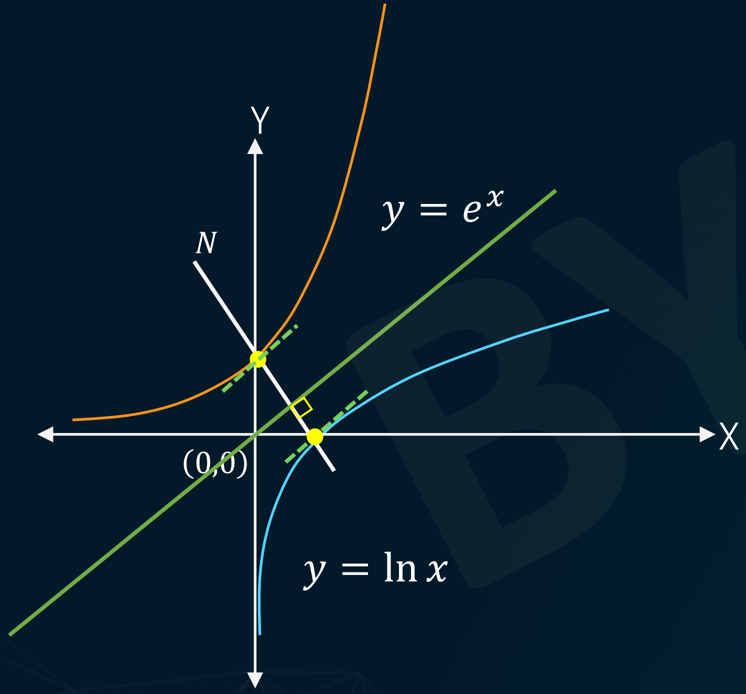
Hence, option (d) is the correct answer

Illustration

The shortest distance between the curves $y = \ln x$ and $y = e^x$ is :

$a.\ \sqrt{2}$ $\quad b.\ 2\sqrt{2}$ $\quad c.\ 3\sqrt{2}$ $\quad d.\ 4\sqrt{2}$

Solution:



Clearly, The curves e^x & $\ln x$ are inverse of each other.

So, they are symmetric about $y = x$ line.

Thus, the common normal to these curves is perpendicular to the line $y = x$.

Let any point on the curve $y = \ln x$ be $P(x_1, \ln x_1)$.

Slope of normal at $P = -\dfrac{1}{\left(\frac{dy}{dx}\right)_P} = -\dfrac{1}{\frac{1}{x_1}} = -x_1$

$\Rightarrow -x_1 = -1$ [slope of normal to line $y = x$]

$\Rightarrow x_1 = 1$; $y_1 = 0$

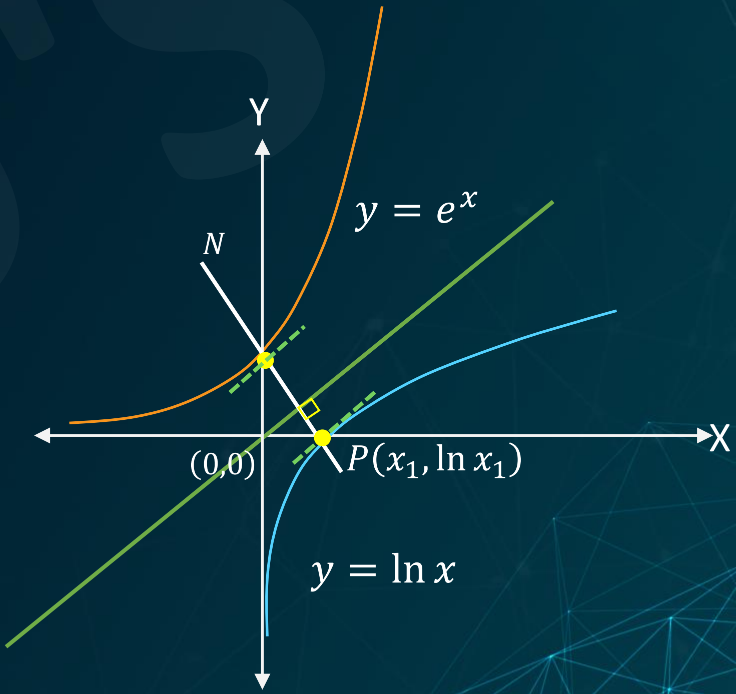
Distance of the point $(1,0)$ from the line

$y = x$ is $= \dfrac{|1-0|}{\sqrt{1^2+1^2}} = \dfrac{1}{\sqrt{2}}$

Shortest distance between the curve is

$= 2 \times \dfrac{1}{\sqrt{2}} = \sqrt{2}$

Hence, option (a) is the correct answer



Summary Sheet

- Two curves are said to be orthogonal to each other, if angle between them is $90°$ at their point of intersection.

- Shortest distance between two continuous, differentiable & non – intersecting curves occurs along the common normal.

- For orthogonality, the angle between the curves is 90^0 at every point of intersection.

BYJU'S Classes

Application of Derivatives

Common Tangent and Mean Value Theorem

Road Map

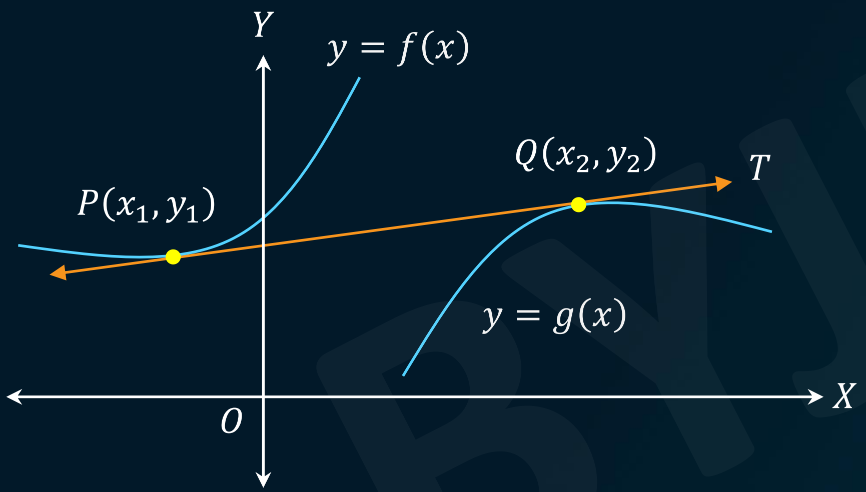
Mean Value Theorems

Common Tangents

Shortest Distance Between Two Curves

Common Tangents

Let $y = f(x)$ & $y = g(x)$ have a common tangent T as shown below:



Here, tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$, tangent to the curve $y = g(x)$ at point $Q(x_2, y_2)$ *is* the same line.

$$\boxed{\left.\frac{df}{dx}\right|_{(x_1,y_1)} = \left.\frac{dg}{dx}\right|_{(x_2,y_2)} = m_{PQ}}$$

For some given curves, condition for existence of tangents /normal can also be applied to get the common tangent/normal.

Illustration

JEE Main – 2019 (April)

The equation of common tangent to the curves $y^2 = 16x$ and $xy = -4$, is :

$a.\ x - y + 4 = 0$

$b.\ x + y + 4 = 0$

$c.\ x - 2y + 16 = 0$

$d.\ 2x - y + 2 = 0$

Solution:

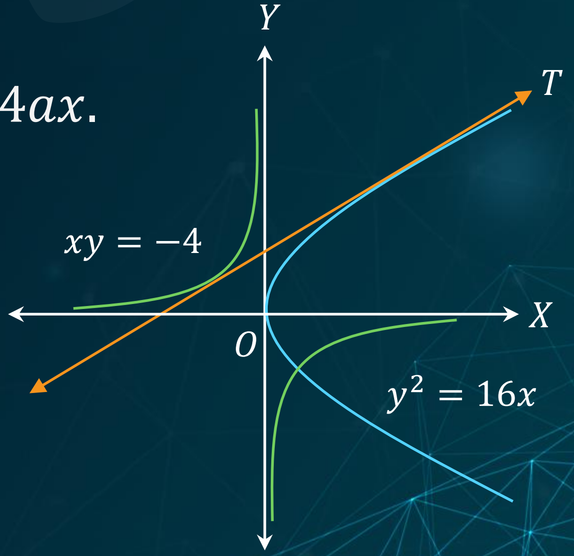
We know, line $y = mx + \frac{a}{m}$ is tangent to the parabola $y^2 = 4ax$.

$\therefore$ Line $y = mx + \frac{4}{m}$ is tangent to $y^2 = 16x,\ m \in \mathbb{R}$

For it to be tangent to curve $xy = -4$,

On substituting $y = mx + \frac{4}{m}$, we get

$$x\left(mx + \frac{4}{m}\right) = -4 \Rightarrow mx^2 + \frac{4x}{m} + 4 = 0$$



Line touches the curve only if, $D = 0$

$$\Rightarrow b^2 - 4ac = 0$$

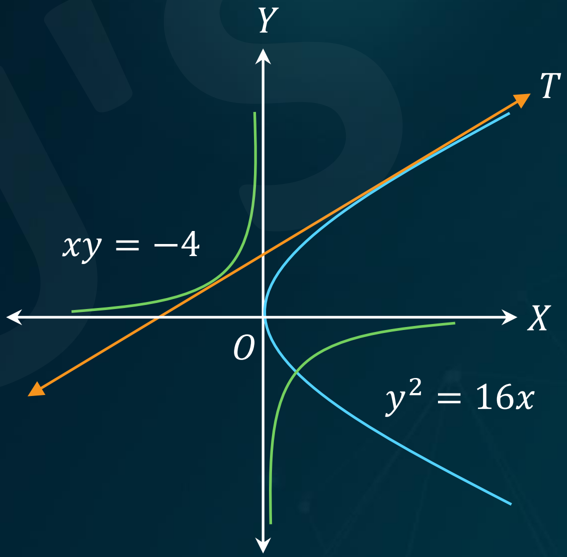
$$\Rightarrow \left(\frac{4}{m}\right)^2 - 4 \times m \times 4 = 0$$

$$\Rightarrow \frac{16}{m^2} - 16m = 0$$

$$\Rightarrow m^3 - 1 = 0$$

$$\Rightarrow (m-1)(m^2 + m + 1) = 0$$

$$\Rightarrow m = 1 \text{ as } m \in \mathbb{R}$$



On substituting $m = 1$ in $y = mx + \frac{4}{m}$, we get

Equation of common tangent: $y = x + 4$

So, option (a) is the correct answer.

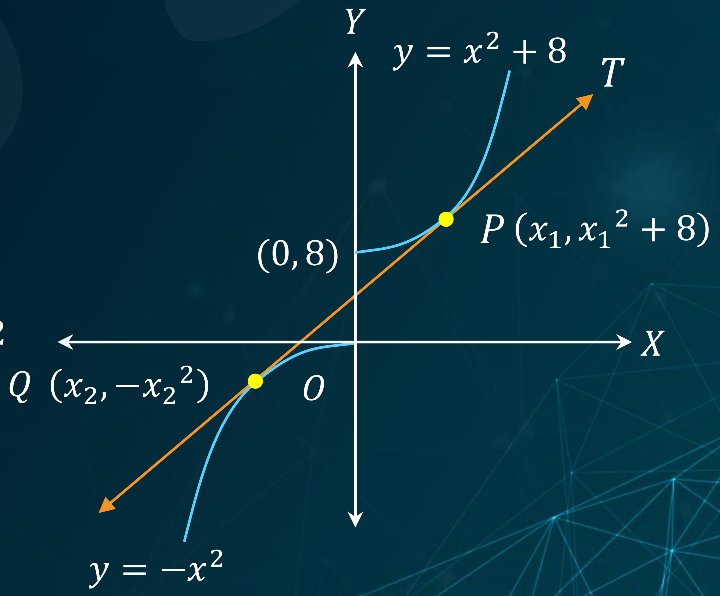
Illustration

Find the equation of a tangent line touching both branches of the function:

$$f(x) = \begin{cases} -x^2, x < 0 \\ x^2 + 8\,, x \geq 0 \end{cases}$$

Solution:

The given function can be plotted as shown.



Slope of tangent at point $P = \left(\dfrac{d\,(x^2+8)}{dx}\right)_{(x_1,y_1)} = 2x_1$

Slope of tangent at point Q $= \left(\dfrac{d\,(-x^2)}{dx}\right)_{(x_2,y_2)} = -2x_2$

Slope of the line joining $P(x_1, {x_1}^2 + 8)$

and Q$(x_2, -{x_2}^2) = m_{PQ} = \dfrac{{x_1}^2+8+{x_2}^2}{x_1-x_2}$

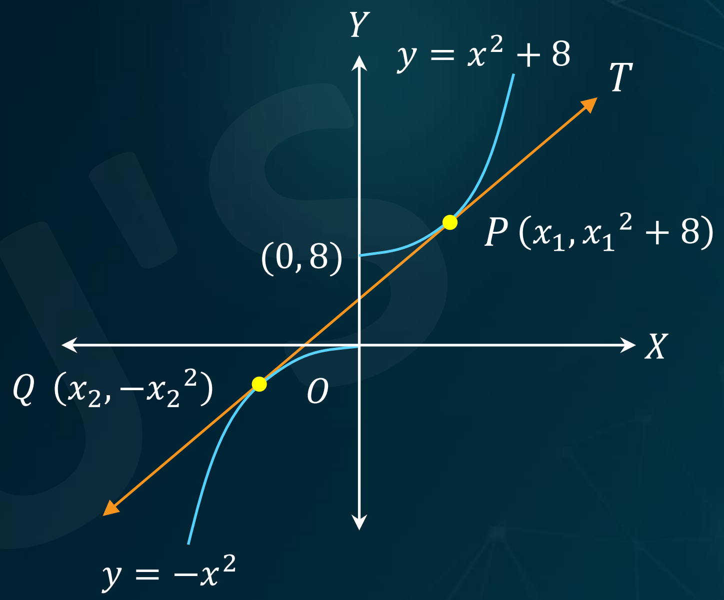
$$m_P = m_Q = m_{PQ}$$

$$\Rightarrow 2x_1 = -2x_2 = \frac{{x_1}^2+8+{x_2}^2}{x_1-x_2}$$

$$\Rightarrow x_1 = -x_2 \text{ and } 2x_1 = \frac{{x_1}^2+8+{x_2}^2}{x_1-x_2}$$

$$\Rightarrow 2x_1 = \frac{2{x_1}^2+8}{2x_1}$$

$$\Rightarrow 2x_1^2 = 8 \quad \Rightarrow x_1 = 2 \qquad (\because x_1 > 0)$$



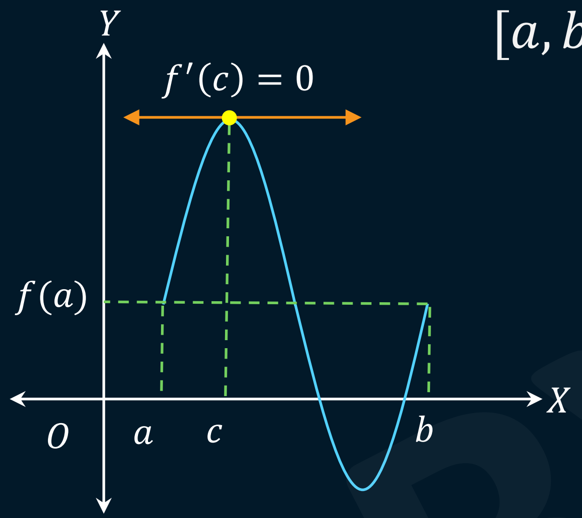
On substituting $x = 2$ in the curve $y = x^2 + 8$, we get $y = 12 \Rightarrow y_1 = 12$

Now, equation of tangent at the point $(2, 12)$: $(y - 12) = 4(x - 2)$

$$\Rightarrow y = 4x + 4$$

Mean Value Theorems

Rolle's Theorem : Let f be a real – valued function defined on the closed interval $[a, b]$ such that:



(i) $f(x)$ is continuous in the closed interval $[a, b]$.

(ii) $f(x)$ is differentiable in the open interval (a, b).

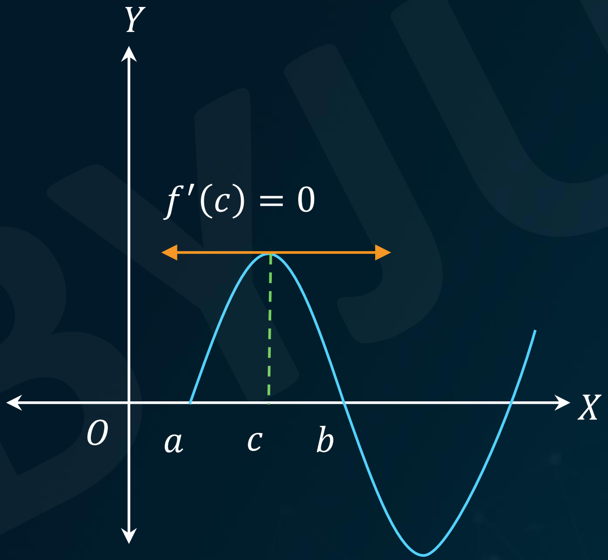
(iii) $f(a) = f(b)$

Then there exists at least one $c \in (a, b)$, such that $f'(c) = 0$.

Geometrically, there will be at least one $c \in (a, b)$, where tangent will be parallel to X–axis.

Note:

Also, we can say that between any two real consecutive roots of $f(x) = 0$ there will be at least one root of $f'(x) = 0$.



$$f(a) = f(b) = 0$$

Illustration

Verify Rolle's theorem for function $f(x) = x^2 - 4x + 3\ , x \in [0,4]$.

Solution: $f(x) = x^2 - 4x + 3\ , x \in [0,4]$

We can see that $f(x)$ is a polynomial function.

So, f is continuous and differentiable in $[0,4]$ & $(0,4)$ respectively.

$$f(0) = 0 - 0 + 3 = 3$$

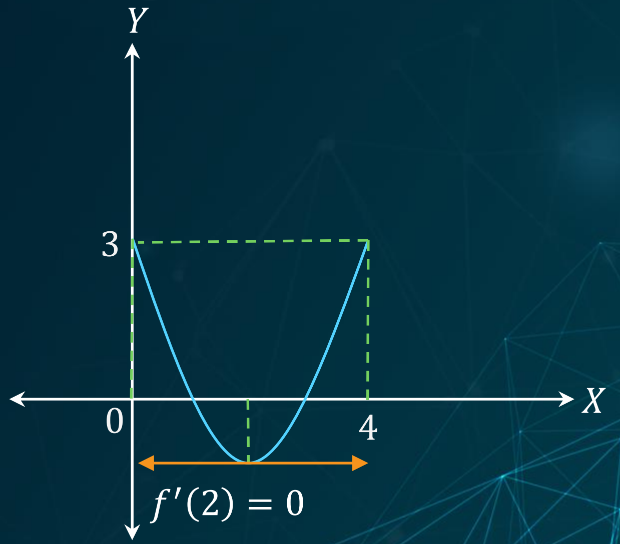
$$f(4) = 4^2 - 16 + 3 = 3 \Rightarrow f(0) = f(4)$$

According to Rolle's Theorem, there exists at least one c in $(0,4)$ such that $f'(c) = 0$.

$$\Rightarrow \left(\frac{d(x^2-4x+3)}{dx}\right)_{x=c} = 0 \Rightarrow 2c - 4 = 0$$

$$\Rightarrow c = 2 \ \in \ [0,4]$$

Thus, Rolle's theorem is verified.



Illustration

JEE Main – 2020 (Sept)

For all twice differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, with $f(0) = f(1) = f'(0) = 0$.

a. $f''(x) = 0$, for some $x \in (0,1)$

b. $f''(0) = 0$

c. $f''(x) \neq 0$, at every point $x \in (0,1)$

d. $f''(x) = 0$, at every point $x \in (0,1)$

Solution: Given, $f(0) = f(1) = 0$

Hence, Rolle's theorem can be applied to the function $f(x)$ in the interval $(0,1)$

By Rolle's theorem, $f'(c_1) = 0$, where $c_1 \in (0,1)$

Applying Rolle's theorem for $\underbrace{y = f'(x)}_{\text{continuous \& differentiable } (\because f \text{ is twice differentiable})}$

$f'(0) = f'(c_1) = 0$

Hence, Rolle's theorem can be applied to the function $f'(x)$ in the interval $(0,1)$.

By Rolle's theorem $f''(c_2) = 0$, for some $c_2 \epsilon (0, c_1)$

$\Rightarrow c_2 \epsilon (0,1)$

$\therefore f''(x) = 0$, for some $x \epsilon (0,1)$

So, option (a) is the correct answer.

Illustration

AIEEE - 2004

If $f(x) = x^{\alpha} \ln x$, and $f(0) = 0$. If Rolle's theorem can be applied to f in $[0,1]$, then value of α can be :

$a.\ -2$ $\quad$ $b.\ -1$ $\quad$ $c.\ 0$ $\quad$ $d.\ \frac{1}{2}$

Solution: $f(x) = x^{\alpha} \ln x$

Since, Rolle's theorem can be applied in the given interval.

$\Rightarrow f$ is continuous and differentiable

$$\Rightarrow f(0) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \lim_{x \to 0^+} x^{\alpha} \ln x = 0$$

$$\lim_{x \to 0^+} x^{\alpha} \ln x = 0$$

When $x \to 0^+, \ln x \to -\infty.$ For the limit to exist, $x^{\alpha} \ln x$ must be of $0 \times \infty$ form.

Case 1: $\alpha < 0$: $x^{\alpha} \ln x$ is of $\infty \times \infty$ form, so limit does not exist.

Case 2: $\alpha = 0$: $x^{\alpha} \ln x$ is of $1 \times \infty$ form, so limit does not exist.

Case 3: $\alpha > 0$: $x^{\alpha} \ln x$ is of $0 \times \infty$ form, so limit may exist.

$$\lim_{x \to 0^+} x^{\alpha} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-\alpha}} = \lim_{x \to 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}}$$

(Applying L'Hospital's rule)

$$= \lim_{x \to 0^+} \frac{x^{\alpha}}{-\alpha} = 0 \Rightarrow \alpha > 0$$

So, option (d) is the correct answer.

Illustration

If f be a continuous function on $[0,1]$, differentiable in $(0,1)$ such that $f(1) = 0$, then there exists some $c \in (0,1)$, such that :

a. $cf'(c) - f(c) = 0$

b. $f'(c) + cf(c) = 0$

c. $f'(c) - cf(c) = 0$

d. $cf'(c) + f(c) = 0$

Solution: Let $g(x) = xf(x)$ (By hit and trial)

Then $g(x)$ is continuous in $[0,1]$ and differentiable in $(0,1)$.

Also, $g(0) = g(1) = 0$

So, by Rolle's theorem , $g'(c) = 0$ for some c in $(0, 1)$

$\Rightarrow cf'(c) + f(c) = 0$ $(\because g'(x) = xf'(x) + f(x)\ )$

So, option (d) is the correct answer.

Illustration

If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$, has _____.

$a.$ At least one root in $(0, 1)$

$b.$ One root in $[2,3]$ and the other in $(-2, -1)$

$c.$ Imaginary roots

$d.$ At least one root in $(1, 2)$

Solution: Let $f(x) = 3ax^2 + 2bx + c$ and $g(x) = \int f(x)$,

$g(x) = ax^3 + bx^2 + cx + d$ $\rightarrow$ Continuous and Differentiable

$$\left.\begin{array}{l} g(0) = d \\ g(1) = a + b + c + d \quad = d \end{array}\right\} g(0) = g(1)$$

Hence, conditions for Rolle’s theorem is satisfied.

$\therefore\ g'(x) = 0$ in $(0, 1)$

$\Rightarrow f(x) = 0$, at least once in $(0, 1)$. Hence the quadratic $3ax^2 + 2bx + c = 0$ has at least one root in $(0, 1)$.

So, option (a) is the correct answer.

Summary Sheet

- Let $y = f(x)$ and $y = g(x)$ have a common tangent, and the common tangent touches the graphs of $f(x)$ and $g(x)$ at points $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively, then,

$$\frac{df}{dx}_{(x_1,y_1)} = \frac{dg}{dx}_{(x_2,y_2)} = m_{PQ}$$

- Rolle's Theorem :

Let f be a real – valued function defined on the closed interval $[a, b]$ such that:

(i) $f(x)$ is continuous in the closed interval $[a, b]$,

(ii) $f(x)$ is differentiable in the open interval (a, b) and

(iii) $f(a) = f(b)$.

Then there exists at least one $c \in (a, b)$, such that $f'(c) = 0$.

BYJU'S Classes

Application of Derivatives

Lagrange's Mean Value Theorem

Road Map

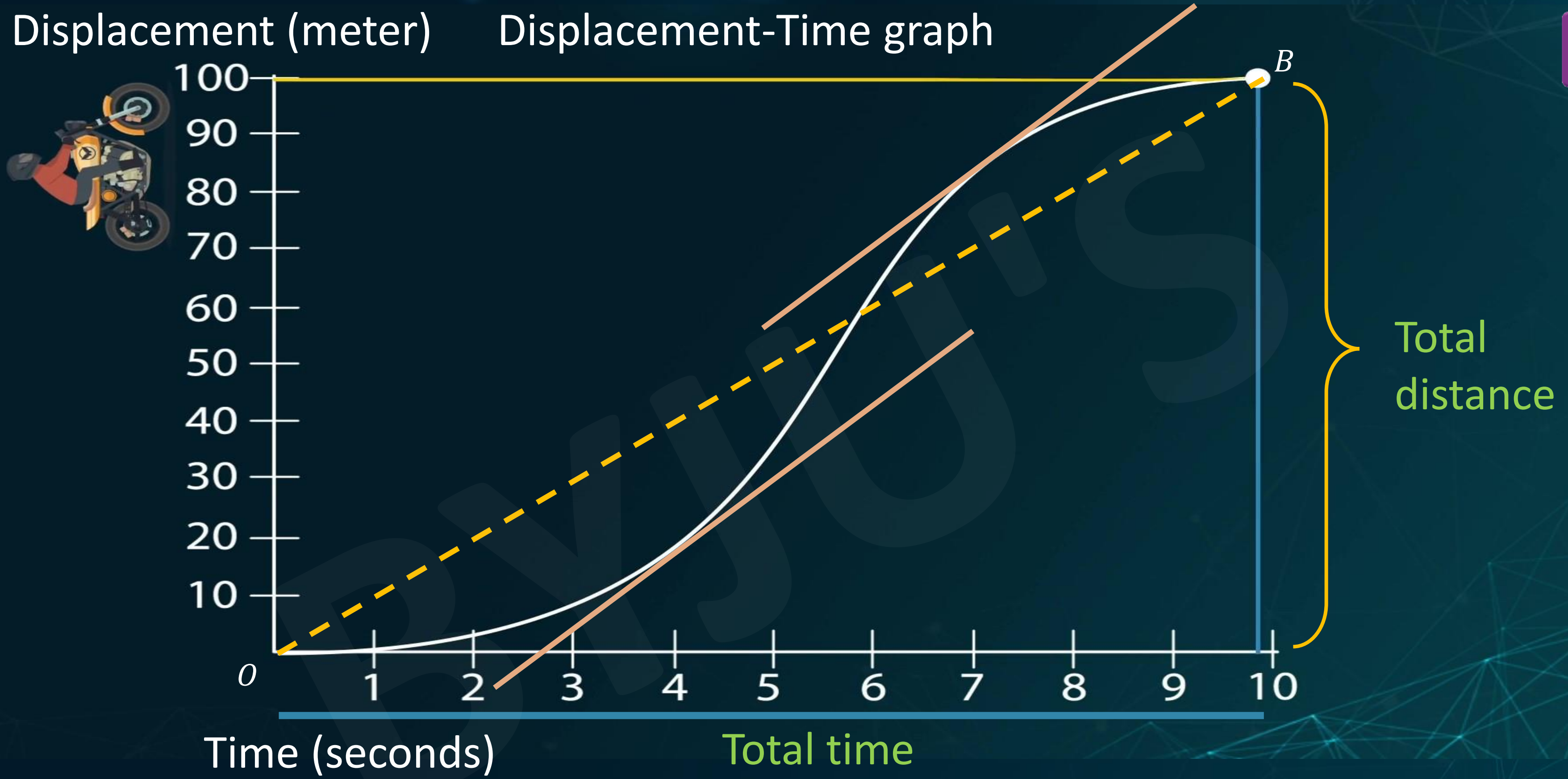
Applications

Cauchy’s Mean Value Theorem

Lagrange’s Mean Value Theorem

Observe the motion of biker for first 10 seconds for which displacement-time graph is shown next.

Time (seconds)



The idea of Lagrange's Mean Value Theorem says that in continuous and differentiable curve slope of the chord joining initial and final points of the curve is equal to the slope of tangent of the curve on at least one points.

Lagrange's Mean Value Theorem (L.M.V.T)

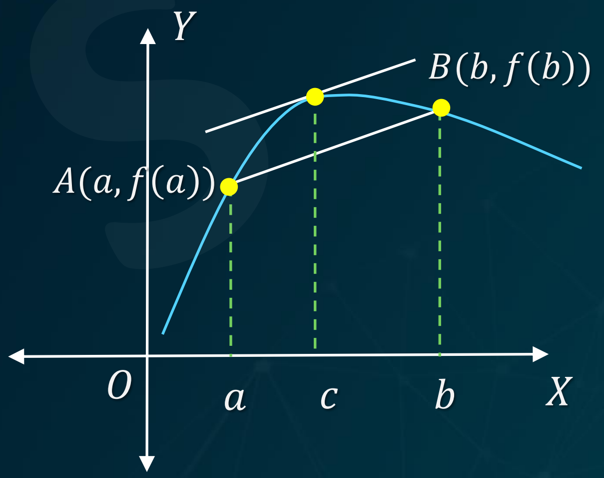
If a function $f(x)$,

(i) Is continuous in the closed interval $[a, b]$

(ii) Is differentiable in the open interval (a, b)

then , there exists at least one $c \in (a, b)$, such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



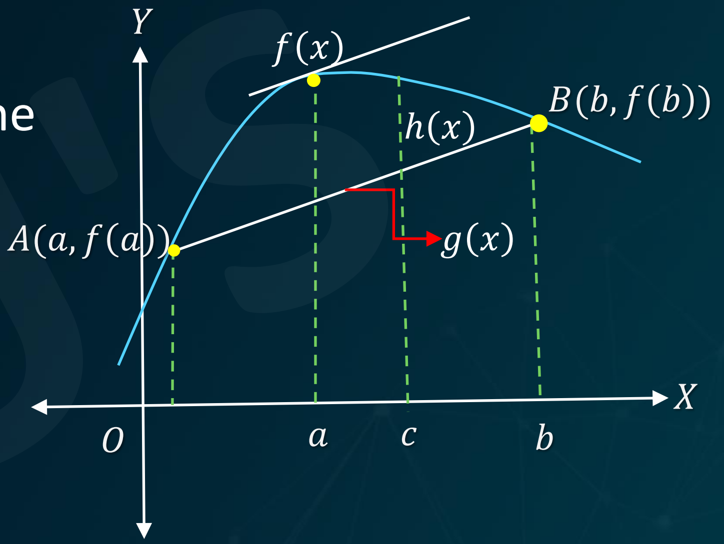
Geometrically, there exists at least one $c \in (a, b)$, where tangent is parallel to line joining points A & B.

Lagrange's Mean Value Theorem (L.M.V.T.)

Proof:

Let $A(a, f(a))$ and $B\big(b, f(b)\big)$ be the points on the taken curve $y = f(x)$

Let $g(x)$ be the secant line to $f(x)$ passing through A, B.



$$\therefore\ g(x) - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\{y - y_1 = m(x - x_1)\}$$

$$\Rightarrow g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Let $h(x) = f(x) - g(x)$

$$\Rightarrow f(x) - h(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

$$\Rightarrow h(x) = f(x) - \left\{\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right\}$$

Now, $h(a) = 0, h(b) = 0$, $\{h(x)$ passing through $(a, f(a)$ and $(b, f(b)\}$
$h(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

$\therefore$ Rolle's theorem is applicable

$$\Rightarrow\ h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$\therefore$ There exists at least one $c \in (a, b)$, such that $h'(c) = 0$

$$\Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\boxed{\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Hence, L.M.V.T. is proved.

Illustration

Verify L.M.V.T for the function $f(x) = -x^2 + 4x + 5 , x \in [-1,1]$

Solution:

Given, $f(x) = -x^2 + 4x + 5$

$$f'(x) = -2x + 4$$

Since $f(x)$ is a polynomial function.

$\Rightarrow f(x)$ is continuous and differentiable in $[-1,1]$ & $(-1,1)$ respectively.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow -2c + 4 = \frac{f(1)-f(-1)}{1-(-1)} = \frac{(-1^2+4(1)+5)-(-(-1^2)+4(-1)+5))}{1-(-1)} = \frac{8-0}{2}$$

$$\Rightarrow -2c + 4 = 4 \Rightarrow c = 0 \in [-1,1]$$

Thus, L.M.V.T. is verified.

Illustration

The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0,1]$ is :

$a.\ \frac{\sqrt{7}-2}{3}$ $b.\ \frac{4-\sqrt{7}}{3}$ $c.\ \frac{4-\sqrt{5}}{3}$ $d.\ \frac{2}{3}$

Solution: $f'(x) = 3x^2 - 8x + 8$

Since $f(x)$ is a polynomial function.

$f(x)$ is continuous and differentiable in $[0,1]$ & $(0,1)$ respectively.

By L.M.V.T $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \frac{f(1)-f(0)}{1-0} = \frac{(1-4+8+11)-(0+11)}{1-0}$

$\Rightarrow 3c^2 - 8c + 8 = 5 \Rightarrow 3c^2 - 8c + 3 = 0$

$\Rightarrow c = \frac{8\pm2\sqrt{7}}{6} = \frac{4\pm\sqrt{7}}{3}$

$$\boxed{c = \frac{4-\sqrt{7}}{3}}$$

Hence, option (b) is the correct answer.

Illustration

IIT JEE - 2003

In [0,1] Lagrange's mean value theorem is not applicable to :

a. $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$

b. $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

c. $f(x) = x|x|$

d. $f(x) = |x|$

Solution:

a. $f\left(\frac{1}{2}^-\right) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{2}^+\right) = 0 \Rightarrow f$ is continuous

$f'\left(\frac{1}{2}^-\right) = -1$ and $f'\left(\frac{1}{2}^+\right) = -2\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

$\therefore$ f is not differentiable at $\frac{1}{2} \in (0,1)$.

$\therefore$ L.M.V.T is not applicable for thus function in [0,1].

$b.$ $$f(x)=\begin{cases}\dfrac{\sin x}{x}, x\neq 0\\ 1, x=0\end{cases}$$

$$f(0)=1 \qquad \lim_{h\to 0^+}\frac{\sin x}{x}=1 \qquad \lim_{h\to 0^-}\frac{\sin x}{x}=1$$

$L.H.L = f(0) = R.H.L \;\Rightarrow f$ is continuous

$$f'(0^+)=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\frac{\sin(h)}{(h)}-f(0)}{h}=\lim_{h\to 0^+}\frac{\frac{\sin(h)}{(h)}-1}{h}=\lim_{h\to 0^+}\frac{h-\frac{h^3}{3!}+\frac{h^5}{5!}+\cdots-h}{h^2}=0$$

$$f'(0^-)=\lim_{h\to 0^+}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0^+}\frac{\frac{\sin(-h)}{(-h)}-f(0)}{-h}=\lim_{h\to 0^+}\frac{\frac{\sin(h)}{(h)}-1}{-h}=\lim_{h\to 0^+}\frac{h-\frac{h^3}{3!}+\frac{h^5}{5!}+\cdots-h}{-h^2}=0$$

$\Rightarrow f$ is differentiable

$\Rightarrow$ L.M.V.T. is applicable

c. $f(x) = x|x|$

$$f(x) = \begin{cases} x^2, x \geq 0 \\ -x^2, x < 0 \end{cases}$$

$f(0) = 0 \quad \lim_{h \to 0^+} x^2 = 0 \quad \lim_{h \to 0^-} - x^2 = 0$

$L.H.L = f(0) = R.H.L \Rightarrow f$ is continuous at $x = 0$

$$f'(x) = \begin{cases} 2x, x \geq 0 \\ -2x, x < 0 \end{cases}$$

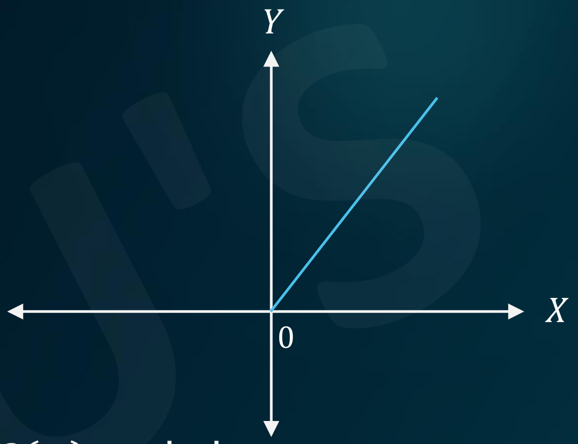
$$f'(0^+) = \lim_{h \to 0^+} 2x = 0$$

$$f'(0^-) = \lim_{h \to 0^+} - 2x = 0$$

$\Rightarrow f(x)$ is differentiable at $x = 0$

$\Rightarrow$ L.M.V.T. is applicable.

d. $f(x) = |x|$



$f(x) = |x|$ is continuous everywhere

We can see from the graph,

$f(x) = x$ is differentiable $\forall\, x \in (0,1)$

$\Rightarrow$ L.M.V.T. is applicable.

Hence, option (a) is the correct answer.

Illustration

Let f be a twice differentiable function on $(1,6)$.
If $f(2) = 8$, $f'(2) = 5$, $f'(x) \geq 1$ and $f''(x) \geq 4, \forall\, x \in (1,6)$, then :

a. $f(5) + f'(5) \geq 28$

b. $f(5) + f'(5) \leq 26$

c. $f'(5) + f''(5) \leq 20$

d. $f(5) \leq 10$

Solution: Given : $f(2) = 8$, $f'(2) = 5$,

$f'(x) \geq 1$ and $f''(x) \geq 4, \forall\, x \in (1,6)$

Given, $f(x)$ is a twice differentiable function on $(1,6)$.

Hence L.M.V.T is applicable for $x \in (1,6)$.

There exists at least one $c \in (a, b)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(x) = \frac{f(5)-f(2)}{5-2} = \frac{f(5)-8}{5-2}$$

$$f'(x) \geq 1$$

$$\Rightarrow \frac{f(5)-8}{5-2} \geq 1$$

$$\Rightarrow f(5) \geq 11 \quad \cdots (i)$$

$$f''(x) = \frac{f'(5)-f'(2)}{5-2} \quad \geq 4$$

$$\Rightarrow f'(5) \geq 17 \cdots (ii)$$

Adding (i) & (ii)

$$\boxed{f(5) + f'(5) \geq 28}$$

Hence, option (a) is the correct answer.

Illustration

Which of the following is true ?

a. $\frac{1}{1+a^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+b^2}$, if $0 < a < b$

b. $\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$, if $0 < a < b$

c. $\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+b^2}$, if $0 < a < b$

d. $\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+a^2}$, if $0 < a < b$

Solution:

Let $f(x) = \tan^{-1} x\,, x \in [a, b]$

$$f'(x) = \frac{1}{1+x^2}$$

$f(x)$ is continuous in $[a, b]$ and differentiable in (a, b).

By L.M.V.T, $f'(c) = \frac{f(b)-f(a)}{b-a}$, $c \in [a, b]$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

$$a < c < b$$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1 + a^2 < 1 + c^2 < 1 + b^2$$

$$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\boxed{\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}}$$

Hence, option (a) is the correct answer.

Cauchy's Mean Value Theorem

Also known as the extended mean value theorem, is a generalization of the mean value theorem.

It states that if the functions $f(x)$ and $g(x)$ are both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists some $c \in (a, b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Illustration

JEE Main - 2014

If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some c ∈ $(0,1)$:

$a.\ f'(c) = g'(c)$

$b.\ f'(c) = 2g'(c)$

$c.\ 2f'(c) = g'(c)$

$d.\ 2f'(c) = 3g'(c)$

Solution:

By Cauchy's theorem $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$; for some $c \in (1,2)$

$$\frac{f'(c)}{g'(c)} = \frac{f(1)-f(0)}{g(1)-g(0)} \Rightarrow \frac{f'(c)}{g'(c)} = \frac{6-2}{2-0} = 2 \quad \boxed{\Rightarrow f'(c) = 2g'(c)}$$

Hence, option (b) is the correct answer.

Illustration

Prove that equation $\cos x = \frac{3x^2}{7}(\sin 2 - \sin 1)$ has at least one root in (1,2).

Solution: $(\sin 2 - \sin 1) \Rightarrow f(2) - f(1)$

This clearly tells $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$g'(c)$ should be variable $\Rightarrow g(x) = x^3 \Rightarrow g'(x) = 3x^2$

Since the functions $f(x)$ and $g(x)$ are both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), there exists some $c \in (a, b)$, such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \text{ ; for some } c \in (1,2)$$

$$\Rightarrow \frac{\cos c}{3c^2} = \frac{f(2)-f(1)}{g(2)-g(1)} = \frac{\sin 2-\sin 1}{2^3-1^3}$$

$$\boxed{\Rightarrow \cos c = \frac{3c^2}{7}(\sin 2 - \sin 1)}$$

Summary Sheet

- **Lagrange's Mean Value Theorem (L.M.V.T)**

If $f(x)$ is continuous and differentiable in $[a, b]$ & (a, b) respectively then , there exists at least one $c \in (a, b)$, such that:

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

- **Cauchy's Mean Value Theorem**

If the functions $f(x)$ and $g(x)$ are both continuous and differentiable in $[a, b]$ & (a, b), then there exists some $c \in (a, b)$, such that:

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

BYJU'S Classes

Application of Derivatives

Monotonicity

Road Map

Applications of Monotonicity

Monotonicity in an Interval

Monotonicity at a Point

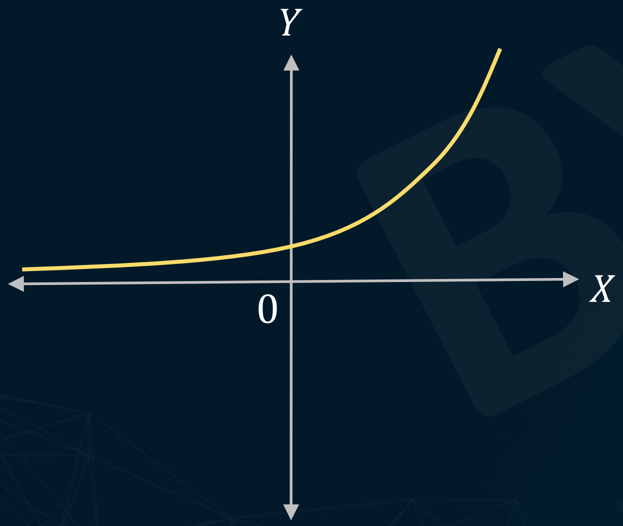
Monotonicity

Monotonocity

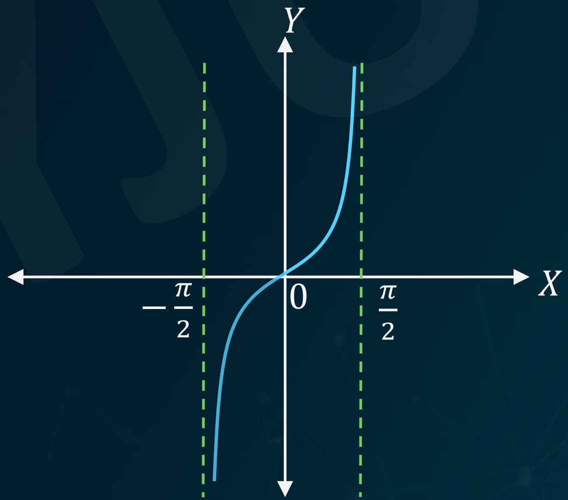
- A function f is said to be monotonic if it is either increasing or decreasing in it's domain.

Let us take a few examples of increasing functions.

$f(x) = e^x$



$f(x) = \tan x, x \in \left(0, \frac{\pi}{2}\right)$

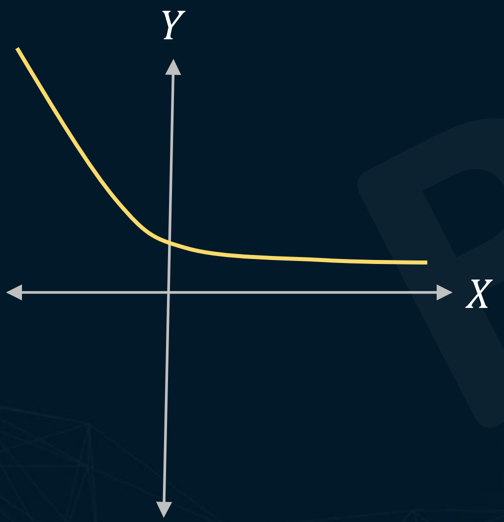


$f(x) = 2x + 3$

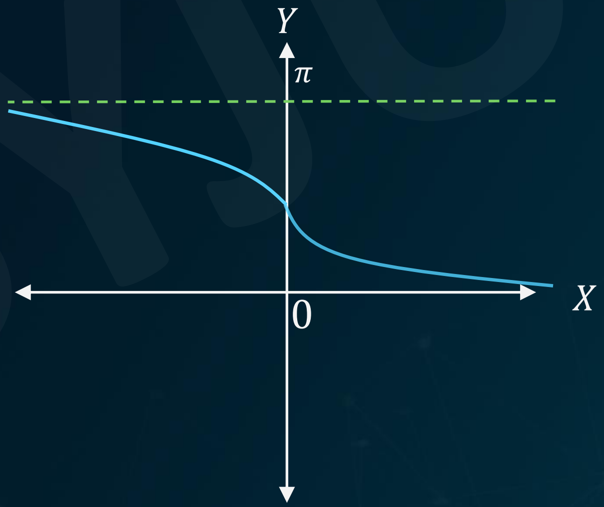
Monotonocity

Now, let us take a few examples of decreasing functions.

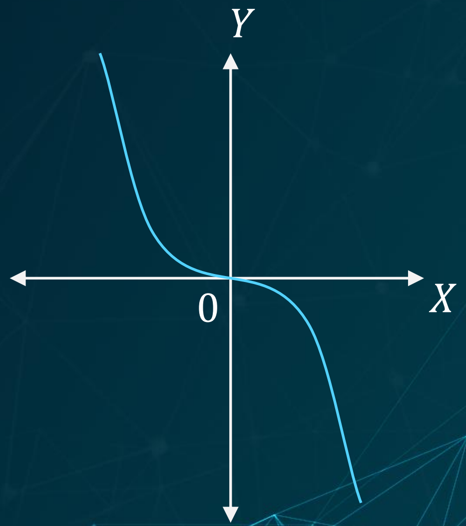
$f(x) = e^{-x}$



$f(x) = cot^{-1}x$



$f(x) = -x^3$

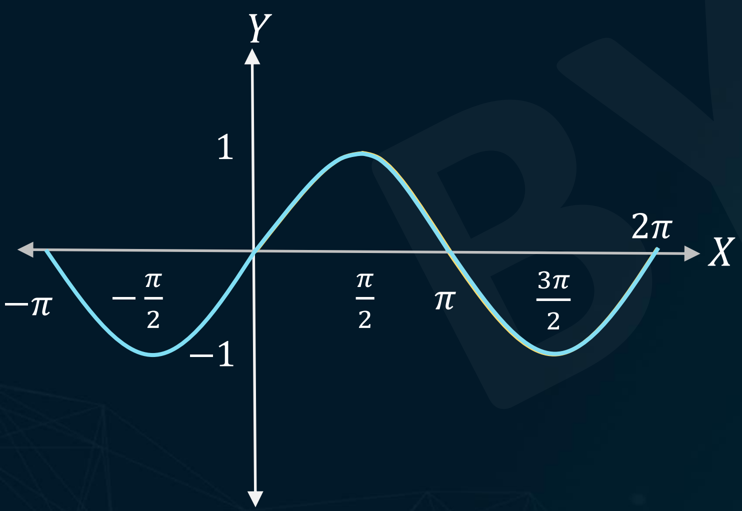


Monotonocity

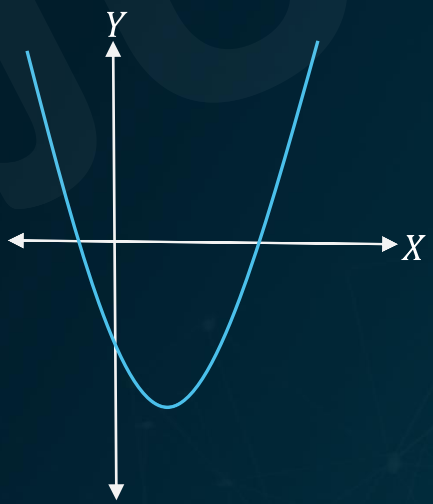
- A function which is increasing as well as decreasing in its domain is called a non-monotonic function.

For example,

$f(x) = \sin x$



$f(x) = ax^2 + bx + c, a \neq 0$

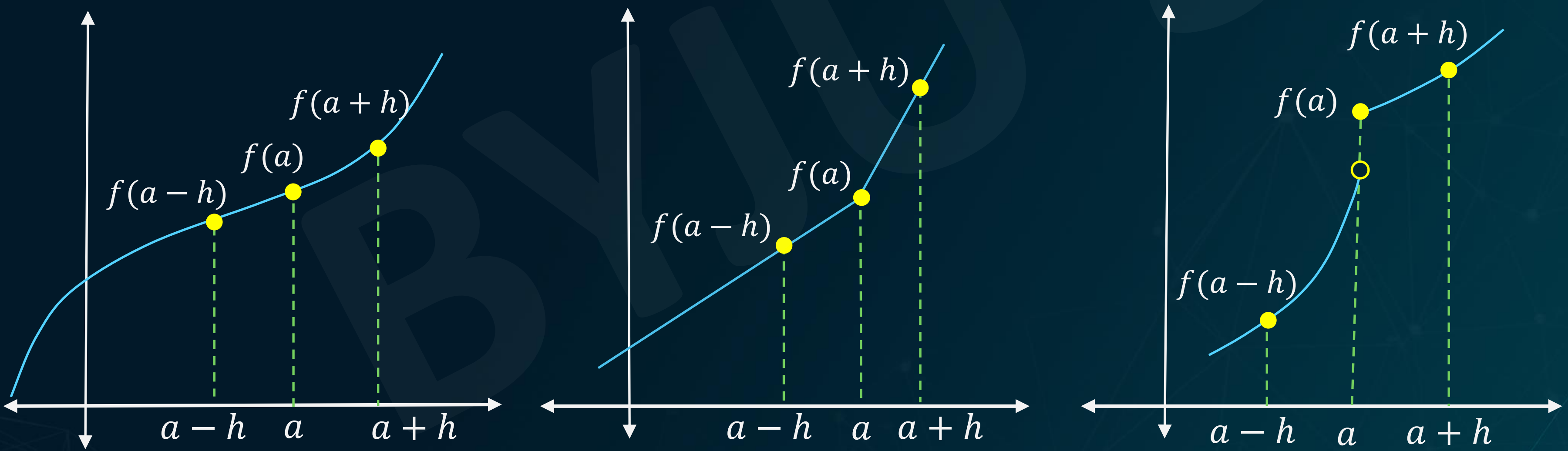


$f(x) = |x|$

Monotonicity at a Point

A function f is said to be strictly increasing at a point $x = a$,

If $a - h < a < a + h$, then $f(a - h) < f(a) < f(a + h)$ as $h \to 0^+$ i.e.,
If, Left neighbourhood value < Value at that point < Right neighbourhood value, then f is said to be strictly increasing at $x = a$.



Function is continuous and differentiable at $x = a$

Function is continuous, non differentiable at $x = a$

Function is discontinuous at $x = a$

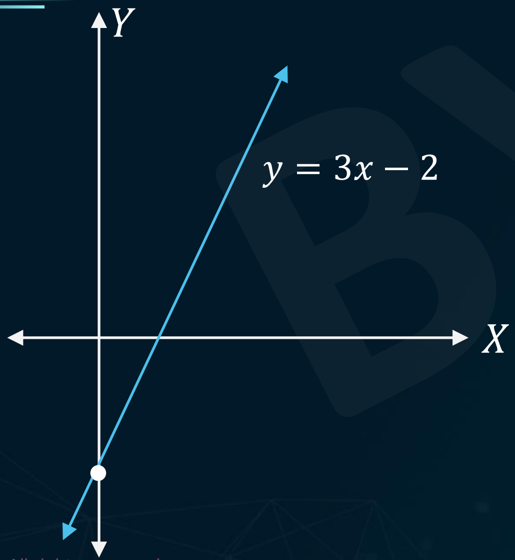
Monotonicity at a Point (using derivatives)

Let f be a differentiable function at point $x = a$

If $f'(a) > 0$, then function is increasing at $x = a$.

Example: Check monotonicity of function $f(x) = 3x - 2$, at $x = 0$.

Solution: As $f(x)$ is a linear polynomial, $f(x)$ is differentiable everywhere



Given, $f(x) = 3x - 2$,

$f'(x) = 3$

$\because f'(x) > 0$

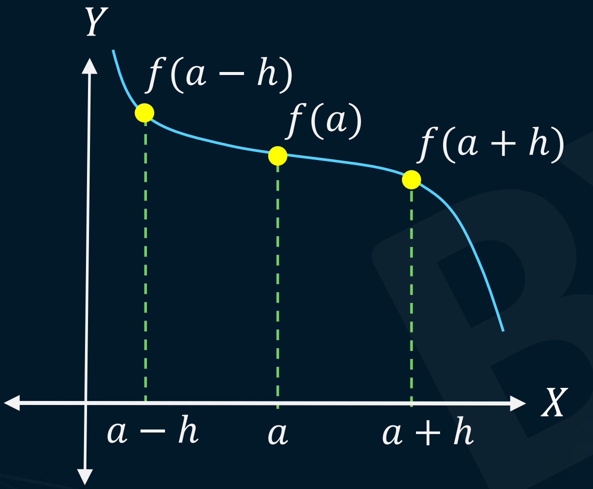
$\therefore f$ is increasing at $x = 0$

Monotonicity at a Point

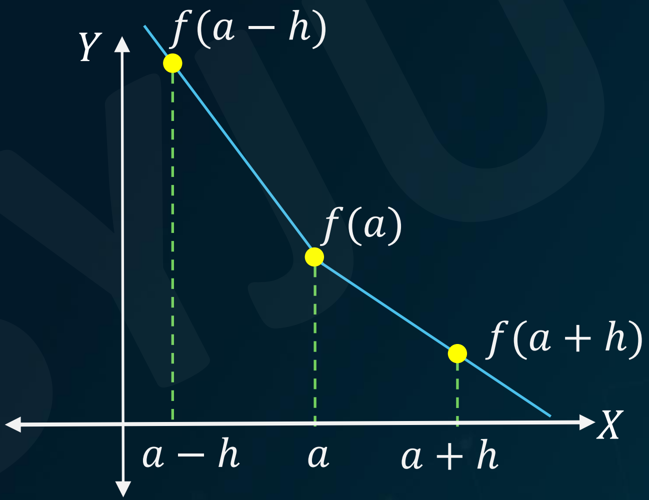
A function f is said to be strictly decreasing at a point $x = a$,

If $a - h < a < a + h$, then $f(a-h) > f(a) > f(a+h)$ as $h \to 0^+$ i.e.,

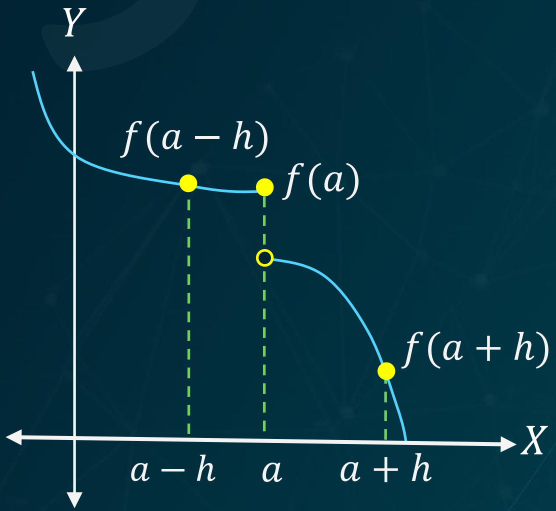
If, Left neighbourhood value > Value at that point > Right neighbourhood value, then f is said to be strictly decreasing at $x = a$.



Function is continuous and differentiable at $x = a$



Function is continuous, non differentiable at $x = a$



Function is discontinuous at $x = a$

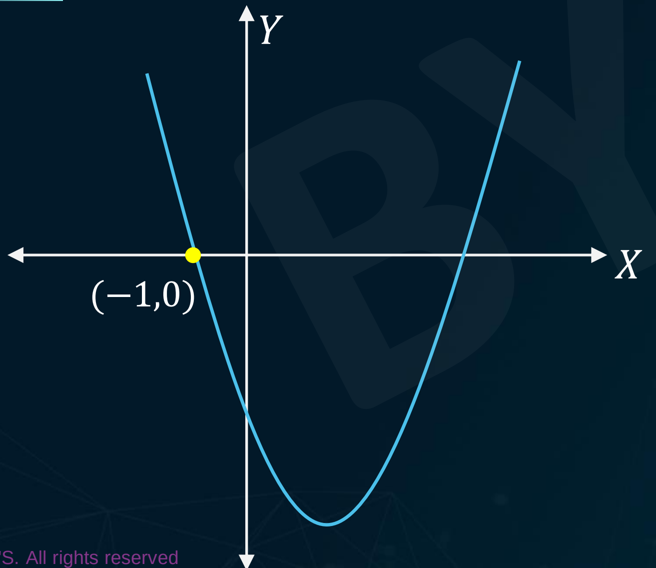
Monotonicity at a Point (using derivatives)

Let f be a differentiable function at point $x = a$

If $f'(a) < 0$, then function is decreasing at $x = a$.

Example: Check monotonicity of function $f(x) = x^2 - 2x - 3$, at $x = -1$.

Solution: As $f(x)$ is a quadratic polynomial, $f(x)$ is differentiable everywhere.



Given, $f(x) = x^2 - 2x - 3$,

$f'(x) = 2x - 2$

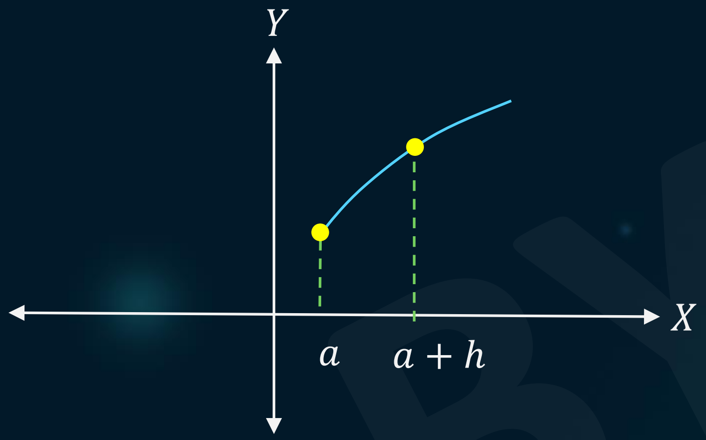
$f'(-1) = -2 - 2 = -4$

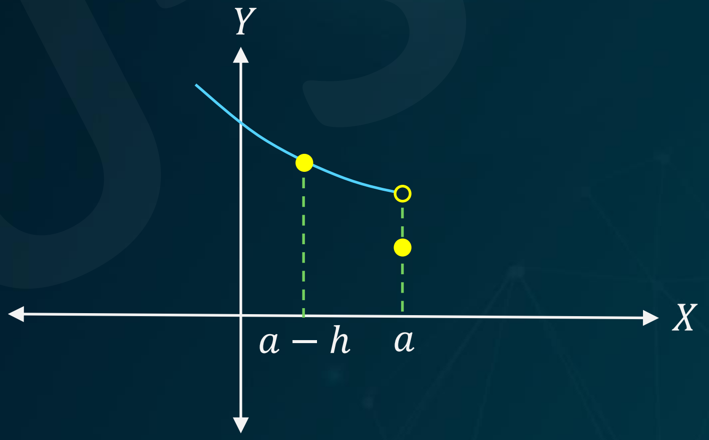
$\because f'(-1) < 0$

$\therefore f$ is decreasing at $x = -1$

Monotonicity at Boundary points

Note: If $x = a$ is a boundary point, appropriate one – sided inequality is applied to check monotonicity.



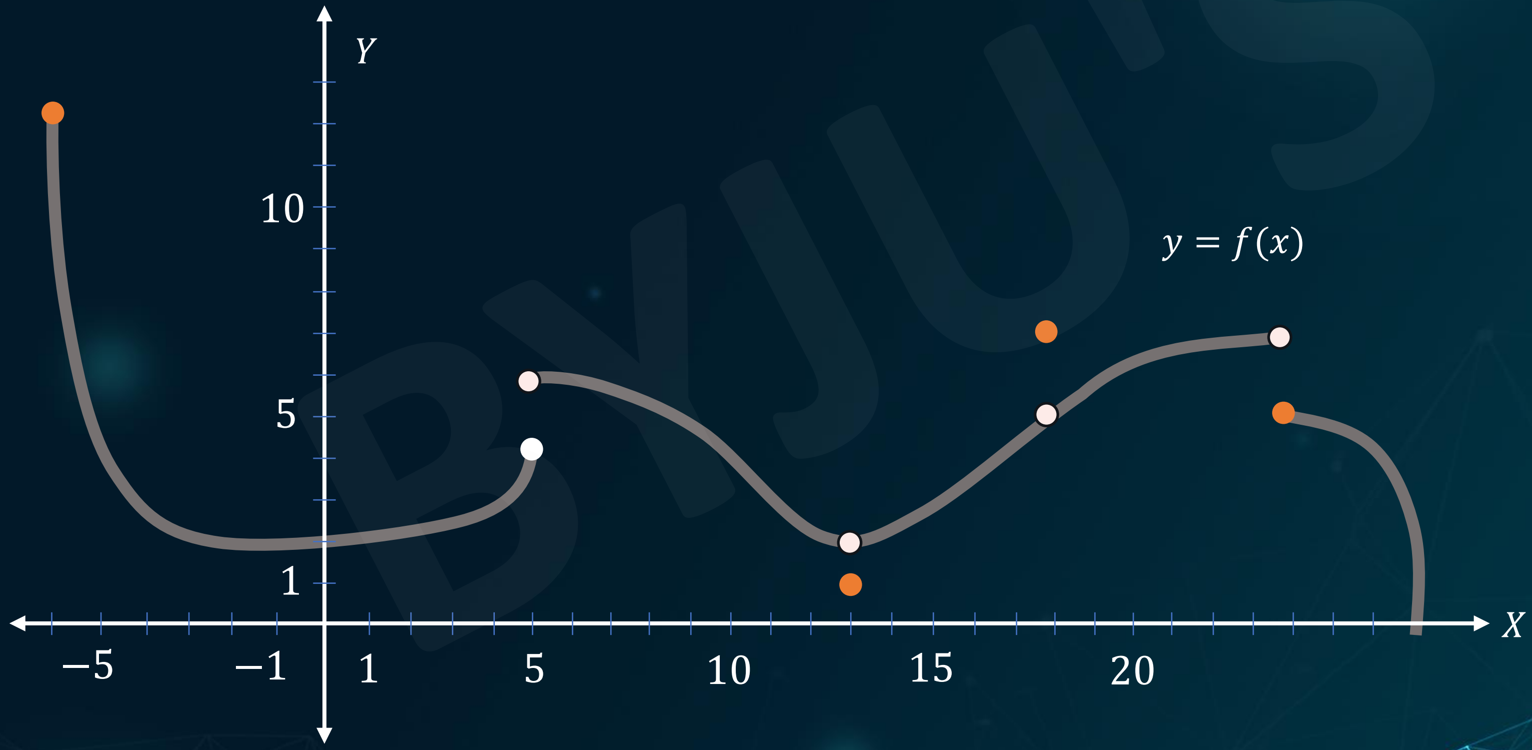


At the left boundary i.e., $x = a$, if $a < a + h$ then, if $f(a) < f(a+h)$ as $h \to 0^+$, we can say that function is increasing at $x = a$.

At the right boundary i.e. $x = a$, if $a - h < a$ then, if $f(a-h) > f(a)$ as $h \to 0^+$, we can say that function is decreasing at $x = a$.

Illustration

A person goes for trekking and the path taken by him is represented in the form of a graph as shown below. Identify the monotonicity of $f(x)$ at $x = -6, 5, 13, 18, 24$



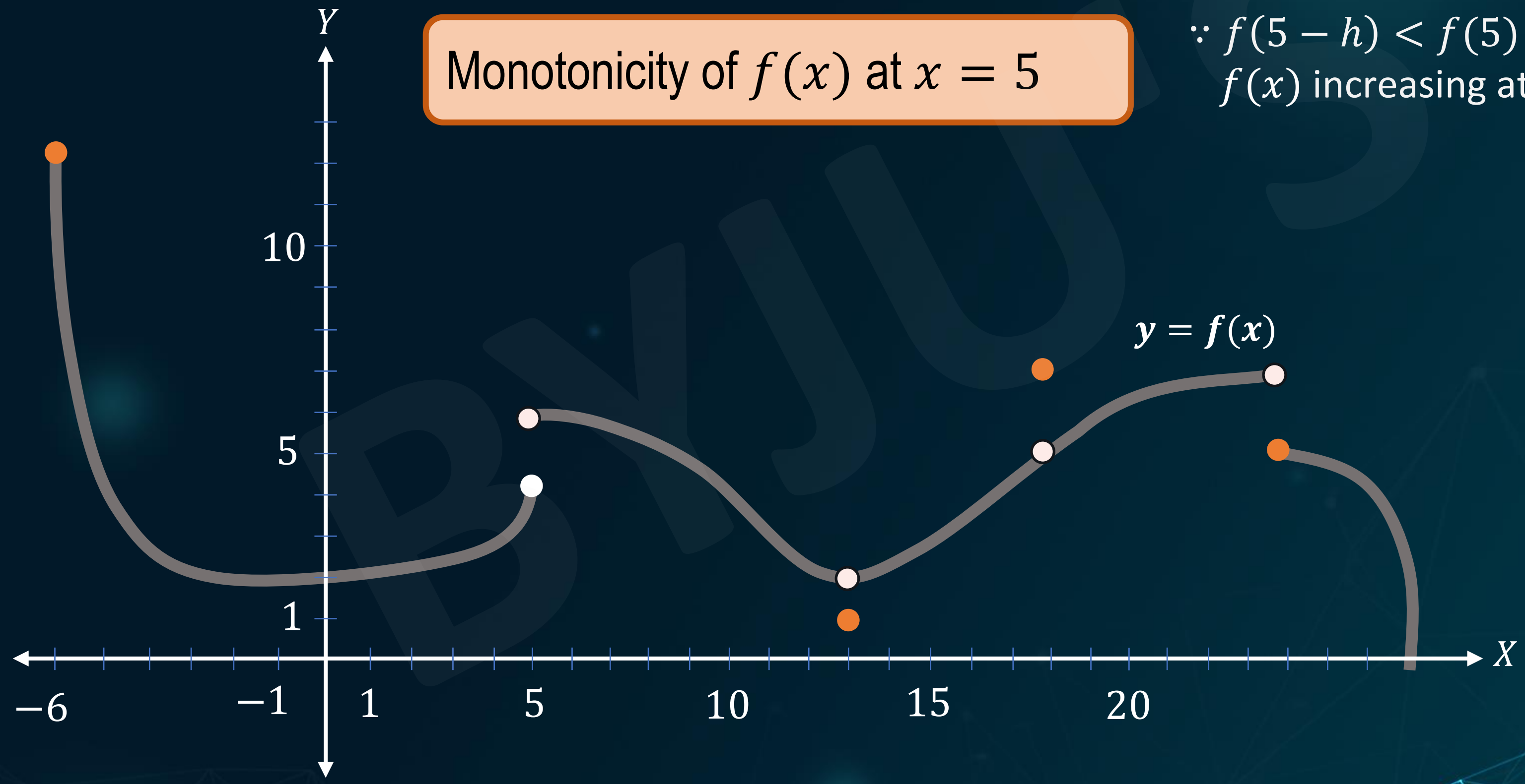
Solution:

Monotonicity of $f(x)$ at $x = -6$

$\because f(-6) > f(-6+h)$,
$f(x)$ decreasing at $x = -6$

Monotonicity of $f(x)$ at $x = 5$

$\because f(5-h) < f(5) < f(5+h)$,
$f(x)$ increasing at $x = 5$

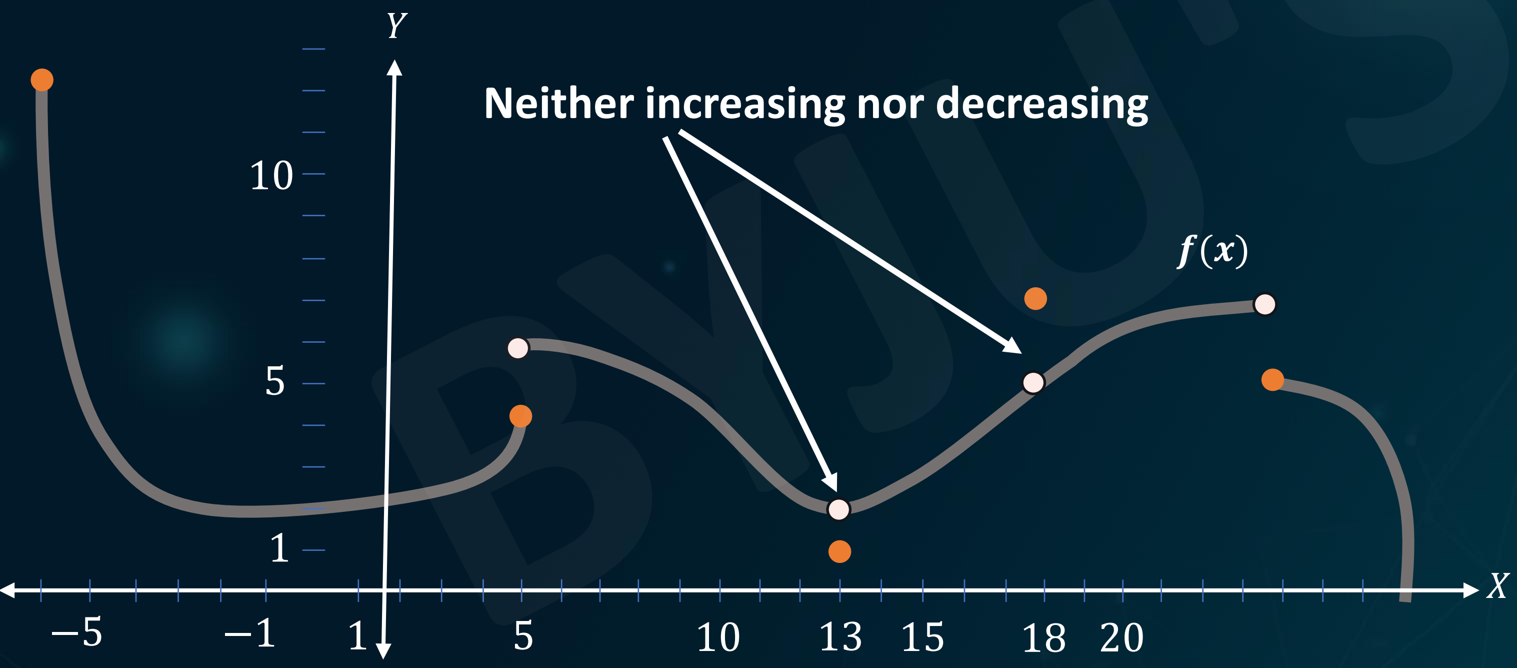


Monotonicity of $f(x)$ at $x = 13$

$\because f(13-h) > f(13) < f(13+h)$,
$f(x)$ is neither increasing nor decreasing at $x = 13$

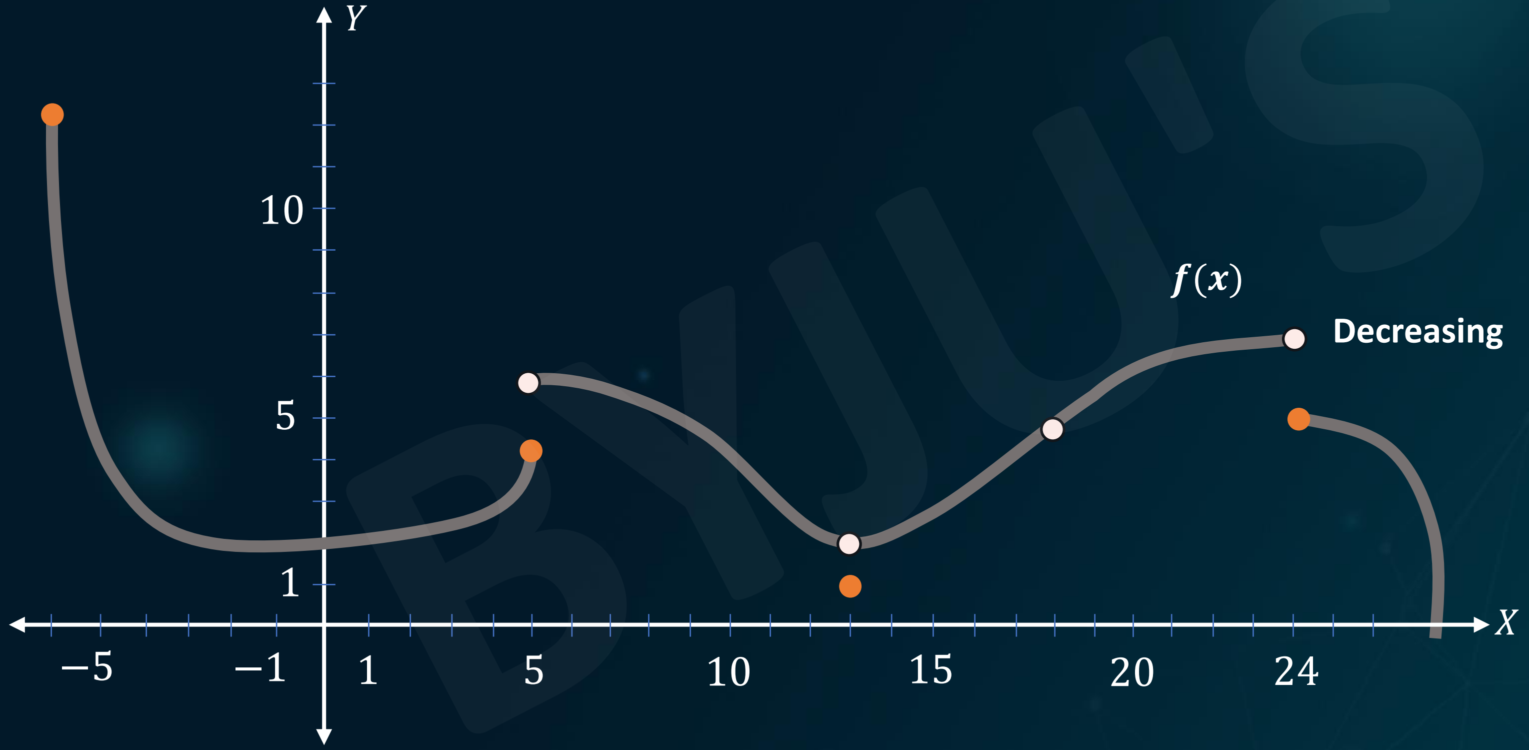
Monotonicity of $f(x)$ at $x = 18$

$\because f(18-h) < f(18) > f(18+h)$,
$f(x)$ is neither increasing nor decreasing at $x = 18$



Monotonicity of $f(x)$ at $x = 24$

$\because f(24 - h) > f(24) > f(24 + h)$,
$f(x)$ is decreasing at $x = 24$



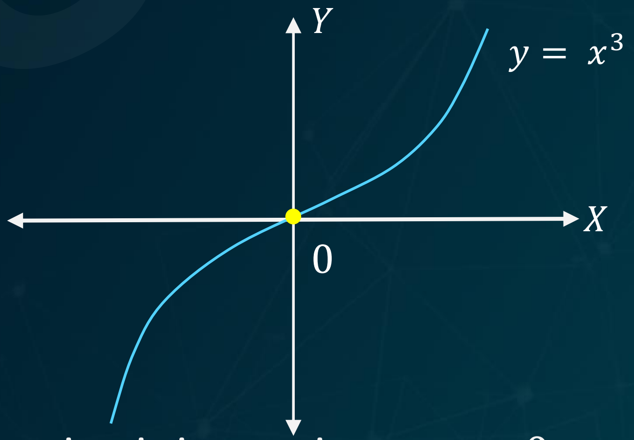
Monotonicity at a Point

- If $f'(a) = 0$, then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a,

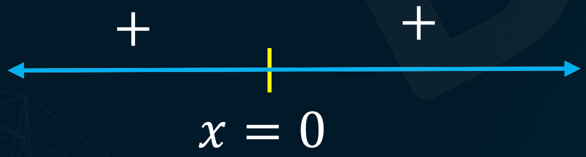
$i)$ If $f'(a^+)$ and $f'(a^-)$ are both positive, then function is increasing at $x = a$.

Example: Check the monotonicity of $f(x) = x^3$ at $x = 0$

$$f'(x) = 3x^2$$



Sign scheme for $f'(x)$ at $x = 0$



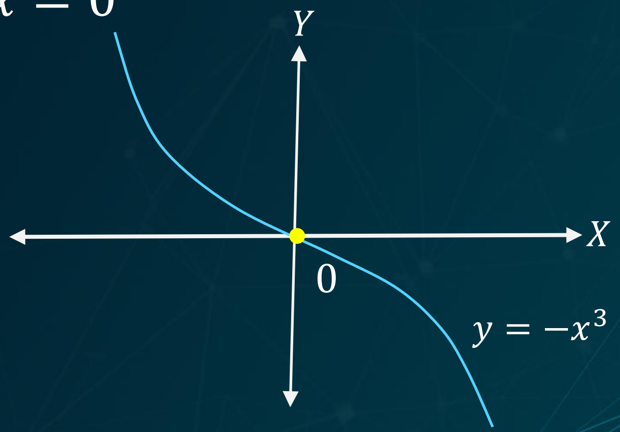
Since $f'(0^+)$ and $f'(0^-)$ are both positive, hence function is increasing at $x = 0$.

Monotonicity at a Point

- If $f'(a) = 0$, then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a,

$ii)$ If $f'(a^+)$ and $f'(a^-)$ are both negative, then function is decreasing at $x = a$.

Example: Check the monotonicity of $f(x) = -x^3$ at $x = 0$



$$f'(x) = -3x^2$$

Sign scheme for $f'(x)$ at $x = 0$

$$\begin{array}{ccc} - & & - \\ \hline & x = 0 & \end{array}$$

$\because f'(0^+)$ and $f'(0^-)$ are both negative, hence function is decreasing at $x = 0$.

Monotonicity at a Point

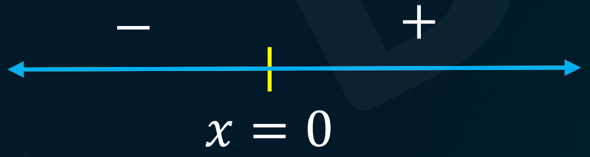
- If $f'(a) = 0$, then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a,

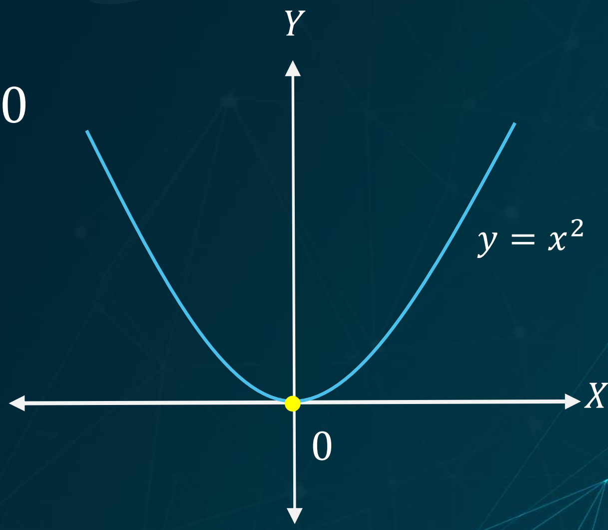
$iii)$ If $f'(a^+)$ and $f'(a^-)$ are of opposite sign, then function is neither increasing nor decreasing at $x = a$.

Example: Check the monotonicity $f(x) = x^2$ at $x = 0$

$$f'(x) = 2x$$

Sign scheme of $f'(x)$ at $x = 0$





$f'(0^+)$ and $f'(0^-)$ are of opposite sign, then function is neither increasing nor decreasing at $x = 0$.

Monotonicity at a Point

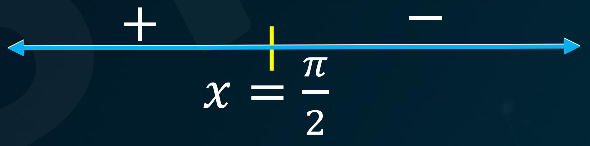
- If $f'(a) = 0$, then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a,

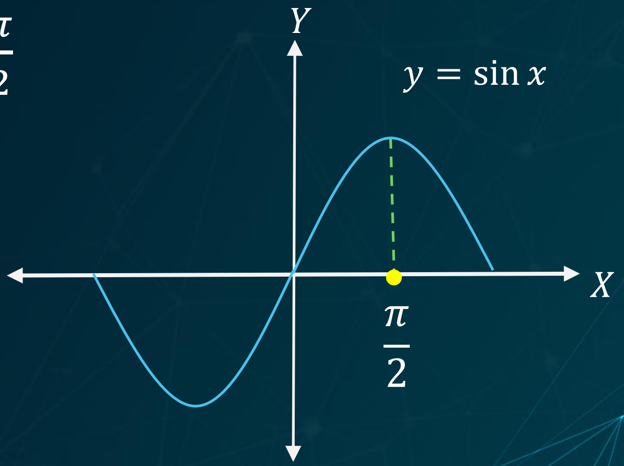
$iii)$ If $f'(a^+)$ and $f'(a^-)$ are of opposite sign, then function is neither increasing nor decreasing at $x = a$.

Example: Check the monotonicity of $y = \sin x$ at $x = \frac{\pi}{2}$

$$y' = \cos x$$

Sign scheme of y'





$f'\left(\frac{\pi^+}{2}\right)$ and $f'\left(\frac{\pi^-}{2}\right)$ are of opposite sign, hence function is neither increasing nor decreasing at $x = \frac{\pi}{2}$.

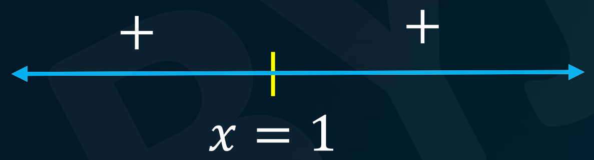
Illustration

Check monotonicity of the function : $f(x) = (x-1)^3$, at $x = 1$

Solution: i) Given, $f(x) = (x-1)^3$

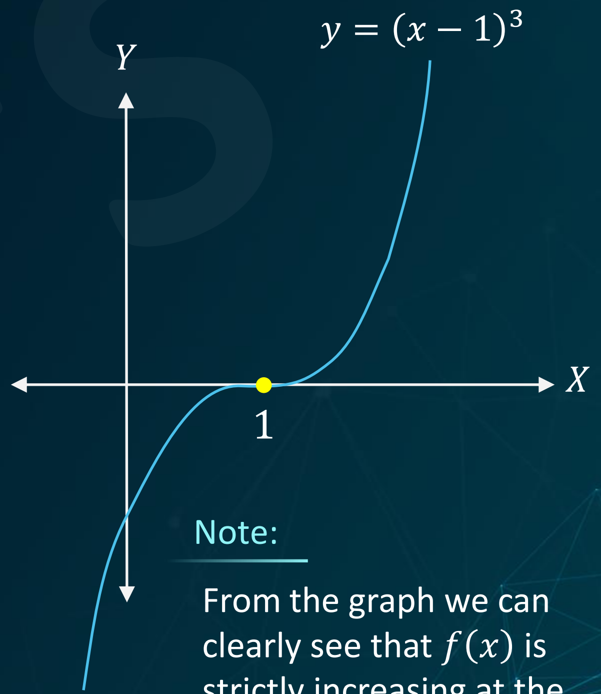
$f'(x) = 3(x-1)^2$

Now at $x = -1$, $f'(1) = 0$



$\because$ $f'(1^+)$ and $f'(1^-)$ are both positive,

Thus, function is increasing at $x = 1$.



Note:

From the graph we can clearly see that $f(x)$ is strictly increasing at the point $x = 1$

Illustration

Check the monotonicity of $f(x) = -\ln x + \tan^{-1} x$, about $x = e$:

Solution:

Given: $f(x) = -\ln x + \tan^{-1} x$

$$f'(x) = -\frac{1}{x} + \frac{1}{1+x^2}$$

$$f'(e) = -\frac{1}{e} + \frac{1}{1+e^2}$$

$$f'(e) = \frac{-1-e^2+e}{e(1+e^2)}$$

$\because$ $(-1 - e^2 + e)$ is negative & $e(1 + e^2)$ is positive.

Hence, $f'(e) < 0$

$\Rightarrow f(x)$ is decreasing at $x = e$.

Illustration

Find the complete set of values of α for which the function :

$$f(x) = \begin{cases} x-2\,; x<1 \\ \alpha\,; x=1 \\ x^2+1\,; x>1 \end{cases}$$ is strictly increasing at $x=1$

a. $\alpha \in (-\infty, 2]$

b. $\alpha \in [-1,2]$

c. $\alpha \in [-1, \infty)$

d. $\alpha \in (-1,2)$

Solution:

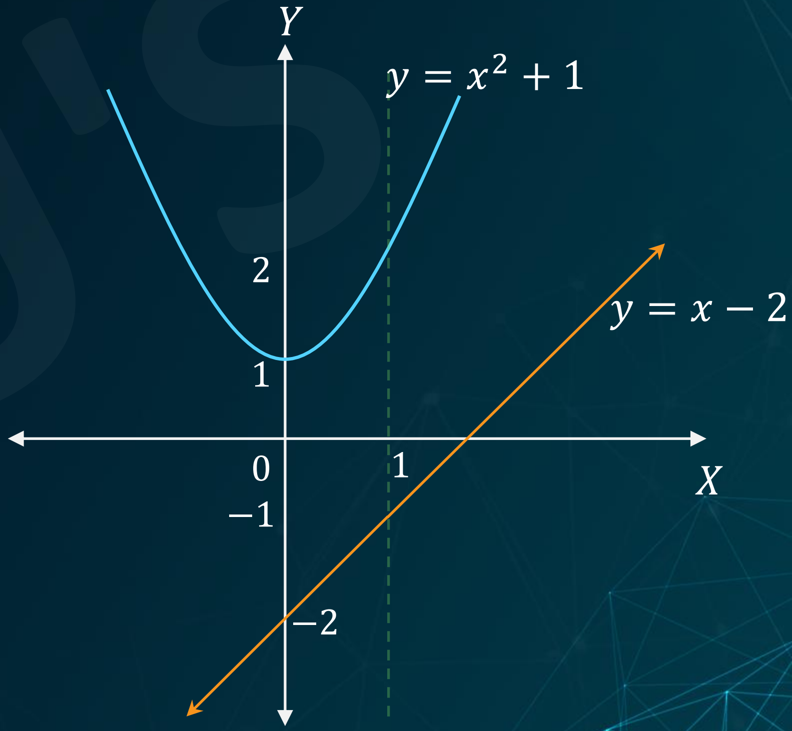
Step 1:

Draw the graphs of $y = x - 2$ and $y = x^2 + 1$

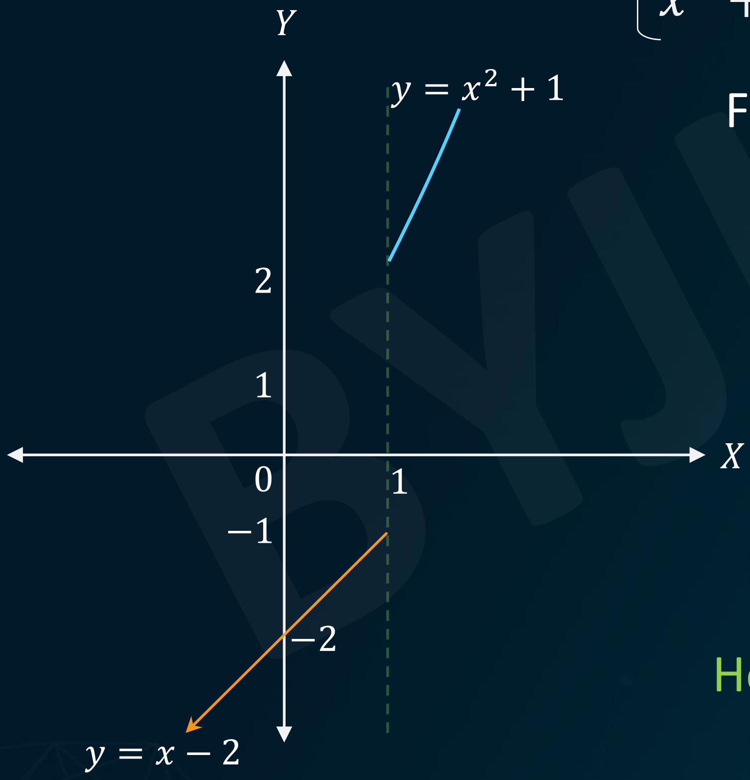
Step 2:

For, obtaining $f(x) = \begin{cases} x - 2\,; x < 1 \\ \alpha\,; x = 1 \\ x^2 + 1\,; x > 1 \end{cases}$

remove the undesirable portion from the graph.



Hence, the graph for $f(x) = \begin{cases} x-2 \,; x<1 \\ \alpha \,; x=1 \\ x^2+1 \,; x>1 \end{cases}$ is drawn below.



For $f(x)$ to be increasing at $x = 1$,

$$f(1^-) < f(1) < f(1^+)$$

$$f(1^-) < \alpha < f(1^+)$$

$$-1 < \alpha < 2$$

$$\boxed{\alpha \in [-1,2]}$$

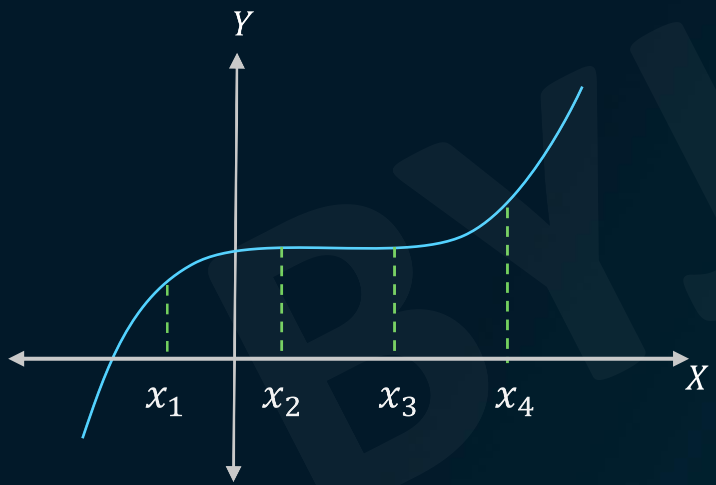
Hence, option (b) is the correct answer.

Monotonicity in an Interval

Increasing functions :

A function f is said to be increasing/non – decreasing in a set S of its domain

if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$



From the graph, we can see that $x_1 < x_2$

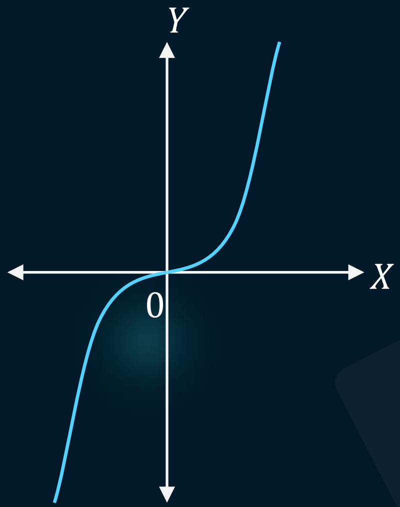
$$f(x_1) < f(x_2)$$

From the graph, we can see that $x_2 < x_3$

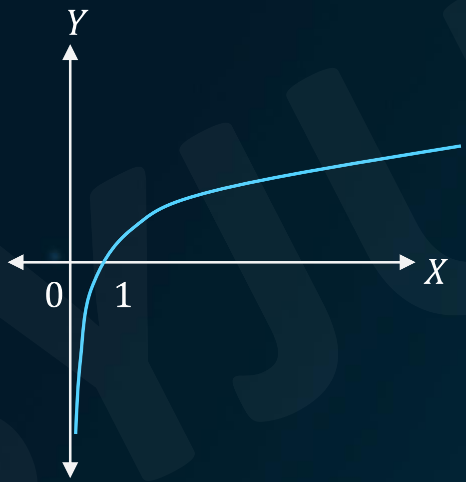
$$f(x_2) = f(x_3)$$

Increasing functions : $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

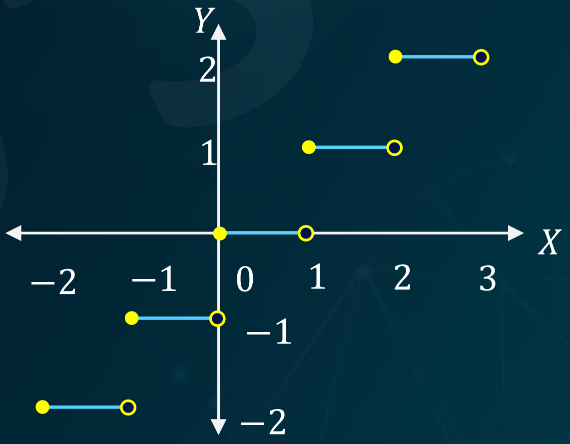
Example:



$y = x^3$



$y = \ln x$

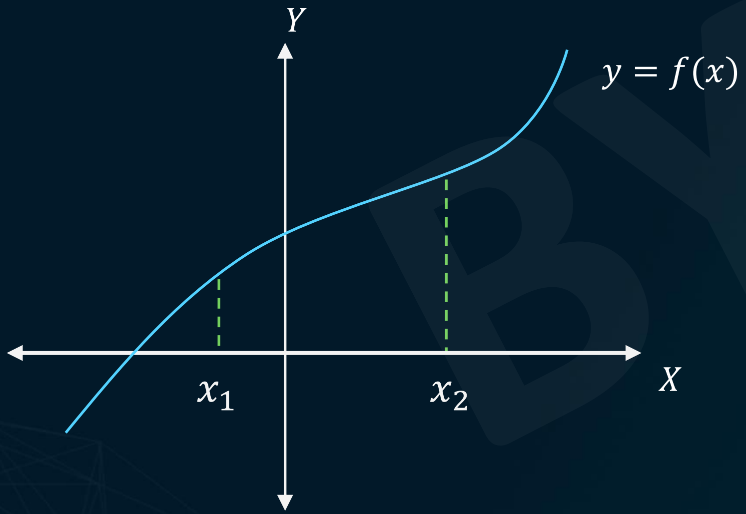


$y = [x]$

Monotonicity in an Interval

Strictly increasing functions:

A function f is said to be strictly (monotonically) increasing in a set S of its domain, if $\forall\, x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

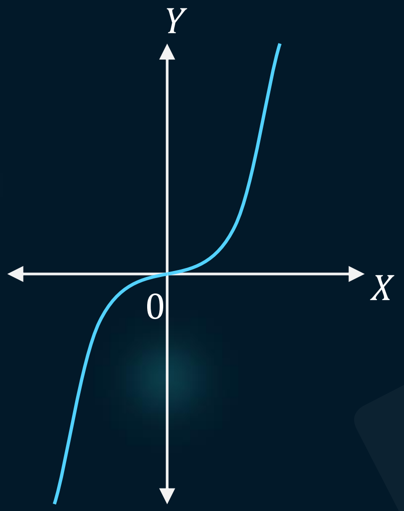


From the graph, we can see that $x_1 < x_2$

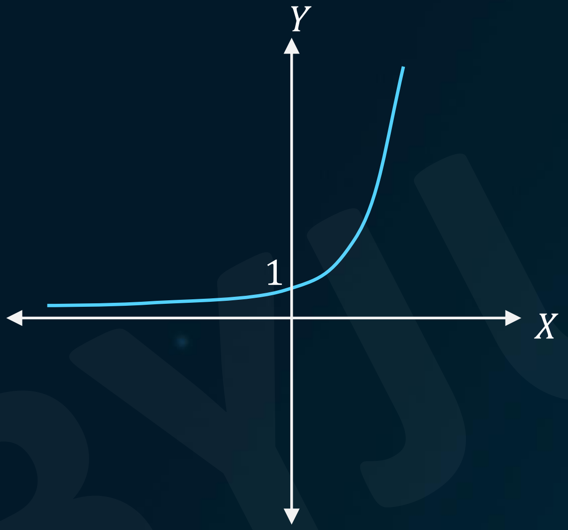
$$f(x_1) < f(x_2)$$

Strictly increasing: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

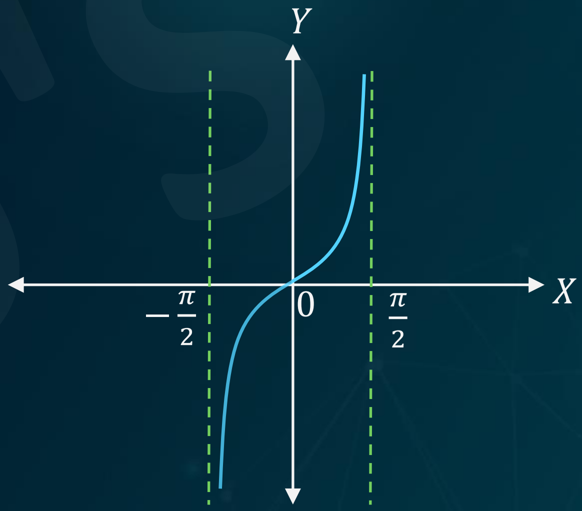
Example:



$y = x^3$



$y = e^x$



$f(x) = \tan x \; ; \; x \in \left(0, \frac{\pi}{2}\right)$

Monotonicity in an Interval

Decreasing function:

A function f is said to be decreasing/non-increasing in a set S of its domain,

if $\forall\ x_1, x_2 \in S$,

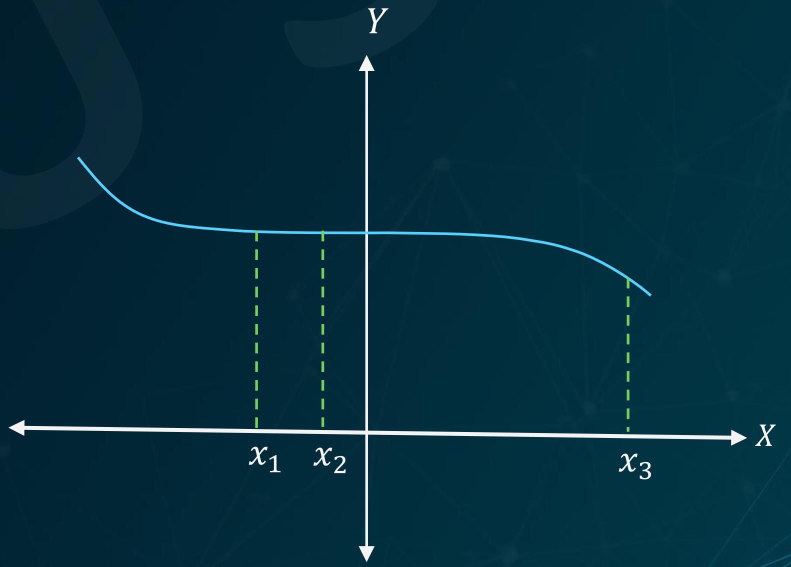
$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

From the graph, we can see that $x_1 < \ x_3$

$$f(x_1) > f(x_3)$$

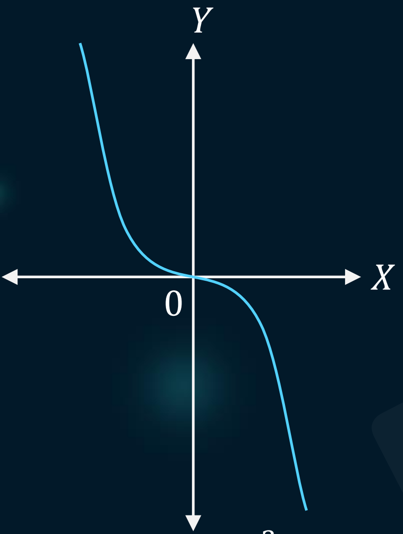
From the graph, we can see that $x_1 < \ x_2$

$$f(x_1) = f(x_2)$$

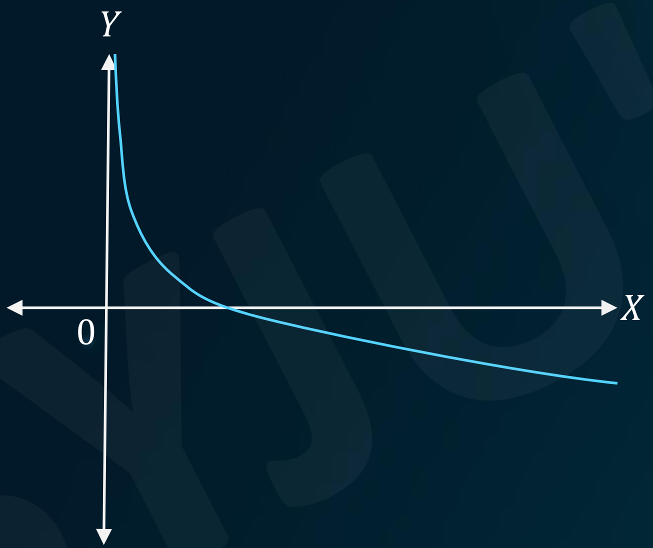


Decreasing functions: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

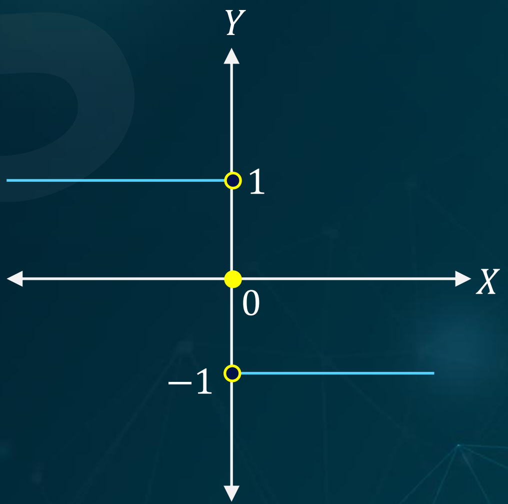
Example:



$y = -x^3$



$y = log_{0.1}\, x$

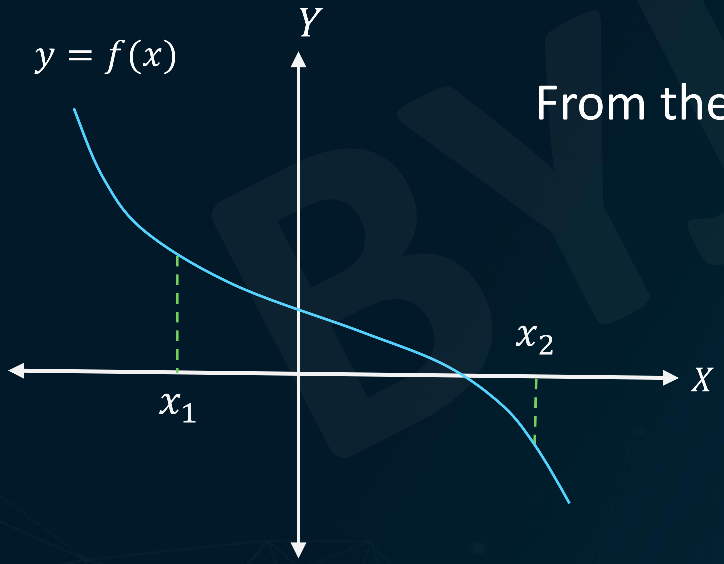


$y = \text{sgn}(-x)$

Monotonicity in an Interval

Strictly decreasing functions :

A function f is said to be strictly (monotonically) decreasing in a set S of its domain , if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

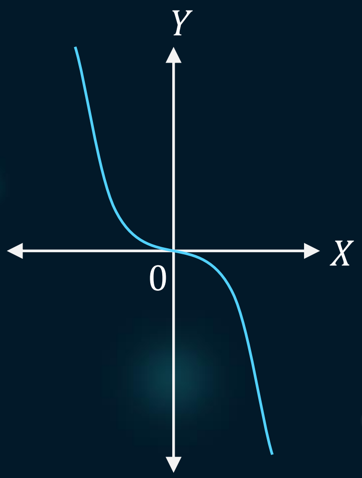


From the graph, we can see that $x_1 < x_2$

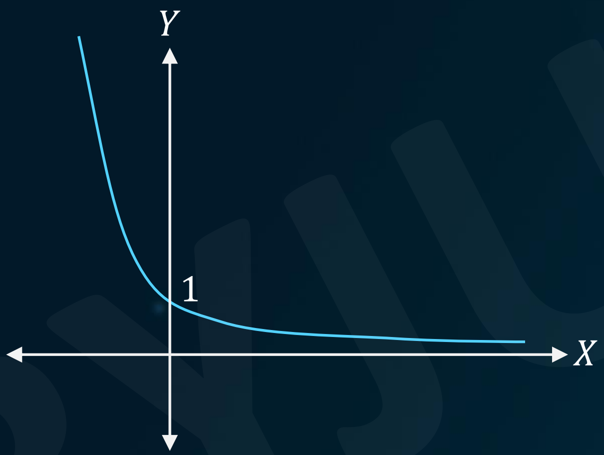
$$\Rightarrow f(x_1) > f(x_2)$$

Strictly decreasing functions : $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

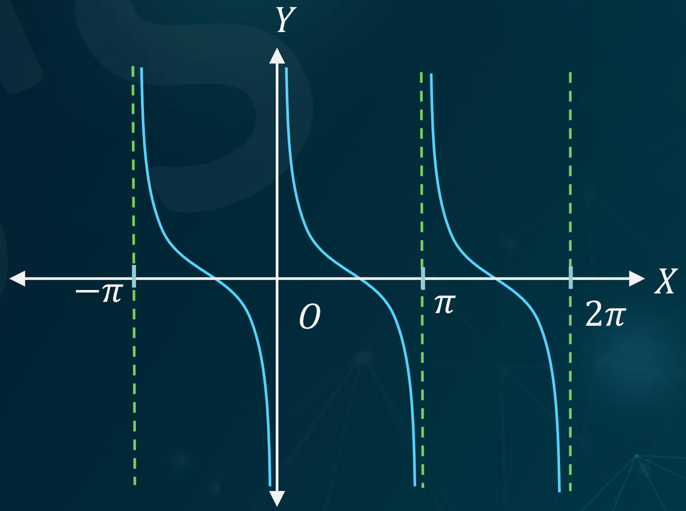
Example:



$y = -x^3$



$y = e^{-x}$



$y = \cot x$

Illustration

JEE Main – 2020 (Jan)

Let f be any function on $[a, b]$ and twice differentiable on (a, b). If for all $x \in (a, b), f'(x) > 0, f''(x) < 0$, then for any $c \in (a, b), \frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than :

a. $\frac{c-a}{b-c}$

b. $\frac{b-c}{c-a}$

c. $\frac{b+c}{b-a}$

d. 1

Solution:

Step 1:

Applying L.M.V.T in the interval (a, c), we obtain,

$$f'(\alpha) = \frac{f(c) - f(a)}{c - a}, a < \alpha < c \cdots (i)$$

Step 2:

Now, applying L.M.V.T in the interval (c, b) , we obtain

$$f'(\beta) = \frac{f(b) - f(c)}{b - c}, c < \beta < b \cdots (ii)$$

Also $f''(x) < 0 \quad \Rightarrow f'(x)$ is decreasing

Hence, $f'(\alpha) > f'(\beta)$

$$\Rightarrow \frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(c)}{b - c}$$, Using (i) & (ii)

$$\boxed{\Rightarrow \frac{f(c) - f(a)}{f(b) - f(c)} > \frac{c - a}{b - c}}$$

Hence, option (a) is the correct answer.

Summary Sheet

- A function f is said to be monotonic if it is either increasing of decreasing in its entire domain.
- A function which is increasing as well as decreasing in its domain is called non-monotonic.
- A function f is said to be strictly increasing at a point $x = a$, if $a - h < a < a + h$, and $f(a-h) < f(a) < f(a+h)$, as $h \to 0^+$.
- If $f'(a) > 0$, then function is increasing at $x = a$.
- A function f is said to be strictly decreasing at a point $x = a$, if $a - h < a < a + h$, and $f(a-h) > f(a) > f(a+h)$, as $h \to 0^+$.
- If $f'(a) < 0$, then function is decreasing at $x = a$.
- If $x = a$ is a boundary point, appropriate one – sided inequality is applied to check monotonicity.

Summary Sheet

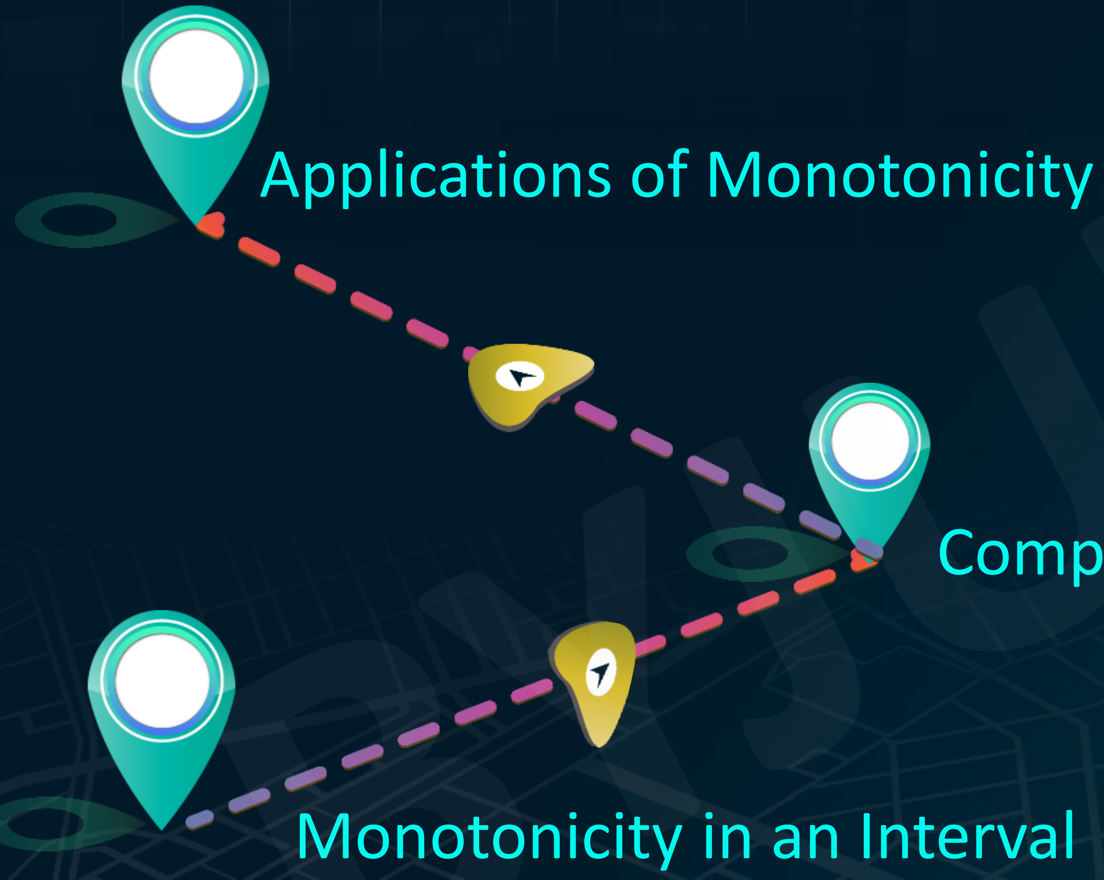
- If $f'(a^+)$ and $f'(a^-)$ are both positive, then function is increasing at $x = a$.
- If $f'(a^+)$ and $f'(a^-)$ are both negative, then function is decreasing at $x = a$.
- If $f'(a^+)$ and $f'(a^-)$ are of opposite signs, then function is neither increasing nor decreasing at $x = a$.
- A function f is said to be increasing/non – decreasing in a set S of its domain if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$
- A function f is said to be strictly (monotonically) increasing in a set S of its domain, if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- A function f is said to be decreasing/non – increasing in a set S of its domain if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$
- A function f is said to be strictly (monotonically) decreasing in a set S of its domain, if $\forall\ x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

BYJU'S Classes

Application of Derivatives

Monotonicity of Differentiable Functions

Road Map


Applications of Monotonicity
Composite Function Monotonicity
Monotonicity in an Interval

Monotonicity for Differentiable Functions

If f is continuous and differentiable function in its entire domain, then

- $f'(x) > 0, \forall\, x \in D_f$

  $\Rightarrow f$ is strictly (monotonically) increasing.

- $f'(x) < 0, \forall\, x \in D_f$

  $\Rightarrow f$ is strictly (monotonically) decreasing.

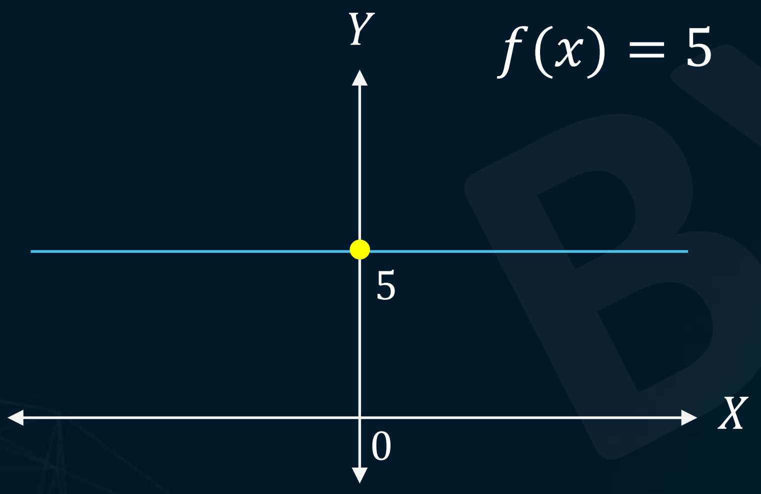
Monotonicity for Differentiable Functions

If f is continuous and differentiable function in its entire domain, then

$$f'(x) \geq 0$$

f is increasing

(If equality occurs in an interval)



$f'(x) = 0, \forall\, x \in \mathbb{R}$

f is strictly increasing

(If equality occurs at discrete point(s))

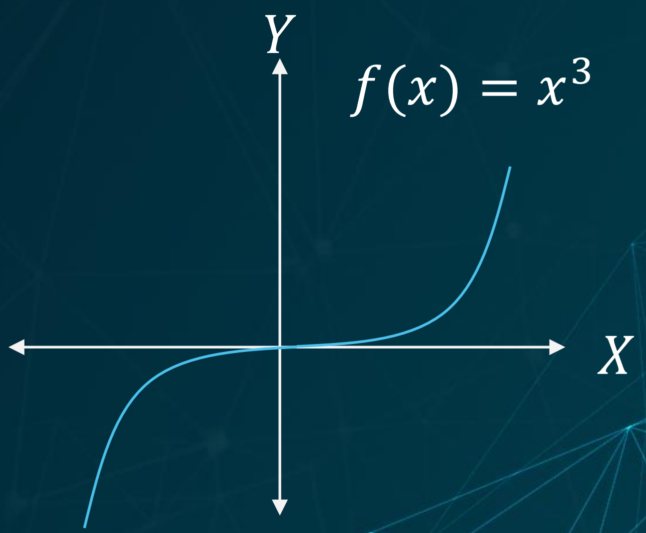
$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow 3x^2 = 0$

$\Rightarrow x = 0$

$f'(x) = 0$, for only $x = 0$

$f(x)$ is strictly increasing



Monotonicity for Differentiable Functions

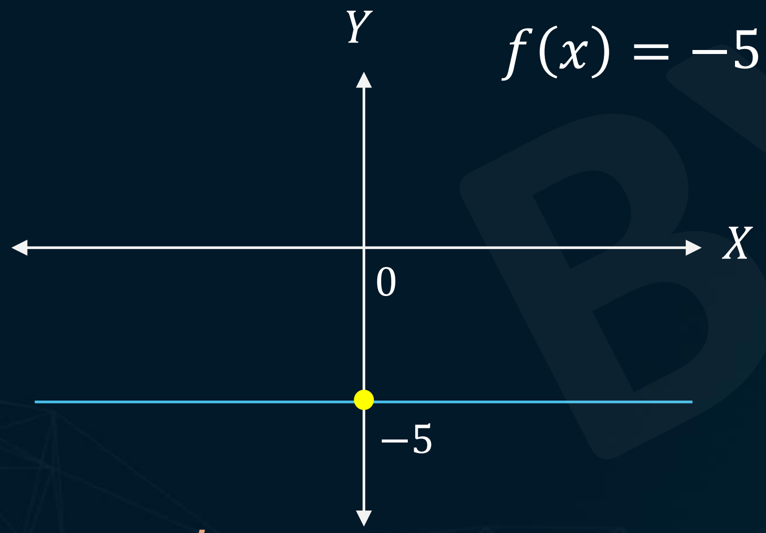
If f is continuous and differentiable function in its entire domain, then

$$f'(x) \leq 0$$

f is decreasing

(If equality occurs in an interval)

$f(x) = -5$



$f'(x) = 0, \forall\, x \in \mathbb{R}$

f is strictly decreasing

(If equality occurs at discrete point(s))

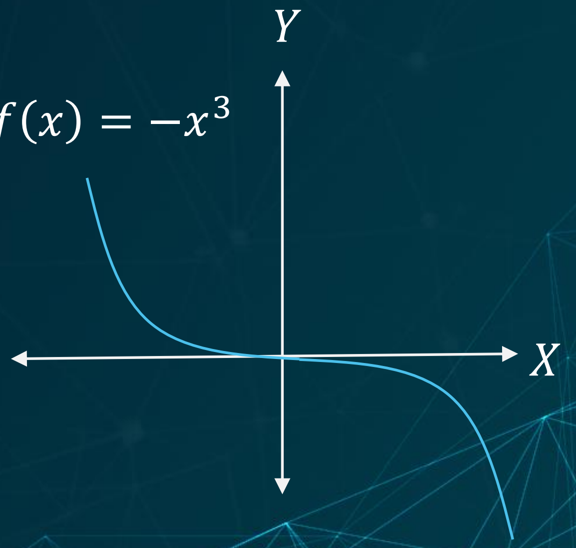
$f(x) = -x^3 \Rightarrow f'(x) = -3x^2$

$f'(x) = 0 \Rightarrow -3x^2 = 0$

$\Rightarrow x = 0$

$f'(x) = 0$, for only $x = 0$

$f(x)$ is strictly decreasing



Illustration

AIEEE - 2001

If function $f(x) = xe^{x(1-x)}$, then $f(x)$ is :

$a.$ increasing on $\left[-\frac{1}{2}, 1\right]$

$b.$ decreasing on $\mathbb{R}$

$c.$ increasing on $\mathbb{R}$

$d.$ decreasing on $\left[-\frac{1}{2}, 1\right]$

Solution:

$$f(x) = xe^{x(1-x)}$$

$$f'(x) = x(1-2x)e^{x(1-x)} + e^{x(1-x)}$$

For $f(x)$ to be increasing , $f'(x) \geq 0$

$$\Rightarrow (x(1-2x)+1)e^{x(1-x)} \geq 0$$

$$\Rightarrow (x(1-2x)+1)e^{x(1-x)} \geq 0$$

$$\Rightarrow (x-2x^2+1)e^{x(1-x)} \geq 0$$

$\{$As $e^{x(1-x)} > 0 \ \forall x \in \mathbb{R}\}$

$$\Rightarrow 2x^2 - x - 1 \leq 0$$

$$\Rightarrow (2x+1)(x-1) \leq 0$$

$$\boxed{\Rightarrow x \in \left[-\frac{1}{2}, 1\right]}$$

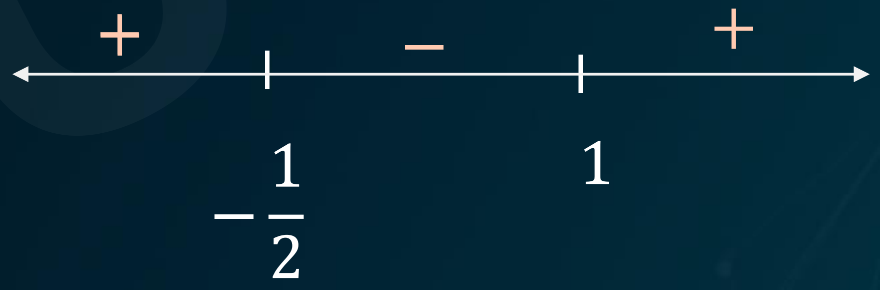
For $f(x)$ to be decreasing , $f'(x) \leq 0$

$$\Rightarrow (x(1-2x)+1)e^{x(1-x)} \leq 0$$

$$\Rightarrow (x-2x^2+1)e^{x(1-x)} \leq 0$$

$\Rightarrow 2x^2 - x - 1 \geq 0$ $\quad$ $\{$As $e^{x(1-x)} > 0 \ \forall x \in \mathbb{R}\}$

$$\Rightarrow (2x+1)(x-1) \geq 0$$



$$\Rightarrow \boxed{x \in \left(-\infty, -\frac{1}{2}\right] \cup [1, \infty)}$$

We can see that $-\frac{1}{2}$ & 1 are included in both the increasing and decreasing intervals

Hence, option (a) is the correct answer.

Illustration

JEE Main – 2020 (Jan)

Let $f(x) = x\cos^{-1}(-\sin|x|), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true ?

$a.$ $f'(0) = -\frac{\pi}{2}$

$b.$ f is not differentiable at $x = 0$

$c.$ f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

$d.$ f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

Solution:

$$f(x) = x\cos^{-1}(-\sin|x|), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \begin{cases} x\left(\frac{\pi}{2} - \sin^{-1}(\sin x)\right), -\frac{\pi}{2} \le x \le 0 \\ x\left(\frac{\pi}{2} - \sin^{-1}(-\sin x)\right), 0 < x \le \frac{\pi}{2} \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x\left(\frac{\pi}{2} - x\right), \ -\frac{\pi}{2} \le x \le 0 \\ x\left(\frac{\pi}{2} + x\right), 0 < x \le \frac{\pi}{2} \end{cases}$$

$$\boxed{\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}}$$

$$\boxed{\sin^{-1}(\sin x) = x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

$$\Rightarrow f'(x) = \begin{cases} \frac{\pi}{2} - 2x, -\frac{\pi}{2} \le x \le 0 \\ \frac{\pi}{2} + 2x, \quad 0 < x \le \frac{\pi}{2} \end{cases}$$

$$f'(0) = \frac{\pi}{2}$$

$$\Rightarrow f''(x) = \begin{cases} -2, & -\frac{\pi}{2} \le x \le 0 \\ 2, & 0 < x \le \frac{\pi}{2} \end{cases}$$

$f''(x) < 0$ in $\left(-\frac{\pi}{2}, 0\right) \Rightarrow f'(x)$ is decreasing in $\left(-\frac{\pi}{2}, 0\right)$

$f''(x) > 0$ in $\left(0, \frac{\pi}{2}\right) \Rightarrow f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$

$\therefore f'(x)$ is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

Hence, option (c) is the correct answer.

Illustration

Find the set of values of a and b, for which the function

$f(x) = \sin^2 x + \sin 2x + ax + b$, is monotonically increasing ?

Solution:

$$f(x) = \sin^2 x + \sin 2x + ax + b$$

$$f'(x) = 2\sin x \cdot \cos x + 2\cos 2x + a$$

$$f'(x) = \sin 2x + 2\cos 2x + a$$

We learnt that $f(x)$ is monotonically increasing for $f'(x) \geq 0$, if equality occurs at discrete point(s)

$$\Rightarrow \sin 2x + 2\cos 2x + a \geq 0$$

We know,

$$-\sqrt{A^2 + B^2} \leq A\cos x + B\sin x \leq \sqrt{A^2 + B^2}$$

$$-\sqrt{1^2 + 2^2} \leq \sin 2x + 2\cos 2x \leq \sqrt{1^2 + 2^2}$$

$$a - \sqrt{5} \leq \sin 2x + 2\cos 2x + a \leq a + \sqrt{5}$$

$$\because \sin 2x + 2\cos 2x + a \geq 0$$

$$\Rightarrow a - \sqrt{5} \geq 0$$

$$\boxed{\therefore a \geq \sqrt{5}, b \in \mathbb{R}}$$

Composite Function Monotonicity

Case 1: When both the functions are monotonically increasing.

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$	$f(f(x))$	$g(g(x))$
M.I.	M.I.	M.I.	M.I.	M.I.	M.I.

For $f(g(x))$

If $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ as $g(x)$ is monotonically increasing function

$\Rightarrow f(g(x_1)) < f(g(x_2))$ as $f(x)$ is monotonically increasing function

$\Rightarrow f(g(x))$ is monotonically increasing function

Similarly, $f(f(x))$ and $g(g(x))$ are also monotonically increasing function

Example: $f(x) = x + 2, g(x) = \ln x$

As $f(x)$ and $g(x)$ are monotonically increasing function

$\Rightarrow f(g(x)) = \ln x + 2$ is monotonically increasing function

Composite Function Monotonicity

Case 2: When both the functions are monotonically decreasing function

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$	$f(f(x))$	$g(g(x))$
M.D.	M.D.	M.I.	M.I.	M.I.	M.I.

If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ as $f(x)$ is monotonically decreasing function

$\Rightarrow\ g(f(x_1)) < g(f(x_2))$ as $g(x)$ is monotonically decreasing function

$\Rightarrow g(f(x))$ is monotonically increasing function

Similarly, $f(g(x))$, $f(f(x))$ and $g(g(x))$ are also monotonically increasing function

Example: $f(x) = -x, g(x) = e^{-x}$

As $f(x)$ and $g(x)$ are monotonically decreasing function

$\Rightarrow g(f(x)) = e^{-(-x)} = e^{x}$ is monotonically increasing function

Composite Function Monotonicity

Case 3: When one is monotonically increasing and other is monotonically decreasing function

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$	$f(f(x))$	$g(g(x))$
M.I.	M.D.	M.D.	M.D.	M.I.	M.I.

If $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ as $g(x)$ is monotonically decreasing function

$\Rightarrow f(g(x_1)) > f(g(x_2))$ as $f(x)$ is monotonically increasing function

$\Rightarrow f(g(x))$ is monotonically decreasing function

Similarly $g(f(x))$ is also monotonically decreasing function

Example: $f(x) = \ln x, g(x) = -x$

As $f(x)$ and $g(x)$ are monotonically increasing and decreasing function respectively

$\Rightarrow g(f(x)) = -\ln x$ is monotonically decreasing function

Composite Function Monotonicity

Case 3: When one is monotonically increasing and other is monotonically decreasing function

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$	$f(f(x))$	$g(g(x))$
M.I.	M.D.	M.D.	M.D.	M.I.	M.I.

If $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ as $g(x)$ is monotonically decreasing function

$\Rightarrow g(g(x_1)) < g(g(x_2))$ as $g(x)$ is monotonically decreasing function

$\Rightarrow g\ (g(x))$ is monotonically increasing function

$\Rightarrow$ Similarly $f\ (f(x))$ is also monotonically increasing function

Illustration

JEE Main – 2019 (April)

Let $f(x) = e^x - x$ and $g(x) = x^2 - x$. Then the set of all $x \in \mathbb{R}$, where the function $h(x) = (fog)(x)$ is increasing is :

a. $[0, \infty)$

b. $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

c. $\left[-\frac{1}{2}, 0\right] \cup [1, \infty)$

d. $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

Solution:

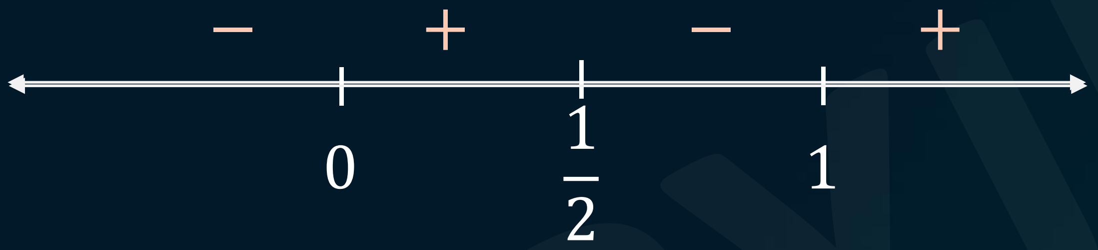
$f(x) = e^x - x$ and $g(x) = x^2 - x$

$f'(x) = e^x - 1 \qquad g'(x) = 2x - 1$

$$h(x) = (fog)(x) = f(g(x)) \Rightarrow h'(x) = f'\big(g(x)\big).g'(x)$$

$$\Rightarrow h'(x) = f'\big(g(x)\big).g'(x) \geq 0$$ {As $h(x) = (fog)(x)$ is increasing function}

$$\Rightarrow \underbrace{\left(e^{x^2-x} - 1\right)}_{x^2 - x = 0}.(2x-1) \geq 0$$



$$\therefore \boxed{x \in \left[0, \frac{1}{2}\right] \cup [1, \infty)}$$

Hence, option (d) is the correct answer.

Summary Sheet

If f is continuous and differentiable function in its entire domain, then

- $f'(x) > 0, \forall\, x \in D_f \;\Rightarrow f$ is strictly (monotonically) increasing
- $f'(x) < 0, \forall\, x \in D_f \;\Rightarrow f$ is strictly (monotonically) decreasing

Composite Function Monotonicity:

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$	$f(f(x))$	$g(g(x))$
M.I.	M.I.	M.I.	M.I.	M.I.	M.I.
M.D.	M.D.	M.I.	M.I.	M.I.	M.I.
M.I.	M.D.	M.D.	M.D.	M.I.	M.I.

BYJU'S Classes

Application of Derivatives

Inflection Point of a Curve

Road Map

Point of Inflection

Monotonicity and Curvature of a curve and its Inverse

Concavity/Convexity of a curve

Illustration

If $fogoh(x)$ is an increasing function, then which of the following is not possible ?

a. $f(x), g(x)$ and $h(x)$ are increasing.

b. $f(x), g(x)$ are decreasing and $h(x)$ is increasing.

c. $f(x), g(x)$ and $h(x)$ are decreasing.

d. $f(x), g(x)$ are increasing and $h(x)$ is decreasing.

Solution:

$fogoh(x)$ is an increasing function

Let $p(x) = fogoh(x)$

Option a: $f(x), g(x)$ and $h(x)$ are increasing.

Since, $f(x), g(x)$ and $h(x)$ are increasing, we have, $f'(x) \geq 0, \quad g'(x) \geq 0$ and $h'(x) \geq 0$

$$\Rightarrow p'(x) = f'\left(g(h(x))\right) \cdot g'(h(x)) \cdot h'(x) \geq 0$$

$\Rightarrow p(x)$ i.e, $fogoh(x)$ is an increasing function which is in accordance with question.

Option b: $f(x), g(x)$ are decreasing and $h(x)$ is increasing.

Since, $f(x), g(x)$ are decreasing and $h(x)$ is increasing, we have, $f'(x) \leq 0$, $g'(x) \leq 0$ and $h'(x) \geq 0$

$$\Rightarrow p'(x) = f'\left(g(h(x))\right) \cdot g'(h(x)) \cdot h'(x) \geq 0$$

$\Rightarrow p(x)$ i.e, $fogoh(x)$ is an increasing function which is in accordance with question.

Option c: $f(x), g(x)$ and $h(x)$ are decreasing.

Since, $f(x), g(x)$ and $h(x)$ are decreasing, we have, $f'(x) \leq 0, \quad g'(x) \leq 0$ and $h'(x) \leq 0$

$$\Rightarrow p'(x) = f'\left(g(h(x))\right) \cdot g'(h(x)) \cdot h'(x) \leq 0$$

$\Rightarrow p(x)$ i.e, $fogoh(x)$ is a decreasing function.

Option d: $f(x), g(x)$ are increasing and $h(x)$ is decreasing.

Since, $f(x), g(x)$ are increasing and $h(x)$ is decreasing, we have, $f'(x) \geq 0$, $g'(x) \geq 0$ and $h'(x) \leq 0$

$$\Rightarrow p'(x) = f'\left(g(h(x))\right) \cdot g'(h(x)) \cdot h'(x) \leq 0$$

$\Rightarrow p(x)$ i.e, $fogoh(x)$ is a decreasing function.

So, options (c),(d) are the correct answers.

Illustration

Let $f : [0,2] \to \mathbb{R}$ be a twice differentiable function such that $f''(x) > 0, \forall\, x \in (0,2)$. If $\phi(x) = f(x) + f(2-x)$, then ϕ is :

$a.$ Increasing in $(0,2)$

$b.$ Increasing in $(0,1)$ and decreasing in $(1,2)$

$c.$ Decreasing in $(0,2)$

$d.$ Decreasing in $(0,1)$ and increasing in $(1,2)$

Solution: $f''(x) > 0, \forall\, x \in (0,2) \Rightarrow f'(x)$ is increasing function.

$$\phi(x) = f(x) + f(2-x)$$

Case 1: $\phi(x)$ is increasing function

$$\phi'(x) = f'(x) - f'(2-x)$$

$$\Rightarrow f'(x) > f'(2-x)$$

As $f'(x)$ is a strictly increasing function, we get,

$$\Rightarrow x > 2 - x$$

$$\Rightarrow x > 1$$

Case 2: $\phi(x)$ is decreasing function

$$\phi'(x) = f'(x) - f'(2-x)$$

$$\Rightarrow f'(x) < f'(2-x)$$

As $f'(x)$ is a strictly increasing function, we get,

$$\Rightarrow x < 2 - x$$

$$\Rightarrow x < 1$$

But, $x \in (0,2)$

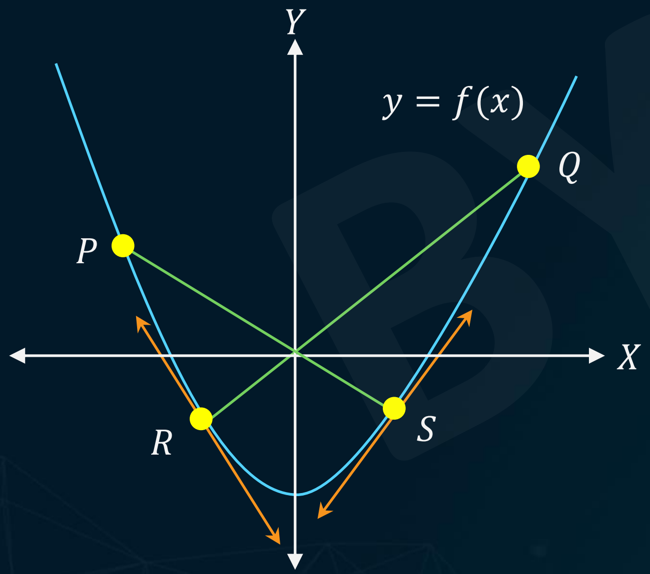
$\therefore$ $\phi(x)$ is increasing in $(1,2)$ and decreasing in $(0,1)$

So, option (d) is the correct answer.

Concavity / Convexity of a Curve

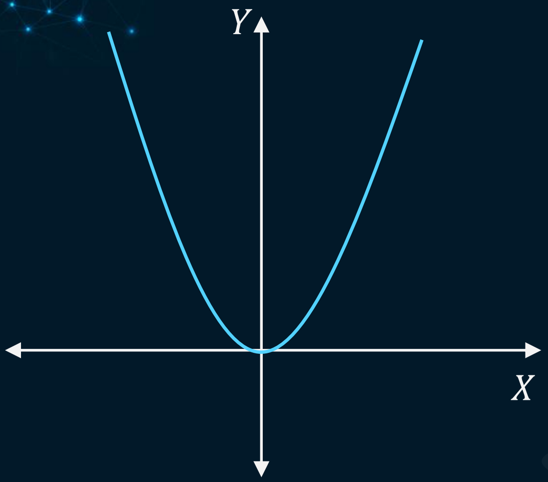
(a) Concave Upward

A function $f(x)$ is said to be concave upwards (convex) in interval (a, b), if tangent drawn at every point $\left(x_0, f(x_0)\right)$, for $x_0 \in (a, b)$ lie below the curve, Or, if we join any two points on the curve, then line segment lies above the curve.

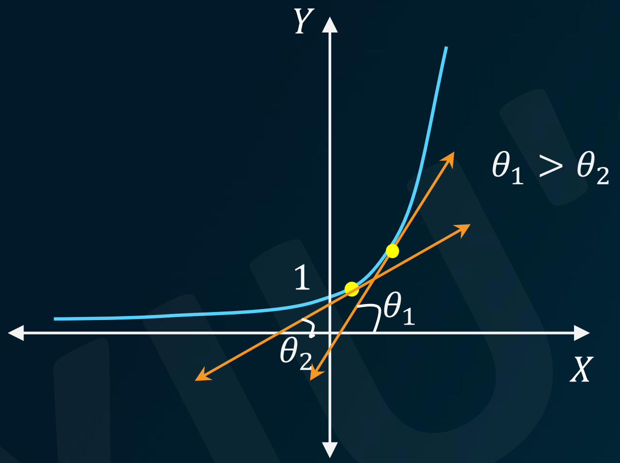


From the given figure, we can see that the curve is concave upwards. Also, the line segments PS and QR lie above the curve and the tangents drawn at points R and S lie below the curve.

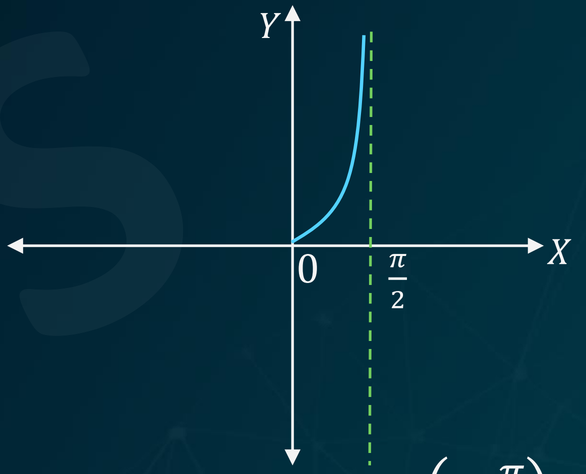
Example:



$f(x) = x^2$



$f(x) = e^x$



$f(x) = \tan x \; ; \; x \in \left(0, \frac{\pi}{2}\right)$

Also, for all concave upward curves, we can say that the slope of the tangent keeps on increasing as we increase the value of x i.e., $f'(x)$ is increasing.

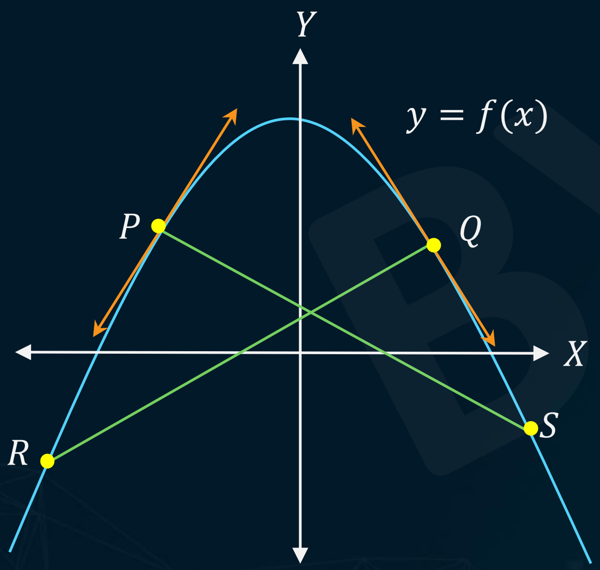
Note:

If $f(x)$ is concave upwards in $x \in (a, b)$ then, $f''(x) > 0$, $\forall \, x \in (a, b)$.

Concavity / Convexity of a Curve

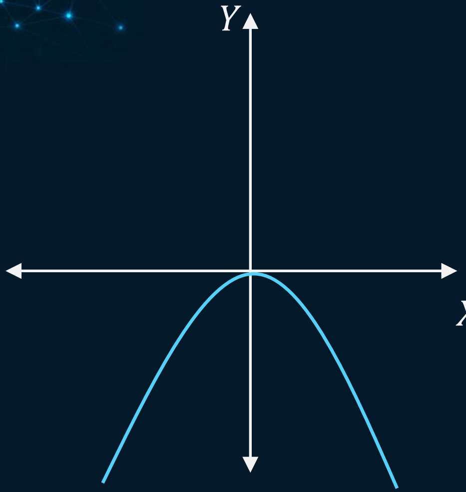
(b) Concave Downward

A function $f(x)$ is said to be concave downwards (concave) in interval (a, b), if tangent drawn at every point $\left(x_0, f(x_0)\right)$, for $x_0 \in (a, b)$ lie above the curve, Or, if we join any two points on the curve, then the line segment lies below the curve.

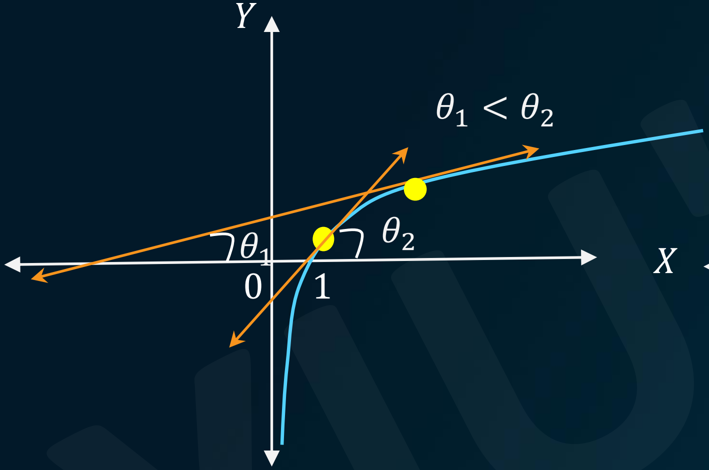


From the given figure, we can see that the curve is concave downwards. Also, the line segments PS and QR lie below the curve and the tangents drawn at points P and Q lie above the curve.

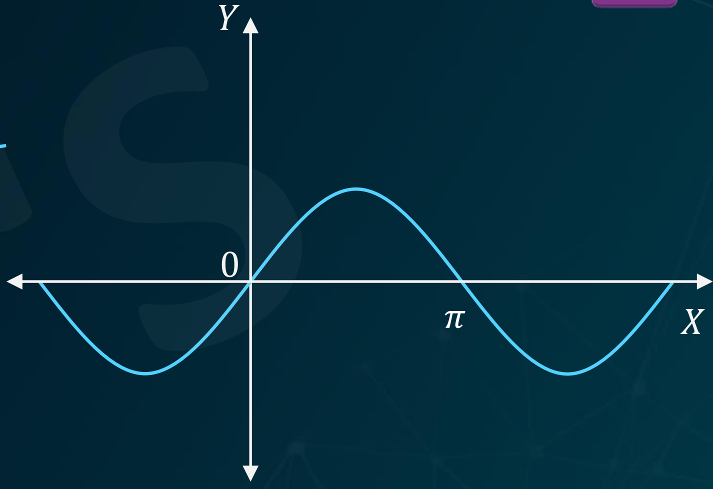
Example:



$$f(x) = -x^2$$



$$f(x) = \ln x$$



$$f(x) = \sin x \; ; x \in (0, \pi)$$

Also, for all concave downwards curves, we can say that the slope of the tangent keeps on decreasing as we increase the value of x i.e., $f'(x)$ is decreasing.

Note:

If $f(x)$ is concave downwards in $x \in (a, b)$ then, $f''(x) < 0 \, , \forall \, x \in (a, b)$.

Relation between $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right)$$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{\frac{d\left(\frac{dx}{dy}\right)}{dx}}{\frac{dy}{dx}}$$

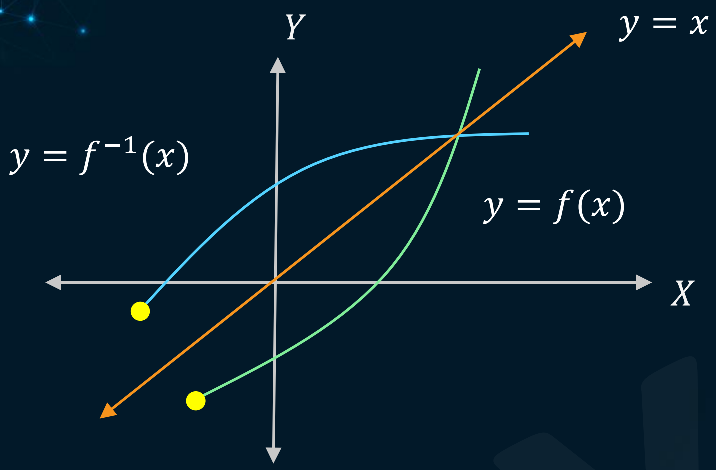
$$= \frac{\frac{d\left(\frac{1}{\frac{dy}{dx}}\right)}{dx}}{\frac{dy}{dx}}$$

$$= -\frac{\frac{d^2y}{dx^2}\cdot\left(\frac{dy}{dx}\right)^{-2}}{\frac{dy}{dx}}$$

Note:

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

Monotonicity and Curvature of a Function and its Inverse



Let a function $f(x)$ be differentiable and invertible.
We can see that $f(x)$ is Monotonically Increasing (M.I) and concave upwards. On plotting the graph of $f^{-1}(x)$, we can see that $f^{-1}(x)$ is Monotonically Increasing but concave downwards.

Note:

The above facts can also be seen with the help of calculus.

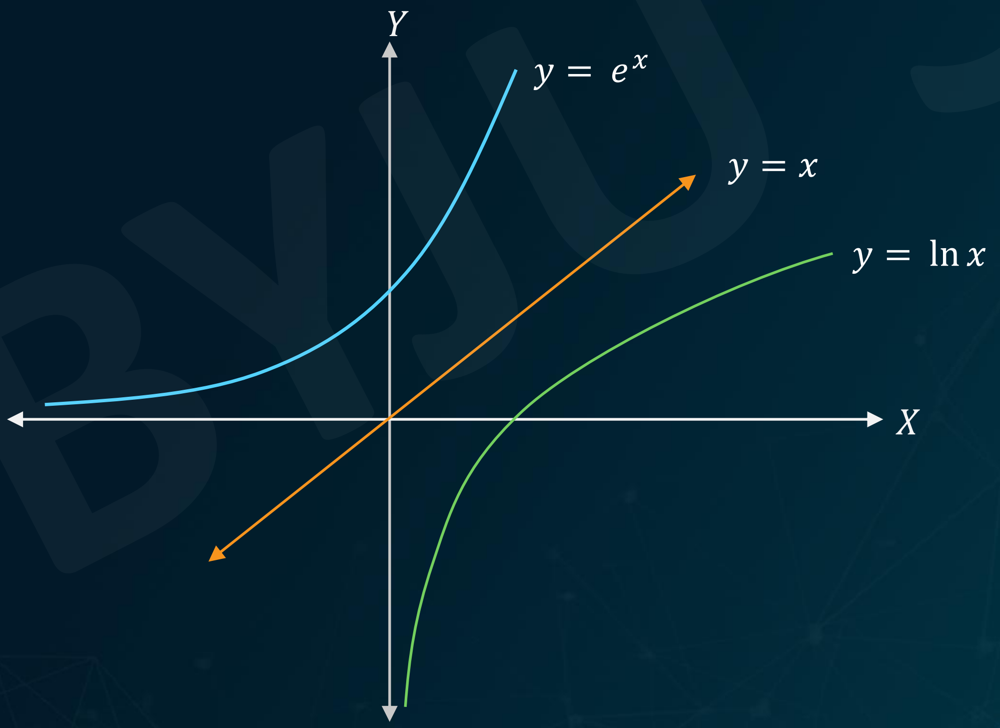
If $f(x)$ is M.I and concave upwards then, $f^{-1}(x)$ will be M.I and concave downwards

For $y = f(x)$: $\dfrac{dy}{dx} > 0$, $\dfrac{d^2y}{dx^2} > 0$

For $y = f^{-1}(x)$: $\dfrac{dx}{dy} > 0, \dfrac{d^2x}{dy^2} < 0$

Monotonicity and Curvature of a Function and its Inverse

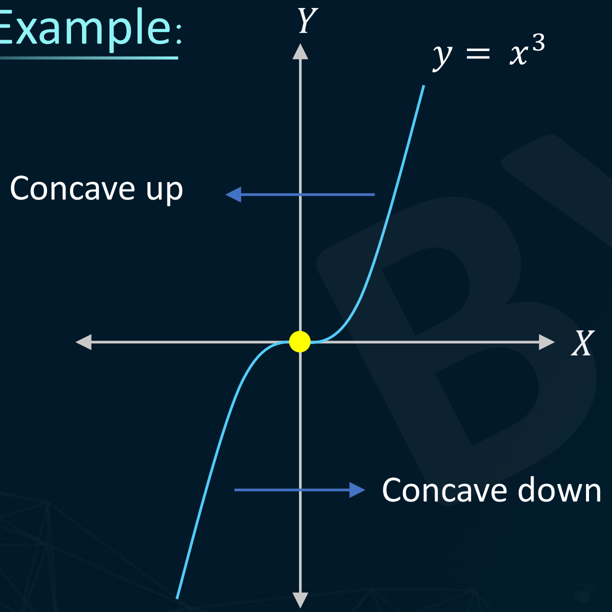
$f(x)$	e^x	M.I.	Concave upward
$f^{-1}(x)$	$\ln x$	M.I.	Concave downward



Point of Inflection

If a function $f(x)$ is continuous at $x = c$, and tangent exists at this point, such that $f''(x)$ has opposite sign on either side of $'c'$, then the point $(c, f(c))$ is known as point of inflection.

Example:



$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$f''(x) = 6x$$

For $x > 0$,

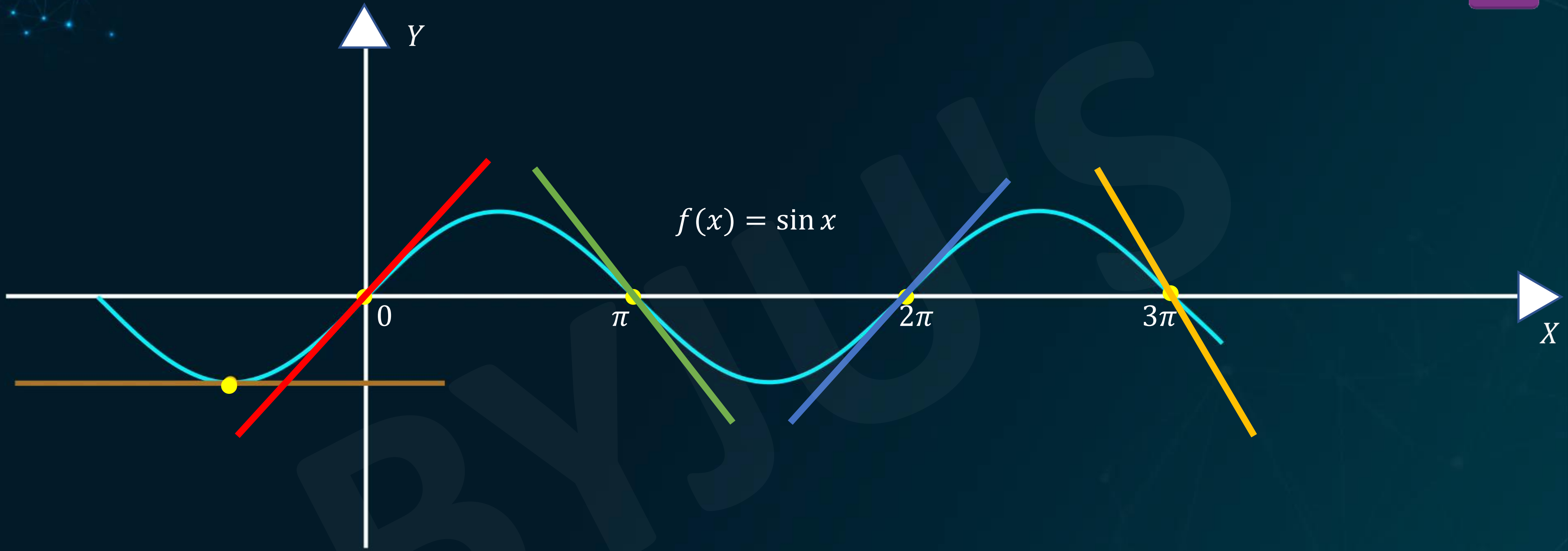
$$f''(x) = 6x > 0$$

For $x < 0$,

$$f''(x) = 6x < 0$$

Since, $f''(0^-)$ and $f''(0^+)$ have opposite signs, $x = 0$ is the point of inflection.

Point of Inflection



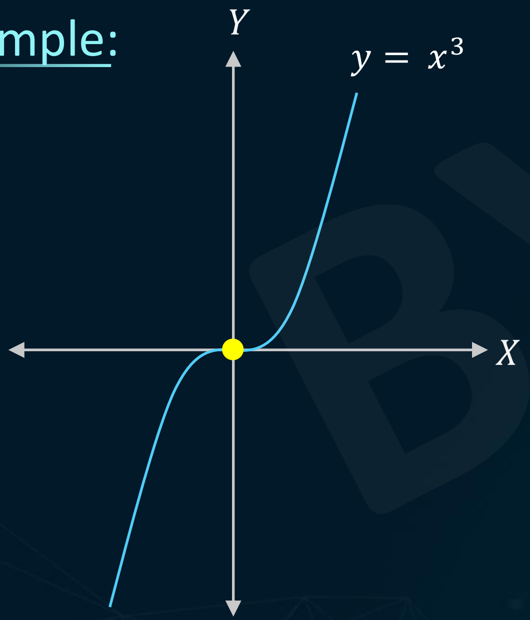
Geometrically, curvature of graph changes about the inflection point .

At $x = 0, \pi, 2\pi, 3\pi$, the curvature of graph is changing as the graph is concave upwards on one side and concave downwards on the other side. So, $x = n\pi, n \in \mathbb{Z}$ are the points of inflection.

Point of Inflection

Mathematically, inflection point may occur at point where $f''(x) = 0$ (but $f'''(x) \neq 0$) or not defined (but tangent should exist for $f(x)$) and sign of $f''(x)$ should change about that point.

Example:



$f(x) = x^3$

$f''(x) = 6x = 0 \Rightarrow x = 0$

And $f'''(x) = 6 \neq 0$

For $x > 0$,

$f''(x) = 6x > 0$

For $x < 0$,

$f''(x) = 6x < 0$

Since, $f''(0^-)$ and $f''(0^+)$ have opposite signs, $x = 0$ is the point of inflection.

Point of Inflection

Example:

$i)\ f(x) = x^4$

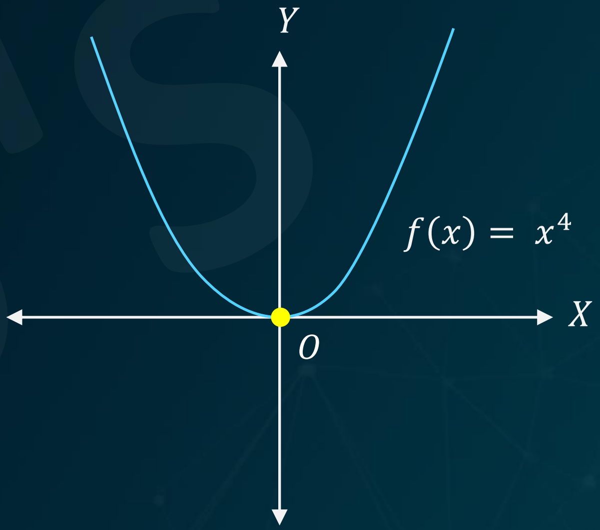
$f'(x) = 4x^3$

$f''(x) = 12x^2 = 0 \Rightarrow x = 0$

So, $x = 0$ can be the point of inflection.

But, $f''(0^+)$ and $f''(0^-)$ are both positive. So, the curve will not change its curve at $x = 0$

$\therefore f(x) = x^4$ does not have any inflection point.



Point of Inflection

Example:

$ii)\ f(x) = x^{\frac{1}{3}}$

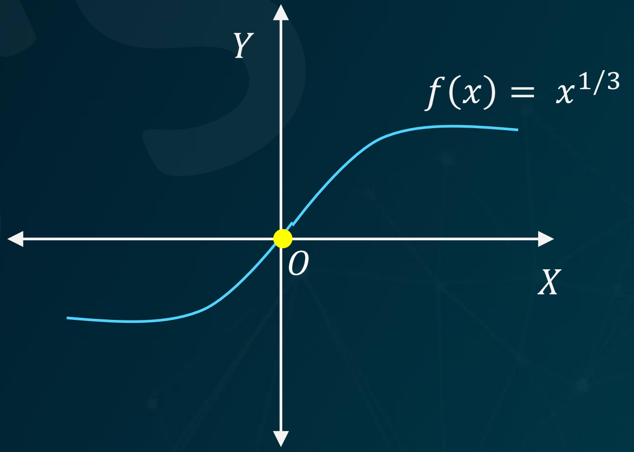
$f'(x) = \frac{1}{3}.x^{-\frac{2}{3}}$, Vertical tangent exists at $x = 0$

$f''(x) = -\frac{2}{9}.x^{-\frac{5}{3}}$ is not defined at $x = 0$

So, $x = 0$ can be the point of inflection.

Also, $f''(0^+) < 0$ and $f''(0^-) > 0$. So, the curve will change its curvature at $x = 0$

$\therefore\ x = 0$ is an inflection point for $f(x) = x^{\frac{1}{3}}$



Illustration

Find intervals of concavity of the function

$f(x) = x^4 - 6x^3 - 108x^2 + 57x + 2$. Also find the inflection point .

Solution:

$$f(x) = x^4 - 6x^3 - 108x^2 + 57x + 2$$

$$f'(x) = 4x^3 - 18x^2 - 216x + 57$$

$$f''(x) = 12\,(x^2 - 3x - 18) = 12(x+3)(x-6)$$

$$f''(x) = 0 \Rightarrow x = -3, 6$$

For $f(x)$ to be Concave upwards $f''(x) > 0 \Rightarrow x \in (-\infty, -3) \cup (6, \infty)$

For $f(x)$ to be Concave downwards $f''(x) < 0 \Rightarrow x \in (-3, 6)$

At $x = -3, 6$ sign of $f''(x)$ changes.

$\therefore x = -3,6$ are the inflection points of $f(x)$

Illustration

What conditions must the coefficients a, b, c satisfy for the curve $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, to have points of inflection ?

a. $b^2 = 8ac$

b. $3b^2 > 8ac$

c. $3b^2 = 8ac$

d. $3b^2 \geq 8ac$

Solution:

At inflection point , $f''(x) = 0$, and its sign should change.

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d,$

$f''(x) = 12ax^2 + 6bx + 2c$

$f''(x)$ is a quadratic expression and a quadratic expression changes its sign only when its discriminant is greater than zero.

$\Rightarrow 36b^2 - 96ac > 0$

$\therefore\ 3b^2 > 8ac$

So, option (b) is the correct answer.

Summary Sheet

- A function $f(x)$ is said to be concave upwards (convex) in interval (a, b), if tangent drawn at every point $\left(x_0, f(x_0)\right)$, for $x_0 \in (a, b)$ lie below the curve, Or, if we join any two points on the curve, then line segment lies above the curve.
- A function $f(x)$ is said to be concave downwards (concave) in interval (a, b), if tangent drawn at every point $\left(x_0, f(x_0)\right)$, for $x_0 \in (a, b)$ lie above the curve, Or, if we join any two points on the curve, then the line segment lies below the curve.
- If $f(x)$ is M.I and concave upwards then, $f^{-1}(x)$ will be M.I and concave downwards.
- If a function $f(x)$ is continuous at $x = c$, and tangent exists at this point, such that $f''(x)$ has opposite sign on either side of $'c'$, then the point $\left(c, f(c)\right)$ is known as point of inflection. Geometrically, curvature of graph changes about the inflection point .

BYJU'S Classes

Application of Derivatives

Application of Monotonicity

Road Map

Maxima and Minima

Inequality using Curvature

Inequality using Monotonicity

Inequality using Monotonicity

Comparison of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of new function $h(x) = f(x) - g(x)$.

Example:

Let $f(x) = \log(1+x)$, $g(x) = x$ where $x \in (0, \infty)$

To find which function is having more value in given interval of x, follow these steps:

Step 1: Assume a function $h(x) = \log(1+x) - x$

Step 2: Find derivative of $h(x)$

Step 3: If $h'(x) > 0$ then $h(x) > h(0)$ where $x \in (0, \infty)$ ($h(x)$ is strictly increasing)

If $h'(x) < 0$ then $h(x) < h(0)$ where $x \in (0, \infty)$ ($h(x)$ is strictly decreasing)

Illustration

Prove the following inequalities :

$i)\ \sin x < x < \tan x \ , x \in \left(0, \frac{\pi}{2}\right)$ $\quad ii)\ \frac{x}{1+x} < \ln(1+x) < x \ , x \in (0, \infty)$

Solution: $i)\ \sin x < x < \tan x\ ,\ \ x \in \left(0, \frac{\pi}{2}\right)$

Let $f(x) = \sin x - x$

$f'(x) = \cos x - 1 < 0$ $\quad \left(\because \text{For } x \in \left(0, \frac{\pi}{2}\right), \cos x \in (0,1)\right)$

$\Rightarrow f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

$x > 0 \Rightarrow f(x) < f(0)$ $\quad$ (For $x \in \left(0, \frac{\pi}{2}\right), f(x)$ is decreasing)

Also, $f(0) = 0$

$\Rightarrow \sin x - x < 0$ $\quad \boxed{\Rightarrow \sin x < x \cdots (i)}$

Now, let's prove $x < \tan x\,, x \in \left(0, \frac{\pi}{2}\right)$

Let $g(x) = x - \tan x$

$\Rightarrow g'(x) = 1 - \sec^2 x < 0 \qquad (\because \sec^2 x \in [1, \infty))$

$\Rightarrow g(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

$x > 0 \Rightarrow g(x) < g(0)$ (For $x \in \left(0, \frac{\pi}{2}\right)$, $g(x)$ is decreasing)

Also, $g(0) = 0$

$\Rightarrow x - \tan x < 0$ $\boxed{\Rightarrow x < \tan x \cdots (ii)}$

From (i) and (ii), $\boxed{\therefore\ \sin x < x < \tan x}$

$ii)$ $\boxed{\dfrac{x}{1+x} < \ln(1+x) < x \ , x \in \ (0, \infty)}$

Let $f(x) = \frac{x}{1+x} - \ln(1+x)$

$f'(x) = \frac{1}{(1+x)^2} - \frac{1}{1+x} \ \Rightarrow f'(x) = \frac{1-(1+x)}{(1+x)^2}$

$\Rightarrow f'(x) = -\frac{x}{(1+x)^2} \ < 0 \qquad (\because x \in \ (0, \infty))$

$\therefore \ f(x)$ is decreasing

$x > 0 \Rightarrow f(x) < f(0)$ (For $x \in (0, \infty)$, $f(x)$ is decreasing)

Also, $f(0) = 0$

$\Rightarrow \frac{x}{1+x} - \ln(1+x) < 0$

$\Rightarrow \frac{x}{1+x} < \ln(1+x) \cdots (i)$

Now, Let $g(x) = \ln(1 + x) - x$

$$g'(x) = \frac{1}{1+x} - 1$$

$$g'(x) = \frac{-x}{1+x} < 0 \quad (\because x \in (0, \infty))$$

$\therefore g(x)$ is decreasing.

$x > 0 \Rightarrow g(x) < g(0)$ (For $x \in (0, \infty), g(x)$ is decreasing)

Also, $g(0) = 0$

$\Rightarrow \ln(1 + x) - x < 0$ $\boxed{\Rightarrow \ln(1 + x) < x \cdots (ii)}$

$$\Rightarrow \frac{x}{1+x} < \ln(1+x) \cdots (i)$$

and

$$\ln(1+x) < x \cdots (ii)$$

From equation (i) and (ii)

$$\boxed{\frac{x}{1+x} < \ln(1+x) < x}$$

Illustration

Prove that $\sin x \tan x > x^2, x \in \left(0, \frac{\pi}{2}\right)$, hence evaluate $\lim\limits_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right]$, $[.]$ denotes G.I.F.

Solution: Let $f(x) = \sin x . \tan x - x^2$, $f(0) = 0$

$$\Rightarrow f'(x) = \cos x . \tan x + \sin x . \sec^2 x - 2x$$

$$\Rightarrow f'(x) = \cos x . \tan x + \tan x . \sec x - 2x$$

$$\Rightarrow f'(x) = \underbrace{\tan x}_{> x} . \underbrace{(\cos x + \sec x)}_{> 2} - 2x$$

Also, $\tan x > x$ and $\cos x + \frac{1}{\cos x} > 2$ (A.M $\geq$ G.M) For $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore f'(x) > 0 \Rightarrow f(x)$ is increasing. So, $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow f(x)$ is increasing and $f(x) > f(0)$

$$\Rightarrow \sin x \cdot \tan x - x^2 > 0$$

$$\Rightarrow \sin x \cdot \tan x > x^2$$

$$\Rightarrow \frac{\sin x \tan x}{x^2} > 1$$

Value of $\frac{\sin x \tan x}{x^2}$ is slightly greater than 1 in rightneighbourhood of 0.

$$\boxed{\lim_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right] = 1} \quad (\because [1^+] = 1)$$

Illustration

Which is greater?

$(i) e^{\pi}$ or π^{e} $\quad (ii) \tan^{-1} e + \frac{1}{\sqrt{1+e^2}}$ or $\tan^{-1}\frac{1}{e} + \frac{e}{\sqrt{1+e^2}}$

Solution: Let us assume that $e^{\pi} > \pi^{e}$

$$e^{\pi} > \pi^{e} \Rightarrow (e^{\pi})^{\frac{1}{\pi e}} > (\pi^{e})^{\frac{1}{\pi e}} \Rightarrow e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$$

From above expression clearly

$$\Rightarrow f(x) = x^{\frac{1}{x}}$$

$$\Rightarrow \ln f(x) = \frac{1}{x} \cdot \ln x$$

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{1}{x^2} + \ln x \cdot \left(-\frac{1}{x^2}\right)$$ (Differentiating on both sides)

$$\Rightarrow f'(x) = f(x)\left[\frac{1-\ln x}{x^2}\right] \Rightarrow f'(x) = x^{\frac{1}{x}} . \left(\frac{1-\ln x}{x^2}\right)$$

$$\Rightarrow f'(x) = x^{1/x} . \left(\frac{1-\ln x}{x^2}\right)$$

For $0 < x < e$, $\ln x < 1$ so $1 - \ln x > 0$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing

For $x > e$, $\ln x > 1$ so $1 - \ln x < 0$

$\Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing

Thus , $f(e) > f(\pi)$

$$e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$$

$$\boxed{\Rightarrow e^{\pi} > \pi^{e}}$$

$(ii)\ \tan^{-1} e + \frac{1}{\sqrt{1+e^2}}$ or $\tan^{-1}\frac{1}{e} + \frac{e}{\sqrt{1+e^2}}$

Let $f(x) = \tan^{-1} x + \frac{1}{\sqrt{1+x^2}}$

$$\Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{x}{(1+x^2)^{\frac{3}{2}}} = \frac{\sqrt{1+x^2} - x}{(1+x^2)^{\frac{3}{2}}}$$

$$\left(\because \sqrt{1+x^2} > \sqrt{x^2}, \sqrt{1+x^2} - \sqrt{x^2} > 0\right)$$

$f'(x) > 0 \Rightarrow f(x)$ is increasing.

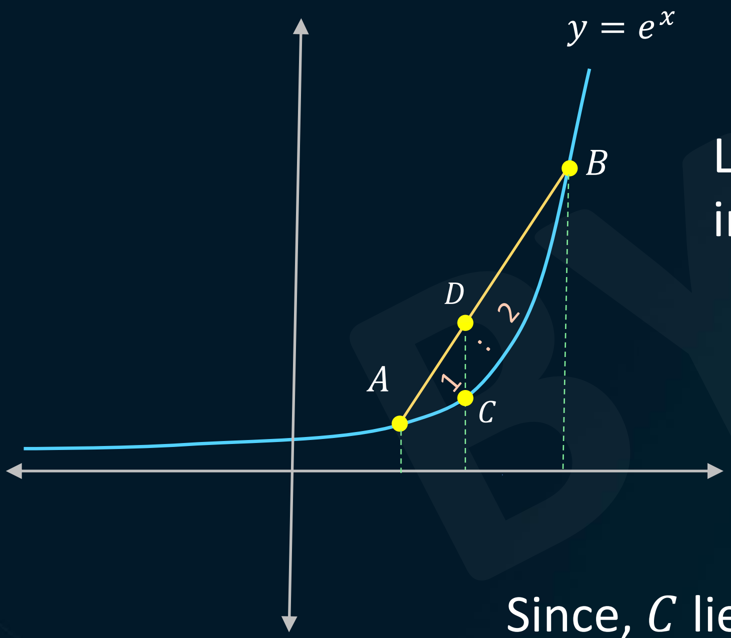
$$\because e > \frac{1}{e} \Rightarrow f(e) > f\left(\frac{1}{e}\right)$$

$$\boxed{\therefore \tan^{-1} e + \frac{1}{\sqrt{1+e^2}} > \tan^{-1}\frac{1}{e} + \frac{e}{\sqrt{1+e^2}}}$$

Illustration

If $x_1 \neq x_2$, then which is greater $e^{\frac{2x_1+x_2}{3}}$ or $\frac{2\,e^{x_1}+e^{x_2}}{3}$?

Solution: Consider the function $f(x) = e^x$



Let $A \equiv (x_1, e^{x_1})$

and $B \equiv (x_2, e^{x_2})$ on the curve $y = e^x$

Let D be the point which divide AB in the ratio 1: 2 internally. So, $D \equiv \left(\frac{2x_1+x_2}{3}, \frac{2\,e^{x_1}+e^{x_2}}{3}\right)$

Corresponding point of D on the curve is C.

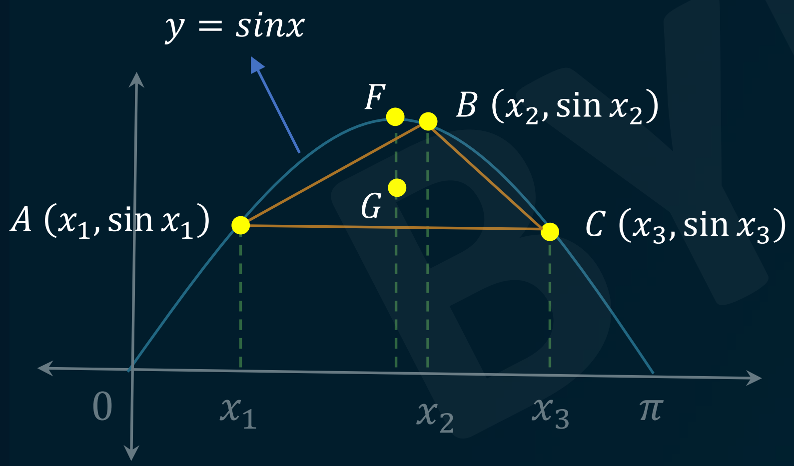
$$C \equiv \left(\frac{2x_1+x_2}{3}, e^{\frac{2x_1+x_2}{3}}\right)$$

Since, C lies below D so we can say that $\boxed{\frac{2\,e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}}$

Illustration

If $0 < x_1, x_2, x_3 < \pi$, then which is greater $\sin\left(\frac{x_1+x_2+x_3}{3}\right)$ or $\frac{\sin x_1+\sin x_2+\sin x_3}{3}$. Hence prove that : if A, B, C are angles of triangle then maximum value of $\sin A+\sin B+\sin C$ is $\frac{3\sqrt{3}}{2}$

Solution:



Consider ΔABC ,

Centroid $G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{\sin x_1+\sin x_2+\sin x_3}{3}\right)$

Now, corresponding point of G on the curve is F which can be written as

$$F \equiv \left(\frac{x_1+x_2+x_3}{3}, \sin\left(\frac{x_1+x_2+x_3}{3}\right)\right)$$

Since, F lies above G so, we can say that

$$\sin\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{\sin x_1+\sin x_2+\sin x_3}{3}$$

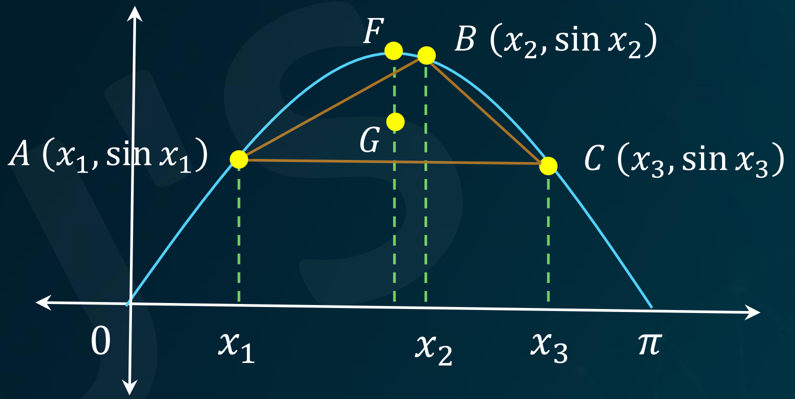
$$\sin\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{\sin x_1+\sin x_2+\sin x_3}{3}$$

If $A + B + C = \pi$

$$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \sin\frac{\pi}{3} \geq \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$



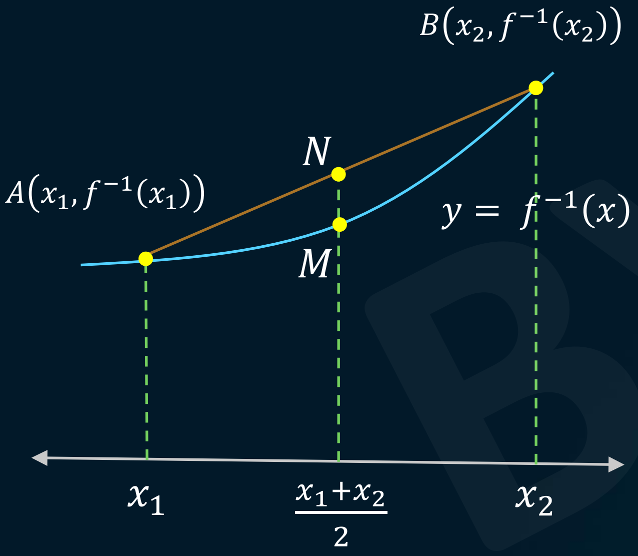
$\therefore$ Maximum value of $\sin A + \sin B + \sin C = \dfrac{3\sqrt{3}}{2}$

Note: Maximum value is possible only when ΔABC is an equilateral triangle.

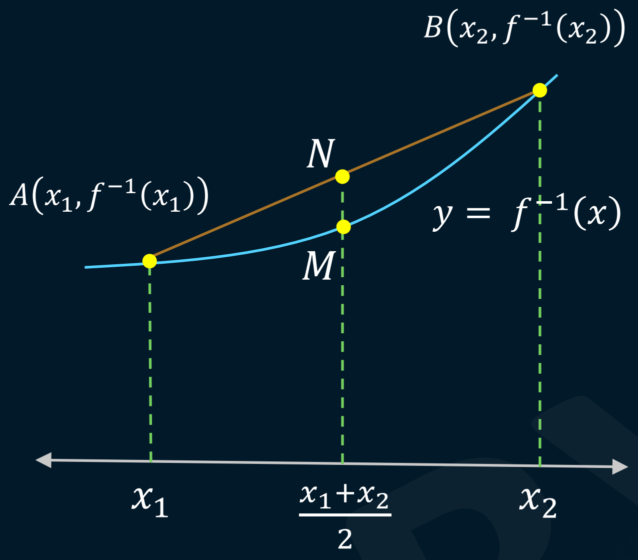
Illustration

If $f(x)$ is concave downwards and $f'(x) > 0$, then for $x_1 \neq x_2$, which of the following is greater : $f^{-1}\left(\frac{x_1+x_2}{2}\right)$ or $\frac{f^{-1}(x_1)+f^{-1}(x_2)}{2}$?

Solution:



$f(x)$ is Monotonically Increasing and concave downwards then $f^{-1}(x)$ will be Monotonically Increasing and concave upwards.



Let $A(x_1, f^{-1}(x_1))$ and $B(x_2, f^{-1}(x_2))$ be the two points on $y = f^{-1}(x)$

N is the mid point of AB

$$N \equiv \left(\frac{x_1+x_2}{2}, \frac{f^{-1}(x_1)+f^{-1}(x_2)}{2}\right)$$

$$M \equiv \left(\frac{x_1+x_2}{2}, f^{-1}\left(\frac{x_1+x_2}{2}\right)\right)$$

Since, M lies below N so, we can say that

$$\boxed{f^{-1}\left(\frac{x_1+x_2}{2}\right) < \frac{f^{-1}(x_1)+f^{-1}(x_2)}{2}}$$

Illustration

For $0 < x < \frac{\pi}{2}$, prove that $\cos(\sin x) > \sin(\cos x)$.

Solution:

Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x > 0$ $\left(\because \text{For } \left(0, \frac{\pi}{2}\right),\ \cos x \in (0,1)\right)$

Hence $f(x)$ is an increasing function in $x \in \left(0, \frac{\pi}{2}\right)$

For $x > 0$, $f(x) > f(0)$ or $x - \sin x > 0$

$\Rightarrow x > \sin x \ldots (i)$

Again, $0 < x < \frac{\pi}{2}$, We have, $0 < \cos x < 1$

Now we replace x by $\cos x$ in equation (i)

$\cos x > \sin(\cos x) \ \ldots (ii)$

$\cos x > \sin(\cos x) \ldots (ii)$

Now in $\left(0, \frac{\pi}{2}\right)$, $\cos x$ is monotonically decreasing.

So, if we apply cos on both sides in equation (i), the inequality will change.

$\Rightarrow \cos x < \cos(\sin x) \ldots (iii)$

From equations (ii) and (iii) , we get

$\sin(\cos x) < \cos x < \cos(\sin x)$

Hence, $\sin(\cos x) < \cos(\sin x)$

Summary Sheet

- Comparison of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of new function $h(x) = f(x) - g(x)$.

- If $h'(x) > 0$ for $x \in (a, b)$ then $h(x)$ is strictly increasing in $x \in (a, b)$ and $h(x) > h(a)$ for $\forall\ x \in (a, b)$

- If $h'(x) < 0$ for $x \in (a, b)$ then $h(x)$ is strictly decreasing in $x \in (a, b)$ and $h(x) < h(a)$ for $\forall\ x \in (a, b)$

- $f(x)$ is Monotonically Increasing and concave downwards then $f^{-1}(x)$ will be Monotonically Increasing and concave upwards and vice versa.

BYJU'S Classes

Application of Derivatives

Maxima and Minima

Road Map

Application of First derivative test

Method to find Extrema (First derivative test)

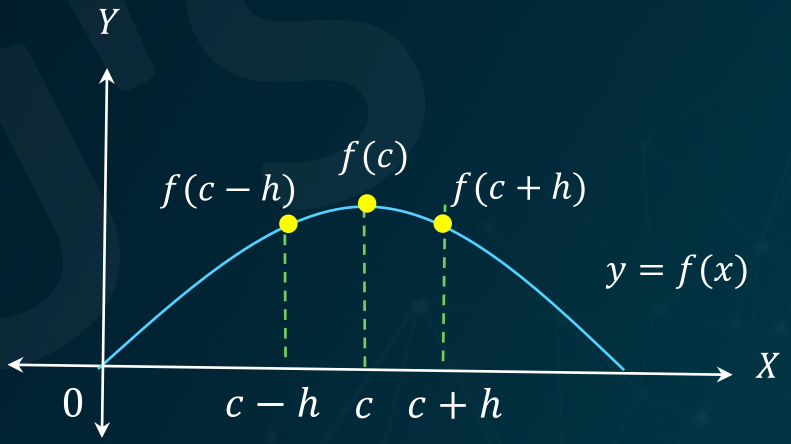
Stationary and critical points

Maxima & Minima (Extrema)

Maxima & Minima (Extrema)

Local or relative extrema:

A function $f(x)$ is said to have a maxima or minima at $x = c$, if $f(c)$ is maximum or minimum respectively in comparison to it's small neighbourhood.

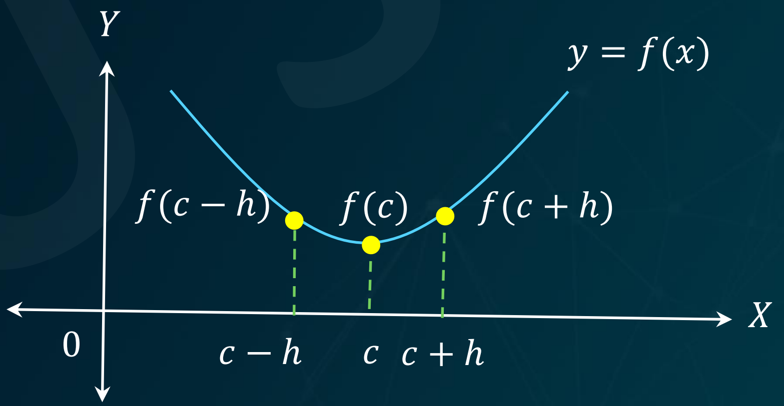


- $c - h < c < c + h, f(x)$ is increasing in $(0, c)$ and decreasing from $c \Rightarrow f(c-h) < f(c) > f(c+h)$, where $h \to 0^+$: then $x = c$ is a point of local maxima.

Maxima & Minima (Extrema)

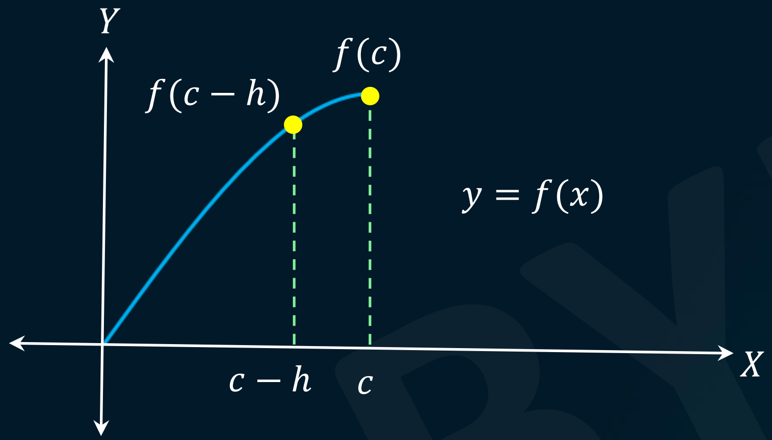
Local or relative extrema:

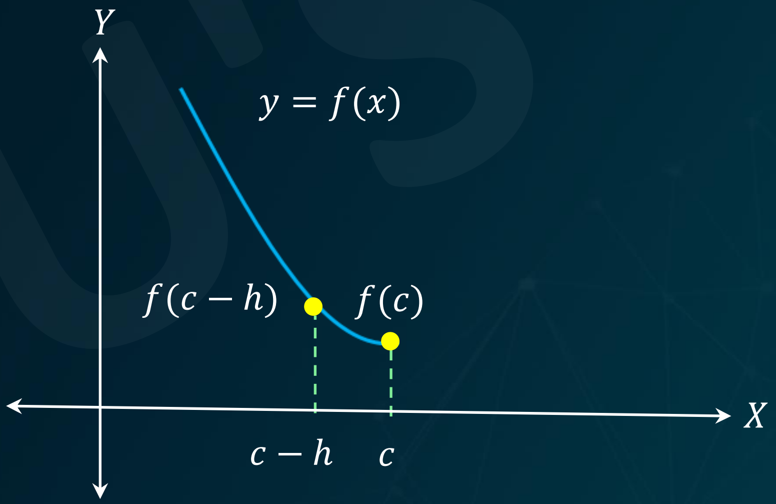
- $c-h<c<c+h, f(x)$ is decreasing in $(0,c)$ and increasing from c
  $\Rightarrow f(c-h)>f(c)<f(c+h)$, where $h\to 0^{+}$: then $x=c$ is a point of local minima.



Maxima & Minima (Extrema)

One – sided extrema:





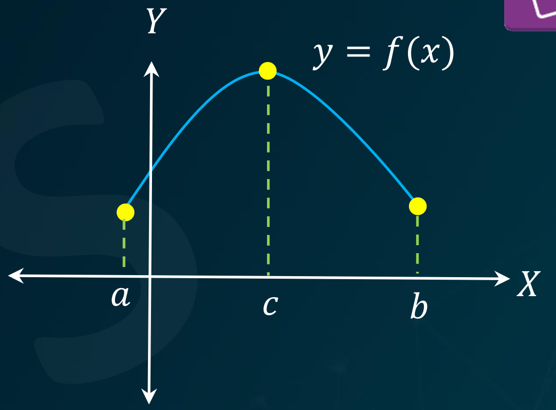
- If $c - h < c \Rightarrow f(c-h) < f(c), h \to 0^+$ then $x = c$ is a point of local maxima.
- If $c - h < c \Rightarrow f(c-h) > f(c), h \to 0^+$ then $x = c$ is a point of local minima.

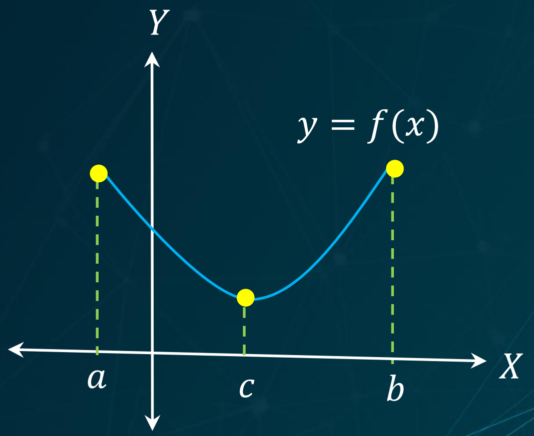
Maxima & Minima (Extrema)

Global or absolute extrema:

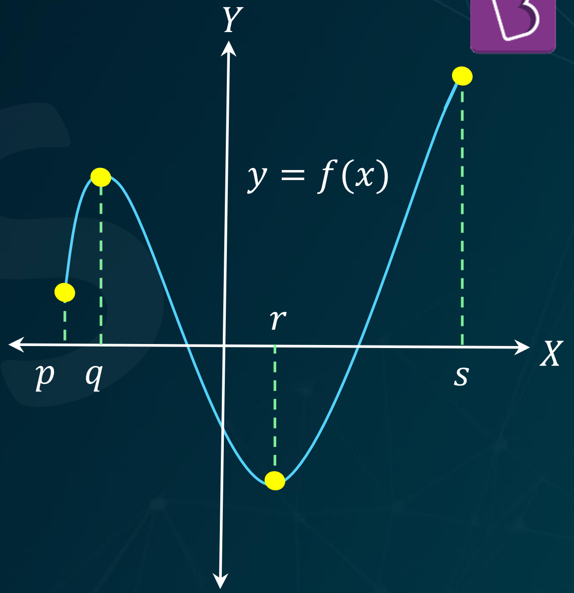
A function $f(x)$ is said to have a global extrema in an interval I, if there exists at least one c such that $f(c)$ is either greatest or least in the entire interval.

- $f(c) \geq f(x), \forall\, x \in [a, b] \Rightarrow x = c$ is a point of global maxima.
- $f(c) \leq f(x), \forall\, x \in [a, b] \Rightarrow x = c$ is a point of global minima.





Maxima & Minima (Extrema)



Example: Find points of extrema of given graph

For $x = p$:

p is left boundary point so we only need to check the right neighborhood.

Here, for $p < p + h \Rightarrow f(p) < f(p + h)$, where $h \to 0^+$

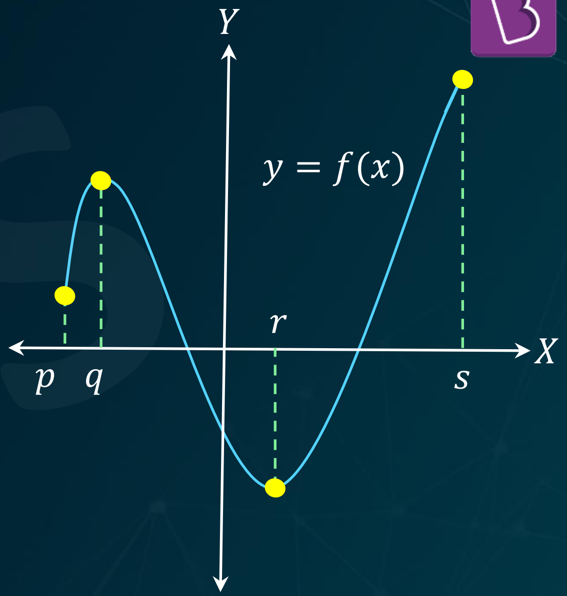
So, $x = p$ is a point of local minima.

For $x = q$:

Here, for $q - h < q < q + h \Rightarrow f(q - h) < f(q) > f(q + h)$, where $h \to 0^+$

So, $x = q$ is a point of local maxima.

Maxima & Minima (Extrema)



For $x = r$:

Here, for $r - h < r < r + h$

$\Rightarrow f(r-h) > f(r) < f(r+h)$ where $h \to 0^+$.

Also, $f(r) \leq f(x), \forall\, x \in [p, s]$

So, $x = r$ is a point of local minima and global minima.

For $x = s$:

Here, for $s - h < s$

$\Rightarrow f(s-h) < f(s)$ where $h \to 0^+$.

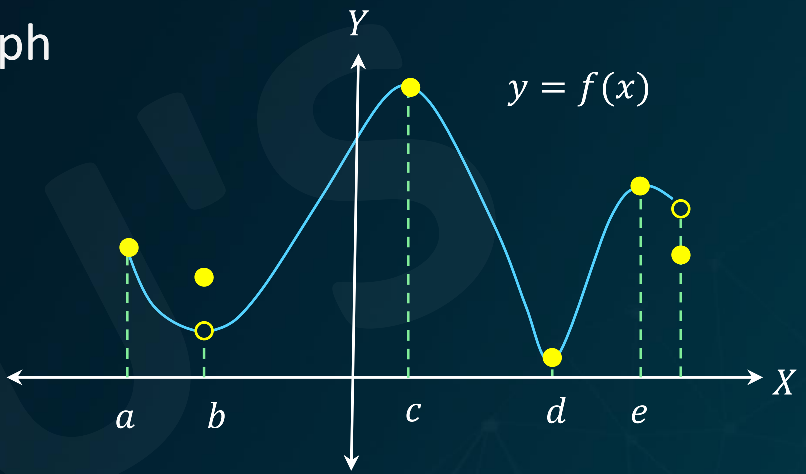
Also, $f(s) \geq f(x), \forall\, x \in [p, s]$

So, $x = s$ is a point of local maxima and global maxima.

Note: For continuous and non constant function, the points of maxima and minima lies alternately.

Maxima & Minima (Extrema)

Example: Find points of extrema of given graph from $x \in [a, g]$



For $x = a$:

We only check right neighborhood

Here, for $a < a + h \Rightarrow f(a) > f(a + h)$

where $h \to 0^+$

So, $x = a$ is a point of local maxima.

For $x = b$:

Here, for $b - h < b < b + h \Rightarrow f(b - h) < f(b) > f(b + h)$

where $h \to 0^+$. So, $x = b$ is a point of local maxima.

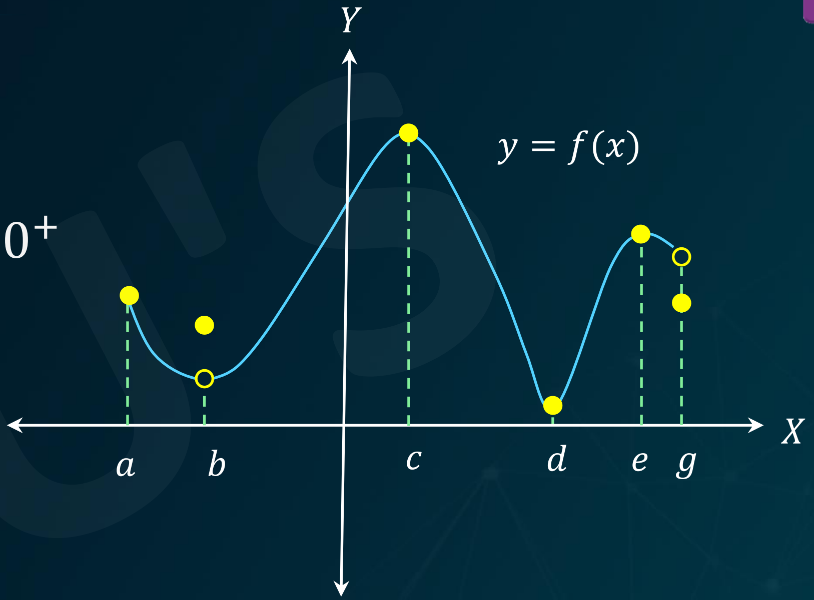
For $x = c$:

Here, for $c - h < c < c + h \Rightarrow f(c - h) < f(c) > f(c + h)$ where $h \to 0^+$

Also, $f(c) \geq f(x), \forall\, x \in [a, g]$

So, $x = c$ is a point of local maxima and global maxima.

Maxima & Minima (Extrema)



For $x = d$:

Here, for $d - h < d < d + h$

$\Rightarrow f(d-h) > f(d) < f(d+h)$ where $h \to 0^+$

Also, $f(d) \leq f(x), \forall\, x \in [a, g]$

So, $x = d$ is a point of local minima

and global minima.

For $x = e$:

Here, for $e - h < e < e + h$

$\Rightarrow f(e-h) < f(e) > f(e+h)$ where $h \to 0^+$

So, $x = e$ is a point of local maxima.

For $x = g$:

Here, for $g - h < g \Rightarrow f(g) < f(g-h)$ where $h \to 0^+$

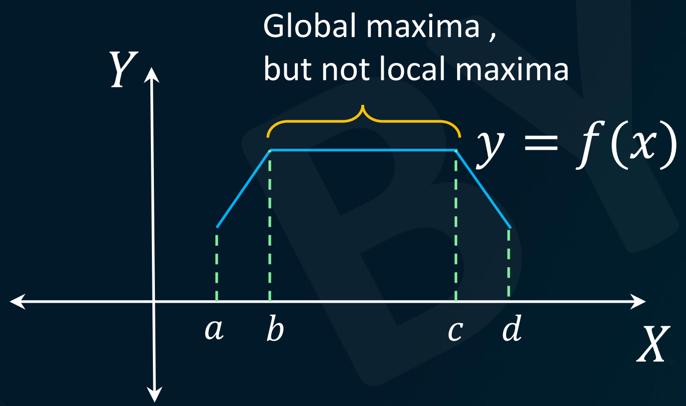
So, $x = f$ is a point of local minima.

Maxima & Minima (Extrema)

Global or absolute extrema:

- Normally , global maxima / minima occurs at points of local maxima or minima , but there can be exception.

Example: Let us consider the given function $f(x)$ for $x \in [b, c]$



- Here, $f(b) = f(b+h)$
  $f(c) = f(c-h)$

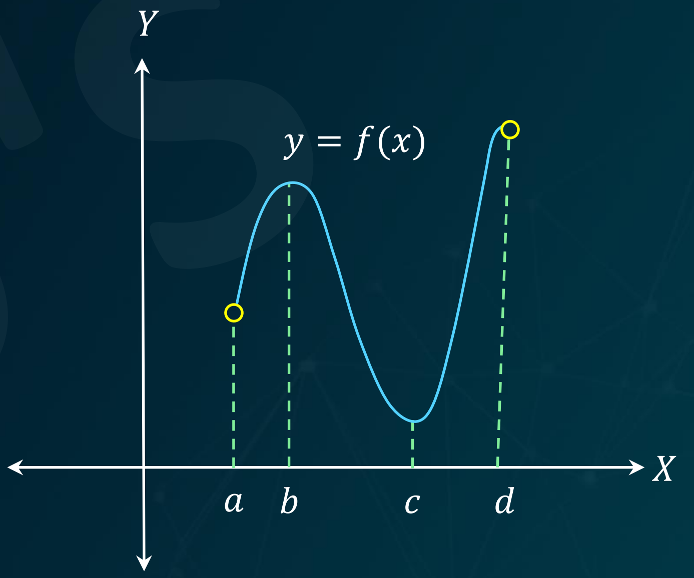
b and c does not satisfy the condition of local maxima. b and c are points of global maxima but not points of local maxima .

Maxima & Minima (Extrema)

Global or absolute extrema:

Global extrema may or may not exist for a function. For existence of global extrema, the value of the function must be attainable/achievable at global extrema point.

Example:



Local maxima : $x = b$

Local minima : $x = c$

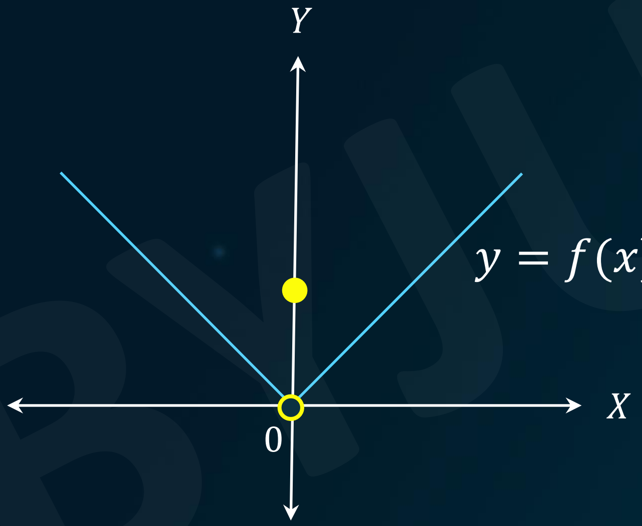
Global maxima : not defined ($\because$ value at $x = d$ is not achievable)

Global minima : $x = c$

Illustration

Let $f(x)=\begin{cases}|x|, 0<|x|\le 2\\ 1, \quad x=0\end{cases}$. Check for extrema at $x=0$.

Solution: To check extrema at $x=0$, let's draw the graph of $y=f(x)$



Here, for $0-h<0<0+h \Rightarrow f(0-h)<f(0)>f(0+h)$, where $h\to 0^+$

$y=f(x)$ has local maxima at $x=0$

Illustration

Let $f(x) = \begin{cases} -x^3 + \dfrac{b^3-b^2+b-1}{b^2+3b+2}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$.

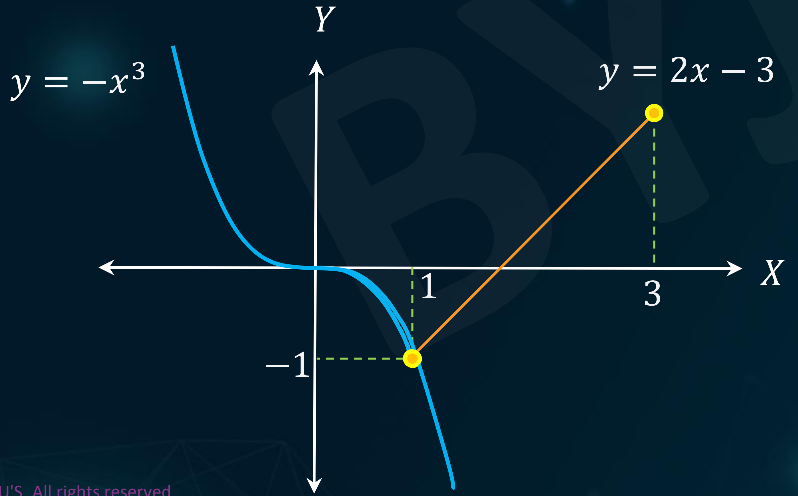
Find all possible values of b such that $f(x)$ has smallest value at $x = 1$.

A. $b \in (-2,-1) \cup [1,\infty)$

B. $b \in (-2,\infty)$

C. $b \in (-1,1) \cup [2,\infty)$

D. $b \in [-1,\infty)$



Graph of $f(x) \pm a$ can be obtained by shifting the graph of $f(x)$ by a units in vertical direction.

Solution:

$$f(x) = \begin{cases} -x^3 + \frac{b^3-b^2+b-1}{b^2+3b+2}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$$

For $f(x)$ to have minima at $x = 1$

$$\lim_{x\to 1}\left(-x^3 + \frac{b^3-b^2+b-1}{b^2+3b+2}\right) \ge -1$$

$$\Rightarrow \frac{b^3-b^2+b-1}{b^2+3b+2} \ge 0$$

$$\Rightarrow \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0$$

$$\boxed{\Rightarrow b \,\epsilon (-2,-1) \cup [1,\infty)}$$

So, option (A) is the correct answer.

Let $f(x)$ be a continuous function.

Stationary points : A point at which $f'(x) = 0$ is called stationary points.

Critical points: A point at which $f'(x) = 0$ or $f'(x)$ does not exist is called critical points.

Stationary points	$f'(x) = 0$
Critical points	$f'(x) = 0$ or $f'(x)$ does not exist.

Example: Find critical points & stationary points of the function:

$(i)\ f(x) = x^3 - 6x^2 - 36x + 7$

$$f'(x) = 3x^2 - 12x - 36 = 0$$

$\Rightarrow x = -2, 6$ (both stationary & critical points)

$(ii)\ f(x) = (x-1)x^{\frac{2}{3}}$

$\Rightarrow f(x) = x^{\frac{5}{3}} - x^{\frac{2}{3}}$

$\Rightarrow f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} = \frac{5x-2}{3x^{\frac{1}{3}}}$

$\Rightarrow$ Critical points : $x = \frac{2}{5}, 0$ $\quad$ ($\because f'(x) = 0$ or $f'(x)$ does not exist)

$\Rightarrow$ Stationary points : $x = \frac{2}{5}$ $\quad$ ($\because f'(x) = 0$)

Methods to find Extrema

First Derivative Test:

Let $f(x)$ is a continuous function and $x = c$ is the critical point ($f'(c) = 0$ or not defined) Observe sign change of $f'(x)$ about $x = c$.

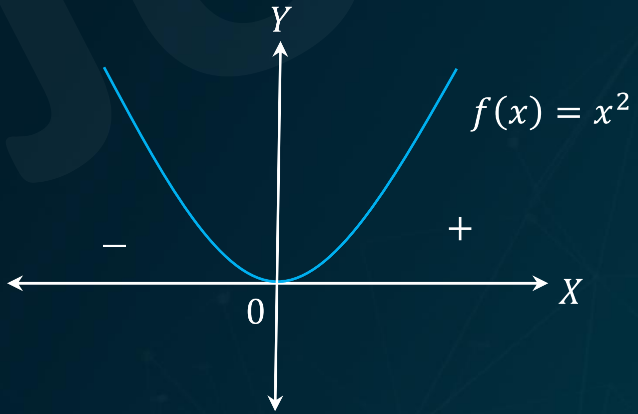
- Case I : If sign of $f'(x)$ changes sign from negative to positive as x crosses c from left to right, then $x = c$ is a point of local minima.

Example:

$$f(x) = x^2$$

$$f'(x) = 2x = 0 \quad at\ x = 0$$

$$f'(0^-) < 0\,;\, f'(0^+) > 0$$



Thus , minima at $x = 0$.

Method to find Extrema

First Derivative Test:

- Case II : If sign of $f'(x)$ changes sign from positive to negative as x crosses c from left to right, then $x = c$ is a point of local maxima.

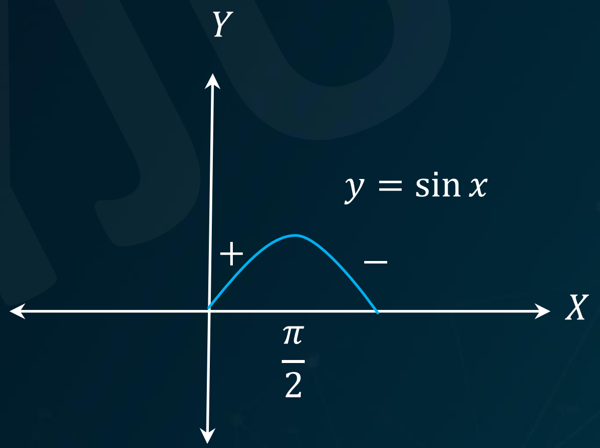
Example:

$$f(x) = \sin x \quad , \quad x \in (0, \pi)$$

$$f'(x) = \cos x = 0 \quad at\ x = \frac{\pi}{2}$$

$$f'\left(\frac{\pi}{2}^-\right) > 0 \,;\, f'\left(\frac{\pi}{2}^+\right) < 0$$

Thus , maxima at $x = \frac{\pi}{2}$



Method to find Extrema

First Derivative Test:

- Case III : If $f'(x)$ does not changes sign as x crosses c, then $x = c$ is neither a point of maxima nor minima.

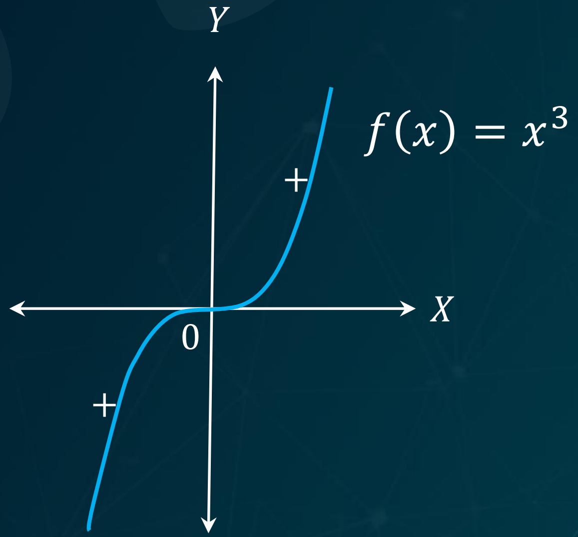
Example:

$f(x) = x^3$

$f'(x) = 3x^2 = 0,$ at $x = 0$

But , sign of $f'(x)$ does not change at $x = 0.$

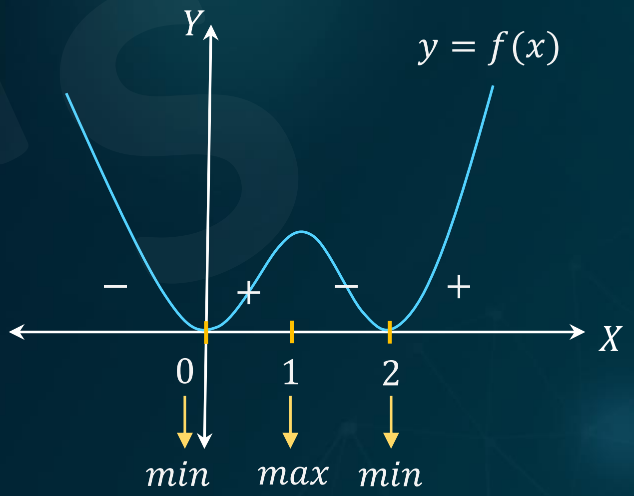
Thus , neither maxima nor minima.



Illustration

Find points of extrema of the function $f(x) = x^2(x-2)^2$.

Solution:



$$f'(x) = 2x(x-2)^2 + 2x^2(x-2)$$

$$= 4x(x-1)(x-2)$$

Critical points : $x = 0\,,1\,,2$ $\left(\begin{array}{c}\because f'(x) = 0 \text{ or} \\ f'(x) \text{ does not exist}\end{array}\right)$

local maxima : $x = 1$

local minima : $x = 0\,,2$

Global maxima : not exist (∴ as $x \to \infty$ or $x \to -\infty \Rightarrow f(x) \to \infty$)

Global minima : $x = 0\,,2$ ($f(0) = f(2) = 0$, least value)

Illustration

If S_1 and S_2 are respectively the sets of local minimum & local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in \mathbb{R}$, then ____.

a. $S_1 = \{-2,0\}$; $S_2 = \{1\}$

b. $S_1 = \{-1\}$; $S_2 = \{0,2\}$

c. $S_1 = \{-2,1\}$; $S_2 = \{0\}$

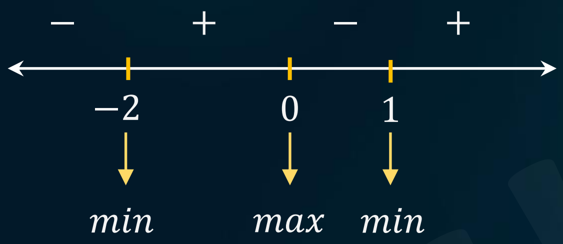
d. $S_1 = \{-2\}$; $S_2 = \{0,1\}$

Solution:

$$f'(x) = 36x^3 + 36x^2 - 72x = 36x(x^2 + x - 2)$$

$$f'(x) = 36x(x+2)(x-1)$$

Sign scheme of $f'(x)$



Here, sign of $f'(x)$ changes its sign from negative to positive as x crosses -2 from left to right, so $x = -2$ is a point of local minima

Sign of $f'(x)$ changes its sign from negative to positive as x crosses 1 from left to right, so $x = 1$ is a point of local minima.

Sign of $f'(x)$ changes its sign from positive to negative as x crosses 0 from left to right, so $x = 0$ is a point of local maxima

$\boxed{S_1 = \{-2,1\}, S_2 = \{0\}}$

So, option (c) is the correct answer.

Summary Sheet

- For $c - h < c < c + h \Rightarrow f(c-h) < f(c) > f(c+h),$ where $h \to 0^+$: then $x = c$ is a point of local maxima.

- For $c - h < c < c + h \Rightarrow f(c-h) > f(c) < f(c+h),$ where $h \to 0^+$: then $x = c$ is a point of local minima.

- $f(c) \geq f(x)$, for given Interval of $x,$ then $x = c$ is a point of global maxima.

- $f(c) \leq f(x),$ for given Interval of x, then $x = c$ is a point of global minima.

Summary Sheet

- Normally , global maxima / minima occurs at points of local maxima or minima , but every time this is not true there can be exception.

Stationary points	$f'(x) = 0$
Critical points	$f'(x) = 0$ or $f'(x)$ does not exist.

BYJU'S Classes

Application of Derivatives

n^{th} Derivative Test

Road Map

n^{th} Derivative Test

Second Derivative Test

Application of Maxima and Minima

Illustration

JEE Main - 2020 (Sept)

The set of all real values of λ for which the function $f(x) = (1-\cos^2 x)(\lambda + \sin x), x \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, has exactly one maxima and one minima, is _____.

a. $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$

b. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

c. $\left(-\frac{3}{2}, \frac{3}{2}\right)$

d. $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

Solution:

$$f(x) = (1-\cos^2 x)(\lambda + \sin x) = \lambda \sin^2 x + \sin^3 x \quad (\because 1 - \cos^2 x = \sin^2 x))$$

$$f'(x) = 2\lambda \sin x \cos x + 3\sin^2 x \cos x$$

$$= \sin x \cos x \,(2\lambda + 3\sin x)$$

For critical points, $f'(x) = 0$

$$f'(x) = \sin x \cos x \,(2\lambda + 3\sin x) = 0$$

$$\Rightarrow \sin x = 0, -\frac{2\lambda}{3}, (\lambda \neq 0) \qquad (\text{Since, } x \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \cos x \neq 0)$$

For exactly one maxima and minima : $-\frac{2\lambda}{3} \epsilon (-1,1)$ and $\lambda \neq 0 (\because \lambda = 0, \sin x \ = 0)$

$\Rightarrow -1 < -\frac{2\lambda}{3} < 1$ and $\lambda \neq 0$

$\Rightarrow -\frac{3}{2} < \lambda < \frac{3}{2}$ and $\lambda \neq 0$

$\Rightarrow \ \lambda \epsilon \left(-\frac{3}{2}, \frac{3}{2}\right)$ and $\lambda \neq 0$

$\therefore \boxed{\lambda \epsilon \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}}$

So, option (d) is the correct answer.

Illustration

JEE Main - 2019 (Jan)

The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$, on the set $S = \{x \epsilon \mathbb{R}: x^2 + 30 \leq 11x\}$ is :

a. -122

b. -222

c. 122

d. 222

Solution:

$$S = \{x \in \mathbb{R}: x^2 + 30 - 11x \leq 0\}$$

$$S = \{x \in \mathbb{R}: (x-5)(x-6) \leq 0\} \Rightarrow x \in [5,6]$$

$$f(x) = 3x^3 - 18x^2 + 27x - 40$$

$$f'(x) = 9x^2 - 36x + 27 = 9(x^2 - 4x + 3)$$

$$f'(x) = 9(x-1)(x-3)$$

For $x \in [5,6]$, $f'(x) > 0 \Rightarrow f(x)$ is increasing for $x \in [5,6]$

Thus, $f_{max.}$ will occur at $x = 6$

Maximum value: $f(6) = 3 \cdot 6^3 - 18 \cdot 6^2 + 27 \cdot 6 - 40$

$$= 122$$

So, option (c) is the correct answer.

Illustration

JEE Main - 2019 (April)

If $f(x)$ is a non–zero polynomial of degree four, having local extreme points at $x = -1,0,1$: then the set $S = \{x \in \mathbb{R}: f(x) = f(0)\}$ contains exactly :

a. Four rational numbers

b. Two irrational and two rational numbers

c. Four irrational numbers

d. Two irrational and one rational number

Solution:

$f(x)$ has local extreme points at $x = -1, 0, 1$

$\Rightarrow f'(x) = 0$ at $x = -1, 0, 1$

Let $f'(x) = a(x+1)x(x-1) = a(x^3 - x), a \neq 0$

On integrating, we get,

$f(x) = a\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + b$, where b is an arbitrary constant

Since , $f(x) = f(0)$

$$\Rightarrow a\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + b = b$$

$$\Rightarrow a\left(\frac{x^4}{4} - \frac{x^2}{2}\right) = 0$$

$$\Rightarrow \left(\frac{x^4}{4} - \frac{x^2}{2}\right) = 0, \text{since } a \neq 0$$

$$\Rightarrow \frac{x^2}{2}\left(\frac{x^2}{2} - 1\right) = 0$$

$$\Rightarrow \frac{x^2}{2} = 0 \text{ or } \left(\frac{x^2}{2} - 1\right) = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

$$\Rightarrow x = 0, \pm\sqrt{2}$$

So, option (d) is the correct answer.

Illustration

Find all possible values of a for which the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$, has a positive point of maximum .

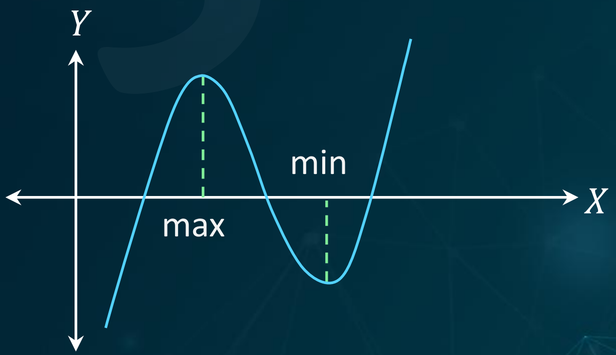
a. $(-\infty, -3) \cup (3, \infty)$ b. $\left(-\infty, \frac{29}{7}\right)$ c. $(-\infty, -3) \cup \left(3, \frac{29}{7}\right)$ d. $(-\infty, \infty)$

Solution:

Since, coefficient of $x^3 > 0 \Rightarrow$ As $x \to \infty,\ f(x) \to \infty$

$$f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$$

$$f'(x) = 3\,x^2 + 6(a-7)x + 3(a^2-9) = 0$$



we can see from the graph that to get positive point of maxima, $f'(x) = 0$ has both roots positive

$$D > 0 \qquad -\frac{b}{2a} > 0 \qquad f'(0) > 0$$

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

Case 1: $D > 0 \Rightarrow b^2 - 4ac > 0$

$\Rightarrow 36(a-7)^2 - 4 \cdot 3 \cdot 3(a^2-9) > 0$

$\Rightarrow a^2 + 49 - 14a - a^2 + 9 > 0$

$\Rightarrow 58 - 14a > 0 \Rightarrow a < \frac{29}{7} \cdots (i)$

Case 2: $-\frac{b}{2a} > 0$

$\Rightarrow \frac{6(a-7)}{2 \cdot 3} < 0 \Rightarrow a < 7 \cdots (ii)$

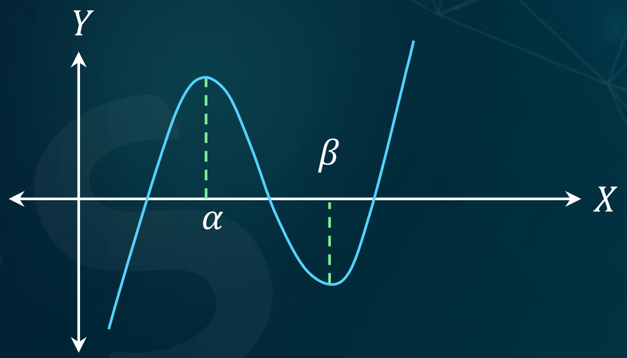
Case 3: $f'(0) > 0 \Rightarrow a^2 - 9 > 0$

$\Rightarrow (a-3)(a+3) > 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty) \cdots (iii)$

By taking intersection of $(i), (ii), (iii)$, we get, $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

So, option (c) is the correct answer.

Alternate Method



Since coefficient of $x^3 > 0 \Rightarrow$ As $x \to \infty,\ f(x) \to \infty$

$$f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$$

$$f'(x) = 3\,x^2 + 6(a-7)x + 3(a^2-9)$$

Let the roots of $f'(x)$ be α, β

Since $f(x)$ has a positive point of maxima $\Rightarrow 0 < \alpha < \beta$

$$\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 0$$

$$\alpha = \frac{-6(a-7) - \sqrt{36(a-7)^2 - 36(a^2-9)}}{6} > 0$$

$$\alpha = \frac{-6(a-7) - \sqrt{36(a-7)^2 - 36(a^2-9)}}{6} > 0$$

$$(7-a) - \sqrt{a^2 - 14a + 49 - a^2 + 9} > 0$$

$$(7-a) > \sqrt{58 - 14a} \quad \cdots (i)$$

$7 - a > 0$ and $58 - 14a > 0$

$\Rightarrow a < 7$ and $a < \frac{29}{7} \Rightarrow a < \frac{29}{7} \quad \cdots (ii)$

On squaring (i), we get,

$$a^2 - 14a + 49 > 58 - 14a \Rightarrow a^2 - 9 > 0$$

$$\Rightarrow (a-3)(a+3) > 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty) \quad \cdots (iii)$$

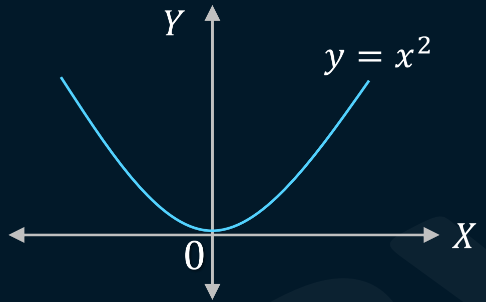
By taking intersection of $(ii), (iii)$, we get, $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

So, option (c) is the correct answer.

Second Derivative Test

If a function $f(x)$ is continuous and differentiable & $f'(x) = 0$, at $x = c$.

◇ If $f''(x) > 0$ at $x = c \Rightarrow f'(x)$ is increasing at $x = c$
$\Rightarrow x = c$ is a point of local minima

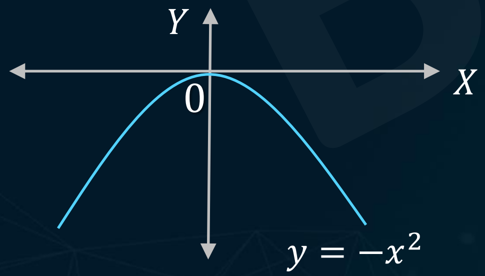


$$f(x) = x^2$$

$$f'(x) = 2x = 0 \Rightarrow x = 0$$

$f''(x) = 2 > 0 \Rightarrow x = 0$ is a point of local minima

◇ If $f''(x) < 0$ at $x = c \Rightarrow f'(x)$ is decreasing at $x = c$
$\Rightarrow x = c$ is a point of local maxima



$$f(x) = -x^2$$

$$f'(x) = -2x = 0 \Rightarrow x = 0$$

$f''(x) = -2 < 0 \Rightarrow x = 0$ is a point of local maxima

Second Derivative Test

If a function $f(x)$ is continuous and differentiable & $f'(x) = 0$, at $x = c$.

- If $f''(x) > 0$ at $x = c$, $\Rightarrow x = c$ is a point of local minima

- If $f''(x) < 0$ at $x = c$, $\Rightarrow x = c$ is a point of local maxima

- If $f''(x) = 0$ at $x = c$, then proceed to the higher derivative test.

Illustration

Find the points of extrema for the function $f(x) = 2x^3 - 9x^2 + 12x + 6$

Solution:

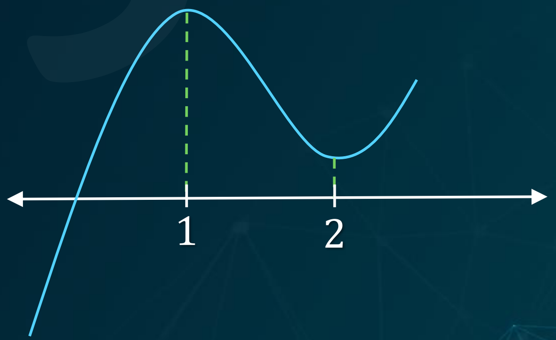
$$f(x) = 2x^3 - 9x^2 + 12x + 6$$

$$f'(x) = 6x^2 - 18x + 12$$

For critical points, $f'(x) = 0$

$$\Rightarrow f'(x) = 6(x^2 - 3x + 2) = 0$$

$$\Rightarrow (x-1)(x-2) \Rightarrow x = 1, 2$$



$$f''(x) = 6(2x - 3)$$

$f''(1) = 6(2-3) < 0 \quad \Rightarrow x = 1$ is a point of local maxima

$f''(2) = 6(4-3) > 0 \quad \Rightarrow x = 2$ is a point of local minima

Illustration

Find the points of extrema for the function $f(x) = x + \sin 2x$ for $0 \leq x < 2\pi$

Solution:

$$f(x) = x + \sin 2x \text{ for } 0 \leq x < 2\pi$$

$$f'(x) = 1 + 2\cos 2x$$

For critical points, $f'(x) = 0$

$$\Rightarrow f'(x) = 1 + 2\cos 2x = 0$$

$$\Rightarrow \cos 2x = -\frac{1}{2} \Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{I}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, n \in \mathbb{I}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$f''(x) = -4\sin 2x$$

$$f''\left(\frac{\pi}{3}\right) = -4\sin\left(\frac{2\pi}{3}\right) < 0 \Rightarrow x = \frac{\pi}{3}$$ is a point of local maxima

$$f\left(\frac{\pi}{3}\right) = x + \sin 2x = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$ local maximum value

$$f''\left(\frac{2\pi}{3}\right) = -4\sin\left(\frac{4\pi}{3}\right) = 4\sin\left(\frac{\pi}{3}\right) > 0 \Rightarrow x = \frac{2\pi}{3}$$ is a point of local minima

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\left(\frac{4\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$ local minimum value

$$f''\left(\frac{4\pi}{3}\right) = -4\sin\left(\frac{8\pi}{3}\right) = -4\sin\left(\frac{2\pi}{3}\right) < 0 \Rightarrow x = \frac{4\pi}{3}$$ is a point of local maxima

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\left(\frac{8\pi}{3}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$ local maximum value

$$f''\left(\frac{5\pi}{3}\right) = -4\sin\left(\frac{10\pi}{3}\right) = 4\sin\left(\frac{\pi}{3}\right) > 0 \Rightarrow x = \frac{5\pi}{3}$$ is a point of local minima

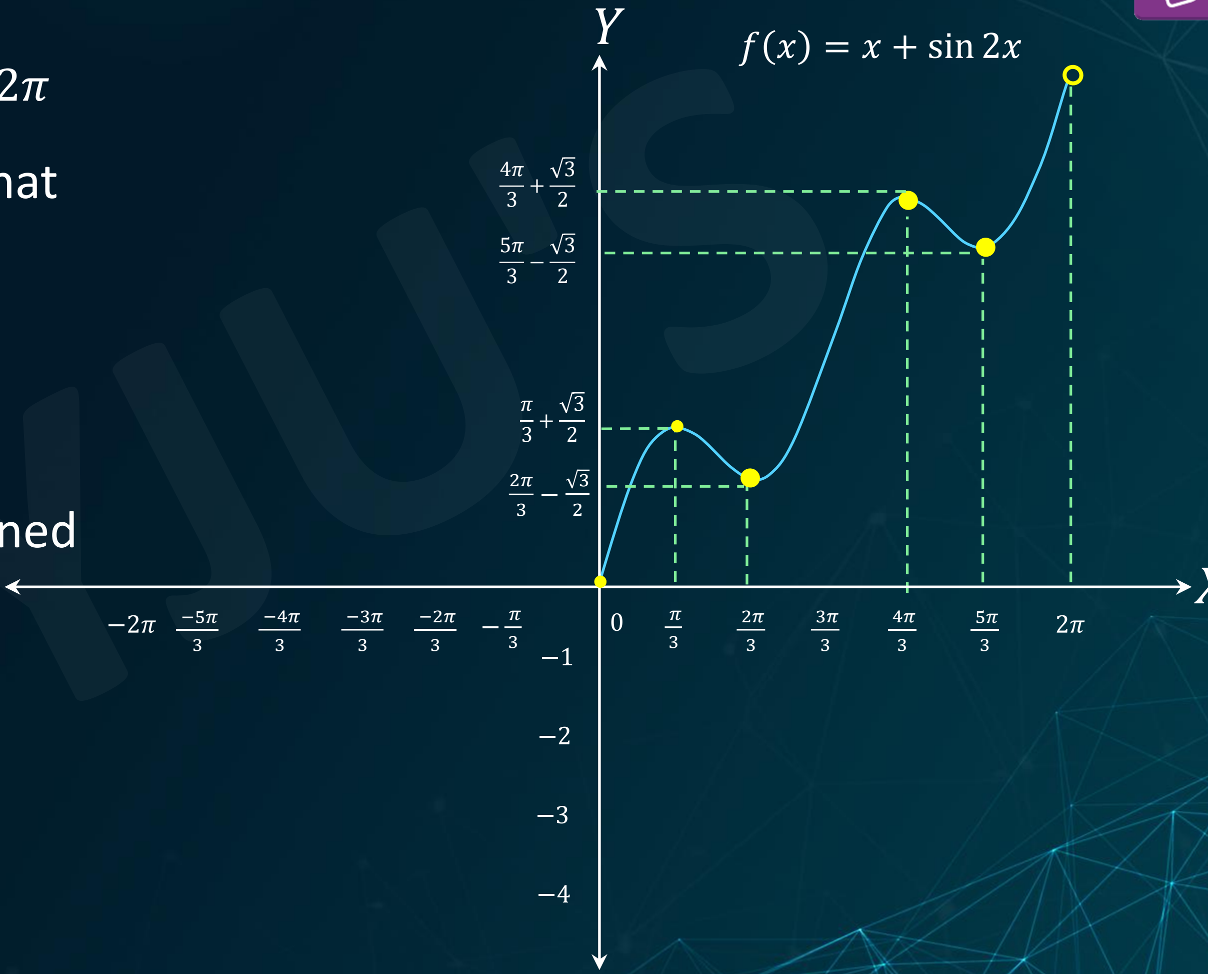
$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\left(\frac{10\pi}{3}\right) = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$ local minimum value

Observations using graph

$f(x) = x + \sin 2x$ for $0 \le x < 2\pi$

We can see from the graph that points of extrema are $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

- $x = 0$ is a global minima
- Global maxima is not defined



n^{th} Derivative Test

Let $f(x)$ have derivatives up to n^{th} order ,

If $f'(c) = f''(c) = \cdots = 0$, then find the first non – zero higher derivative.

Let $f^n(c)$ be the first non – zero derivative.

- If n is even , and $f^n(c) > 0 \;\Rightarrow x = c$ is a local minima
- If n is even , and $f^n(c) < 0 \;\Rightarrow x = c$ is a local maxima

Example: $f(x) = (x-3)^4$

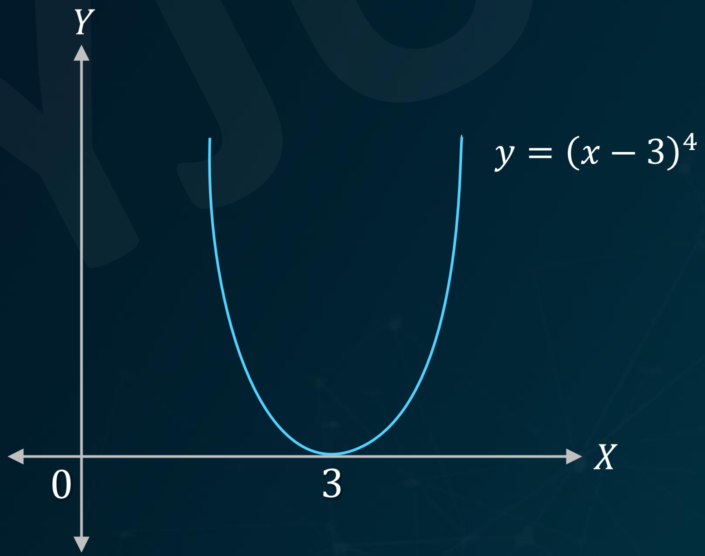
$f'(x) = 4(x-3)^3 = 0 \Rightarrow x = 3$: critical point

$f''(x) = 12(x-3)^2 \qquad f''(3) = 0$

$f'''(x) = 24(x-3) \qquad f'''(3) = 0$

$f^{iv}(x) = 24 > 0$

$\Rightarrow x = 3$ is point of minima



◇ If n is even , and $f^n(c) > 0$

$\Rightarrow x = c$ is a local minima

◇ If n is even , and $f^n(c) < 0$

$\Rightarrow x = c$ is a local maxima

n^{th} Derivative Test

Let $f(x)$ have derivatives up to n^{th} order ,

If $f'(c) = f''(c) = \cdots = 0$, then find the first non – zero higher derivative

Let $f^{n}(c)$ be the first non – zero derivative

- If n is odd , and $f^{n}(c) > 0$, $\Rightarrow f(x)$ is increasing at $x = c$
- If n is odd , and $f^{n}(c) < 0$, $\Rightarrow f(x)$ is decreasing at $x = c$

Example: $f(x) = (2x-1)^3, x \in \mathbb{R}$

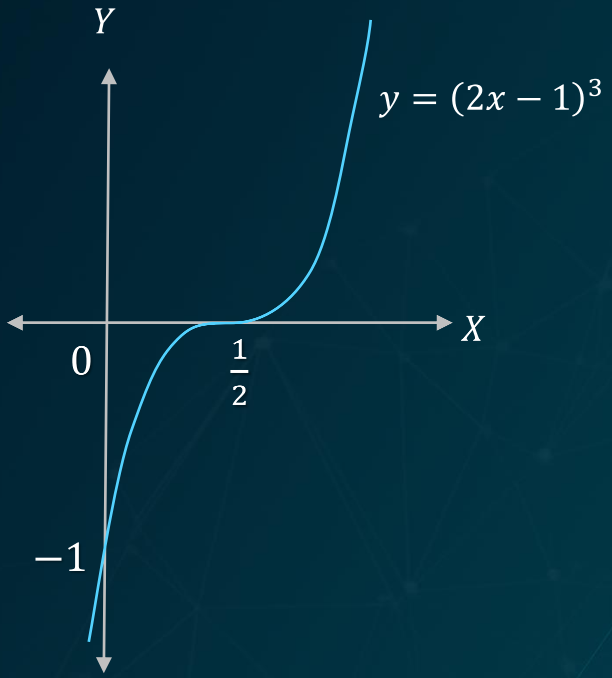
$$f'(x) = 6(2x-1)^2 = 0 \Rightarrow x = \frac{1}{2} \text{ is a critical point}$$

$$f''(x) = 24(2x-1) \Rightarrow f''\left(\frac{1}{2}\right) = 0$$

$$f'''(x) = 48 > 0$$

$\Rightarrow f(x)$ is increasing at $x = \frac{1}{2}$

There is no point of local extrema.



Summary Sheet

- **Second Derivative Test**

If a function $f(x)$ is continuous and differentiable & $f'(x) = 0$, at $x = c$.

◇ If $f''(x) > 0$ at $x = c$, $\Rightarrow x = c$ is a local minima

◇ If $f''(x) < 0$ at $x = c$, $\Rightarrow x = c$ is a local maxima

◇ If $f''(x) = 0$ at $x = c$, then proceed to the higher derivative test.

- **n^{th} Derivative Test**

Let $f(x)$ have derivatives up to n^{th} order,

If $f'(c) = f''(c) = \cdots = 0$, then find the first non – zero higher derivative.

Let $f^n(c)$ be the first non – zero derivative.

◇ If n is even, and $f^n(c) > 0 \Rightarrow x = c$ is a local minima

◇ If n is even, and $f^n(c) < 0 \Rightarrow x = c$ is a local maxima

Summary Sheet

Let $f(x)$ have derivatives up to n^{th} order ,

If $f'(c) = f''(c) = \cdots = 0$, then find the first non – zero higher derivative

Let $f^n(c)$ be the first non – zero derivative

- If n is odd , and $f^n(c) > 0$, $\Rightarrow f(x)$ is increasing at $x = c$
- If n is odd , and $f^n(c) < 0$, $\Rightarrow f(x)$ is decreasing at $x = c$

BYJU'S Classes

Application of Derivatives

Application of Maxima and Minima

Road Map

Previous years JEE problems

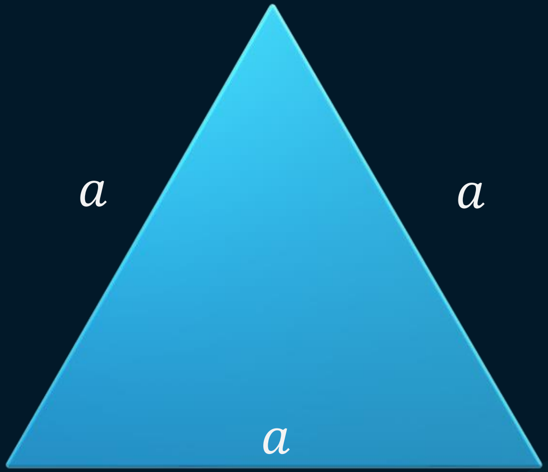
Applications of
Maxima and Minima

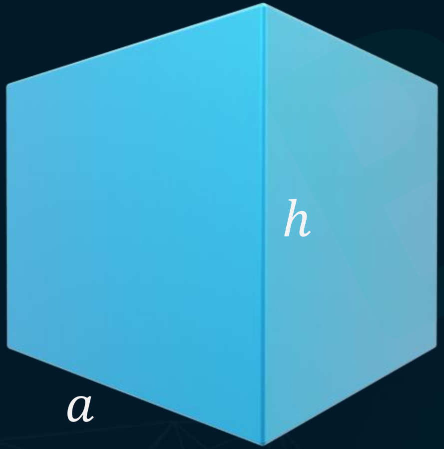
Mensuration Formulae

Mensuration Formulae

3D figures	Volume	3D Curved/lateral surface area	Total surface area
Cube	l^3	$4l^2$	$6l^2$
Cuboid	lbh	$2h(l+b)$	$2(lb+bh+hl)$
Cone	$\frac{1}{3}\pi r^2 h$	πrl	$\pi rl+\pi r^2$
Cylinder	$\pi r^2 h$	$2\pi rh$	$2\pi rh+2\pi r^2$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$	$4\pi r^2$

Note: Triangular Prism





Area of base = Area of equilateral triangle $= \frac{\sqrt{3}}{4} \cdot a^2$

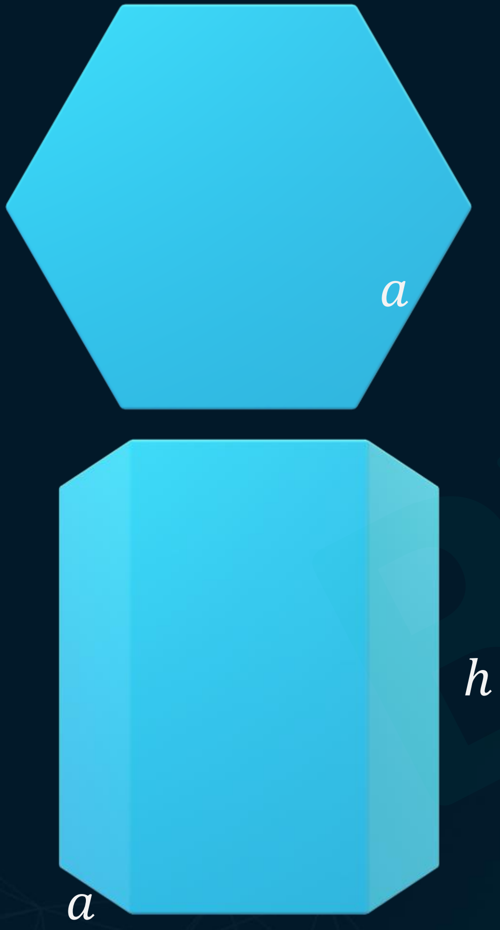
Volume = Area × Height $= \frac{\sqrt{3}}{4} . a^2 . h$

Lateral surface area = Area of 3 lateral rectangle $= 3a . h$

Total surface area = 2 (Area of base) + Lateral surface area

$$= 3ah + 2\left(\frac{\sqrt{3}}{4} . a^2\right)$$

Note: Hexagonal Prism



Area of base $= 6\ \times$Area of equilateral triangle $= 6\ \cdot \frac{\sqrt{3}}{4} \cdot a^2$

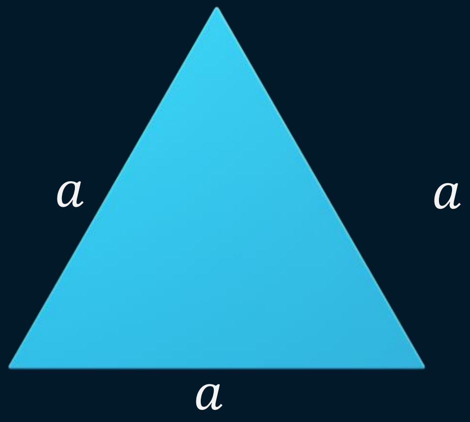
Volume $=$ Area $\times$Height $=\ 6.\frac{\sqrt{3}}{4}.a^2.h$

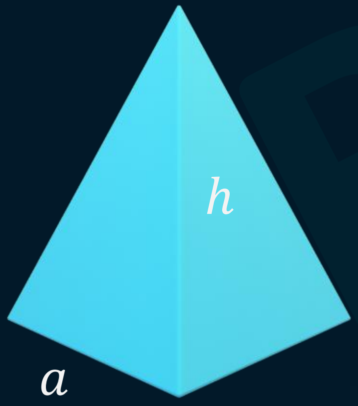
Lateral surface area = Area of 6 lateral rectangle = $6a\ .h$

Total surface area $= 2($Area of base$\}$ $+$ Lateral surface area

$$= 6a\ h + 2(6.\frac{\sqrt{3}}{4}.a^2)$$

Note: Triangular Pyramid (Tetrahedron)





Area of base $=$ Area of equilateral triangle $= \frac{\sqrt{3}}{4} \cdot a^2$

Volume $=$ Area $\times$ Height $= \frac{1}{3} . \frac{\sqrt{3}}{4} . a^2 . h$

Lateral surface area $=$ Area of 3 equilateral triangle

$$= 3 . \frac{\sqrt{3}}{4} . a^2$$

Total surface area $=$ Area of base $+$ Lateral surface area

$$= 4 . \frac{\sqrt{3}}{4} . a^2$$

Illustration

Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Solution:

Given $x > 0, y > 0$ and $x + y = 60 \Rightarrow x = 60 - y$

Let $F = xy^3$

$F = xy^3 = (60 - y)\, y^3 = 60y^3 - y^4$ {Given $x = 60 - y$}

By differentiating w.r.t y, we get,

$F'(y) = 180y^2 - 4y^3$

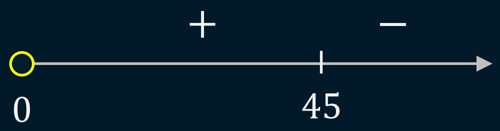
For extrema, $F'(y) = 0$

$\Rightarrow F'(y) = 180y^2 - 4y^3 = 0$

$\Rightarrow 4y^2(45 - y) = 0$

$\Rightarrow y = 0, 45$

Critical points are $0, 45$



We can see here that at $y = 45$, $F'(y)$ is changing its sign from positive to negative.
So, $F(y)$ is increasing from $x = 0$ to $x = 45$

$\Rightarrow$ At $y = 45, F(y)$ will be maxima

Given, $x + y = 60 \Rightarrow$ At $y = 45, x = 60 - 45 = 15$

Thus , required positive numbers are: $y = 45$, $x = 15$

Illustration

JEE Main - 2019

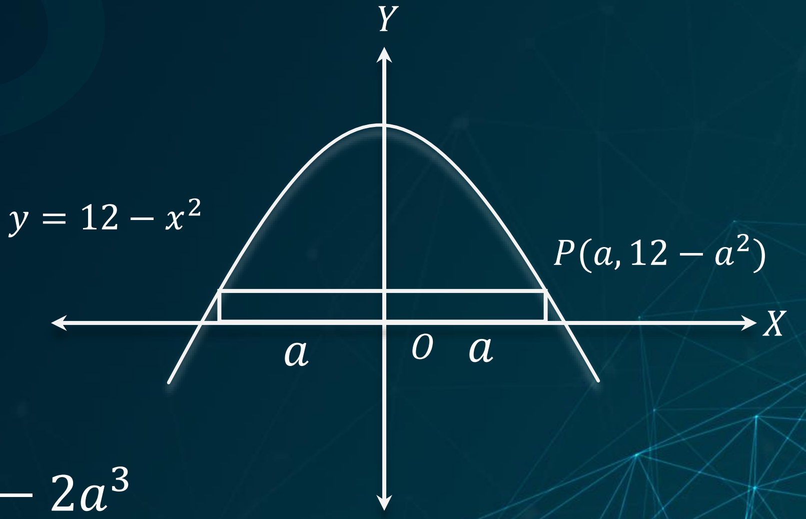
The maximum area (in sq. units) of a rectangle having its base on the X-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is

$a.\ 32$ $\quad$ $b.\ 18\sqrt{3}$ $\quad$ $c.\ 20\sqrt{2}$ $\quad$ $d.\ 36$

Solution:

Given: $y = 12 - x^2$

Let any point $P(a, 12 - a^2)$ be on the parabola $y = 12 - x^2$



Area of rectangle, $A = l \times b$

From figure, $l = 2a, b = 12 - a^2$

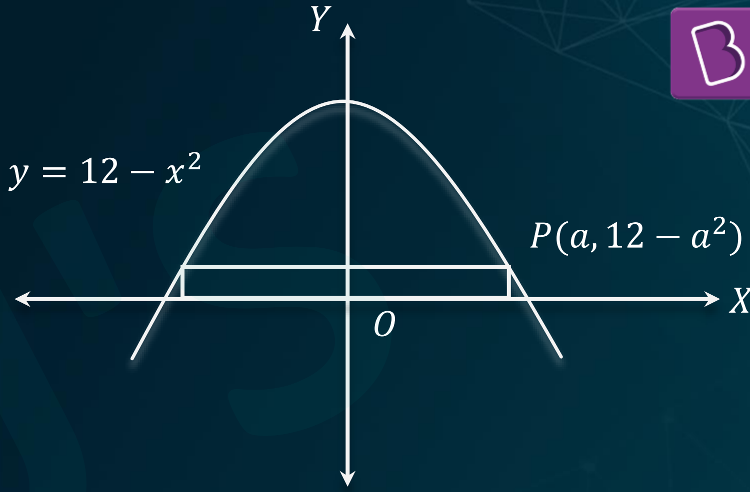
$\Rightarrow$ Area of rectangle, $A = 2a(12 - a^2) = 24a - 2a^3$

Differentiate w.r.t a, we get,

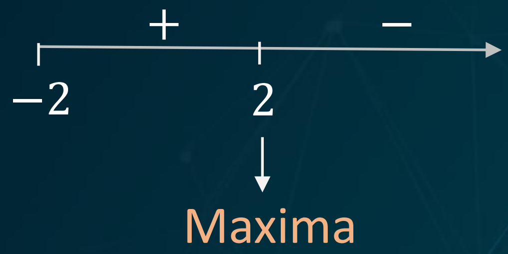
$$\frac{dA}{da} = 24 - 6a^2$$

To maximise area, $\frac{dA}{da} = 0$

$\Rightarrow 24 - 6a^2 = 0 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$



We can see here that at $a > 0 \Rightarrow a = 2$,
Also, at $a = 2$, sign of $f'(y)$ changes from positive to negative.



$\Rightarrow$ Area will be maximum at $a = 2$

Area$_{max}$ $= 2 \times 2 \times (12 - 4) = \boxed{32 \text{ sq. units}}$

Hence, option (a) is the correct answer.

Illustration

The height($in\ units$) of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 units is :

$a. \sqrt{3}$

$b. 2\sqrt{3}$

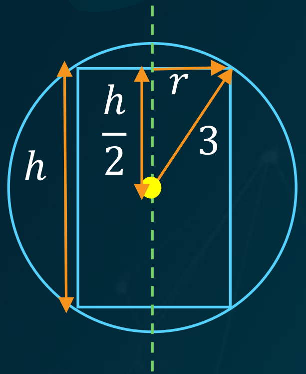
$c. \frac{2\sqrt{3}}{6}$

$d. \sqrt{6}$

Solution:

From right angle triangle,

$$r^2 + \left(\frac{h}{2}\right)^2 = 3^2 = 9 \cdots (i)$$



We know, volume of cylinder = $\pi r^2 h$

By substituting $r^2 = 9 - \left(\frac{h}{2}\right)^2$, we get,

$$V = \pi\left(9 - \left(\frac{h}{2}\right)^2\right).h$$

$$\Rightarrow V = \pi\left(9h - \frac{h^3}{4}\right)$$

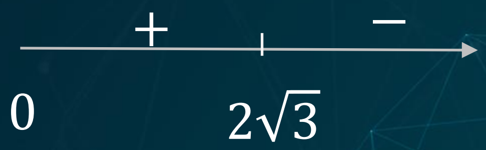
Differentiating w.r.t to h, we get,

$$\frac{dV}{dh} = \pi\left(9 - \frac{3}{4}h^2\right)$$

To maximise the volume, $\frac{dV}{dh} = 0$

$$\Rightarrow \pi\left(9 - \frac{3}{4}h^2\right) = 0 \Rightarrow \frac{3h^2}{4} = 9 \Rightarrow h = 2\sqrt{3}\ (\because h > 0)$$

We can see here that at $h = 2\sqrt{3}$, sign of $\frac{dV}{dh}$ is changing from positive to negative.



$\Rightarrow$ At $h = 2\sqrt{3}$, volume will be maximum

Thus, height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 units is $2\sqrt{3}$ units.

Hence, option (b) is the correct answer.

Illustration

JEE Main - 2016

A wire of length 2 units is cut into two parts which are bent respectively to form a square of side x units and a circle of radius of r units . If the sum of areas of square and the circle so formed is minimum , then :

$a.\ 2x = (\pi + 4)r$

$b.\ (4 - \pi)x = \pi r$

$c.\ x = 2r$

$d.\ 2x = r$

Solution:

Given, sum of perimeter of square and circle $= 2$

$$\Rightarrow 4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1 \cdots (i)$$

$$\Rightarrow x = \frac{1 - \pi r}{2}$$

Sum of areas $S = x^2 + \pi r^2 \cdots (ii)$

Substitute $x = \frac{1-\pi r}{2}$ in (ii), we get,

$$S = \pi r^2 + \left(\frac{1-\pi r}{2}\right)^2$$

Differentiating w.r.t r, we get,

$$\frac{dS}{dr} = 2\pi r - \frac{2\pi(1-\pi r)}{4} = \frac{\pi}{2}\big((4+\pi)r - 1\big)$$

To minimise, $\frac{dS}{dr} = 0$

$$\Rightarrow \frac{\pi}{2}\big((4+\pi)r - 1\big) = 0 \Rightarrow r = \frac{1}{4+\pi}$$

To check maxima and minima, calculate $\frac{d^2S}{dr^2}$

$$\frac{dS}{dr} = \frac{\pi}{2}((4+\pi)r - 1)$$

$$\frac{d^2S}{dr^2} = \frac{\pi}{2}(4+\pi)$$

$\because \frac{d^2S}{dr^2} > 0$, $\therefore S$ is minimum at $r = \frac{1}{4+\pi}$

Substituting $\pi r = 1 - 4r$ in (i)

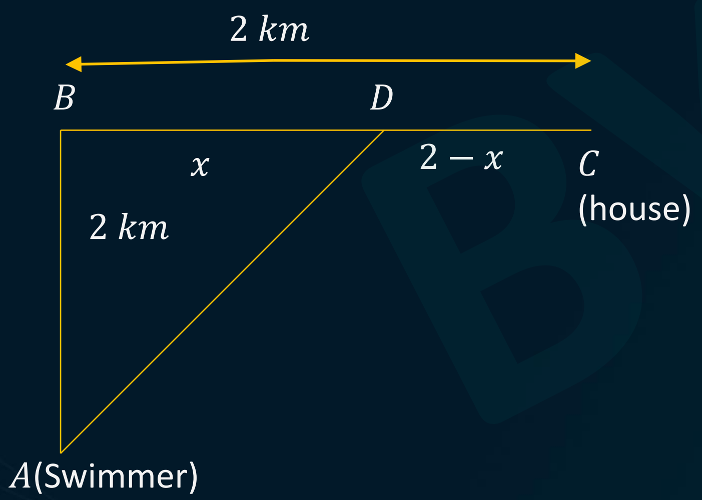
$\Rightarrow 2x + 1 - 4r = 1$ $\boxed{\Rightarrow x = 2r}$

Hence, option (c) is the correct answer.

Illustration

A swimmer across the sea is at the distance $2\ km$ from the closest point on a straight seashore. The house of the swimmer is on the shore at distance $2\ km$ from that closest point. He can swim at a speed of $3\ km$ per hour and walk at a speed of $5\ km$ per hour. At what point on the shore should he land so that he reaches his house in the shortest possible time ?

Solution:



Let swimmer lands on shore at $x\ km$ distance from straight seashore and walks $(2-x)\ km$ to reach the house.

Total time(t) = time taken from A to D(t_{AD}) + time taken from D to $C(t_{DC})$

$$Time = \frac{Distance}{Speed}$$

Total time : $t = t_{AD} + t_{DC}$

$$t = \frac{\sqrt{4+x^2}}{3} + \frac{2-x}{5} \quad ,0 \le x \le 2$$

$$\frac{dt}{dx} = \frac{1}{2\sqrt{4+x^2}} \times \frac{2x}{3} - \frac{1}{5} = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{5}$$

To minimise, $\frac{dt}{dx} = 0$

$$\Rightarrow \frac{x}{3\sqrt{4+x^2}} - \frac{1}{5} = 0$$

$$\Rightarrow 25x^2 = 9(4 + x^2)$$

$$\Rightarrow 16x^2 = 36 \Rightarrow x^2 = \frac{36}{16} \Rightarrow x = \frac{3}{2} \ (\because x > 0)$$

We can see here that at $x = \frac{3}{2}$, sign of $\frac{dt}{dx}$ is changing from negative to positive.

$0 \quad - \quad \frac{3}{2} \quad +$

$\Rightarrow$ At $x = \frac{3}{2}$, time will be minimum

So the swimmer should land on the shore at $\frac{3}{2}\,km$ from straight seashore to reaches his house in the shortest possible time.

Illustration

A piece of pipe is being carried down a hallway that is $27\ ft$ wide. At the end of the halfway, there is a right angled turn and the hallway narrows down to $8\ ft$ wide. What is the longest pipe (always keeping it horizontal) that can be carried around the turn in the hallway ?

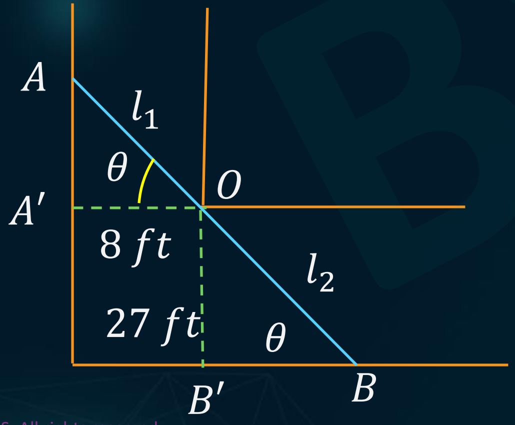
a. $13\sqrt{13}$

b. $33\sqrt{3}$

c. $6\sqrt{13}$

d. $13\sqrt{6}$

Solution: At the time of taking turn from one hallway to other hallway, at that instant, we have,



l_1 is the length of pipe in $8\ ft$ wide hallway

l_2 is the length of pipe in the $27\ ft$ wide hallway

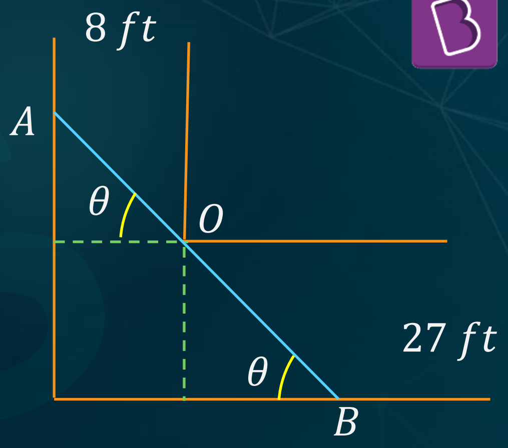
θ is the angle made by the pipe at the instant w.r.t ground

In $\Delta\, AOA'$ $\cos\theta = \dfrac{8}{l_1} \Rightarrow l_1 = 8\sec\theta$

In $\Delta\, BOB'$ $\sin\theta = \dfrac{27}{l_2} \Rightarrow l_2 = 27\operatorname{cosec}\theta$

Length of pipe, $L = l_1 + l_2 = 8\sec\theta + 27\,\text{cosec}\,\theta$

To maximise the length $\frac{dL}{d\theta} = 0$



$$\frac{dL}{d\theta} = 8\sec\theta\tan\theta - 27\,\text{cosec}\,\theta\cot\theta = 0$$

$$\Rightarrow \frac{8\sin\theta}{\cos^2\theta} - \frac{27\cos\theta}{\sin^2\theta} = 0$$

$$\Rightarrow 8\sin^3\theta - 27\cos^3\theta = 0$$

$$\Rightarrow \tan^3\theta = \frac{27}{8} \quad \Rightarrow \tan\theta = \frac{3}{2}$$

$$\frac{dL}{d\theta} = 8\sec\theta\tan\theta - 27\,\text{cosec}\,\theta\cot\theta$$

$$\frac{d^2L}{d\theta^2} = 8(\sec\theta\tan^2\theta + \sec^3\theta) + 27(\text{cosec}\,\theta\cot^2\theta + cosec^3\theta)$$

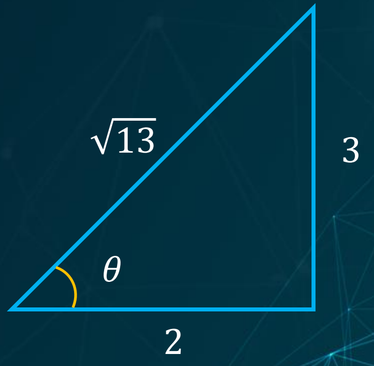
$$\frac{d^2L}{d\theta^2} = 8(\sec\theta\tan^2\theta + \sec^3\theta) + 27(\operatorname{cosec}\theta\cot^2\theta + cosec^3\theta)$$

$\Rightarrow$ At $\tan\theta = \frac{3}{2}, \frac{d^2L}{d\theta^2} > 0$

$\Rightarrow$ At $\tan\theta = \frac{3}{2}$, we get maximum length of the pipe

$$L = 8\sec\theta + 27\ \operatorname{cosec}\theta$$

$$L = 8\cdot\frac{\sqrt{13}}{2} + 27\cdot\frac{\sqrt{13}}{3} \quad \boxed{= 13\sqrt{13}\ ft.}$$



Hence, option (a) is the correct answer.

Illustration

Two towns A and B are situated on the same side of a straight road at distances a and b respectively , perpendiculars drawn from A and B meet the road at point C and D respectively. The distance between C and D is c . A hospital is to be built at a point P on the road such that the distance APB is minimum. Find distance of P from point C .

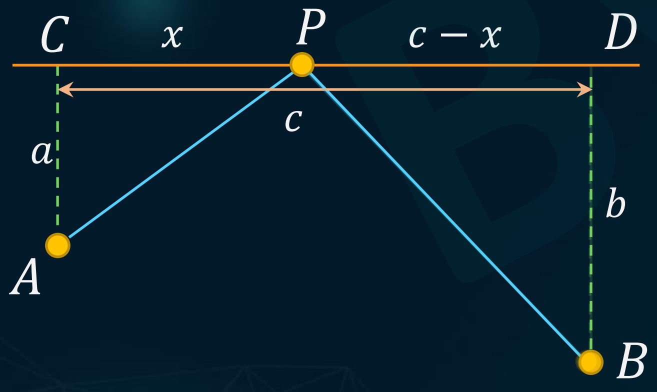
a. $\dfrac{ac}{a+b}$

b. $\dfrac{bc}{a+b}$

c. $\dfrac{a+c}{a+b}$

d. $\dfrac{b+c}{a+b}$

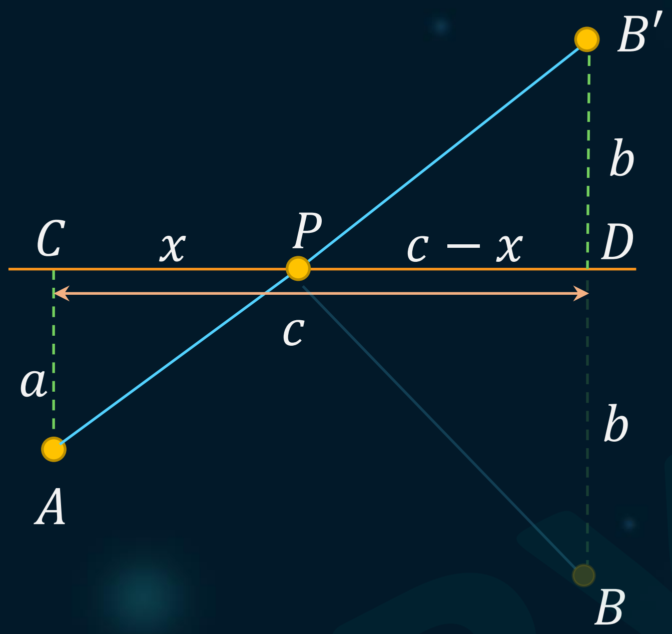
Solution:



Given A and B are situated on the same side of a straight road

$AC = a, \qquad BD = b, \qquad CD = x$

Let x be the distance of P from C



Since APB is minimum

$\Rightarrow APB'$ lies on a straight line where B' is mirror image of B w.r.t line CD

From figure, ΔAPC and $\Delta DPB'$ are similar.

$$\Rightarrow \frac{x}{a} = \frac{c-x}{b}$$

$$\Rightarrow xb = ca - ax$$

$$\Rightarrow (a+b)x = ac \quad \boxed{\Rightarrow x = \frac{ac}{a+b}}$$

Hence, option (a) is the correct answer.

Summary Sheet

Mensuration Formulae:

3D figures	Volume	3D Curved/lateral surface area	Total surface area
Cube	l^3	$4l^2$	$6l^2$
Cuboid	lbh	$2h(l+b)$	$2(lb+bh+hl)$
Cone	$\frac{1}{3}\pi r^2 h$	πrl	$\pi rl+\pi r^2$
Cylinder	$\pi r^2 h$	$2\pi rh$	$2\pi rh+2\pi r^2$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$	$4\pi r^2$

BYJU'S Classes

Application of Derivatives

Curve Tracing

Road Map

Curve Tracing

Asymptotes

Application of Maxima and Minima

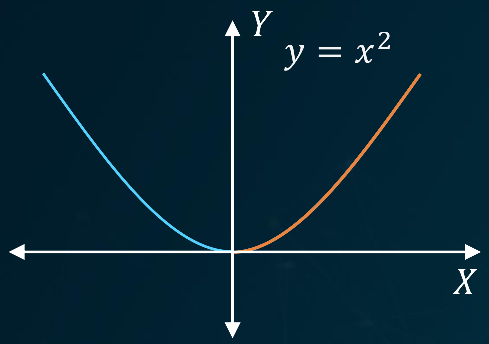
Curve Tracing

Symmetry:

- If on replacing $x \to -x$, equation of curve doesn't change, then the curve is symmetric about y –axis.

For plotting such curves, draw the curve for $x \geq 0$ and then take mirror image with respect to $Y -$ axis.

Example: $y = x^2$

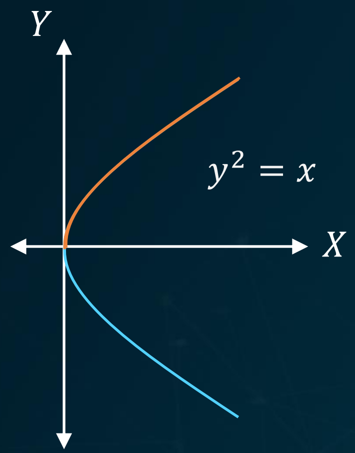


Curve Tracing

- If on replacing $y \to -y$, equation of curve doesn't change, then the curve is symmetric about x −axis.

  For plotting such curves, plot the curve for $y \geq 0$ and then take mirror image with respect to $X -$ axis.

Example: $y^2 = x$

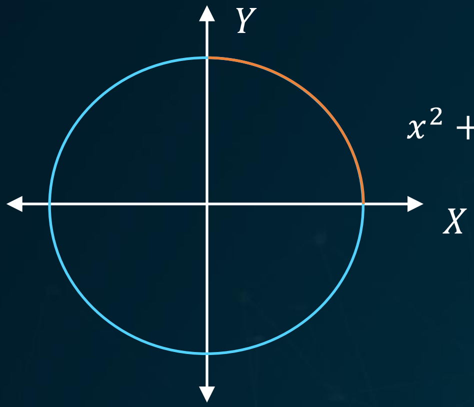


Curve Tracing

- If on replacing $y \to -y$ & $x \to -x$, equation of curve doesn't change then the curve is symmetric in all four quadrants

For plotting such curves, plot the curve for 1^{st} quadrant and then take mirror image with respect to $X -$ axis as well as $Y -$ axis.

Example: $x^2 + y^2 = 4$

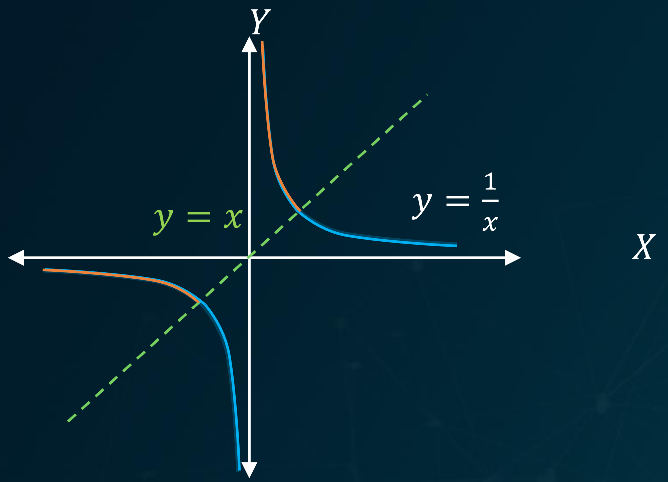


Curve Tracing

- If on interchanging x & y, equation of curve doesn't change, then the curve is symmetric about $y = x$.

The curve is mirror image with respect to line $y = x$.

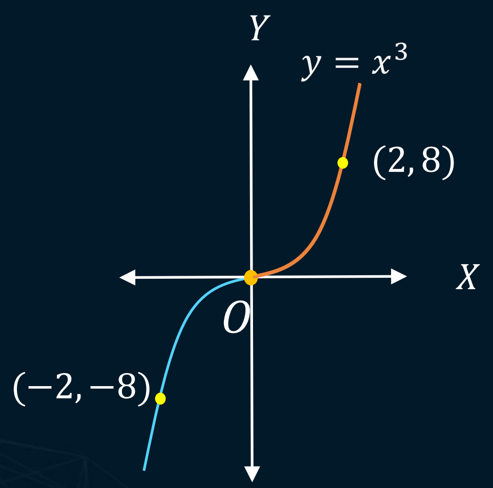
Example: $y = \frac{1}{x}$



Curve Tracing

- If $f(x)$ is an odd function then the graph of $f(x)$ is symmetric about the origin.

Example: $y = x^3$



Note:

For odd function, $f(-x) = -f(x)$

Substitute $x = 0$,

$\Rightarrow f(0) = -f(0)$

$\Rightarrow f(0) = 0$

$\therefore$ If an odd function is defined at $x = 0$, then $f(0) = 0$.

Curve Tracing

Steps to draw curve:

- Check symmetry (if any).
- Get idea about domain and range of function to be drawn.
- Identify discontinuity (if any).
- Get critical points & increasing/decreasing intervals.
- Estimate value of function at $x = 0$ (Points of intersection with $Y - axis$), $\pm\infty$ (behavior of curve at extreme points) and at critical points.
- Find roots (if any)

Curve Tracing

Shape of Curve: Consider a curve $y = f(x)$

- $\frac{dy}{dx} > 0 \Rightarrow f(x)$ is increasing
- $\frac{dy}{dx} < 0 \Rightarrow f(x)$ is decreasing
- $\frac{d^2y}{dx^2} > 0 \Rightarrow f(x)$ will be concave upwards
- $\frac{d^2y}{dx^2} < 0 \Rightarrow f(x)$ will be concave downwards

Illustration

Plot graph of $y = (x-1)^2(x-2)$

Solution:

$$y = (x-1)^2(x-2)$$

i) On replacing $y \to -y$ or $x \to -x$ equation of curve changes.

$\Rightarrow$ Curve is not symmetric.

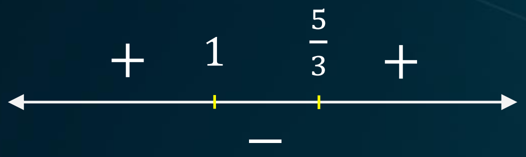
ii) Now, since the given function is polynomial,

$\Rightarrow$ Domain is $\mathbb{R}$

$\Rightarrow$ Range is $\mathbb{R}$

iii) For roots, $y = 0 \Rightarrow x = 1,2$ (1 is the repeated root)

iv) For critical points,



$$\frac{dy}{dx} = (x-1)(3x-5) = 0 \Rightarrow x = 1, \frac{5}{3}$$

$$\frac{dy}{dx} > 0 \Rightarrow \underbrace{x \in (-\infty, 1) \cup \left(\frac{5}{3}, \infty\right)}_{\text{function is increasing}}$$

$$\frac{dy}{dx} < 0 \Rightarrow \underbrace{x \in \left(1, \frac{5}{3}\right)}_{\text{function is decreasing}}$$

v) Value of function at critical points

Since function is increasing from $(-\infty, 1) \cup \left(\frac{5}{3}, \infty\right)$

$\Rightarrow$ Local maxima: $y(1) = (1-1)^2(1-2) = 0$

Since function is decreasing from $\left(1, \frac{5}{3}\right)$

$\Rightarrow$ Local minima: $y\left(\frac{5}{3}\right) = \left(\frac{5}{3} - 1\right)^2 \left(\frac{5}{3} - 2\right) = -\frac{4}{27}$

vi) Shape of curve

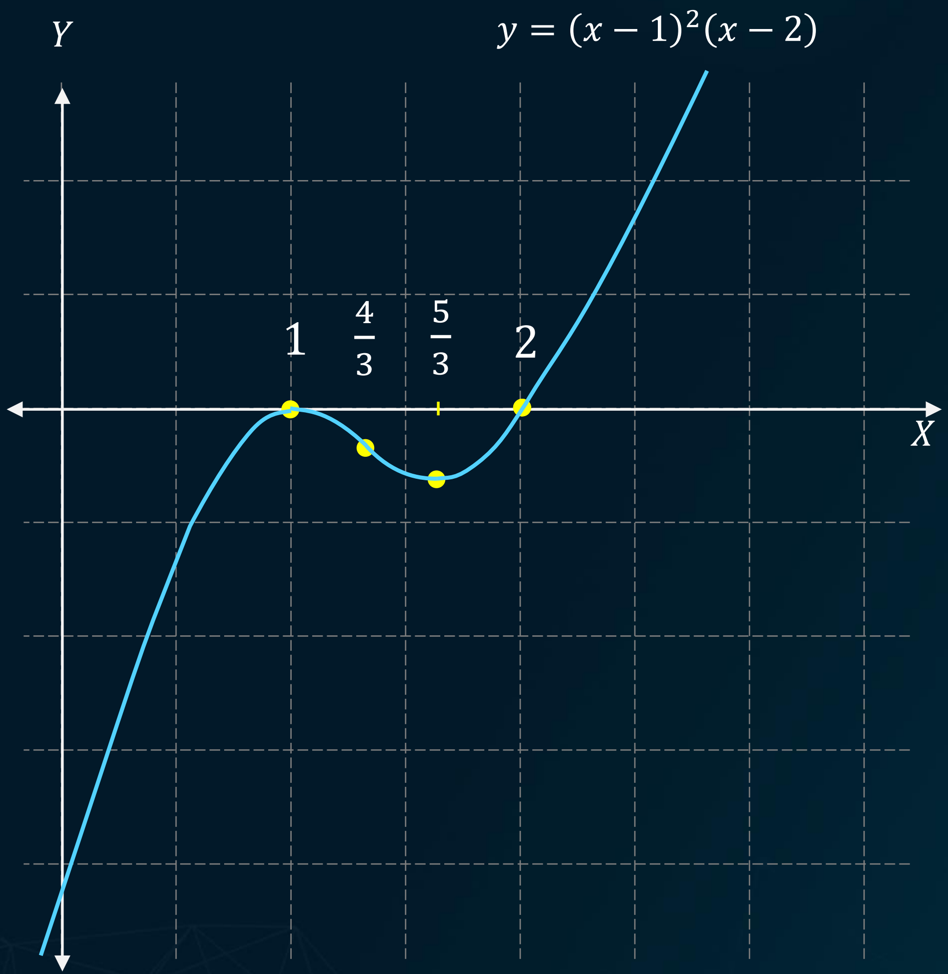
$$\frac{dy}{dx} = (x-1)(3x-5) = 3x^2 - 8x + 5$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 8 = 0 \Rightarrow x = \frac{4}{3} \text{ is point of inflection.}$$

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{concave upwards; } x > \frac{4}{3}$$

$$\frac{d^2y}{dx^2} < 0 \Rightarrow \text{concave downwards; } x < \frac{4}{3}$$

$$y\left(\frac{4}{3}\right) = \left(\frac{4}{3} - 1\right)^2 \left(\frac{4}{3} - 2\right) = -\frac{2}{27}$$



Start drawing the curve from $-\infty$(As $x \to -\infty \Rightarrow y \to -\infty$)

From $-\infty$ to 1, graph is increasing and concave downwards.

At $x = 1$, we get a local maxima.

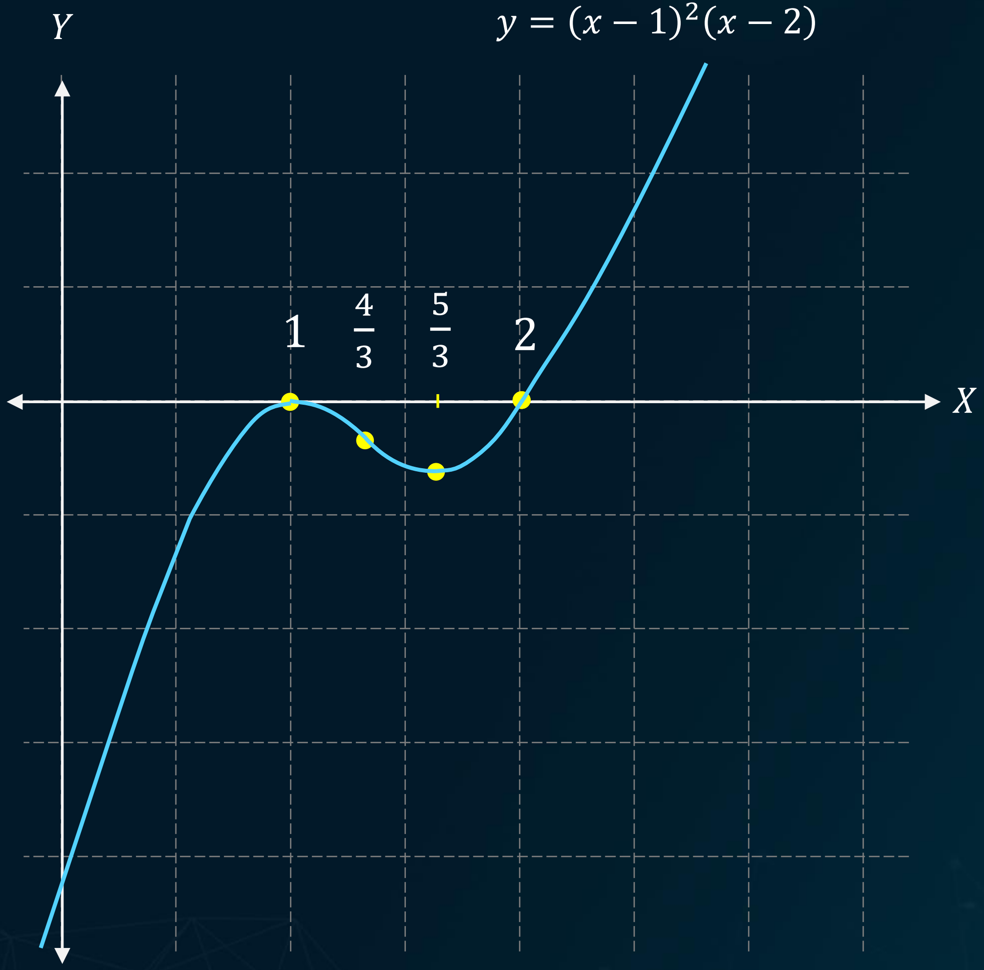
Further from 1 to $\frac{5}{3}$, graph is decreasing.

Also, it is concave downwards till $x = \frac{4}{3}$ and from $\frac{4}{3}$ onwards, it is concave upwards.

At $x = \frac{5}{3}$, we get a local minima.

From $\frac{5}{3}$ to ∞, Graph is increasing and concave upwards.

Clearly, $x = 1$ and 2 are the roots of y



$y = (x-1)^2(x-2)$

$x = 1, 2$ (1 is the repeated root)

$\frac{dy}{dx} = 0 \Rightarrow x = 1, \frac{5}{3}$

$y(1) = 0$ & $y\left(\frac{5}{3}\right) = -\frac{4}{27}$,

$x = \frac{4}{3}$ is point of inflection , $y\left(\frac{4}{3}\right) = -\frac{2}{27}$

$\Rightarrow$ concave upwards; $x > \frac{4}{3}$

$\Rightarrow$ concave downwards; $x < \frac{4}{3}$

Illustration

Plot graph of $ii)\ y = x + \sin x$

Solution:

i) $y = x + \sin x$ is an odd function $(\because (f(-x) = -f(x))$

Therefore, the function is symmetric about origin.

ii) For $y = x + \sin x$,

Domain is $\mathbb{R}$

Range is $\mathbb{R}$

iii) Value of function

Put $x = 0 \Rightarrow y = 0$

Put $x = \pi \Rightarrow y = \pi$

Put $x = 2\pi \Rightarrow y = 2\pi$

iv) For critical points

$\frac{dy}{dx} = 0 \Rightarrow 1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = (2k+1)\pi, k \in \mathbb{I}$ are critical points.

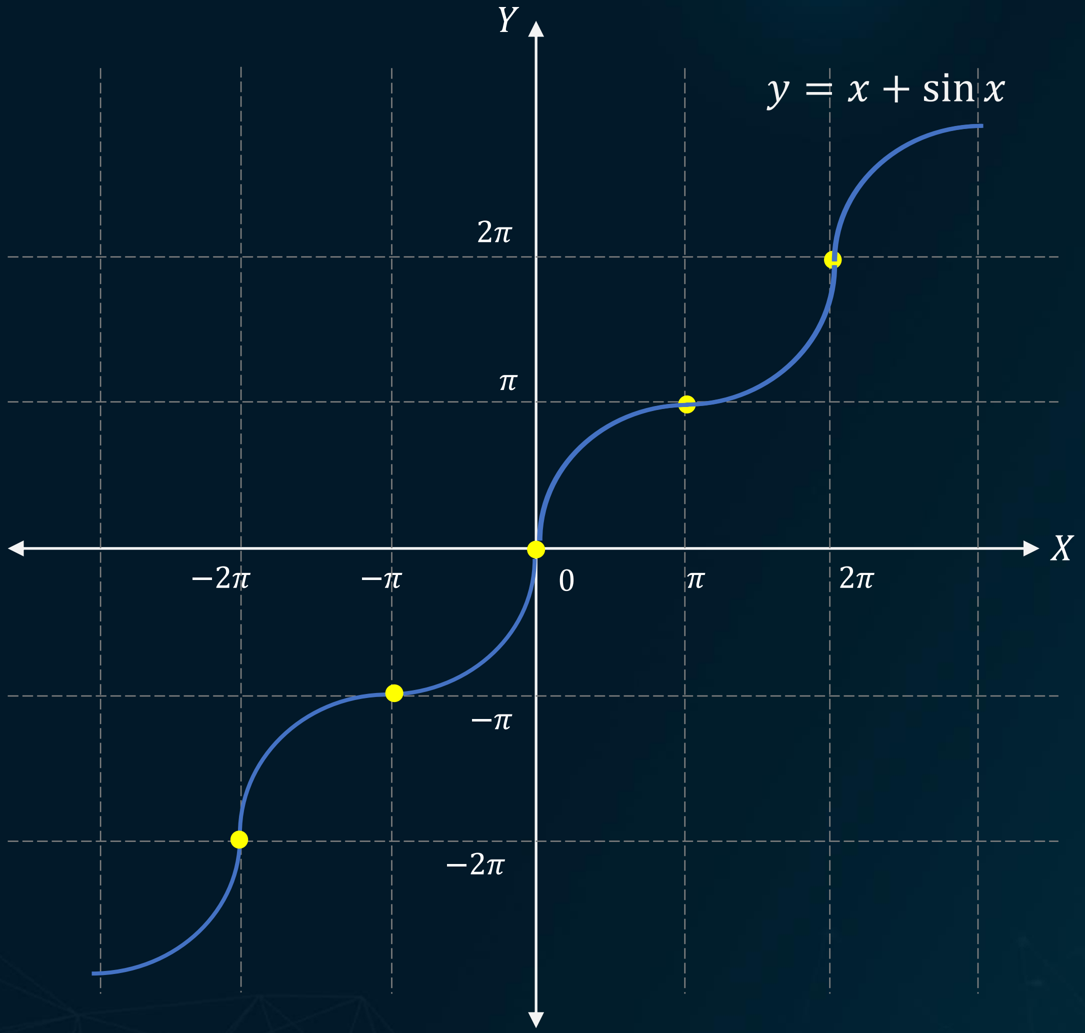
Also, $\frac{dy}{dx} = 1 + \cos x \geq 0 \; \forall \; x \in \mathbb{R} \Rightarrow f(x)$ is increasing $\forall \; x \in \mathbb{R}$

v) Shape of curve

$\frac{d^2y}{dx^2} = -\sin x, \frac{d^2y}{dx^2} = 0 \Rightarrow -\sin x = 0 \Rightarrow x = 2n\pi, n \in \mathbb{I}$

$\frac{d^2y}{dx^2} > 0 \Rightarrow (2n-1)\pi < x < 2n\pi, , n \in \mathbb{I}$ (concave upwards)

$\frac{d^2y}{dx^2} < 0 \Rightarrow 2n\pi < x < (2n+1)\pi, , n \in \mathbb{I}$ (concave downwards)



Since graph is symmetric about the origin so we can it draw for the 1^{st} quadrant and then take mirror image of it about origin

At $x = 0, y = 0$

From $x = 0$ to $x = \pi$, graph is increasing and concave downwards.

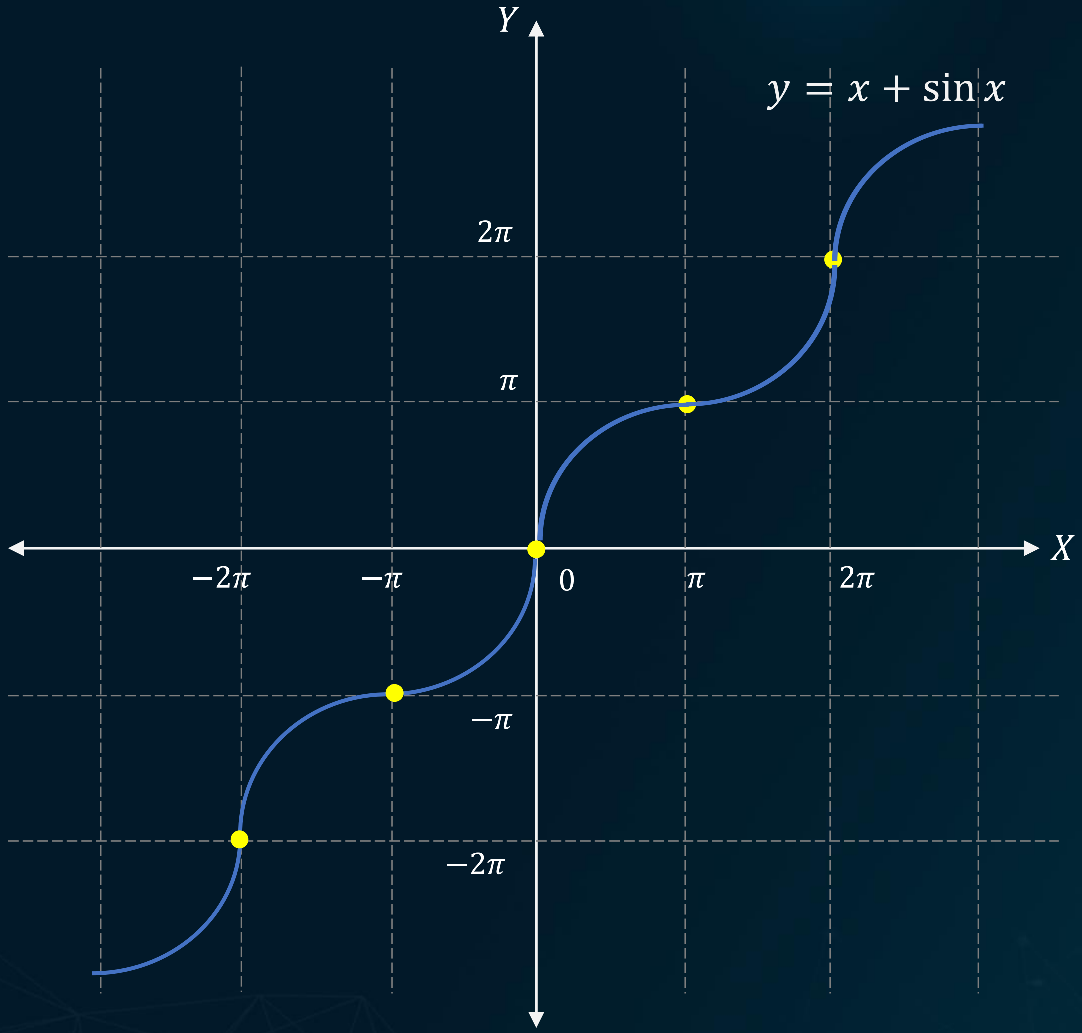
At $x = 2\pi, y = 2\pi$

From $x = \pi$ to $x = 2\pi$, graph is increasing and concave upwards.

Similarly, graph can be drawn for the other intervals in 1^{st} quadrant.

At the end take mirror image w.r.t origin to draw the whole graph.

Note: Here we superimpose $y = \sin x$ over $y = x$.



$$y = x + \sin x$$

$$x = 0 \Rightarrow y = 0$$

$$x = \pi \Rightarrow y = \pi$$

$$x = 2\pi \Rightarrow y = 2\pi$$

$$\frac{dy}{dx} = 1 + \cos x \geq 0 \text{ (increasing)}$$

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\frac{d^2y}{dx^2} > 0 \text{ (concave up)}$$

$$\frac{d^2y}{dx^2} < 0 \text{ (concave down)}$$

Note: Here we superimpose $y = \sin x$ over $y = x$.

Illustration

Plot graph of $iii)\ f(x)=\frac{e^x}{x}$, also find range of $f(x)$.

Solution: $y=\frac{e^x}{x}$

i) On replacing $y\to -y$ or $x\to -x$ equation of curve changes.

$\Rightarrow$ curve is not symmetric.

ii) Domain : $x\in\mathbb{R}-\{0\}$

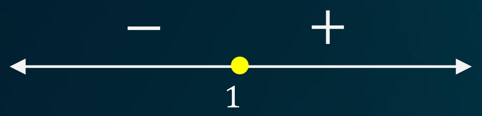
For range,

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{e^x}{x}=\frac{e^{-\infty}}{-\infty}\to 0^-;\ \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{e^x}{x}=\lim_{x\to\infty}\frac{e^x}{1}\to\infty$$

(Using L'Hospital's Rule)

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}\frac{e^x}{x}\to-\infty;\ \lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{e^x}{x}\to\infty$$

iii) For critical points,



$$f'(x) = \frac{e^x(x-1)}{x^2} = 0 \quad \Rightarrow x = 1$$

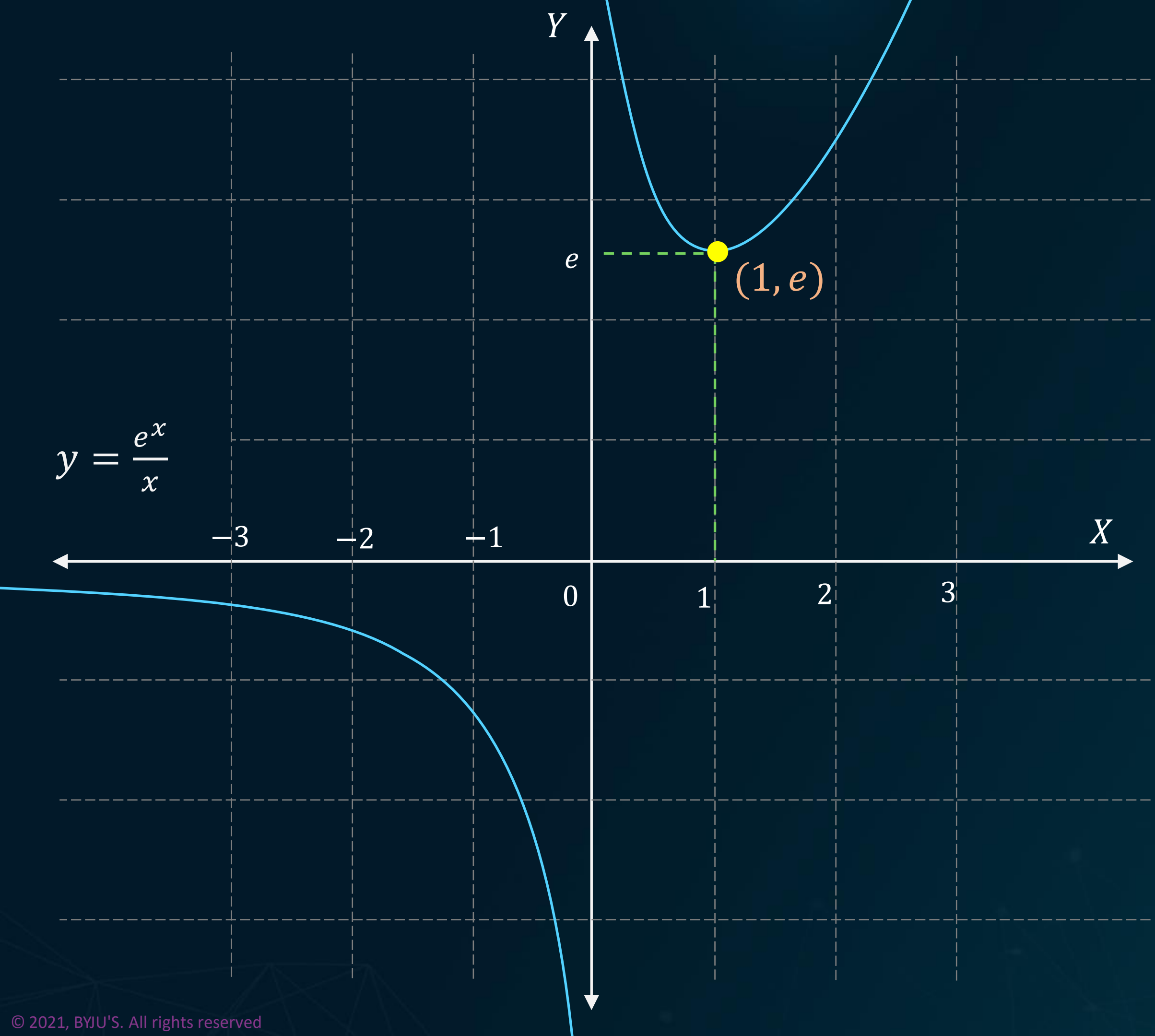
$$f(1) = \frac{e}{1} = e$$

$f'(x)$ is changing its sign from negative to positive as x going from left to right of 1, hence $x = 1$ is the point of local minima

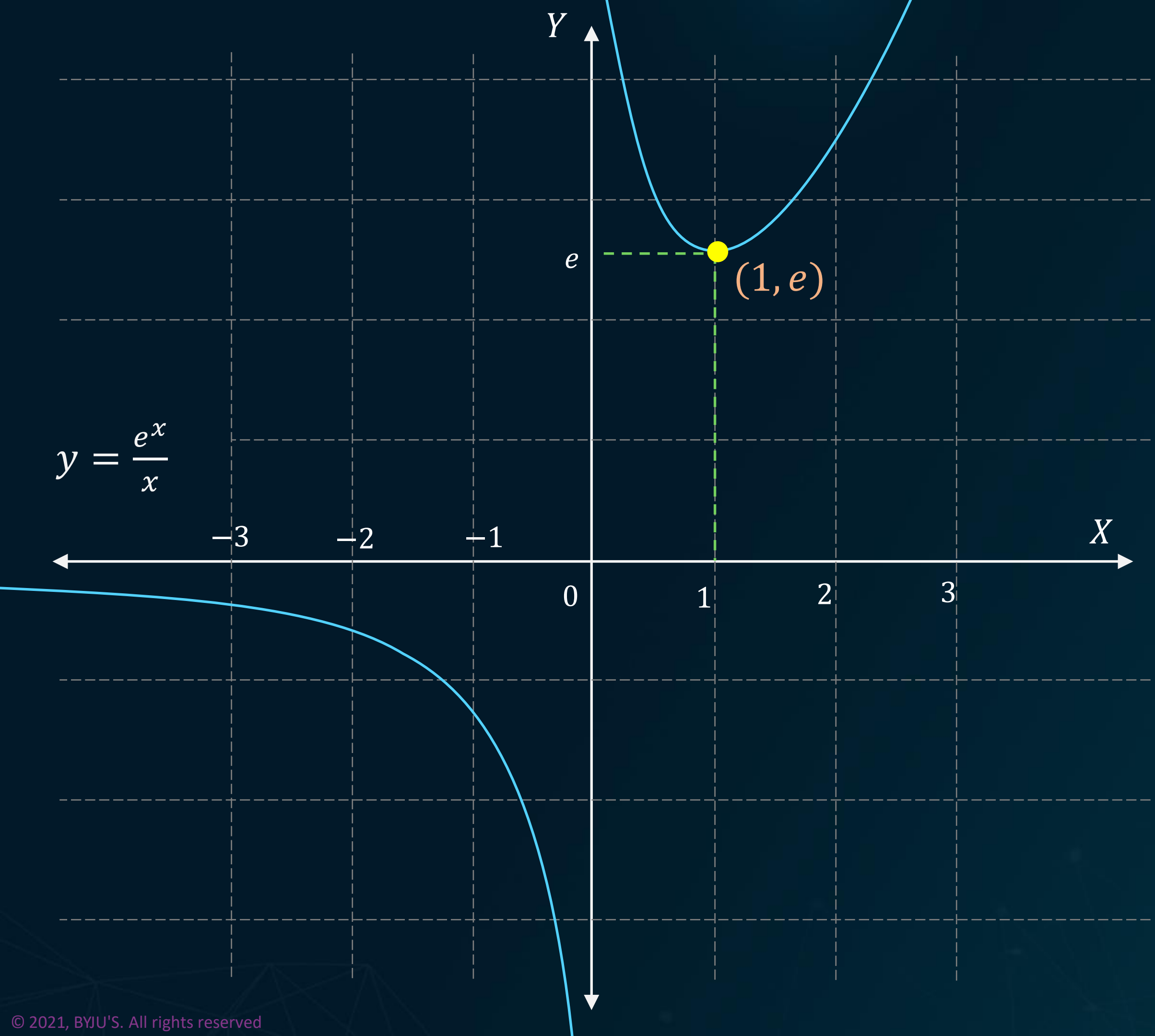
iv) Shape of curve

$$f'(x) > 0 \ \Rightarrow x > 1 \text{ (increasing)}$$

$$f'(x) < 0 \ \Rightarrow x < 1 \text{ (decreasing)}$$



As $x \to -\infty, f(x) \to 0^-$
From $-\infty$ to 0^-, $f(x)$ is decreasing
Further as $x \to 0^+, f(x) \to \infty$ and
from 0^+ to $1, f(x)$ is decreasing.
At $x = 1$, we get local minima.
From 1 to $\infty, f(x)$ is increasing and
as $x \to \infty, f(x) \to \infty$



$$y = \frac{e^x}{x}$$

Domain : $x \in \mathbb{R} - \{0\}$

$$\lim_{x \to -\infty} f(x) \to 0^-; \lim_{x \to \infty} f(x) \to \infty;$$

$$\lim_{x \to 0^-} f(x) \to -\infty; \lim_{x \to 0^+} f(x) \to \infty$$

$$f'(x) = \frac{e^x(x-1)}{x^2} = 0 \Rightarrow x = 1$$

$f(1) = e$ (local minima)

$f'(x) > 0 \Rightarrow x > 1$ (increasing)

$f'(x) < 0 \Rightarrow x < 1$ (decreasing)

Illustration

Plot graph of $iv)\ x^{2/3} + y^{2/3} = a^{2/3}$

Solution:

i) On replacing $y \to -y\,, x \to -x$ equation of curve doesn't change.

$\Rightarrow$ Symmetric in all four quadrants. Plot the curve for first quadrant i.e $x > 0, y > 0$

ii) Value of function

Put $x = 0 \Rightarrow y^{2/3} = a^{2/3} \Rightarrow y = \pm a$

Put $y = 0 \Rightarrow x^{2/3} = a^{2/3} \Rightarrow x = \pm a$

iii) For critical points, differentiate $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

For first quadrant $x > 0, y > 0$; $\frac{dy}{dx} < 0 \Rightarrow$ function is decreasing

iv) Shape of curve

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \Rightarrow x^{\frac{1}{3}}\frac{dy}{dx} = -y^{\frac{1}{3}}$$

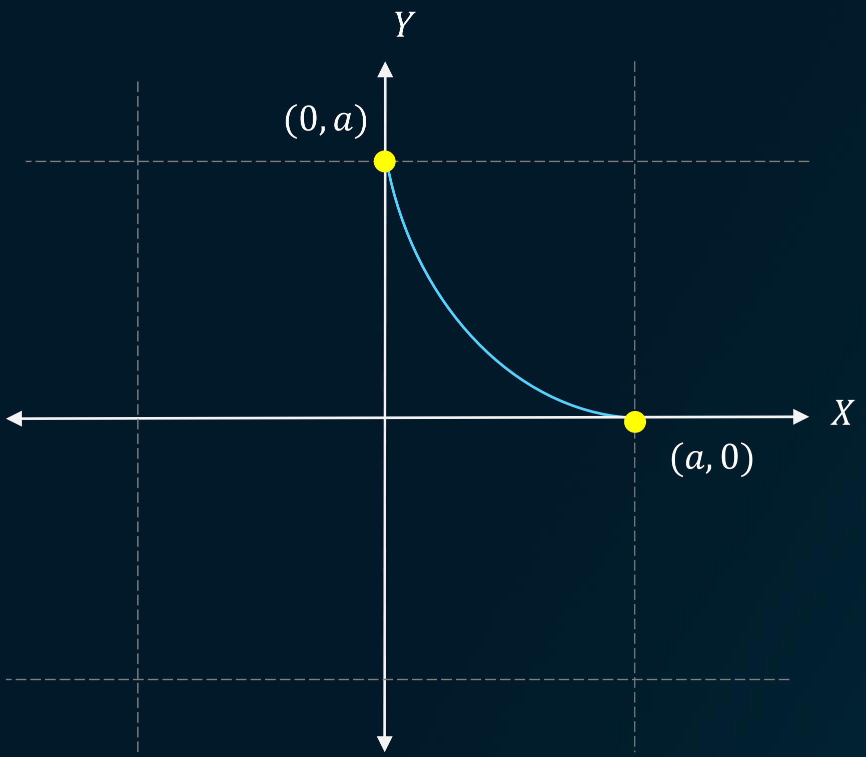
Differentiating w.r.t x

$$x^{\frac{1}{3}}\frac{d^2y}{dx^2} + \frac{1}{3}x^{\frac{-2}{3}}\frac{dy}{dx} = -\frac{1}{3}y^{\frac{-2}{3}}\frac{dy}{dx} \quad \Rightarrow x^{\frac{1}{3}}\frac{d^2y}{dx^2} = -\frac{1}{3}\left(x^{\frac{-2}{3}} + y^{\frac{-2}{3}}\right)\frac{dy}{dx}$$

Substituting $\frac{dy}{dx}$ and rearranging terms, we get $\frac{d^2y}{dx^2} = \frac{1}{3}\frac{x^{2/3}+y^{2/3}}{x^{4/3}y^{1/3}}$

For first quadrant $x > 0, y > 0$; $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave up

$x^{2/3} + y^{2/3} = a^{2/3}, x > 0, y > 0$

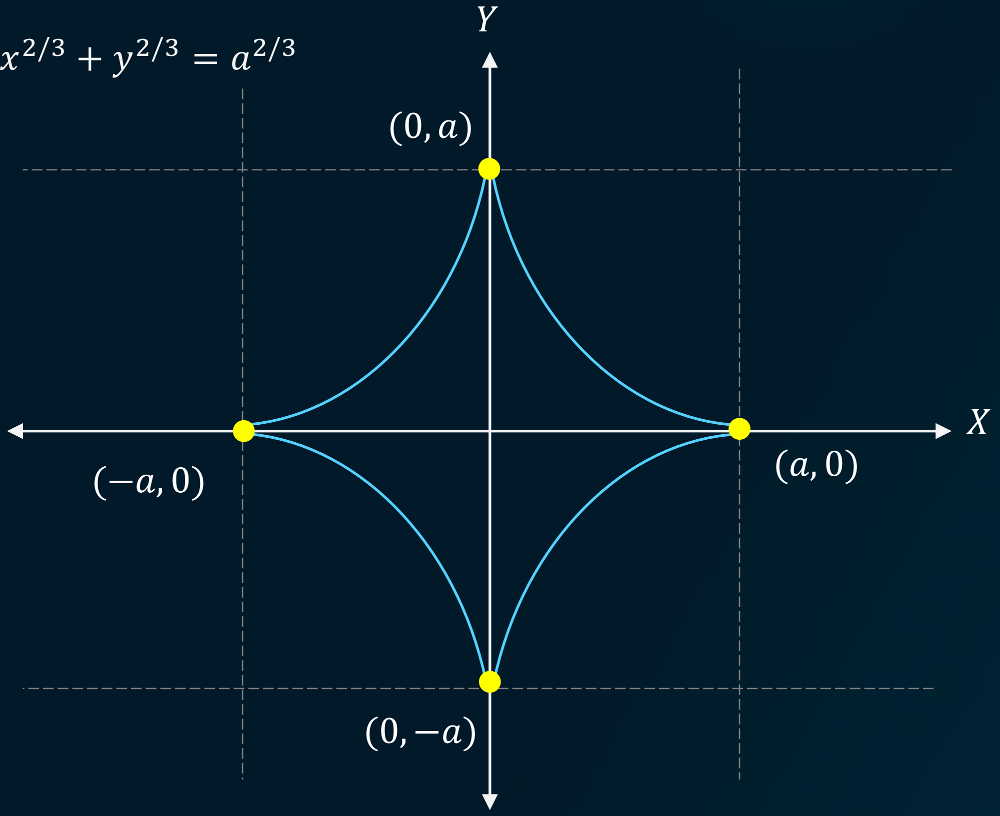


Since $f(x)$ symmetric in all four quadrants. So plot the curve for first quadrant only i.e $x > 0, y > 0$ and then take mirror image about $X - axis$ and $Y - axis$ both.

$x = 0 \Rightarrow y = a, x = a \Rightarrow y = 0,$

Also the graph is decreasing and concave upwards from $x = 0$ to $x = a$

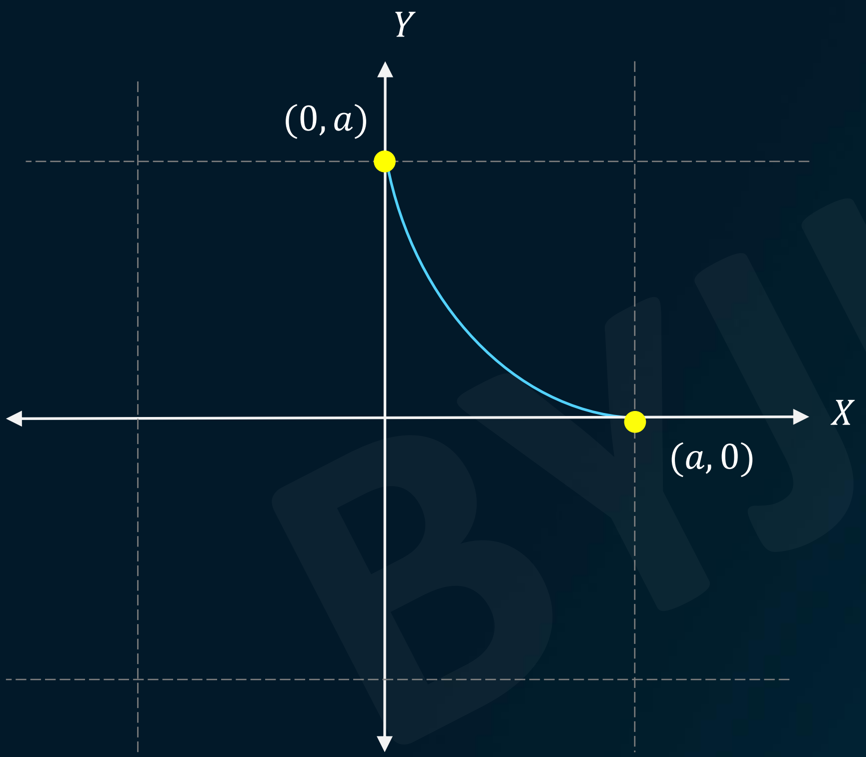
$x^{2/3} + y^{2/3} = a^{2/3}$



We get the final graph after taking mirror image of drawn graph about $X - axis$ and $Y - axis$ both.

This shape is known as asteroid.

$x^{2/3} + y^{2/3} = a^{2/3}, x > 0, y > 0$



$$x^{2/3} + y^{2/3} = a^{2/3}$$

Put $x = 0 \Rightarrow y = \pm a$

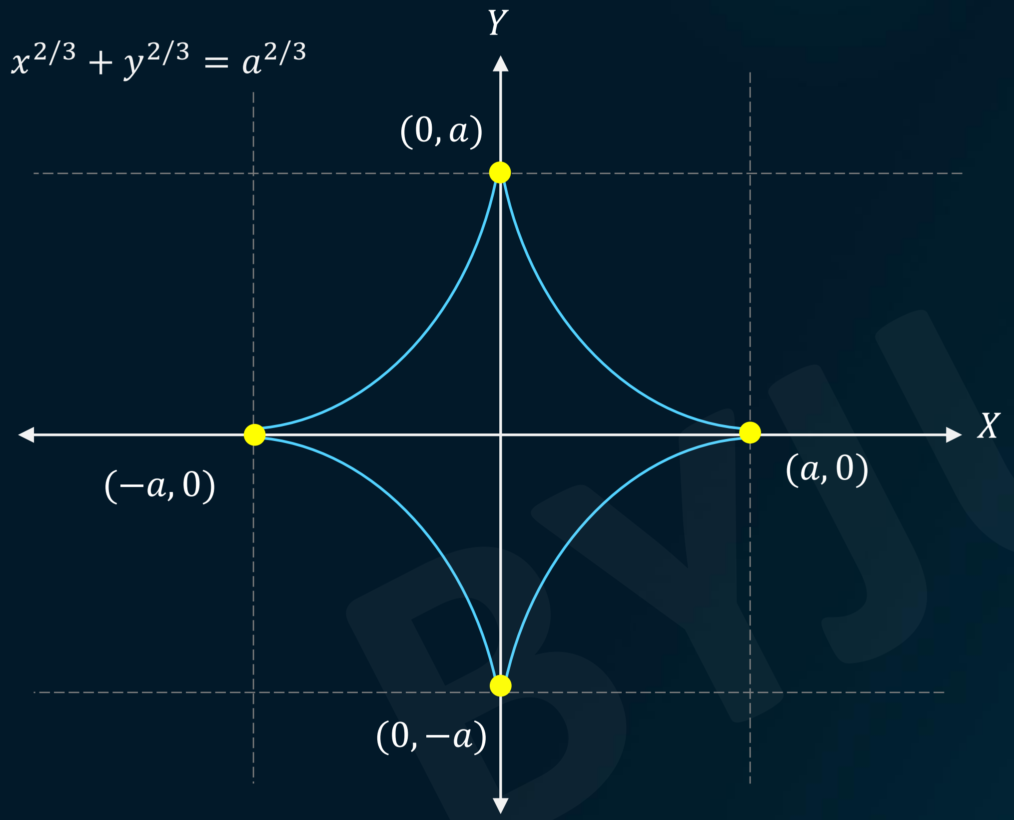
Put $y = 0 \Rightarrow x = \pm a$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

For $x > 0, y > 0$;

$\frac{dy}{dx} < 0 \Rightarrow$ function is decreasing

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave up



$$x^{2/3} + y^{2/3} = a^{2/3}$$

$\because$ Symmetric in all four quadrants

Put $x = 0 \Rightarrow y = \pm a$

Put $y = 0 \Rightarrow x = \pm a$

This shape is known as asteroid.

Illustration

Plot graph of $v)\ y = \frac{2x+3}{x^2+4}$

Solution:

i) On replacing $y \to -y\,, x \to -x$ equation of curve changes.

$\Rightarrow$ curve is not symmetric.

ii) Value of function

$x = 0 \Rightarrow y = \frac{3}{4}$ $\qquad$ $y = 0 \Rightarrow x = -\frac{3}{2}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x+3}{x^2+4} = \lim_{x\to-\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x^2}} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+3}{x^2+4} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x^2}} = 0$$

iii) Domain : $x \in \mathbb{R}$ $(\because x^2 + 4 > 0)$

iv) For critical points, differentiate $y = \frac{2x+3}{x^2+4}$

$$\frac{dy}{dx} = \frac{2(x^2+4) - 2x(2x+3)}{(x^2+4)^2} = -\frac{2(x+4)(x-1)}{(x^2+4)^2}$$

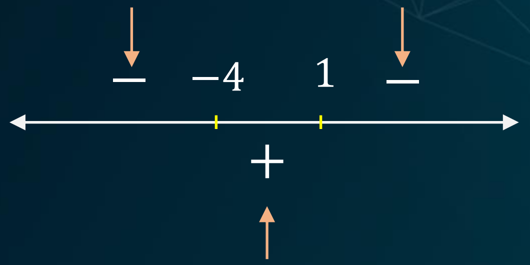
$$\frac{dy}{dx} = 0 \Rightarrow x = -4, 1$$

$$\overset{-}{} \quad -4 \quad \overset{+}{} \quad 1 \quad \overset{-}{}$$

$$\frac{dy}{dx} < 0 \Rightarrow \underbrace{x \in (-\infty, -4) \cup (1, \infty)}_{\text{function is decreasing}}$$

$$\frac{dy}{dx} > 0 \Rightarrow \underbrace{x \in (-4, 1)}_{\text{function is increasing}}$$

v) Value at critical points

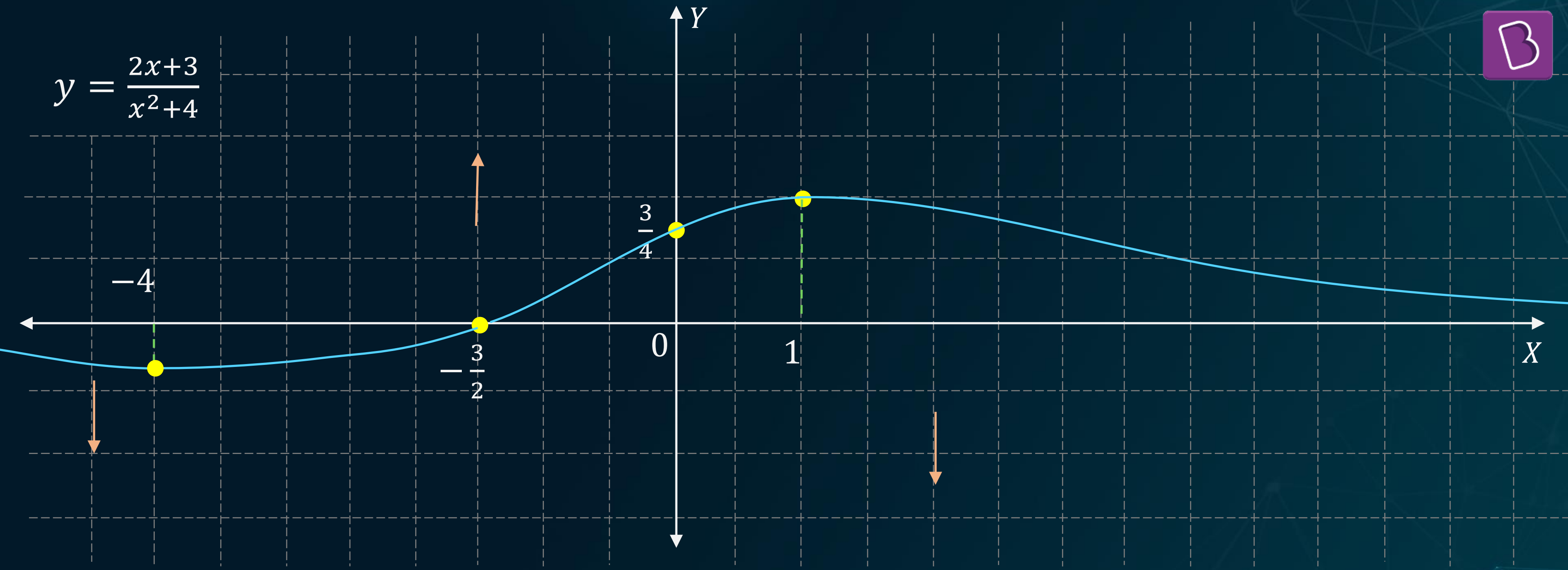


Function is decreasing from $(-\infty, -4) \cup (1, \infty)$

$\Rightarrow$ Local minima: $y(-4) = \frac{2(-4)+3}{(-4)^2+4} = -\frac{1}{4}$

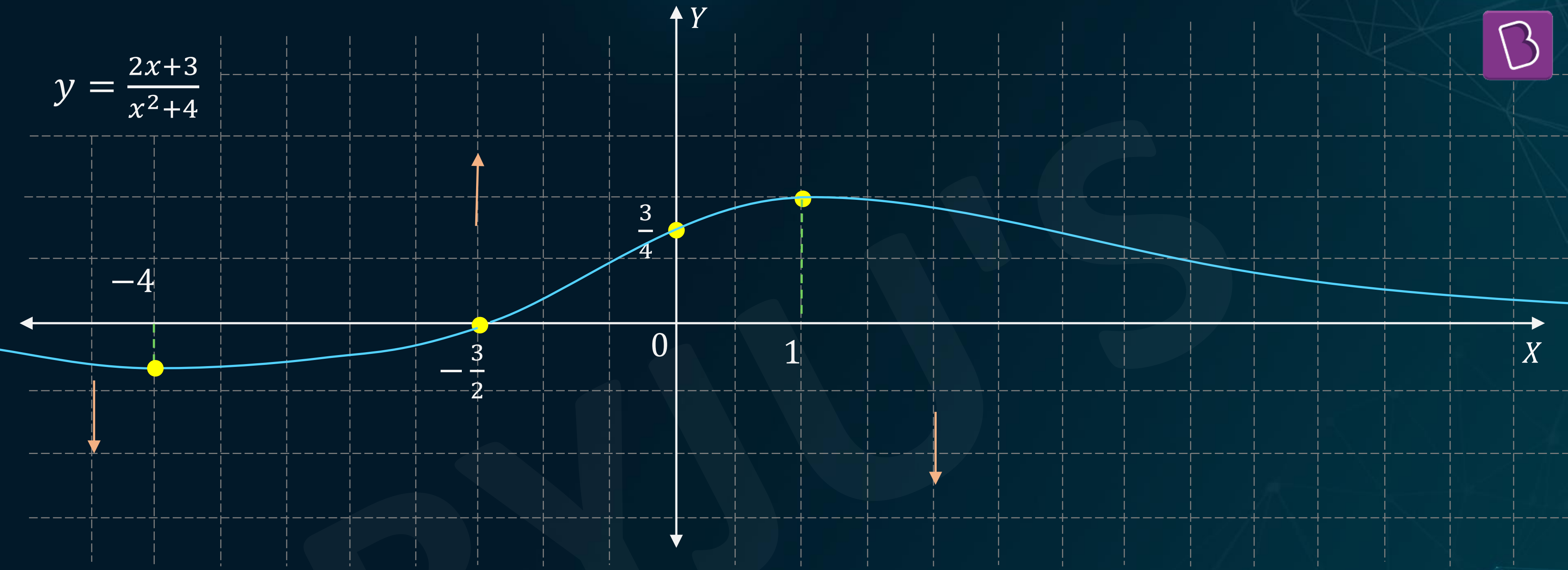
Function is increasing from $(-4, 1)$

$\Rightarrow$ Local maxima: $y(1) = \frac{2(1)+3}{(1)^2+4} = 1$



$x \to -\infty \Rightarrow f(x) \to 0^-$
From x $\to -\infty$ to $x = -4$, $f(x)$ is decreasing and from $x = -4$ to $x = 1$, $f(x)$ is increasing
At $x = -4$, we get local minima.

At $x = 1$,we get local maxima.
From $x = 1$ to $x \to \infty$, $f(x)$ is decreasing and as $x \to \infty$, $f(x) \to 0^+$
Clearly $x = -\frac{3}{2}$ is the roots of $f(x)$



$$y = \frac{2x+3}{x^2+4}$$

$$x = 0 \Rightarrow y = \frac{3}{4}, y = 0 \Rightarrow x = -\frac{3}{2}$$

$$\lim_{x\to-\infty} f(x) \to 0^-; \lim_{x\to\infty} f(x) \to 0^+$$

$$\frac{dy}{dx} = 0 \Rightarrow x = -4, 1$$

Local maxima : $x = 1, y(1) = 1$

Local minima : $x = -4, y(-4) = -\frac{1}{4}$

Summary Sheet

- If on replacing $x \to -x$, equation of curve doesn't change
  $\Rightarrow$ curve is symmetric about y −axis.

- If on replacing $y \to -y$, equation of curve doesn't change
  $\Rightarrow$ curve is symmetric about x −axis.

- If on replacing $y \to -y\ \ \& \, x \to -x$, equation of curve doesn't change
  $\Rightarrow$ curve is symmetric in all four quadrants.

- If on interchanging $x \,\&\, y$, equation of curve doesn't change, then the curve is symmetric about $y = x$.

- If $f(-x) = -f(x)\ \forall\ x$ in domain of 'f', then f is said to be an odd function. The graph of an odd function is symmetric about the origin.

Summary Sheet

Steps to draw curve:

- Check symmetry (if any).
- Get idea about domain and range of function to be drawn.
- Identify discontinuity (if any).
- Get critical points & increasing/decreasing intervals.
- Estimate value of function at $x = 0$ (Points of intersection with $Y - axis$), $\pm\infty$ (behavior of curve at extreme points) and at critical points.
- Find roots (if any)

BYJU'S Classes

Application of Derivatives

Asymptotes of a Function

Road Map

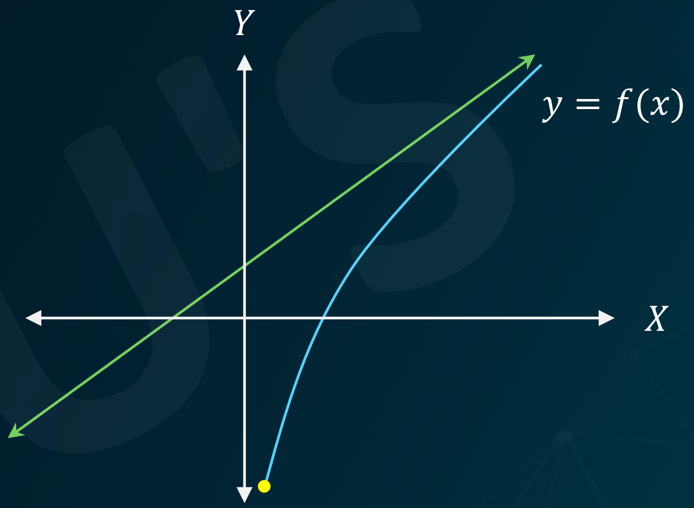
Asymptotes

Nature of roots of a real valued cubic polynomial

Curve tracing

Asymptotes

A straight line is called an asymptote to a curve , if distance between a point on the curve and this line approaches zero as the line tends to infinity (i.e., tangent at infinity)



(i) Horizontal asymptote :

Let the curve be $y = f(x)$

If $\lim\limits_{x \to \infty} f(x) \to a$, or $\lim\limits_{x \to -\infty} f(x) \to a$

then the line $y = a$, will be the horizontal asymptote.

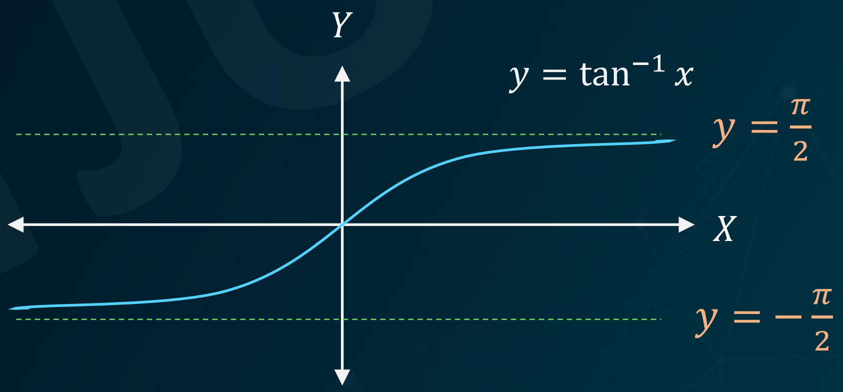
Example: $f(x) = \tan^{-1} x$

Domain: $x \in \mathbb{R}$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\lim_{x \to \infty} f(x) \to \frac{\pi}{2}$$

$$\lim_{x \to -\infty} f(x) \to -\frac{\pi}{2}$$



(ii) Vertical asymptote :

Let the curve be $y = f(x)$.

If $\lim\limits_{x \to a} f(x) \to \infty$ or $-\infty$, then the line $x = a$, will be the vertical asymptote.

Example: $y = \frac{1}{x}$

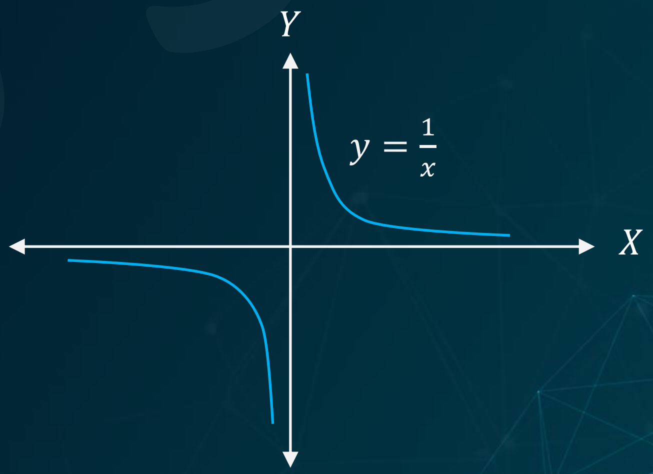
$\lim\limits_{x \to 0^-} f(x) \to -\infty$; $\lim\limits_{x \to 0^+} f(x) \to \infty$

$\Rightarrow x = 0$ $i.e$ $Y - axis$ is a vertical asymptote.

$\lim\limits_{x \to \infty} f(x) \to 0$; $\lim\limits_{x \to -\infty} f(x) \to 0$

$\Rightarrow y = 0$ $i.e$ $X - axis$ is a horizontal asymptote.

(both vertical & horizontal asymptotes exist in this case)



(iii) Inclined / Oblique asymptote : (not parallel to any of axis & $m \neq 0$)

Let $y = mx + c$ be an asymptote of the curve $y = f(x)$ such that as $x \to \infty$ or $x \to -\infty$ we get $y \to \infty$ or $y \to -\infty$.

Then if $m = \lim\limits_{\substack{x\to\infty \\ or\ x\to-\infty}} \left(\frac{f(x)}{x}\right)$ is a finite value, asymptote exists.

Here $m = \lim\limits_{x\to\infty} \left(\frac{f(x)}{x}\right) \Rightarrow \lim\limits_{x\to\infty} f'(x)$ is slope of function at ∞ i.e slope of asymptote.

Now $c = y - m(x)$

$$\Rightarrow c = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} (f(x) - mx)$$

Illustration

Plot graph of $i)\ x^2 - y^2 = a^2$

Solution: $y = f(x) = \pm\sqrt{x^2 - a^2}$

For an inclined asymptote :

$$m = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} \left(\frac{\pm\sqrt{x^2-a^2}}{x}\right) = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} \left(\pm\sqrt{\frac{x^2}{x^2} - \frac{a^2}{x^2}}\right) = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} \left(\pm\sqrt{1 - \frac{a^2}{x^2}}\right) = \pm 1$$

$$c = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} (f(x) - mx)$$

$$\Rightarrow c = \lim_{\substack{x\to\infty \\ or\ x\to-\infty}} \left(\pm\sqrt{x^2 - a^2} - x\right) \text{ for } m = 1.$$

$$\lim_{x\to\infty} \left(\sqrt{x^2 - a^2} - x\right) \qquad \lim_{x\to-\infty} \left(-\sqrt{x^2 - a^2} - x\right)$$

$$c = \lim_{x\to\infty}\left(\sqrt{x^2 - a^2} - x\right)$$

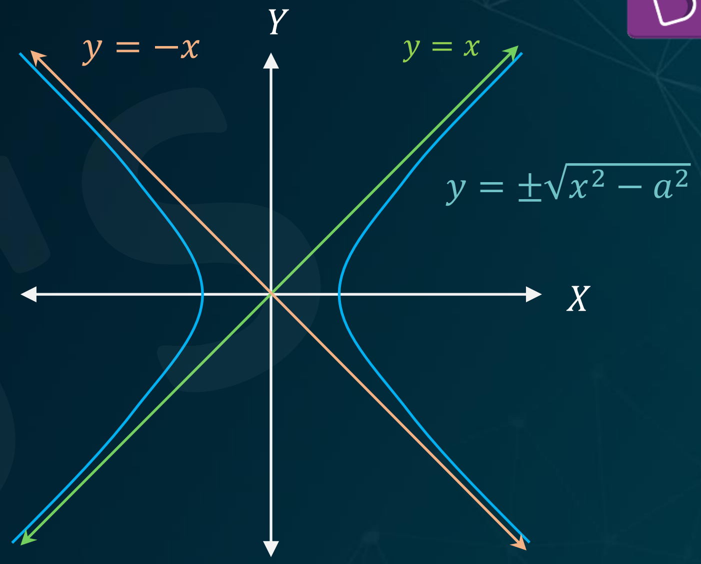
$$\Rightarrow \lim_{x\to\infty} \frac{(x^2 - a^2 - x^2)}{(\sqrt{x^2 - a^2} + x)} \Rightarrow \lim_{x\to\infty} \frac{-a^2}{(\sqrt{x^2 - a^2} + x)} = 0$$

Similarly, $\lim_{x\to -\infty}\left(-\sqrt{x^2 - a^2} - x\right) = 0$

Also, for $m = -1, c = 0$.

$\therefore$ Inclined asymptote : $y = \pm x$

$(\because m = \pm 1$ and $c = 0)$



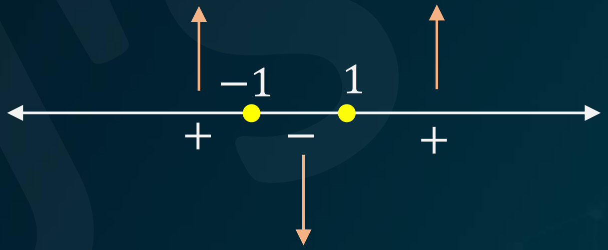
Illustration

Plot graph of $ii)\ y = x + \frac{1}{x}$.

Solution: Domain : $x \in \mathbb{R} - \{0\}$

$$\lim_{x \to -\infty} f(x) \to -\infty; \lim_{x \to \infty} f(x) \to \infty;$$

$$\lim_{x \to 0^-} f(x) \to -\infty; \lim_{x \to 0^+} f(x) \to \infty;$$



$f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$ are critical points.

$\Rightarrow$ Function is decreasing for $x = (-1, 1)$.

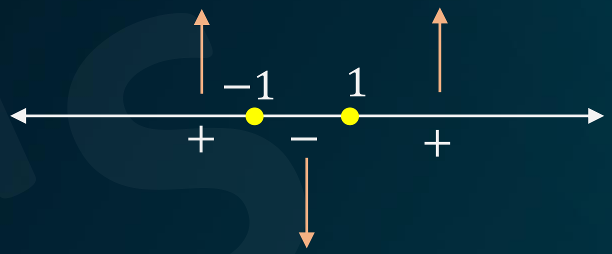
$\Rightarrow$ Local maxima: $x = -1, y(-1) = -2$

$\Rightarrow$ Local minima: $x = 1, y(1) = 2$

$$f'(x) = 1 - \frac{1}{x^2}$$

$f'(x) > 0 \Rightarrow x < -1$ or $x > 1$ (increasing)

$f'(x) < 0 \Rightarrow -1 < x < 1$ (decreasing)

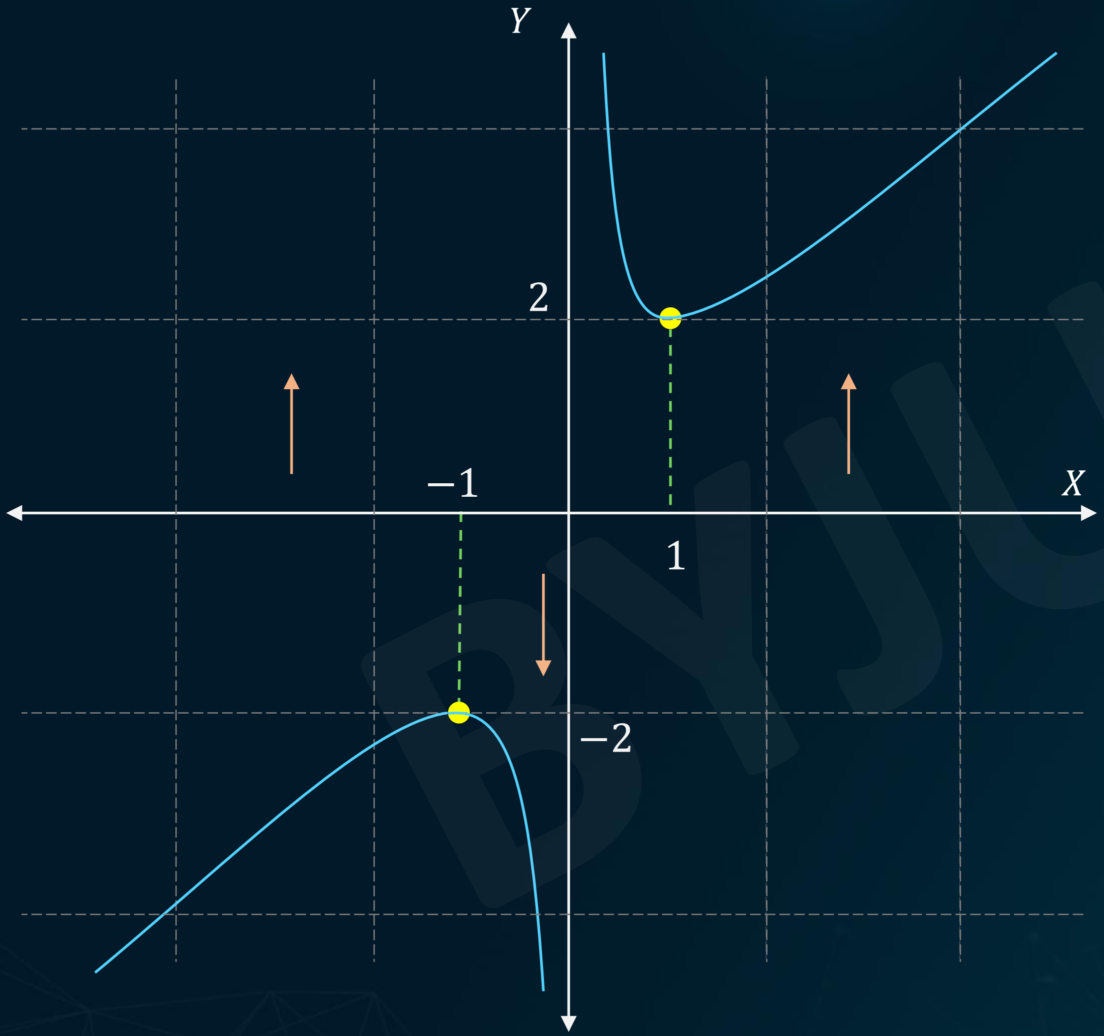


Now for the shape of curve,

$$f''(x) = \frac{2}{x^3}$$

$f''(x) > 0 \Rightarrow x > 0$ (concave upwards)

$f''(x) < 0 \Rightarrow x < 0$ (concave downwards)



$$y = x + \frac{1}{x}$$

$$\lim_{x \to -\infty} f(x) \to -\infty;\ \lim_{x \to \infty} f(x) \to \infty;$$

$$\lim_{x \to 0^-} f(x) \to -\infty;\ \lim_{x \to 0^+} f(x) \to \infty;$$

Local maxima: $x = -1, y(-1) = -2$

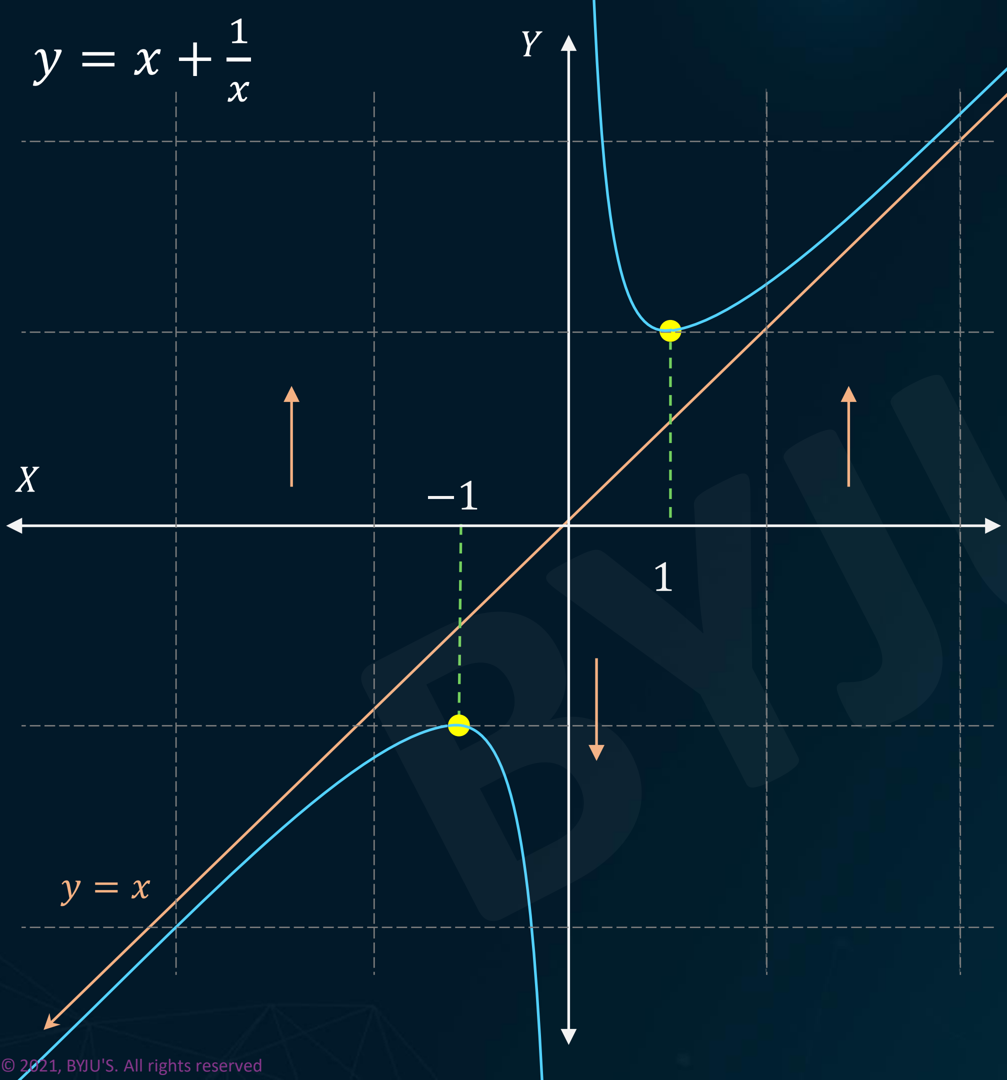
Local minima: $x = 1, y(1) = 2$

$f'(x) > 0 \Rightarrow x < -1$ or $x > 1$ $(\uparrow)$

$f'(x) < 0 \Rightarrow -1 < x < 1$ $(\downarrow)$

$f''(x) > 0 \Rightarrow x > 0$ (concave $\uparrow$)

$f''(x) < 0 \Rightarrow x < 0$ (concave $\downarrow$)



Asymptotes :

As $\lim_{x \to 0^-} f(x) \to -\infty$ & $\lim_{x \to 0^+} f(x) \to \infty$

$\Rightarrow x = 0$ *i.e* $Y - axis$ is a vertical asymptote.

Now for an inclined asymptote : $y = mx + c$,

$$m = \lim_{\substack{x \to \infty \text{ or} \\ x \to -\infty}} \left(\frac{f(x)}{x}\right) = \lim_{\substack{x \to \infty \text{ or} \\ x \to -\infty}} \left(1 + \frac{1}{x^2}\right) = 1$$

$$c = \lim_{\substack{x \to \infty \text{ or} \\ x \to -\infty}} (f(x) - mx)$$

$$c = \lim_{\substack{x \to \infty \text{ or} \\ x \to -\infty}} \left(x + \frac{1}{x} - x\right) = 0$$

Inclined asymptote : $y = x$

Nature of roots of a real valued cubic polynomial equation

Cubic polynomial equation $ax^3 + bx^2 + cx + d = 0$ will have one real root or three real roots.

Consider $f(x) = ax^3 + bx^2 + cx + d$

Case (i) $f'(x) = 3ax^2 + 2bx + c$

If $D = 4b^2 - 12ac < 0$

Then , $f(x)$ is increasing (a>0) or decreasing (a<0).

Thus, $f(x)$ has exactly one real root.

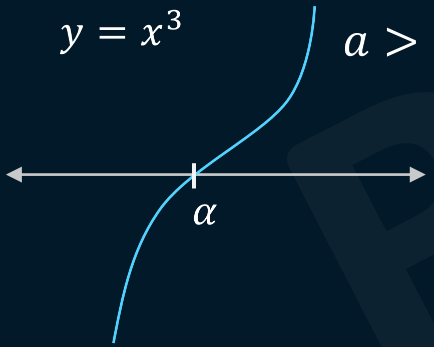
Nature of roots of a real valued cubic polynomial equation

Note:

For $f(x) = ax^3 + bx^2 + cx + d$,

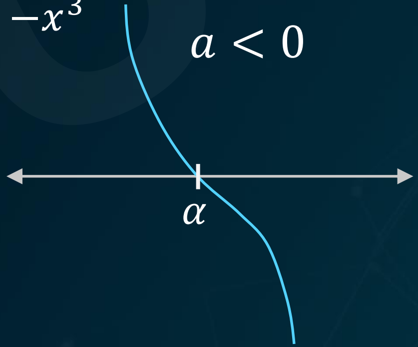
- If $a > 0$, as $x \to \infty$; $f(x) \to \infty$
- If $a < 0$, as $x \to \infty$; $f(x) \to -\infty$

$y = x^3$



$a > 0$

$y = -x^3$



$a < 0$

$f(x)$ has exactly one real root.

Illustration

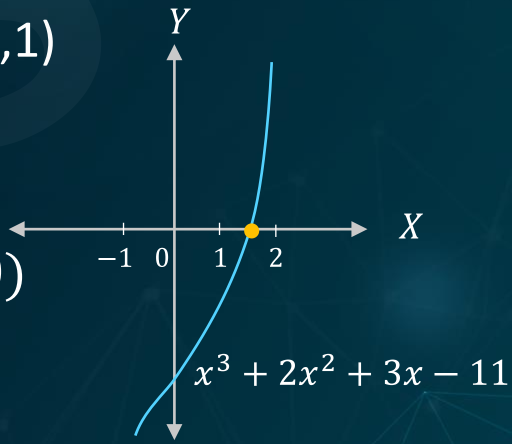
Let a function $f(x) = x^3 + 2x^2 + 3x - 11$, then $f(x) = 0$, has :

$a.$ three real roots

$b.$ one real root between (−1,0)

$c.$ One real root between (1,2)

$d.$ one real root between (0,1)



Solution: $f(x) = x^3 + 2x^2 + 3x - 11$

$f'(x) = 3x^2 + 4x + 3 > 0 \ \forall\, x \in \mathbb{R} \ (\because D = 4^2 - 4(3)(3) < 0)$

Thus , $f(x)$ is increasing.

$f(1) = 1 + 2 + 3 - 11 < 0$, $f(2) = 8 + 8 + 6 - 11 > 0$

So, by Intermediate value theorem, the root lies between $(1, 2)$.

Hence, option (c) is the correct answer.

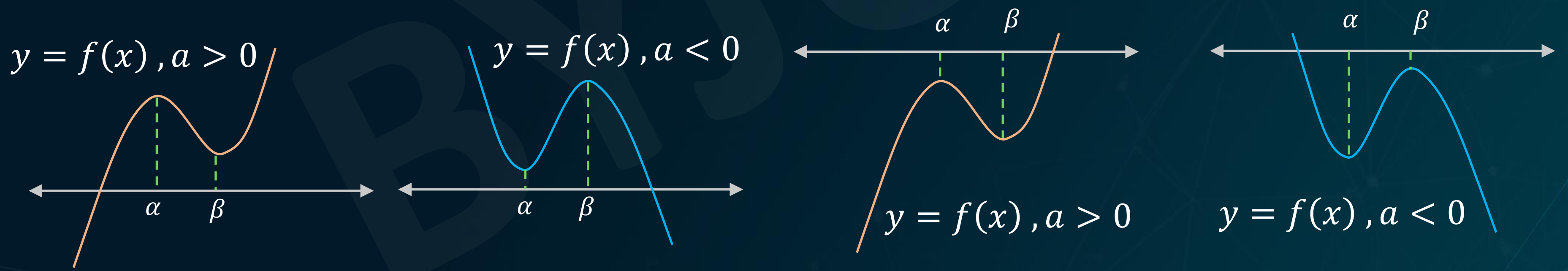
Nature of roots of a real valued cubic polynomial equation

Case (ii) Let $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c = 0, x = \alpha, \beta$ are the roots of $f'(x)$

- If $f(\alpha) \cdot f(\beta) > 0$

$\Rightarrow f(x) = 0$ has exactly one real root



Note: If $a > 0$, first maxima and then minima occurs.

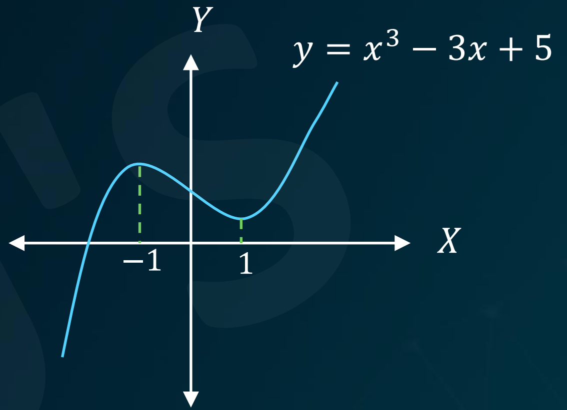
Nature of roots of a real valued cubic polynomial

Example: $f(x) = x^3 - 3x + 5$

$f'(x) = 3(x^2 - 1) = 0 \Rightarrow x = \pm 1$

$f(-1) = 7 > 0\ ,\ f(1) = 3 > 0$

Thus , $f(x) = 0$, has one real root.

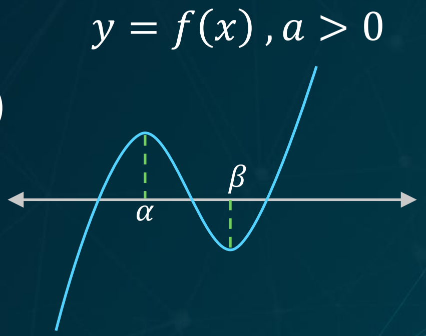


Case (iii) Let $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c = 0\ ,\ x = \alpha, \beta$ are the roots of $f'(x)$

➢ If $f(\alpha) \cdot f(\beta) < 0$

$\Rightarrow f(x) = 0$ has three real and distinct roots.



Nature of roots of a real valued cubic polynomial

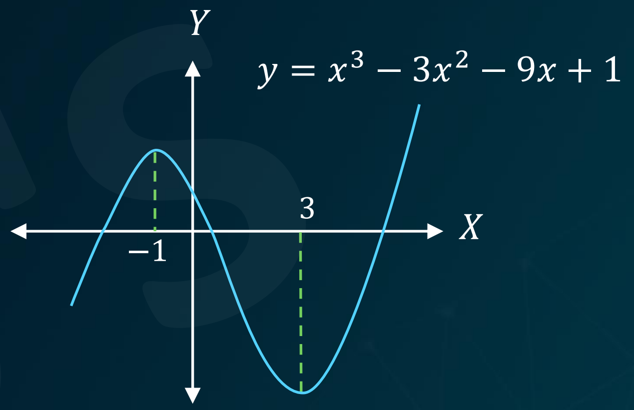
Example: $f(x) = x^3 - 3x^2 - 9x + 1$

$f'(x) = 3(x^2 - 2x - 3) = 0$

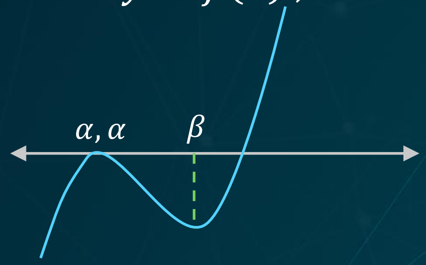
$\Rightarrow x = -1, 3$

$f(-1) = 6 > 0, f(3) = -26 < 0$

Thus , $f(x) = 0$, has three real and distinct roots.



Case (iv) Let $f(x) = ax^3 + bx^2 + cx + d$



$f'(x) = 3ax^2 + 2bx + c = 0, x = \alpha, \beta$ are the roots of $f'(x)$

➢ If $f(\alpha).f(\beta) = 0$ but $f''(\alpha), f''(\beta) \neq 0$

$\Rightarrow f(x) = 0$ has three real roots of which two are equal.

Nature of roots of a real valued cubic polynomial

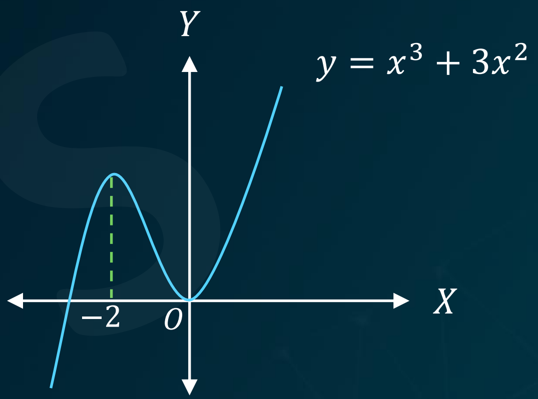
Example: $f(x) = x^3 + 3x^2$

$f'(x) = 3(x^2 + 2x) = 0 \Rightarrow x = 0, -2$

$f(0) = 0, f(-2) = 4 > 0$

$f''(x) = 6(x+1), f''(0) = 6 \neq 0, f''(-2) = -6 \neq 0$

Thus , $f(x) = 0$, has one repeated root $x = 0$.

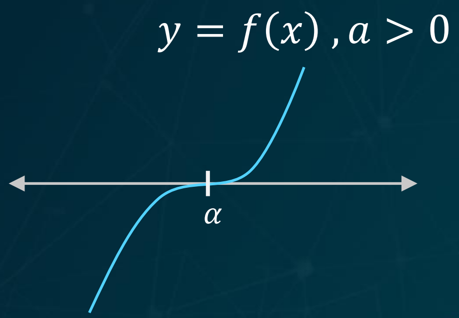


Case (v) Let $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c = 0$, has equal roots $(x = \alpha)$

➢ If $f(\alpha) = 0 = f'(\alpha) = f''(\alpha)$

$\Rightarrow f(x) = 0$ has three equal roots.



Nature of roots of a real valued cubic polynomial

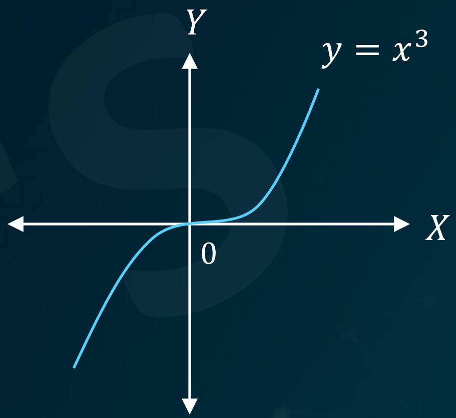
Example: $f(x) = x^3$

$$f'(x) = 3x^2 \Rightarrow f'(0) = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 0$$

$$f(0) = 0 = f'(0) = f''(0)$$

Thus , $f(x) = 0$, has three equal roots at $x = 0$.



Summary Sheet

- A straight line is called asymptote to a curve , if distance of a point on the curve to this line approaches zero as the point tends to infinity.
- Let cubic polynomial equation be $ax^3 + bx^2 + cx + d = 0$

Let $f(x) = ax^3 + bx^2 + cx + d$

case (i) $f'(x) = 3ax^2 + 2bx + c > 0 \; or < 0$

If $D = 4b^2 - 12ac < 0$

Then , $f(x)$ is increasing (a>0) or decreasing (a<0).

Thus, $f(x)$ has exactly one real root.

case (ii) $f'(x) = 3ax^2 + 2bx + c = 0$, $x = \alpha, \beta$ are the roots of $f'(x)$

$f(\alpha) \cdot f(\beta) > 0 \quad \Rightarrow f(x) = 0$ has exactly one real root

Summary Sheet

case (iii) $f'(x) = 3ax^2 + 2bx + c = 0$, $x = \alpha, \beta$ are the roots of $f'(x)$

$f(\alpha) \cdot f(\beta) < 0 \Rightarrow f(x) = 0$ has three real and distinct roots.

case (iv) $f'(x) = 3ax^2 + 2bx + c = 0$, $x = \alpha, \beta$ are the roots of $f'(x)$

$f(\alpha). f(\beta) = 0$ but $f''(\alpha), f''(\beta) \neq 0$

$\Rightarrow f(x) = 0$ has three real roots of which two are equal.

case v) $f'(x) = 3ax^2 + 2bx + c = 0$, has equal roots ($x = \alpha$)

$f(\alpha) \cdot f(\beta) > 0\, f(\alpha) = 0 = f'(\alpha) = f''(\alpha)$

$\Rightarrow f(x) = 0$ has three equal roots.

BYJU'S Classes

Application of Derivatives

Miscellaneous Questions

Road Map

Monotonicity, Maxima and Minima

Rolle’s Theorem and L.M.V.T

Tangents and Normal

Illustration

The slope of tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1,3)$ is :

Solution:

Given, $(y - x^5)^2 = x(1 + x^2)^2$

Differentiating w.r.t x, we get

$2(y - x^5) \times (y' - 5x^4) = x \times 2 \times (1 + x^2) \times 2x + (1 + x^2)^2$

Substituting $x = 1, y = 3$ in the above equation

$2 \times (3 - 1^5) \times (y' - 5 \times 1^5) = 2 \times 2 \times (1 + 1^2) + 4$

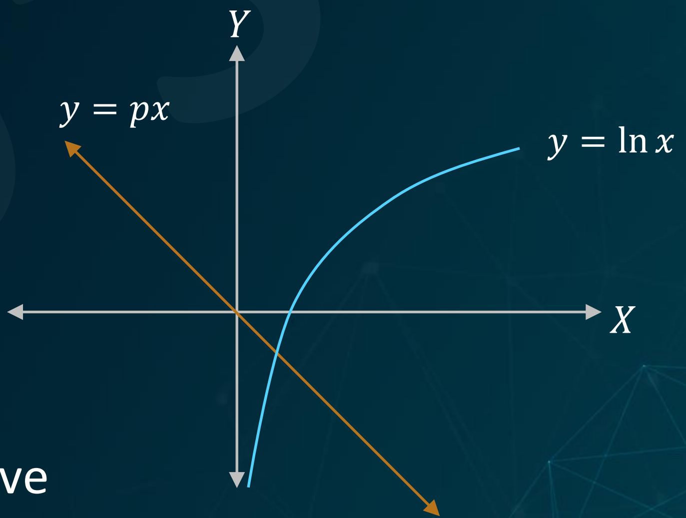
$\Rightarrow 4(y' - 5) = 12 \Rightarrow y' = 8$

$\Rightarrow$ Slope of tangent at the point $(1,3)$ is 8

Illustration

Find the range of values of p for which the equation $px = \ln x$, has exactly one solution .

$a. (-\infty, \infty)$ $b. (-\infty, 0)$ $c. (-\infty, 0] \cup \left\{\frac{1}{e}\right\}$ $d. \left(0, \frac{1}{e}\right)$



Solution:

Given equation, $px = \ln x$

Let $f(x) = px$ and $g(x) = \ln x$

Case 1 : When graphs of $f(x)$ and $g(x)$ intersect at exactly one point

From the figure we can see that for all the non-positive slope of the line $y = px$, graph of the $f(x)$ and $g(x)$ intersects at exactly one point

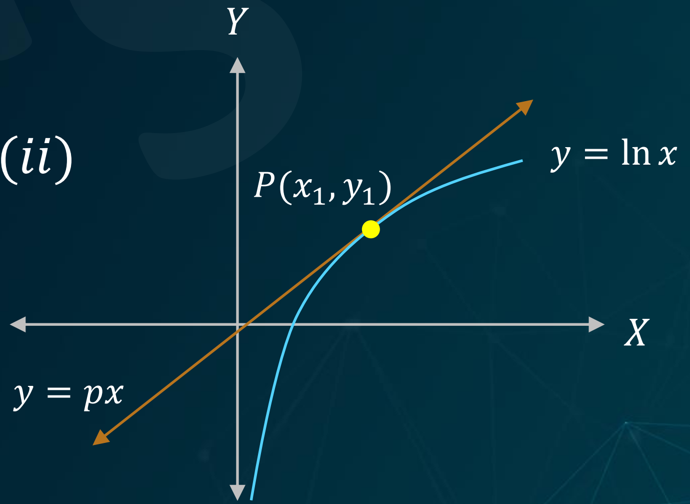
$\therefore px = \ln x$ will have exactly one solution ,for all $p \leq 0$

Case 2 : When graph of $f(x)$ and $g(x)$ touch each other

Let $f(x)$ touches $g(x)$ at the point $P(x_1, y_1)$ as shown in the figure

$P(x_1, y_1)$ lies on the line $y = px \Rightarrow y_1 = px_1 \ldots (i)$

$P(x_1, y_1)$ lies on the curve $y = \ln x \Rightarrow y_1 = \ln x_1 \ldots (ii)$



From (i) and (ii)

$$px_1 = \ln x_1 \cdots (iii)$$

Slope of tangent at the point $P(x_1, y_1)$ to the curve $y = \ln x$,

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \left(\frac{1}{x}\right)_{(x_1, y_1)} = \frac{1}{x_1}$$

This slope m is equal to the slope of the line $y = px$

$\Rightarrow p = m$

$\Rightarrow p = \frac{1}{x_1} \;\; \Rightarrow px_1 = 1$

$\Rightarrow \ln x_1 = 1$ (From (iii))

$\Rightarrow x_1 = e$

Substituting $x_1 = e$ in the equation (i),

$$pe = 1 \;\; \Rightarrow p = \frac{1}{e}$$

Combining the results of case 1 and case 2,

$$p \in (-\infty, 0] \cup \left\{\frac{1}{e}\right\}$$

So, option (b) is the correct answer.

Rolle's Theorem (Recap)

Let f be a real – valued function defined on the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the interval $[a, b]$

(ii) $f(x)$ is differentiable in the interval (a, b)

(iii) $f(a) = f(b)$

Then there exists at least one $c \in (a, b)$, such that $\boxed{f'(c) = 0}$

Lagrange's Mean Value Theorem (L.M.V.T) (Recap)

Let f be a real – valued function defined on the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the interval $[a, b]$

(ii) $f(x)$ is differentiable in the interval (a, b)

then , there exists at least one $c \in (a, b)$, such that $\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$

Illustration

Let f be a twice differentiable function satisfying $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then :

$a. f''(x) = 2, \forall x \in \mathbb{R}$

$b. f'(x) = 5 = f''(x)$, for some $x \in (1,3)$

$c.$ There exists at least one $x \in (1,3)$ such that $f''(x) = 2$

$d. f''(x) = 3, \forall x \in (2,3)$

Solution:

Given, f is twice differentiable function.

Let $g(x) = f(x) - x^2$

As $f(x)$ is twice differentiable function and x^2 is a polynomial function.
So, $g(x)$ will be continuous and differentiable everywhere.

$g(1) = f(1) - 1^2 = 1 - 1 = 0$

$g(2) = f(2) - 2^2 = 4 - 4 = 0$ And $g(3) = f(3) - 3^2 = 9 - 9 = 0$

Thus, by Rolle's theorem

$g'(c_1) = 0$, for some $c_1 \in (1,2)$

Similarly , $g(2) = g(3) = 0$

Thus, by Rolle's theorem

$g'(c_2) = 0$, for some $c_2 \in (2,3)$

So , $g'(c_1) = g'(c_2) = 0$

As $g(x)$ is twice differentiable, $g'(x)$ is continuous and differentiable in the interval (c_1, c_2).

Thus, by Rolle's theorem

$g''(c) = 0$, for some $c \in (c_1, c_2)$

As $c_1 \in (1,2)$ and $c_2 \in (2,3) \Rightarrow c \in (1,3)$

$\Rightarrow f''(c) - 2 = 0$, for some $c \in (1,3)$ $(\because g(x) = f(x) - x^2)$

Now replacing c with x,

$f''(x) = 2$, for some $x \in (1,3)$

So, option (c) is the correct answer.

Illustration

IIT JEE 2006

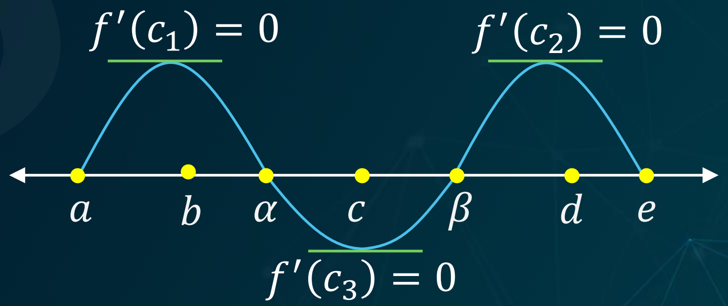
If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2,$ $f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeros of $g(x) = (f'(x))^2 + f''(x).f(x)$ in the interval $[a, e]$ is :

Solution:

Given, $f(b) = 2$ and $f(c) = -1$

Here, sign of $f(x)$ changes.

So at least one zeros of $f(x)$ must lie between b and c.



Similarly, at least one zeros of $f(x)$ must lie between c and d as shown in the figure.

Hence, minimum number of zeros of $f(x)$ is 4.

If $f(a) = f(b) = 0$ where $f(x)$ is continuous and differentiable, then there will exist atleast one c for which $f'(c) = 0$. (Rolle's theorem)

Hence minimum number of zeros of $f'(x)$ is 3

$g(x) = (f'(x))^2 + f''(x).f(x)$. We can see that $g(x)$ is derivative of $f(x).f'(x)$

Let $h(x) = f(x).f'(x) \rightarrow$ 7 zeros ($\because$ $f(x)$ has 4 zeros and $f'(x)$ has 3 zeros)

$$h'(x) = (f'(x))^2 + f''(x).f(x)$$

$$h'(x) = g(x)$$

$\because$ $h(x)$ has 7 zeros $\Rightarrow$ Minimum Number of zeros of $h'(x)$ is 6

$\Rightarrow$ Minimum Number of zeros of $g(x)$ is 6

Illustration

Let the function $f : [-7,0] \to \mathbb{R}$, be continuous on $[-7,0]$ and differentiable on $(-7,0)$. If $f(-7) = -3$ and $f'(x) \leq 2$, for all $x \in (-7,0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval :

$a.$ $(-\infty, 20]$ $b.$ $[-3, 11]$ $c.$ $(-\infty, 11]$ $d.$ $[-6, 20]$

Solution:

Given, the function $f : [-7,0] \to \mathbb{R}$ is continuous on $[-7,0]$ and differentiable on $(-7,0)$.

Hence, L.M.V.T. can be applied to the function $f(x)$ in the interval $(-7,-1)$

$f'(c) = \frac{f(-1)-f(-7)}{-1-(-7)} \cdots (i)$, where $c \in (-7,-1)$ (By L.M.V.T.)

Also given, $f'(x) \leq 2 \cdots (ii)$

From (i) and (ii)

$$\frac{f(-1)-f(-7)}{-1-(-7)} \le 2$$

$$\Rightarrow \frac{f(-1)+3}{6} \le 2 \Rightarrow f(-1) \le 9 \cdots (iii)$$

L.M.V.T. can also be applied to the function $f(x)$ in the interval $(-7, 0)$

$\Rightarrow f'(d) = \frac{f(0)-f(-7)}{0-(-7)}$ where $d \in (-7, 0)$ (By L.M.V.T.)

$$\Rightarrow \frac{f(0)+3}{7} \le 2 \Rightarrow f(0) \le 11 \cdots (iv)$$

Adding (iii) and (iv),

$$f(-1)+f(0) \le 20$$

So, option (a) is the correct answer.

Illustration

If the function $f : [0,4] \rightarrow \mathbb{R}$ is differentiable , then show that for $(a, b) \in [0,4]$; $(f(4))^2 - (f(0))^2 = 8f'(a)f(b)$

Solution:

Given, the function $f : [0,4] \rightarrow \mathbb{R}$ is differentiable

Hence, L.M.V.T. can be applied to the function $f(x)$

Applying L.M.V.T. to $f(x)$ in the interval $(0, 4)$

$f'(a) = \frac{f(4)-f(0)}{4-0}$, for some $a \in (0,4)..(i)$

Also, by Intermediate value theorem,

$f(b) = \frac{f(4)+f(0)}{2}$, for some $b \in (0,4)..(ii)$

Multiplying (i) and (ii)

$$f'(a)f(b) = \left(\frac{f(4) - f(0)}{4}\right)\left(\frac{f(4) + f(0)}{2}\right)$$

$$\Rightarrow 8\, f'(a)f(b) = (f(4) - f(0))(f(4) + f(0))$$

$$\boxed{\Rightarrow (f(4))^2 - \left(f(0)\right)^2 = 8f'(a)f(b)}$$

Illustration

Let $f:(1,3) \to \mathbb{R}$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f :

a. $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

b. $\left(\frac{2}{5}, \frac{4}{5}\right]$

c. $\left(\frac{2}{5}, \frac{3}{5}\right)$

d. $\left(\frac{2}{5}, \frac{3}{5}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right)$

Solution:

Given, $f(x) = \frac{x[x]}{1+x^2}$

$$\Rightarrow f(x) = \begin{cases} \frac{x}{1+x^2}, & x \in (1,2) \\ \frac{2x}{1+x^2}, & x \in [2,3) \end{cases}$$

Differentiating w.r.t. x,

$$f'(x) = \begin{cases} \frac{1+x^2-x(2x)}{(1+x^2)^2} & , x \in (1,2) \\ 2 \cdot \frac{1+x^2-x(2x)}{(1+x^2)^2} & , x \in [2,3) \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \frac{1-x^2}{(1+x^2)^2} & , x \in (1,2) \\ \frac{2(1-x^2)}{(1+x^2)^2} & , x \in [2,3) \end{cases}$$

In the interval $x \in (1,3)$, $\frac{1-x^2}{(1+x^2)^2} < 0$

And In the interval $x \in (2,3)$, $\frac{2(1-x^2)}{(1+x^2)^2} < 0$

$\therefore f(x)$ is decreasing in $(1,3)$

$$1 < x < 2 \Rightarrow f(1) > f(x) > f(2)$$

$$\Rightarrow \frac{1}{1+1^2} > f(x) > \frac{2}{1+2^2}$$

$$\Rightarrow \frac{1}{2} > f(x) > \frac{2}{5} \dots (i)$$

Also, $3 > x \geq 2 \Rightarrow f(3) < f(x) \leq f(2)$ ($\because f$ is decreasing function)

$$\Rightarrow \frac{2 \times 3}{1+3^2} < f(x) \leq \frac{2 \times 2}{1+2^2}$$

$$\Rightarrow \frac{3}{5} < f(x) \leq \frac{4}{5} \dots (ii)$$

From (i) and (ii), we get,

$$\boxed{\text{Range of } f(x) \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]}$$

So, option (a) is the correct answer.

Maxima and Minima (Recap)

First Derivative Test

Let $f(x)$ be a continuous function

Step 1: $x = c$ is the critical point ($f'(c) = 0$ or not defined)

Step 2: Observe sign change of $f'(x)$ about $x = c$.

(i) If sign of $f'(x)$ changes sign from negative to positive as x crosses c from left to right , then $x = c$ is a point of local minima.

(ii) If sign of $f'(x)$ changes sign from positive to negative as x crosses c from left to right , then $x = c$ is a point of local maxima.

Illustration

IIT JEE 2010

Let f be a function defined on $\mathbb{R}$ (the set of real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4 \ \forall\ x \in \mathbb{R}$.
If g is a function defined on $\mathbb{R}$ with values in interval $(0, \infty)$ such that $f(x) = \ln g(x), \forall\ x \in \mathbb{R}$, then the number of points in $\mathbb{R}$ at which g has a local maximum is ______.

Solution: Given $f(x) = \ln g(x)$

Differentiating w.r.t. x

$$f'(x) = \frac{1}{g(x)}.g'(x)$$

$$\Rightarrow g'(x) = f'(x).g(x)$$

$$\Rightarrow g'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4.g(x)$$

Here, critical points are: $2009\ 2010, 2011, 2012$

Sign changes of $g'(x)$ can be shown on the number line

$g'(x)$ changes its sign from positive to negative only at $x = 2009$

Maxima occurs at $x = 2009$

Number of points at which g has a local maxima $= 1$

Illustration

Let $p(x)$ be the polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$. Then the value of $p(2)$ is

Solution: Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

Given, $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$

$$\Rightarrow \lim_{x\to 0}\left(1+\frac{ax^4+bx^3+cx^2+dx+e}{x^2}\right) = 2$$

$$\Rightarrow \lim_{x\to 0}\left(1+ax^2+bx+c+\frac{d}{x}+\frac{e}{x^2}\right) = 2$$

As limit exists finitely,

$d = 0, \mathrm{e} = 0, \mathrm{c} + 1 = 2 \Rightarrow c = 1$ (For finite value in R. H. S.)

$$\Rightarrow p(x) = ax^4 + bx^3 + x^2$$

Differentiating w.r.t to x

$$p'(x) = 4ax^3 + 3bx^2 + 2x$$

Given, $p(x)$ has extremum at $x = 1, 2$

$$\Rightarrow p'(1) = p'(2) = 0$$

$$\Rightarrow 4a + 3b + 2 = 0 \cdots (i)$$

and $32a + 12b + 4 = 0 \cdots (ii)$

On solving (i) and (ii) simultaneously, we get, $a = \frac{1}{4}, b = -1$

Thus , $p(x) = \left(\frac{1}{4}\right)x^4 - x^3 + x^2$

$$\Rightarrow p(2) = \left(\frac{1}{4}\right)16 - 8 + 4 = 0$$

$$\boxed{\Rightarrow p(2) = 0}$$

Summary Sheet

- Rolle's Theorem

Let f be a real – valued function defined on the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the interval $[a, b]$

(ii) $f(x)$ is differentiable in the interval (a, b)

(iii) $f(a) = f(b)$

Then there exists at least one $c \in (a, b)$, such that $f'(c) = 0$

- Lagrange's Mean Value Theorem (L.M.V.T)

Let f be a real – valued function defined on the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the interval $[a, b]$

(ii) $f(x)$ is differentiable in the interval (a, b)

Then, there exists at least one $c \in (a, b)$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

- <u>First Derivative Test</u>

Let $f(x)$ be a continuous function such that $x = c$ is the critical point ($f'(c) = 0$ or not defined), then observe sign change of $f'(x)$ about $x = c$.

(i) If sign of $f'(x)$ changes sign from negative to positive as x crosses c from left to right , then $x = c$ is a point of local minima.

(ii) If sign of $f'(x)$ changes sign from positive to negative as x crosses c from left to right , then $x = c$ is a point of local maxima.