BYJU'S Classes

MATHEMATICS

CONTINUITY AND DIFFERENTIABILITY

INTRODUCTION TO CONTINUITY

NOTES

What you already know

- R.H.L and L.H.L
- Standard limits

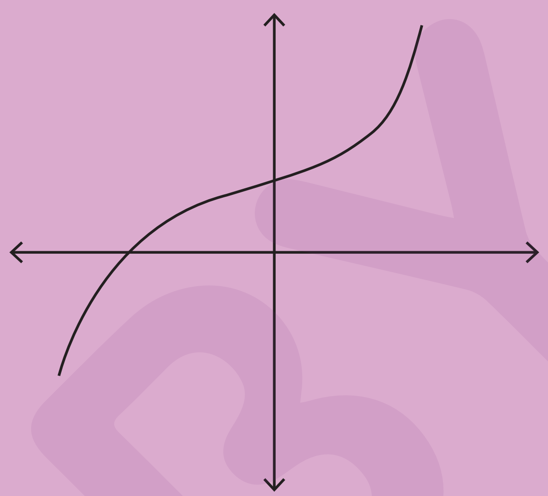
What you will learn

- Continuity of a function
- Discontinuous functions
- One-sided continuity
- Types of discontinuity

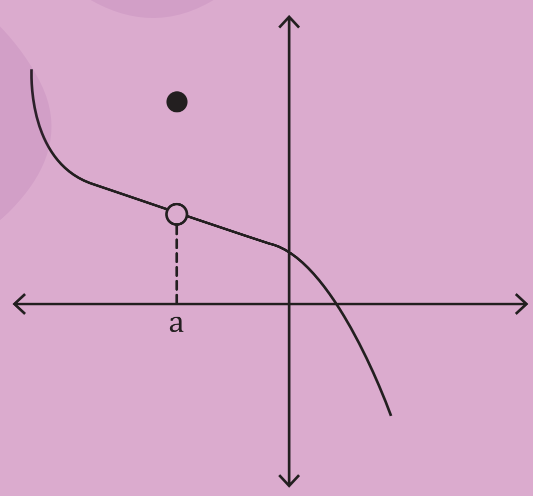
Continuity of a Function

Graphically, a continuous function is a real-valued function whose graph does not have any breaks.

Check whether the functions are continuous or discontinuous.



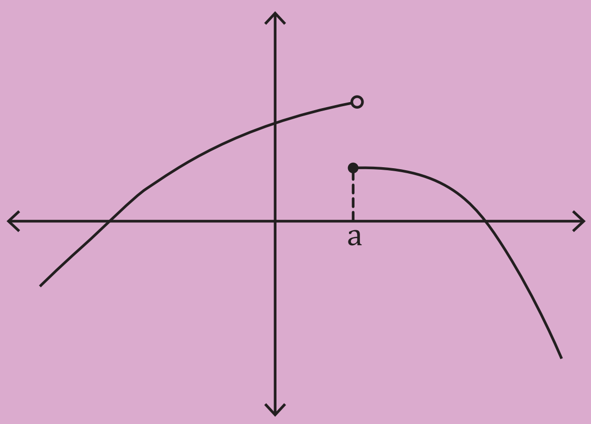
(a) We can see that the function does not have any abrupt changes in value or any breaks. So, the given function is a continuous function.



(b) We can see that the value of function changes abruptly at $x = a$ or the graph is having a break at $x = a$. So, the given function is a discontinuous function.

Mathematically, a function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$, i.e., f is continuous at $x = c$ iff $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

Check whether the functions are continuous or discontinuous.

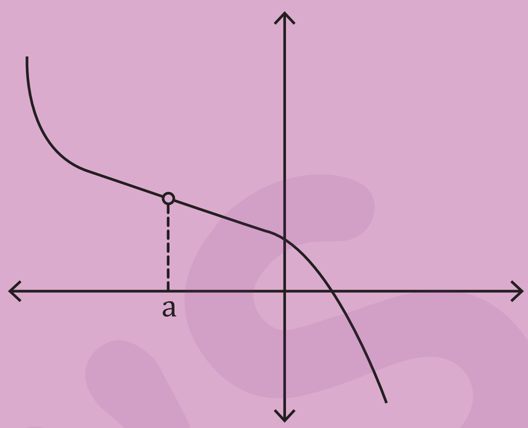


Solution

For a function to be continuous at x = c, $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

For the given function, $\lim\limits_{x\to a} f(x)$ does not exist.

So, the function is not continuous at x = a

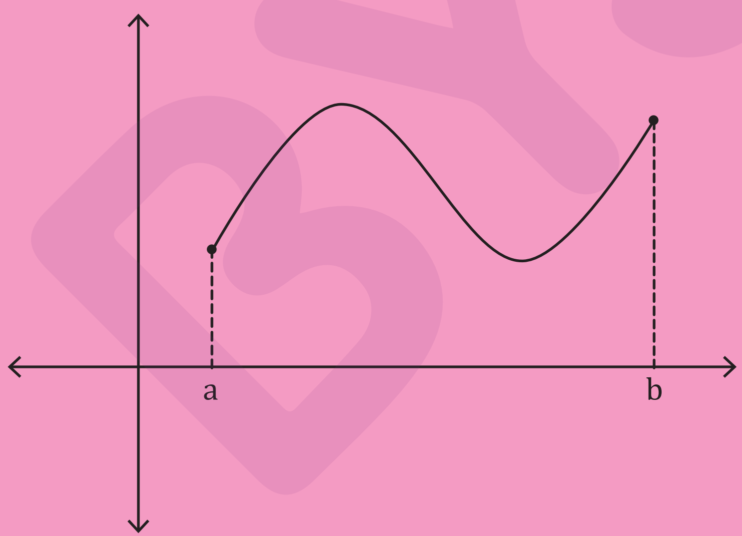


Solution

For a function to be continuous at x = c, $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$. For the given function, f(x) is not defined at x = a. So, the function is not continuous at x = a

One-Sided Continuity

Similar to one-sided limits, we can define one-sided continuity. Let us understand this using the following function:



Continuity at x = a

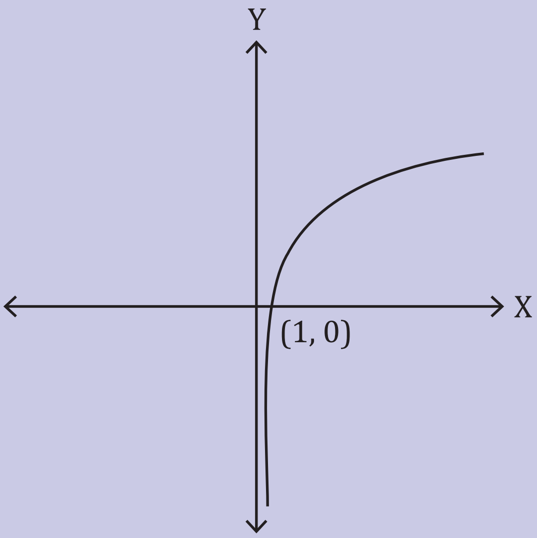
The function f(x) will be continuous at x = a when $\lim\limits_{x\to a^+} f(x) = f(a)$. This means that we are not bothered about the L.H.L as the function is not defined.

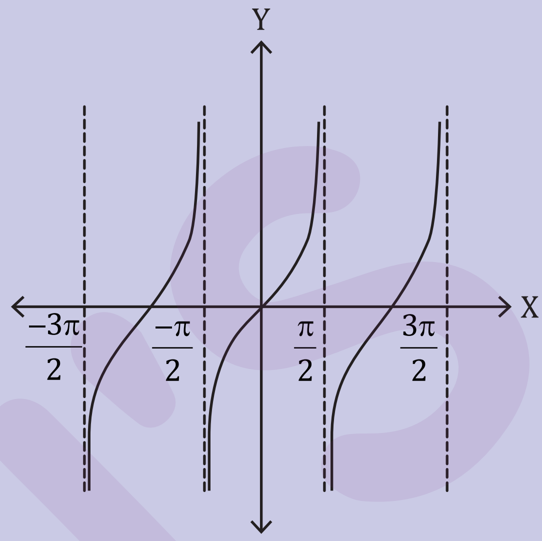
Continuity at x = b

The function f(x) will be continuous at x = b when $\lim\limits_{x\to b^-} f(x) = f(b)$. This means that we are not bothered about the R.H.L as the function is not defined.

Note

Polynomial, logarithmic, trigonometric and exponential functions are continuous in their domains. For example, we can see that y = $\ln x$ and y = $\tan x$ are continuous in their domains.

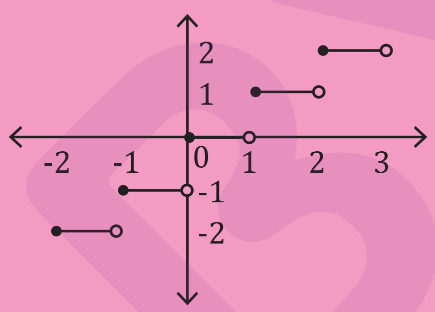




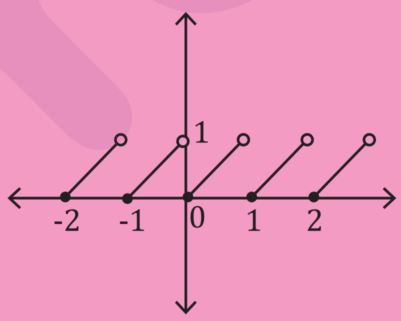
Discontinuous Functions

Let us look at some of the most popular and widely used discontinuous functions.

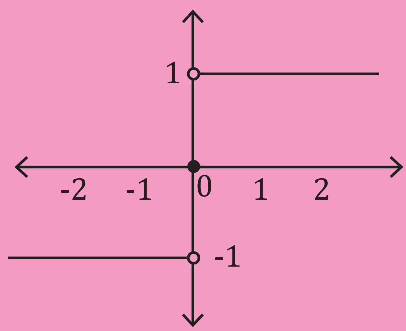
1. $y = [x]$; where [.] denotes G.I.F. Looking at the graph, we can see that $y = [x]$ is discontinuous at all the integral values of x.



2. $y = \{x\}$; where {.} denotes fractional part function Looking at the graph, we can see that $y = \{x\}$ is discontinuous at all the integral values of x.



3. $y = \operatorname{sgn} x$ Looking at the graph, one can see that $y = \operatorname{sgn} x$ is discontinuous at $x = 0$



If $f(x)=\begin{cases} x+1, & x\le 1 \\ 2x-1, & x>1 \end{cases}$ **, then check the continuity of f(x) at x = 1**

Solution

A function f(x) is said to be continuous at x = c if $\lim\limits_{x\to c} f(x)=f(c)$, i.e., f is continuous at x = c iff $\lim\limits_{x\to c^-} f(x)=\lim\limits_{x\to c^+} f(x)=f(c)$

Now,

$$\text{L.H.L} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = 2$$

$$\text{R.H.L} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x-1) = 1$$

$$f(1) = 1+1 = 2$$

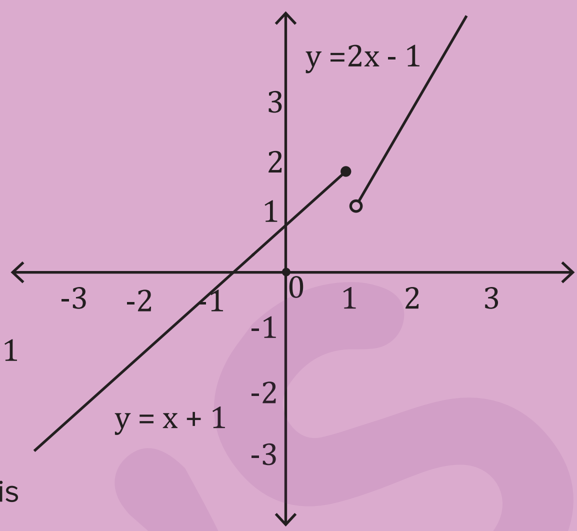
Here, L.H.L ≠ R.H.L

So, the function is not continuous at x = 1

Alternative method

Let us plot the graph of f(x) to check its continuity at x = 1

$$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x-1, & x > 1 \end{cases}$$



In the graph, there is a break at x = 1. So, the function is not continuous at x = 1

If $f(x) = \begin{cases} \dfrac{x^2-x-6}{x+2}, & x \ne -2 \\ k, & x = -2 \end{cases}$ is continuous at x = -2, then find the value of k.

(a) 0 **(b) -2** **(c) -5** **(d) Does not exist**

Solution

Step 1:

Since f(x) is continuous at x = -2, we can say that $\lim_{x\to -2} f(x) = f(-2) = k$

Step 2:

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2-x-6}{x+2}$$

$$= \lim_{x\to -2} \frac{(x+2)(x-3)}{(x+2)}$$

$$= \lim_{x\to -2}(x-3) = -5$$

So, option (c) is the correct answer.

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If $f(x) = \begin{cases} \dfrac{\sin(a+2)x + \sin x}{x}, & x < 0 \\ b, & x = 0 \\ \dfrac{(x+3x^2)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{4}{3}}}, & x > 0 \end{cases}$ is continuous at x = 0, then find the value of a + 2b.

(a) -2 **(b) -1** **(c) 1** **(d) 0**

Solution

Step 1: A function f(x) is said to be continuous at x = c if $\lim\limits_{x \to c} f(x) = f(c)$, i.e., f is continuous at x = c iff $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = f(c)$

Given, f(x) is continuous at x = 0 $\Rightarrow \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = b$

Step 2:

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\left(x + 3x^2\right)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{4}{3}}}$$

$$= \lim_{x \to 0^+} \frac{x^{\frac{1}{3}}\left(1 + 3x\right)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{1}{3}} \cdot x}$$

$$= \lim_{x \to 0^+} \frac{\left(1 + 3x\right)^{\frac{1}{3}} - 1}{x}$$

$$= \lim_{x \to 0^+} \frac{\left(1 + 3x \cdot \frac{1}{3}\right) - 1}{x} \qquad \left(\because x \to 0, \left(1 + x\right)^n = 1 + nx\right)$$

$$= \lim_{x \to 0^+} \frac{x}{x} = 1$$

$\therefore$ b = 1

Step 3:

$$\text{L.H.L} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(a+2)x + \sin x}{x}$$

$$= \lim_{x \to 0^-} \frac{\sin(a+2)x}{x} + \lim_{x \to 0^-} \frac{\sin x}{x}$$

$$= \lim_{x \to 0^-} \frac{\sin(a+2)x}{(a+2)x} \cdot (a+2) + \lim_{x \to 0^-} \frac{\sin x}{x}$$

$$= (a+2) + 1 \qquad \left(\because \text{For } \lim_{x \to 0^-} \frac{\sin x}{x} = 1\right)$$

$$= a + 3$$

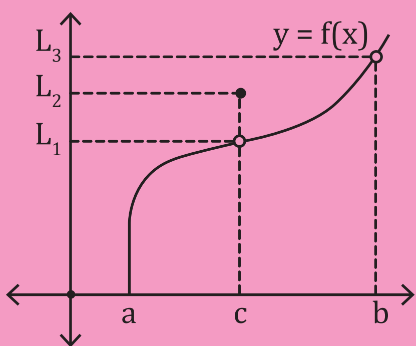
$\Rightarrow$ a + 3 = 1, a = -2

$\therefore$ a + 2b = 0

So, option (d) is the correct answer.

Types of Discontinuity

Removable discontinuity: Limit exist finitely (L.H.L = R.H.L = Finite) and ≠ value of function at that point or the value of function is not defined at that point. In the graph, we can see that both $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to b} f(x)$ are finite values.



There are two types of removable discontinuity.

(a) **Isolated point:** L.H.L = R.H.L ≠ Value of function at that point
At point c, there is an isolated point removable discontinuity.

(b) **Missing point:** L.H.L = R.H.L, but the value of function is not defined at that point. At point b, there is a missing point removable discontinuity.

Non-removable discontinuity: Limit does not exist, i.e., either L.H.L or R.H.L or both are finite but not equal or at least one of them is infinite.
There are three types of non-removable discontinuity.

(a) **Finite discontinuity:** Limit does not exist, i.e., L.H.L and R.H.L are finite but not equal.

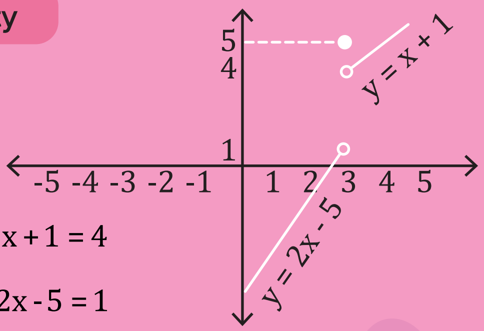
(b) **Infinite discontinuity:** At least one among L.H.L and R.H.L is infinite.

(c) **Oscillatory discontinuity:** Limits oscillate between two values that are finite.

Let us see one example of each of these.

Finite discontinuity

$$f(x)=\begin{cases} x+1, x>3 \\ 5, x=3 \\ 2x-5, x<3 \end{cases}$$



$$\text{R.H.L} = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+1 = 4$$

$$\text{L.H.L} = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} 2x-5 = 1$$

$$f(3)=5$$

Here, L.H.L ≠ R.H.L but both are finite.
So, it is a finite type non-removable discontinuity.
Jump of discontinuity is defined as:
|L.H.L - R.H.L| = |1 - 4| = 3

Infinite discontinuity

$$f(x)=\begin{cases} \dfrac{1}{x-1}, x>1 \\ 2x+3, x\le 1 \end{cases}$$

Here,

$$\text{R.H.L} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} \to \infty$$

$$\text{L.H.L} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x+3 = 5$$

$$f(1)=5$$

∴ It is an infinite-type non-removable discontinuity as R.H.L → ∞

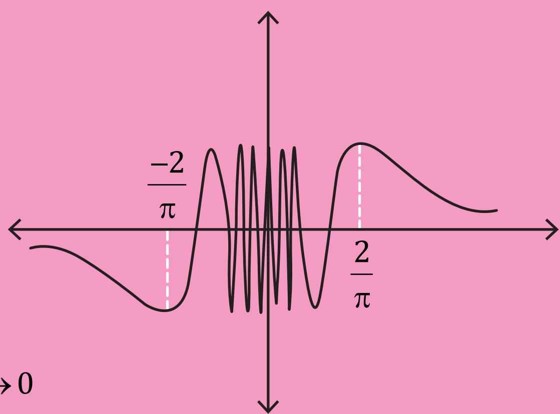
Oscillatory discontinuity

$$f(x)=\sin\frac{1}{x}$$

$f(x)=\sin\dfrac{1}{x}$ becomes 1 at $x=\dfrac{2}{\pi}$ and -1 at $x=-\dfrac{2}{\pi}$

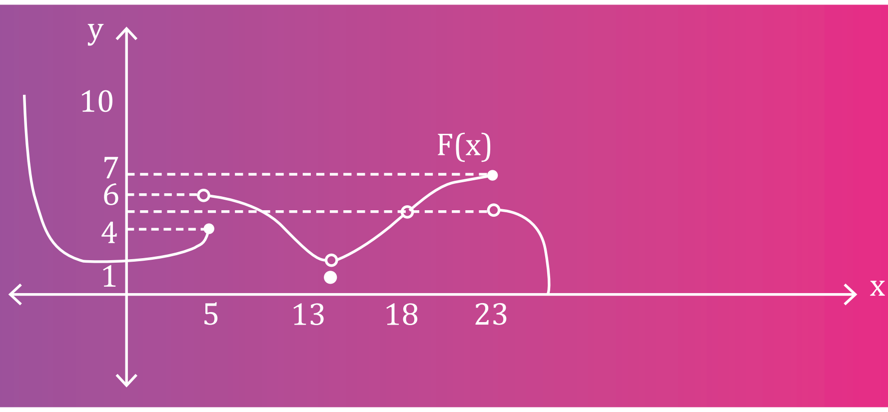
The function $f(x)=\sin\dfrac{1}{x}$ does not attain a limit as $x\to 0$.

As $x\to 0$, $\sin\dfrac{1}{x}$ oscillates between -1 and 1 and the frequency of oscillations is very high.



One cannot find a particular value of $f(x)=\sin\dfrac{1}{x}$ when $x\to 0$

Find the types of discontinuity in the given graph.



Solution

(a) At x = 5, there is a finite non-removable discontinuity as L.H.L ≠ R.H.L, but both are finite.
Jump of discontinuity = |L.H.L - R.H.L| = |6 - 4| = 2

(b) At x = 13, there is an isolated point removable discontinuity as L.H.L = R.H.L but not equal to the Value of function at that point

(c) At x = 18, there is a missing point removable discontinuity as L.H.L = R.H.L, but the value of function is not defined at that point.

(d) At x = 23, there is a finite non-removable discontinuity as L.H.L ≠ R.H.L, but both are finite.
Jump of discontinuity = |L.H.L - R.H.L| = |7 - 5| = 2

Identify the type of discontinuity: $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **at x = 0**

Solution

Step 1:

$$\text{L.H.L} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x\to 0^-} \frac{0-1}{0+1} \quad \left(\because x \to 0^- \Rightarrow \frac{1}{x} \to -\infty, e^{\frac{1}{x}} \to 0\right)$$

$$= -1$$

Step 2:

$$\text{R.H.L} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x\to 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}}$$

$$= \lim_{x\to 0^-} \frac{1-0}{1+0} \quad \left(\because x \to 0^+ \Rightarrow \frac{1}{x} \to \infty, \frac{1}{e^{\frac{1}{x}}} \to 0\right)$$

$$= 1$$

And f(0) = 1

Step 3:

∴ At x = 0, there is a finite non-removable discontinuity as L.H.L ≠ R.H.L, but both are finite.

Concept Check

(1) If $f(x)=[x]-\left[\frac{x}{4}\right], x \in \mathbb{R}$, where [.] denotes G.I.F, then which of the following options is correct?

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(a) $\lim_{x\to 4^-} f(x)$ exists but $\lim_{x\to 4^+} f(x)$ does not exist.

(b) Both $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$ exist but are not equal.

(c) $\lim_{x\to 4^+} f(x)$ exists but $\lim_{x\to 4^-} f(x)$ does not exist.

(d) f is continuous at x = 4

(2) Identify the type of discontinuity: $f(x)=\begin{cases} \frac{1}{2}\sin\left(\frac{\pi x}{2}\right), & x<1 \\ -1, & x=1 \\ -\frac{\ln(\cos(x-1))}{(x-1)^2}, & x>1 \end{cases}$ at x = 1

Summary Sheet

Key Takeaways

Continuity of a function

- A continuous function is a function that does not have any abrupt changes in value.
- Graphically, a continuous function is a real-valued function whose graph does not have any breaks.
- Mathematically, a function f(x) is said to be continuous at x = c if $\lim_{x\to c} f(x)=f(c)$, i.e., f is continuous at x = c iff $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$

Discontinuous functions

- y = [x] is discontinuous at all the integral values of x, where [.] represents the Greatest integer function.
- y = {x} is discontinuous at all the integral values of x, where {.} represents the Fractional part function.
- y = sgn x is discontinuous at x = 0

Types of discontinuity

- **Removable discontinuity:** Limit exist finitely (L.H.L = R.H.L = Finite) and ≠ value of function at that point or the value of function is not defined at that point

  There are two types of removable discontinuity.

  (a) **Isolated point:** L.H.L = R.H.L ≠ Value of function at that point

  (b) **Missing point:** L.H.L = R.H.L, but the value of function is not defined at that point.

- **Non-removable discontinuity:** Limit does not exist, i.e., either L.H.L or R.H.L or both are finite but not equal or at least one of them is infinite.
  There are three types of non-removable discontinuity.
  (a) **Finite-type discontinuity:** Limit does not exist, i.e., L.H.L and R.H.L are finite but not equal.
  (b) **Infinite-type discontinuity:** At least one among L.H.L and R.H.L is infinite.
  (c) **Oscillatory-type discontinuity:** Limits oscillate between two values that are finite.

Mind Map

- Continuity and Differentiability
  - Continuity of a function
    - One sided continuity
  - Discontinuity functions
    - $y = [x]$
    - $y = \{x\}$
    - $y = \text{sgn}\{x\}$
  - Types of Discontinuity
    - Removable
      - Missing point
      - Isolated point
    - Non - Removable
      - Finite
      - Infinite
      - Oscillatory

Self-Assessment

If $f(x) = \dfrac{1 - \cos 7(x - \pi)}{x - \pi}$ $(x \neq \pi)$ is continuous at $x = \pi$, then find $f(\pi)$.

(a) 0 (b) 1 (c) -1 (d) 7

Answers

Concept Check

1. **Step 1:**

$$\text{R.H.L} = \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} [x] - \left[\frac{x}{4}\right] = [4^+] - \left[\frac{4^+}{4}\right]$$

$$= 4 - 1$$

$$= 3$$

Step 2:

$$\text{L.H.L} = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} [x] - \left[\frac{x}{4}\right] = [4^-] - \left[\frac{4^-}{4}\right]$$

$$= 3 - 0$$

$$= 3$$

Step 3:

$$f(4) = [4] - \left[\frac{4}{4}\right] = 4 - 1 = 3$$

We can see that $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$

∴ The function is continuous at x = 4

So, option (d) is the correct answer.

2. **Step 1:**

$$\text{L.H.L} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{2}\sin\left(\frac{\pi x}{2}\right)$$

$$= \frac{1}{2}$$

$$\text{R.H.L} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -\frac{\ln(\cos(x-1))}{(x-1)^2}$$

$$= \lim_{x\to 1^+} \frac{\frac{\sin(x-1)}{\cos(x-1)}}{2(x-1)} \quad \text{(Applying L'Hospital's rule)}$$

$$= \lim_{x\to 1^+} \frac{\sin(x-1)}{2(x-1).\cos(x-1)} \quad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 1^+} \frac{1}{2\cos(x-1)} = \frac{1}{2}$$

Step 2:

At x = 1, there is an isolated point removable discontinuity as L.H.L = R.H.L but not equal to the value of function at that point

Self-Assessment

Step 1:

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} \frac{1-\cos 7(x-\pi)}{x-\pi} \text{ is of the form } \frac{0}{0}.$$

So, L'Hospital's rule can be applied.

Step 2:

$$\Rightarrow \lim_{x\to\pi} \frac{1-\cos 7(x-\pi)}{x-\pi} = \lim_{x\to\pi} \frac{7\sin 7(x-\pi)}{1} \qquad \text{(LH rule)}$$

$$= 0$$

Step 3:

For the function to be continuous at $x = \pi$, $\lim_{x\to\pi} f(x) = f(\pi)$

$$\Rightarrow f(\pi) = 0$$

So, option (a) is the correct answer.

MATHEMATICS

CONTINUITY AND DIFFERENTIABILITY

CONTINUITY IN AN INTERVAL

NOTES

What you already know

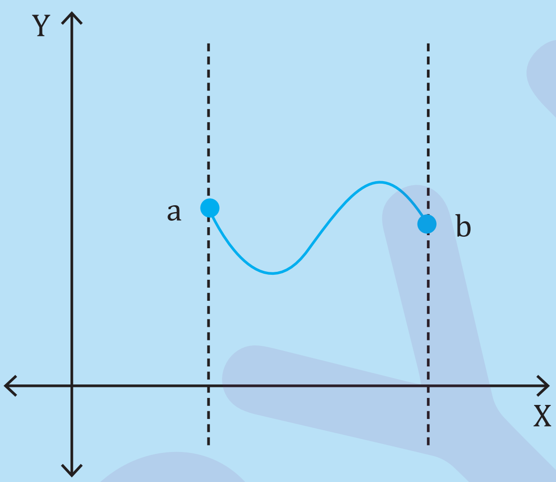
- Continuity of a function
- One-sided continuity
- Discontinuous functions
- Types of discontinuity

What you will learn

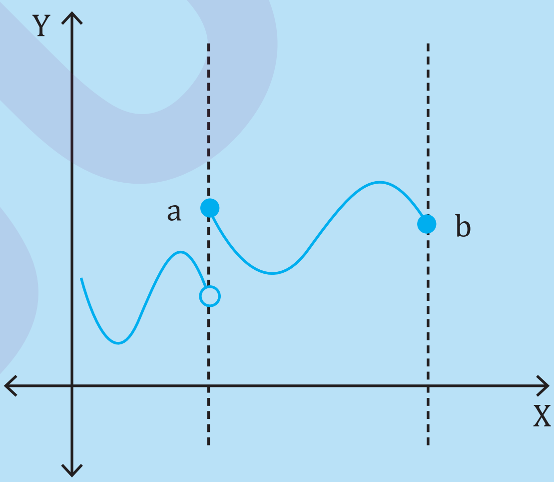
- Continuity in an interval
- Theorems on continuity

Continuity in an Interval

- A function f is said to be continuous in (a, b), if f is continuous at each and every point in the interval (a, b).
- A function f is said to be continuous in [a, b], if:
  - » f is continuous in (a, b).
  - » f is right-continuous at a, i.e., $\text{RHL} = \lim_{x \to a^+} f(x) = f(a)$
  - » f is left-continuous at b, i.e., $\text{LHL} = \lim_{x \to b^-} f(x) = f(b)$



Function is continuous in interval [a, b].



We can see that though, overall function is discontinuous still it is continuous in interval [a, b].

Note

The polynomial, trigonometric, exponential, and logarithmic functions are continuous at every point of their respective domain. Thus, for these functions, the continuity should be checked at:

- » Points where the function changes the definition
- » Boundary points

Comment on the continuity of $f(x) = \begin{cases} 2x+3, & -3 \le x < -2 \\ x+1, & -2 \le x < 0 \\ x+2, & 0 \le x \le 1 \end{cases}$

Solution

Step 1:

Given,

$$f(x) = \begin{cases} 2x+3, & -3 \le x < -2 \\ x+1, & -2 \le x < 0 \\ x+2, & 0 \le x \le 1 \end{cases}$$

Clearly, possible points of discontinuity

$x = -3, -2, 0, 1$

Now, we can see that at point -3 and 1 function is right continous and left continous, respectively.

At $x = -2$,

LHL = $\lim_{x \to -2^-} (2x+3) = -1$

RHL = $\lim_{x \to -2^+} (x+1) = -1$

$f(-2) = x + 1 = -1$

$\Rightarrow$ f is continous at -2.

Step 2:

At $x = 0$

LHL = $\lim_{x \to -0^-} (x+1) = 1$

$f(0) = x + 2 = 2$

Here, LHL $\neq f(0)$

$\Rightarrow$ f is discontinuous at $x = 0$.

Therefore, function f is continuous at every point except at 0.

Note

- For $[f(x)]$, the continuity should be checked at points where $f(x)$ becomes an integer. ([.] represents the greatest integer function.)

Discuss the continuity of $f(x) = [\ln x], x \in [1, e^3]$, where [.] represents the greatest integer function.

Solution

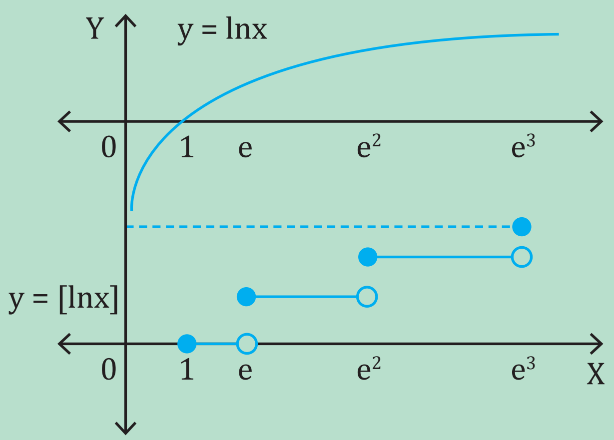
Given, $f(x) = [\ln x], x \in [1, e^3]$

$\Rightarrow 1 \le x \le e^3$

$\Rightarrow 0 \le \ln x \le 3$

$\Rightarrow [\ln x] = 0, 1, 2, 3$

Therefore, we can see that function $f(x)$ is right-continuous at $x = 1$ and discontinuous at $x = e, e^2$, and e^3

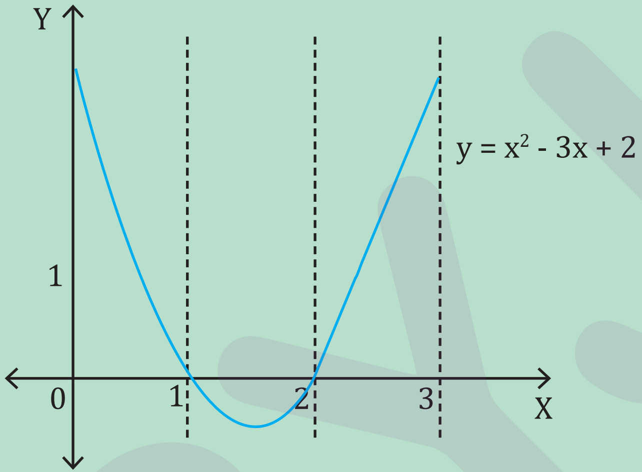


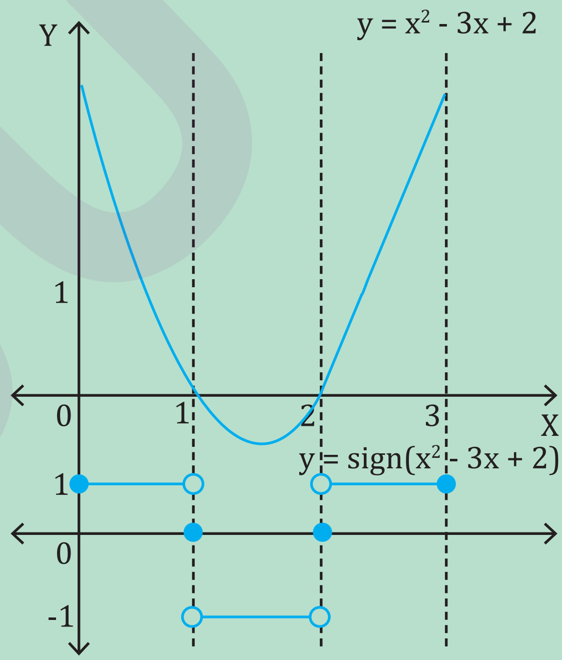
- For $\{f(x)\}$, the continuity should be checked at points where $f(x)$ becomes an integer. $\{.\}$ denotes the fractional part function.
- For sgn $(f(x))$, the continuity should be checked at points where $f(x) = 0$

Discuss the continuity of $f(x) = \text{sgn}\ (x^2 - 3x + 2), 0 \le x \le 3$

Solution

Given, $f(x) = \text{sgn}(x^2 - 3x + 2), 0 \le x \le 3$
As we know that for sgn (x), the continuity should be checked at points where $f(x) = 0$
$\Rightarrow (x - 1)(x - 2) = 0$
Therefore, f is discontinuous at 1 and 2.





$f(x) = \begin{cases} [2x], & 0 \le x < 1 \\ \{x\}\ \text{sgn}(-x), & 1 \le x \le 2 \end{cases}$, where [.] denotes the G.I.F. and { } denotes the fractional part function. Comment on the continuity of function f in interval $[0, 2]$.

Solution

Given, $f(x) = \begin{cases} [2x], & 0 \le x < 1 \\ \{x\}\ \text{sgn}(-x), & 1 \le x \le 2 \end{cases}$, where [.] denotes the G.I.F. and { } denotes the fractional part function

$\Rightarrow 0 \le x < 1$

$\Rightarrow 0 \le 2x < 2$

$\Rightarrow [2x] = 0, 1$

$\Rightarrow$ Possible points of discontinuity $x = 0, \frac{1}{2}, 1, 2$

At $x = 0$

$\text{RHL} = \lim\limits_{x \to 0^+} [2x] = 0$, and $f(0) = 0$

Thus, f is right continous at 0.

At $x = \frac{1}{2}$,

$\text{LHL} = \lim\limits_{x \to \frac{1}{2}^-} [2x] = 0$, and $f\left(\frac{1}{2}\right) = 1$

$\Rightarrow f$ is discontinuous at $x = \frac{1}{2}$.

At $x = 1$

$\text{LHL} = \lim\limits_{x \to 1^-} [2x] = [2^-] = 1$ and $f(1) = \{x\}\text{sgn}(-x) = 0$

$\Rightarrow f$ is discontinuous at 1

At $x = 2$

$\text{LHL} = \lim\limits_{x \to 2^-} \{x\}\text{sgn}(-x) = 1 \times (-1) = -1$, and $f(2) = \{2\}\ \text{sgn}(-2) = 0$

$\Rightarrow f$ is discontinuous at 2.

Theorems on Continuity

- If f and g are continuous at $x = a$, then
  - $f(x) \pm g(x)$ will also be continuous at $x = a$

    Examples: $f(x) = \sin x$ and $g(x) = x + 1$ are continuous functions at $x = \frac{\pi}{2}$
  - $f(x) \,.\, g(x)$ will also be continuous at $x = a$
  - $\frac{f(x)}{g(x)}$ is also continuous at $x = a$ if $g(a) \neq 0$
- If f is continuous and g is discontinuous at $x = a$, then
  - $f(x) \pm g(x)$ will be discontinuous at $x = a$

    Example: If $f(x) = x$, $g(x) = [x]$, then $f(x) - g(x)$ will be discontinuous at integers.
  - $f(x) \,.\, g(x)$ and $\frac{f(x)}{g(x)}$, $[g(a) \neq 0]$ may be continuous at $x = a$

If $h : \mathbb{R} \to \mathbb{R}$ is a function defined by $h(x) = [x] \times \cos((2x - 1)\frac{\pi}{2})$, where [.] denotes the G.I.F., then which of the following is correct for $h(x)$?

(a) Continuous for every real x

(b) Discontinuous only at $x = 0$

(c) Discontinuous only at non-zero integral value of x

(d) Continuous only at $x = 0$

Step 1:

Given, $h(x) = [x] \times \cos\left((2x-1)\frac{\pi}{2}\right)$

Let $f(x) = [x]$, $g(x) = \cos\left((2x-1)\frac{\pi}{2}\right)$

Clearly, $f(x)$ is discontinuous at integers and $g(x)$ is continuous $\forall\ x \in \mathbb{R}$

$g(x) = \cos\left((2x-1)\frac{\pi}{2}\right)$

$\Rightarrow g(x) = \cos\left(x\pi - \frac{\pi}{2}\right) = \sin(\pi x)$

$\Rightarrow h(x) = [x] \times \sin(\pi x)$

Now, the possible points of discontinuity are all the integers $(\mathbb{Z})$

$h(\mathbb{Z}) = [\mathbb{Z}] \times \sin(\pi\mathbb{Z}) = 0$

Step 2:

$h\left(\mathbb{Z}^{+}\right) = \left[\mathbb{Z}^{+}\right] \times \sin\left(\pi\mathbb{Z}^{+}\right) = 0$

$h\left(\mathbb{Z}^{-}\right) = \left[\mathbb{Z}^{-}\right] \times \sin\left(\pi\mathbb{Z}^{-}\right) = 0$

$\therefore h(x)$ is continuous $\forall x \in \mathbb{R}$

Hence, option (a) is the correct answer.

- If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then,
  - » $f(x) \pm g(x)$ may be continuous at $x = a$

    Examples: $f(x) = [x]$, $g(x) = \{x\}$ are discontinuous functions

    $\Rightarrow f(x) + g(x) = [x] + \{x\} = x$

    $\Rightarrow f + g$ is continuous $\forall\ x \in \mathbb{R}$
  - » $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$, $[g(a) \neq 0]$ may be continuous at $x = a$

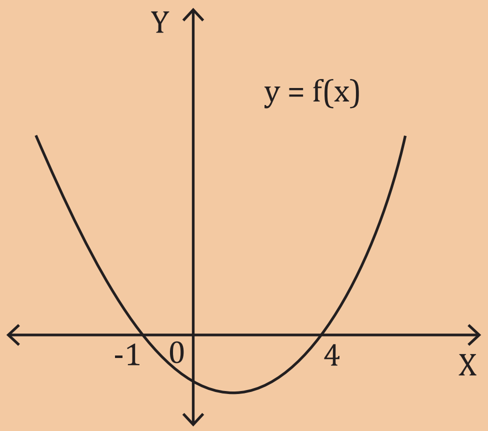
General result

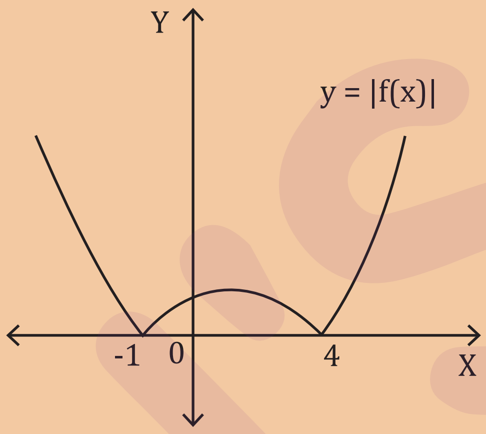
$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \times g(x)$	$\frac{f(x)}{g(x)}$
Continuous	Continuous	Continuous	Continuous	Continuous
Continuous	Discontinuous	Discontinuous	Continuous/ Discontinuous	Continuous/ Discontinuous
Discontinuous	Discontinuous	Continuous/ Discontinuous	Continuous/ Discontinuous	Continuous/ Discontinuous

Note

If a function f is continuous, then |f| is always continuous.

Example: If $f(x) = x^2 - 3x - 4$, then $g(x) = |f(x)| = |x^2 - 3x - 4|$ is continuous $\forall\ x \in \mathbb{R}$





Let $f(x) = [x] + \sqrt{x - [x]}$, where [.] denotes the G.I.F., then

(a) $f(x)$ is continuous for $\mathbb{R}^+$.

(b) $f(x)$ is continuous for $\mathbb{R}$.

(c) $f(x)$ is continuous for $\mathbb{R}$ - Z.

(d) None of these

Solution

Given, $f(x) = [x] + \sqrt{x - [x]}$, where [.] denotes the G.I.F.

$\Rightarrow f(x) = [x] + \sqrt{\{x\}}$

Clearly, $[x]$ is discontinuous for all the integers and $\{x\}$ is also discontinuous for all the integers.

$\Rightarrow$ The possible points of discontinuity: $\mathbb{Z}$

Let $x = 1$

$$f(1) = [1] + \sqrt{\{1\}} = 1$$

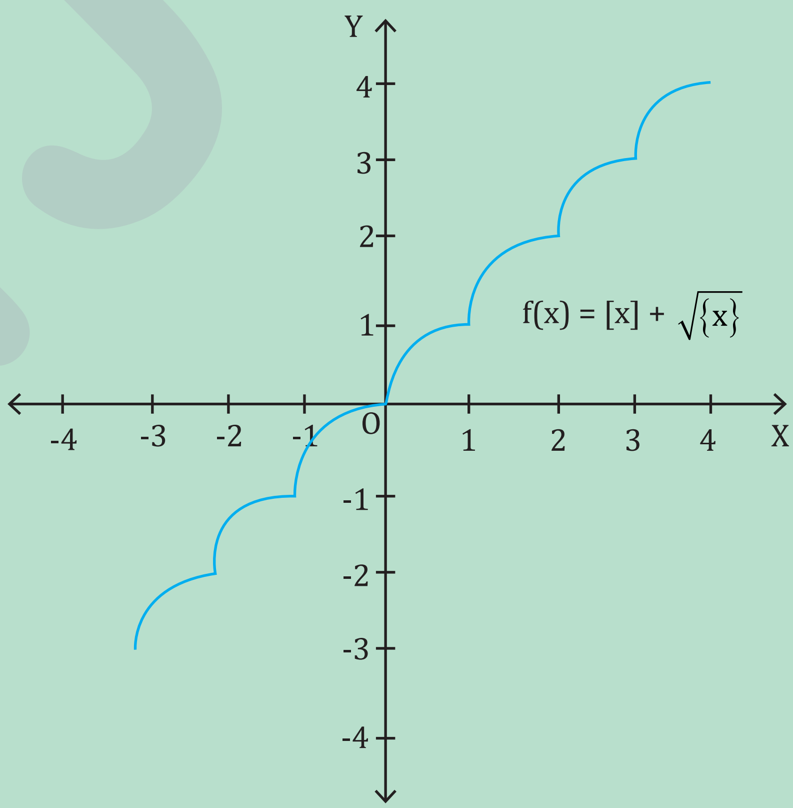
$$\text{LHL} = \lim_{x \to 1^-} [x] + \sqrt{\{x\}} = 0 + 1 = 1$$

$$\text{RHL} = \lim_{x \to 1^+} [x] + \sqrt{\{x\}} = 1 + 0 = 1$$

$$\text{LHL} = \text{RHL} = f(1)$$

$\Rightarrow$ Function f is continuous for all $x \in \mathbb{Z}$

So, option (b) is the correct answer.



Concept Check

1. If the function f is defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by $f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$ **JEE MAIN 2019**

If function is continuous at $x = \frac{\pi}{4}$, then find k.

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\frac{1}{2}$ (d) 2

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & x < -1 \\ |ax^2 + x + b|, & -1 \le x \le 1 \\ \sin(\pi x), & x > 1 \end{cases}$ **JEE MAIN 2021**

If f(x) is continuous on $\mathbb{R}$, then what is a + b equal to :

(a) 3 (b) -1 (c) -3 (d) 1

3. Let $f : [-1, 3] \to \mathbb{R}$ be defined as $f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$, **JEE MAIN 2019**

where [x] denotes the G.I.F. Then, f is discontinuous at:

(a) Four or more points (b) Only three points

(c) Only two points (d) Only one point

Summary Sheet

Key Takeaways

- The polynomial, trigonometric, exponential, and logarithmic functions are continuous at every point of their respective domain. Thus, for these functions, the continuity should be checked at:
  - » Points where the function changes the expression
  - » Boundary points
- For [f(x)], the continuity should be checked at points where f(x) becomes an integer. [.] represents the greatest integer function.
- For {f(x)}, the continuity should be checked at points where f(x) becomes an integer. {.} denotes the fractional part function.
- For sgn f(x), the continuity should be checked at points where f(x) = 0

Mind Map

- Functions
  - Continuity in an interval
  - Theorems on continuity

Self-Assessment

Comment on the continuity of the function $f(x) = \begin{cases} [x] + \{x\}, & x < 1 \\ 2 - |x|, & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$ Where [.] denotes the G.I.F. and {.} denotes the fractional part function

Answers

Concept Check

1.
Given,

$$f(x) = \begin{cases} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \ne \dfrac{\pi}{4} \\ k, & x = \dfrac{\pi}{4} \end{cases}$$

Now,

$$\text{LHL} = \text{RHL} = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow k = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

Using L'hospital's rule, we get

$$k = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}(-\sin x)}{(-\operatorname{cosec}^2 x)} = \frac{1}{2}$$

Hence, option (c) is the correct answer.

2.

Step 1: Given,

$$f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & x < -1 \\ |ax^2 + x + b|, & -1 \le x \le 1 \\ \sin(\pi x), & x > 1 \end{cases}$$

Now, given that f is continuous at points $x = -1, 1$

At $x = -1$,

$LHL = RHL = f(-1)$

$\Rightarrow LHL = \lim\limits_{x \to (-1)^-} 2\sin\left(\frac{-\pi x}{2}\right) = 2$

$\Rightarrow f(-1) = |a(-1)^2 + (-1) + b|$

Now, we have $|a + b - 1| = 2$

$\Rightarrow a + b - 1 = 2$ or $a + b - 1 = -2$

$\Rightarrow a + b = 3$ or $a + b = -1$

Step 2:

At $x = 1$,

$LHL = \lim\limits_{x \to 1^-} |ax^2 + x + b| = |a + 1 + b|$

$RHL = \lim\limits_{x \to 1^+} \sin(\pi x) = 0$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b + 1 = 0$

$\Rightarrow a + b = -1$

Hence, option (b) is the correct answer

3.

Step 1: Given,

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases} \text{ and } f : [-1, 3] \to \mathbb{R}$$

Clearly, the possible points of discontinuity: $x = -1, 0, 1, 2, 3$

As we can see that at $x = -1$, the function is $|x| + [x]$ and both (x and $[x]$) are right-continuous at -1, so the function is continuous at -1.

At $x = 0$,

$|x|$ is always continous but

$[x]$ is discontinous.

$\Rightarrow$ Function is discontinuous at 0.

At $x = 1$

$LHL = \lim\limits_{x \to 1^-} |x| + [x] = 1 + 0 = 1$

$f(1) = 1 + 1 = 2$

$\Rightarrow$ Function is discontinuous at $x = 1$.

Step 2:

At $x = 2$,

$LHL = \lim\limits_{x \to 2^-} (x + |x|) = 2 + 2 = 4$

$f(2) = 2 + 2 = 4$

$RHL = \lim\limits_{x \to 2^+} (x + [x]) = 2 + 2 = 4$

$\Rightarrow$ Function is continuous at 2.

At $x = 3$,

Clearly, x is continuous at $x = 3$ but $[x]$ is not continuous at $x = 3$.

$\Rightarrow$ The function is discontinuous at 3.

Self-Assessment

Given,

$$f(x) = \begin{cases} [x] + \{x\}, & x < 1 \\ 2 - |x|, & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

The possible points of discontinuity : $x = 1, 2$

At $x = 1$

$LHL = \lim\limits_{x \to 1^-} ([x] + \{x\}) = 0 + 1 = 1$

$f(1) = 2 - 1 = 1$

$RHL = \lim\limits_{x \to 1^+} (2 - x) = 1$

$\Rightarrow$ f is continuous at $x = 1$.

At $x = 2$

$LHL = \lim\limits_{x \to 2^-} (2 - |x|) = 0$

$f(2) = 1$

$\Rightarrow$ f is discontinuous at $x = 2$.

BYJU'S Classes

MATHEMATICS

CONTINUITY AND DIFFERENTIABILITY

THEOREMS OF CONTINUITY

NOTES

What you already know

- Continuity of a function
- One-sided continuity
- Continuity in an interval

What you will learn

- Theorems on continuity
- Single point continuous function
- Intermediate value theorem

Theorems on Continuity

For a composite function $f(g(x))$, the continuity must be checked at the following points:

1. Where $g(x)$ is discontinuous
2. Where $g(x) = c$, given that $f(x)$ is discontinuous at $x = c$

If $f(x) = \dfrac{2}{2x-1}$ and $g(x) = \dfrac{2+x}{x}$, find the points of discontinuity of $f(g(x))$.

Solution

Step 1:

$g(x) = \dfrac{2+x}{x}$ is discontinuous at $x = 0$

$f(x) = \dfrac{2}{2x-1}$ is discontinuous at $x = \dfrac{1}{2}$

Step 2:

$f(g(x))$ is discontinuous at $g(x) = \dfrac{1}{2}$

$\Rightarrow \dfrac{2+x}{x} = \dfrac{1}{2}$

$\Rightarrow x = -4$

So, $f(g(x))$ is discontinuous at $x = 0, -4$

If $f(x) = \begin{cases} 1+x, & x < 0 \\ 2x-1, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 4-x, & x < -1 \\ 2x+7, & x \geq -1 \end{cases}$ then check the continuity of $g(f(x))$.

Solution

Step 1:

Let us discuss the continuity of $f(x)$.

$f(x)$ is changing its expression at $x = 0$

$$\text{L.H.L} = \lim_{x \to 0^-}(1 + x) = 1$$

$$\text{R.H.L} = \lim_{x \to 0^+}(2x - 1) = -1$$

$$f(0) = -1$$

So, $f(x)$ is discontinuous at $x = 0$

Step 2:

Let us discuss the continuity of $g(x)$.

$g(x)$ is changing its expression at $x = -1$

$$\text{L.H.L} = \lim_{x \to -1^-}(4 - x) = 5$$

$$\text{R.H.L} = \lim_{x \to -1^+}(2x + 7) = 5$$

$$f(-1) = 5$$

So, $g(x)$ is continuous throughout its domain.

Step 3:

Let us check the continuity of $g(f(x))$ at $x = 0$

$$\text{L.H.L} = \lim_{x \to 0^-} g(f(x)) = \lim_{x \to 0^-} g(1 + x) = g(1^-) = 2(1) + 7 = 9$$

$$\text{R.H.L} = \lim_{x \to 0^+} g(f(x)) = \lim_{x \to 0^+} g(2x - 1) = \lim_{x \to 0^+} g(-1^+) = 2(-1) + 7 = 5$$

$$g(f(0)) = g(-1) = -2 + 7 = 5$$

Since, L.H.L $\neq$ R.H.L, $g(f(x))$ is not continuous at $x = 0$

Single Point Continuous Function

Functions that are continuous at one point and are defined everywhere else are known as single point continuous functions.

Example:

Let us discuss the continuity of $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

$f(x)$ is discontinuous for $x \in \mathbb{R}-\{0\}$ as the function produces a rational number for all the rational values of x and 0 elsewhere.

Let us check the continuity of $f(x)$ at $x = 0$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x) = 0$$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = 0$$

$$f(0) = 0$$

Since, L.H.L = R.H.L = Value of the function, $f(x)$ is continuous at $x = 0$

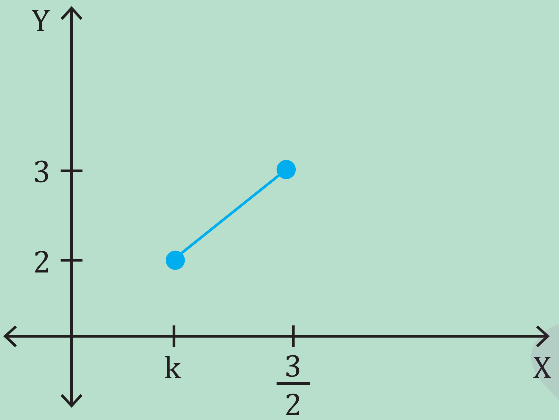
$\therefore$ $f(x)$ is discontinuous for $x \in \mathbb{R} - \{0\}$ ($f(x)$ is discontinuous for $x \in \mathbb{R} - \{0\}$ as there are an infinite number of irrational numbers between two rational numbers)

Let the function $f(x) : \mathbb{R} \to \mathbb{Z}$ be a continuous function such that $f\left(\frac{3}{2}\right) = 3$. Find the value of $f(3) - f(2)$.

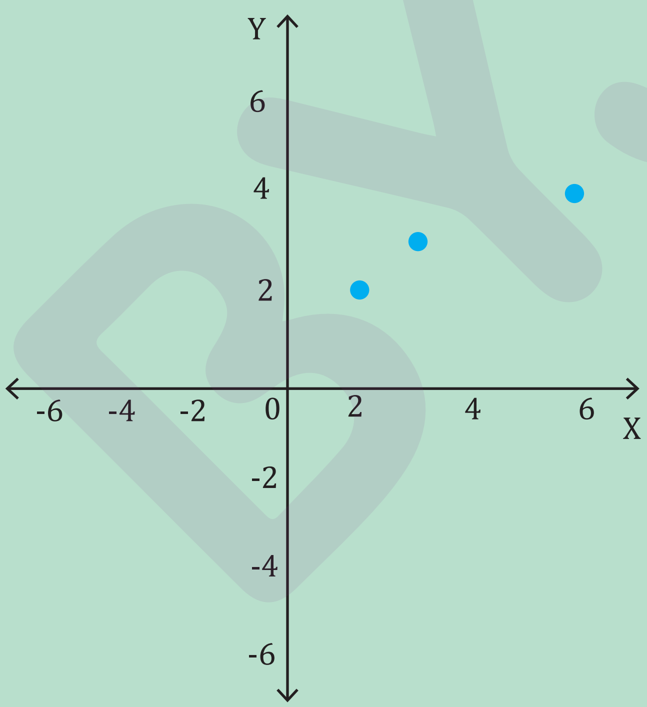
(a) 3 (b) 0 (c) 6 (d) 2

Solution

The function $f(x)$ is continuous for $x \in \mathbb{R}$.
Let us try to draw the graph of $f(x)$.
Let us assume that $f(x) = 2$ for $x = k$

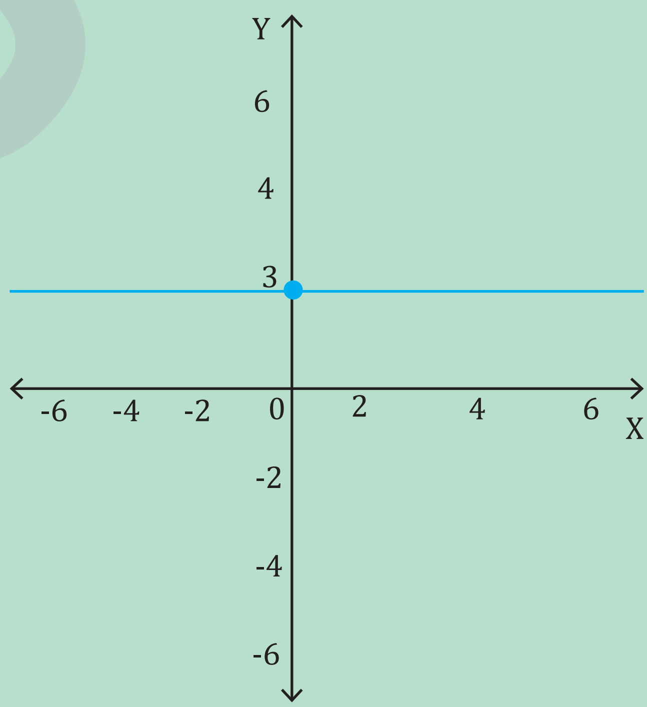


However, the graph is not possible as the range of $f(x)$ contains integers.



In the graph, $f(x)$ is taking integral values, but it is not possible as $f(x)$ is a continuous function.

So, the only possibility is $f(x) = \text{Constant} = f\left(\frac{3}{2}\right)$



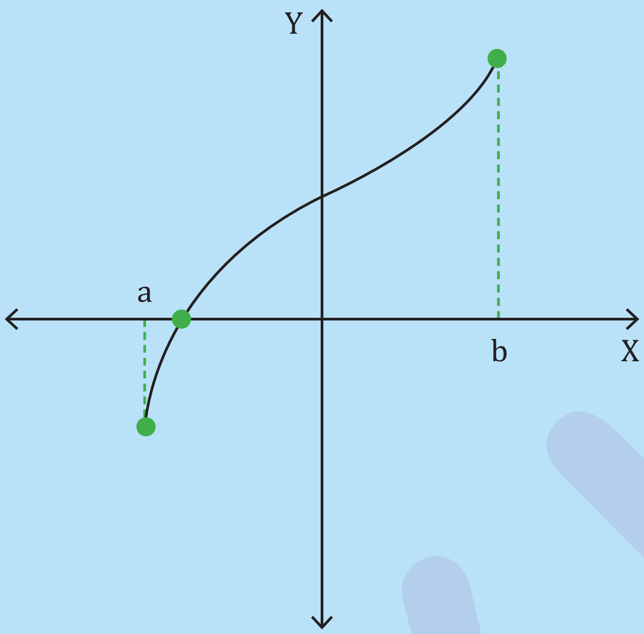
$\Rightarrow f(3) - f(2) = 0$

So, option (b) is the correct answer.

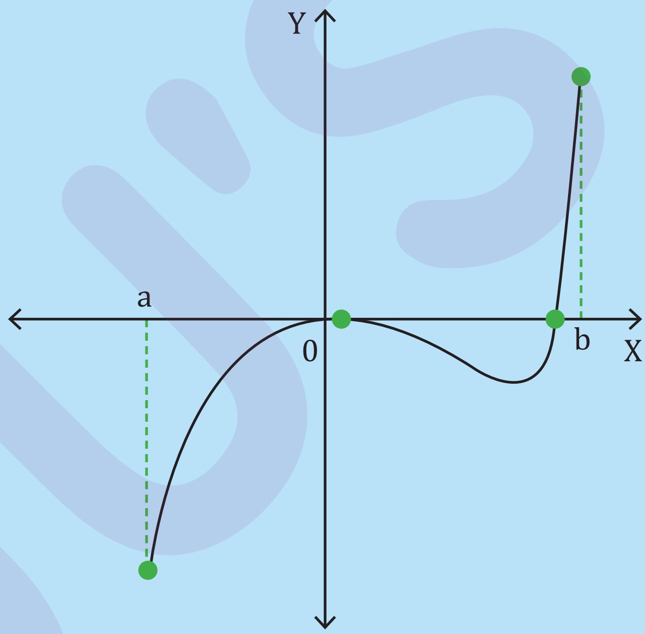
Intermediate Value Theorem

1. If a function f is **continuous** in [a, b] and if f(a) and f(b) are of **opposite signs**, then there exists **at least one solution** of the equation $f(x) = 0$ in the open interval (a,b).
Let us visualise it graphically.

We can see that $f(a) < 0$, $f(b) > 0$, and there exists one solution for the function in (a,b).

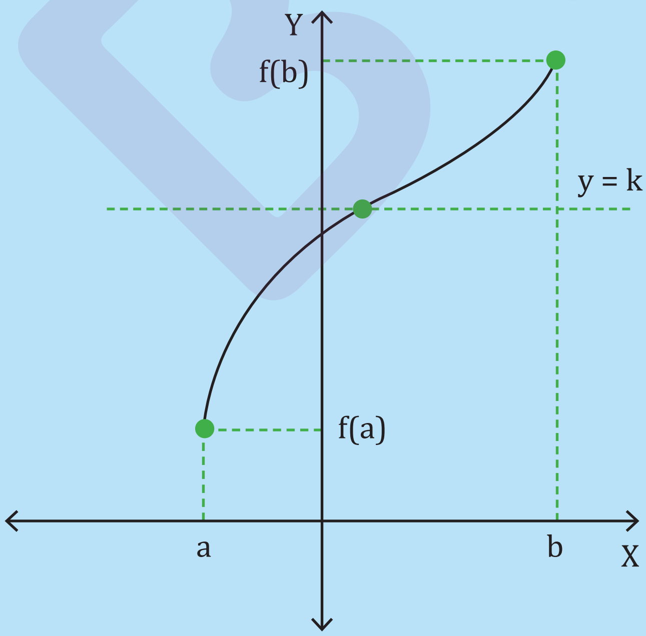


We can see that $f(a) < 0$, $f(b) > 0$, and there exist two solutions or at least one solution for the function in (a,b).

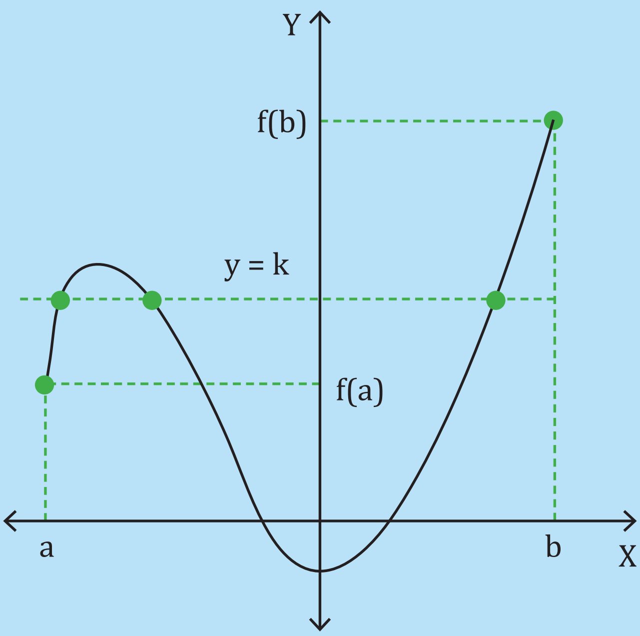


2. If a function f is **continuous** in [a, b] and if k is any real number between f(a) and f(b), then there exists **at least one solution** of the equation $f(x) = k$ in the open interval (a,b).

We can see that k lies between f(a) and f(b) and there exists one solution for $f(x) = k$ in (a,b).



We can see that k lies between f(a) and f(b) and there exist three solutions/at least one solution for $f(x) = k$ in (a,b).



If $f(x) = x \ln x - 2$, then $f(x) = 0$ has

(a) No solution in $(1, e)$
(b) At least one real solution in $(1, e)$
(c) Two real solutions in $(1, e)$
(d) None of these

Solution

$f(x) = x \ln x - 2$
$f(x)$ is a continuous function for $x > 0$
$f(1) = -2 < 0$ and $f(e) = e - 2 > 0$
Using the intermediate value theorem, we can say that $f(x)$ has at least one real solution in $(1, e)$.
So, option (b) is the correct answer.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$

Statement 1: $f(c) = \dfrac{1}{3}$, for some $c \in \mathbb{R}$

Statement 2: $0 < f(x) \le \dfrac{1}{2\sqrt{2}}, \forall x \in \mathbb{R}$

(a) Statement 1 is true, statement 2 is true and statement 2 is not a correct explanation of statement 1.
b) Statement 1 is true and statement 2 is false.
c) Statement 1 is false and statement 2 is true.
d) Statement 1 is true, statement 2 is true and statement 2 is a correct explanation of statement 1.

Solution

We know that $f(x)$ is a continuous function.

And $\dfrac{e^x + \dfrac{2}{e^x}}{2} \ge \sqrt{e^x \cdot \dfrac{2}{e^x}}$ (A.M $\ge$ G.M)

$\Rightarrow e^x + \dfrac{2}{e^x} \ge 2\sqrt{2}$

$\Rightarrow \dfrac{1}{e^x + \dfrac{2}{e^x}} \le \dfrac{1}{2\sqrt{2}}$

Now, $e^x > 0$ and $2e^{-x} > 0$

Hence, $0 < f(x) \le \dfrac{1}{2\sqrt{2}}, \forall x \in \mathbb{R}$

Also, $\dfrac{1}{3}$ lies between 0 and $\dfrac{1}{2\sqrt{2}}$. So, using the intermediate value theorem, we can say that

$f(c) = \dfrac{1}{3}$, for some $c \in \mathbb{R}$

So, option (d) is the correct answer.

Let a function $f(x)=\lim_{n\to\infty}(\sin x)^{2n}$, $n\in\mathbb{N}$. Comment on the discontinuity of f(x).

Solution

$f(x) = \lim_{n\to\infty}(\sin x)^{2n}, n\in\mathbb{N}$

We know that $\sin x \in [-1, 1]$

$\Rightarrow \sin^2 x \in [0, 1]$

When $\sin^2 x \in [0, 1)$, $f(x) = \lim_{n\to\infty}(\sin x)^{2n} = 0$

When $\sin^2 x = 1$, $f(x) = \lim_{n\to\infty}(\sin x)^{2n} = 1$

So, the discontinuity will occur when $\sin x = \pm 1$

$\Rightarrow$ The points of discontinuity are $x = \frac{(2k+1)\pi}{2}$, $k\in\mathbb{Z}$

Concept Check

1. Find the points of discontinuity of $y = f(u)$, where $f(u) = \frac{3}{2u^2 + 5u - 3}$ and $u = \frac{1}{x+2}$

2. Which of the following functions have single point continuity?

(a) $f(x) = \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$ (b) $f(x) = \begin{cases} 1, & x\in\mathbb{Q} \\ 1-x, & x\notin\mathbb{Q}\end{cases}$

3. $f(x) = (x-a)(x-c) + (x-b)(x-d)$, where $a < b < c < d$. Then f(x) has

(a) Exactly one real root (b) No real root

(c) Two real roots (d) Cannot be determined

Summary Sheet

Key Takeaways

- For a composite function $f(g(x))$, the continuity must be checked at the following points:
  - Where $g(x)$ is discontinuous
  - Where $g(x) = c$, given that $f(x)$ is discontinuous at $x = c$
- If a function f is **continuous in** $[a, b]$ and if $f(a)$ and $f(b)$ are of **opposite signs**, then there exists **at least one solution** of the equation $f(x) = 0$ in (a, b).
- If a function f is **continuous** in $[a, b]$ and if k is any real number between $f(a)$ and $f(b)$, then there exists **at least one solution** of the equation $f(x) = k$ in the interval (a, b).

Mind Map

Intermediate value theorem ← Continuity and Differentiability → Theorems on continuity

Self-Assessment

Show that the equation $x^3 - 3x^2 + 1 = 0$ has at least one solution in the interval $(0, 1)$.

Answers

Concept Check

1. Step 1:

u is discontinuous at $x = -2$

f(u) is discontinuous when $2u^2 + 5u - 3 = 0$

$\Rightarrow u = -3, \frac{1}{2}$

Step 2:

$\frac{1}{x+2} = -3$ or $\frac{1}{x+2} = \frac{1}{2}$

$\Rightarrow x = -\frac{7}{3}$ or $x = 0$

So, $f(u)$ is discontinuous at $x = -\frac{7}{3}, -2, 0$

2.

(a) f(x) is discontinuous for $x \in \mathbb{R}$ as there are an infinite number of irrational numbers between two rational numbers.

(b) Let f(x) be continuous at $x = k$

Then, $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

$\Rightarrow k = 1 - k, k = \frac{1}{2}$

$\therefore f(x)$ is discontinuous for $x \in \mathbb{R} - \left\{\frac{1}{2}\right\}$

3.

$f(x) = (x - a)(x - c) + (x - b)(x - d)$

$\Rightarrow f(a) = (a - b)(a - d)$

$\Rightarrow f(b) = (b - a)(b - c)$

$\Rightarrow f(c) = (c - b)(c - d)$

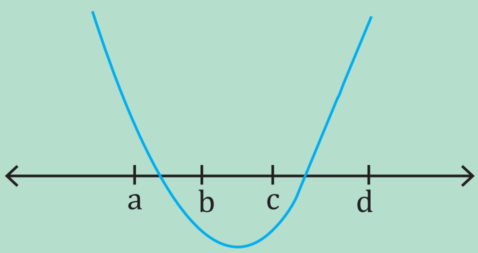
$\Rightarrow f(d) = (d - a)(d - c)$

Since $a < b < c < d$, we have,

$f(a) > 0, f(b) < 0, f(c) < 0, f(d) > 0$

$f(x) = 0$ has one root in (a, b) and another root in (c, d).

So, option (c) is the correct answer.



Self-Assessment

$f(x) = x^3 - 3x^2 + 1$ is continuous in $[0, 1]$.
$f(0) = 1 > 0$
$f(1) = -1 < 0$
Using the intermediate value theorem, we can say that $x^3 - 3x^2 + 1 = 0$ has at least one solution in the interval $(0, 1)$.

CONTINUITY AND DIFFERENTIABILITY

INTRODUCTION TO DIFFERENTIABILITY

What you already know

- Continuity of a function
- Theorems on continuity
- One-sided continuity
- Intermediate value theorem

What you will learn

- Concept of tangent
- Differentiability

Concept of Tangent

Let a function $f(x)$ be defined on the interval (a, b).
Let a point $c \in (a, b)$,
Let us consider three points, $P(c, f(c))$, $Q(c - h, f(c - h))$, and $R(c + h, f(c + h))$ as shown in the figure.

Slope of PR $= \dfrac{f(c+h)-f(c)}{(c+h)-c}$

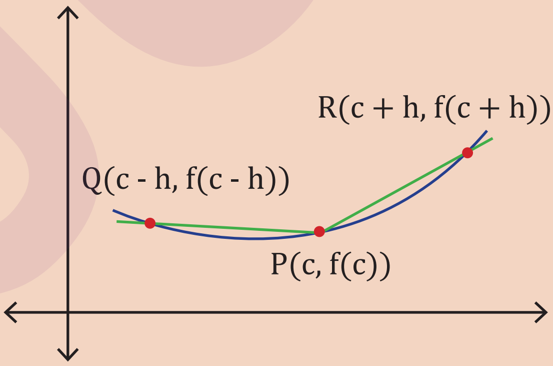
As $h \to 0^+$, $R \to P$

$$\Rightarrow m_{PR} = \lim_{h\to0^+}\frac{f(c+h)-f(c)}{h}$$

Slope of PQ $= \dfrac{f(c-h)-f(c)}{(c-h)-c}$

As $h \to 0^+$, $Q \to P$

$$\Rightarrow m_{PQ} = \lim_{h\to0^+}\frac{f(c-h)-f(c)}{-h}$$



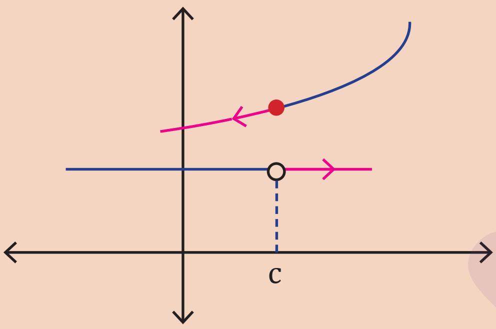
Hence, the tangent is the limiting case of the secant.
So, for a unique tangent at a point P, $m_{PQ} = m_{PR}$

$$\lim_{h\to0^+}\frac{f(c+h)-f(c)}{h} = \lim_{h\to0^+}\frac{f(c-h)-f(c)}{-h}$$

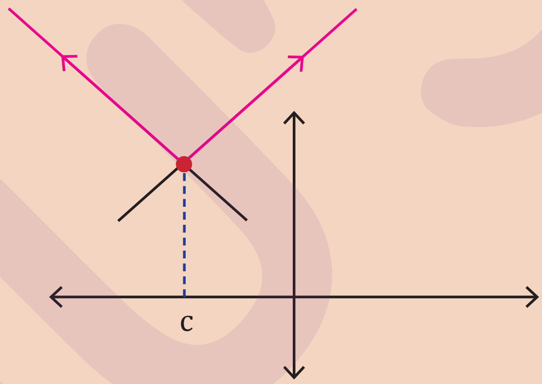
Differentiability

- A function is said to be differentiable at a point P if it has a unique tangent (of finite slope) at point P.
- A function is differentiable if it does not have a jump (break) or a sharp corner in its graph.

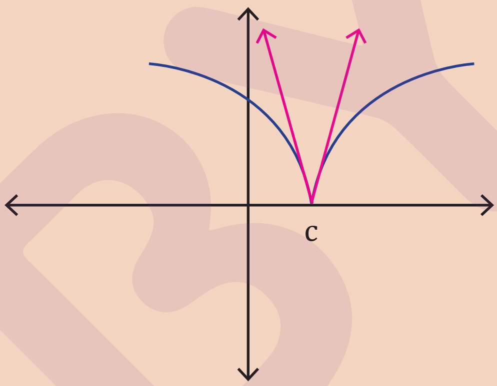
Let us consider the graph of a function.
Here, at $x = c$, the graph has a breakage.
So, a unique tangent is not possible at that point.
Hence, this function is not differentiable at $x = c$.



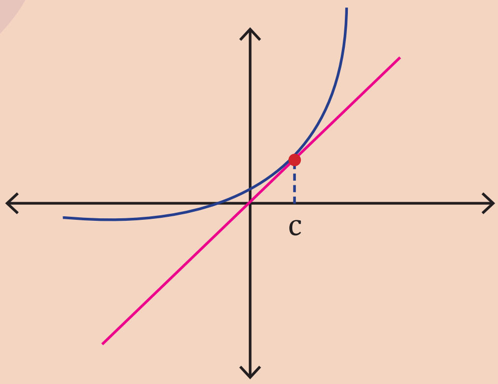
Let us consider the graph of another function.
Here, at $x = c$, the graph has a sharp corner.
So, a unique tangent is not possible at that point.
Hence, this function is not differentiable at $x = c$.



The given graph is continuous but not differentiable because it has a sharp corner at $x = c$



The given graph is continuous as well as differentiable.



Condition for differentiability

So, a function $f(x)$ is said to be differentiable or derivable at $x = c$ if

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c-h)-f(c)}{-h} = \text{a finite value}$$

The slope of the tangent in the left neighbourhood of c is known as the left-hand derivative(L.H.D).

$$\text{L.H.D} = \lim_{h\to 0^+}\frac{f(c-h)-f(c)}{-h}$$

The slope of the tangent in the right neighbourhood of c is known as the right- hand derivative (R.H.D).

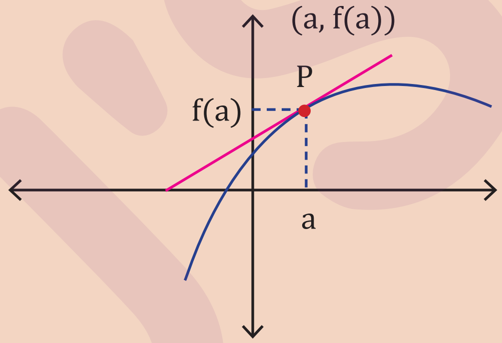
$$\text{R.H.D} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$

So, the derivative of function f(x) is defined as follows:

$$f'(x)(\text{Slope of tangent at } x = c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

Tangent at a point P(a, f(a)) for function f(x)

If L.H.D. = R.H.D. = finite at a point $x = a$, then the function is differentiable and the tangent has slope $f'(a)$.



Slope of the tangent = $f'(a)$
Also, the tangent passes through the point (a, f(a)).
∴ The equation of tangent is given by,
$y - f(a) = f'(a)\,(x - a)$

Find the equation of tangent to the following:

(i) $f(x) = x^2$ at $x = 1$ (ii) $f(x) = x^{\frac{1}{3}}$ at $x = 0$

Solution

(i)
$f(x)= x^2 \Rightarrow f'(x) = 2x \Rightarrow f'(1) = 2$
⇒The slope of the tangent at point $x = 1$ is 2.
At $x = 1, y = f(x) = 1^2 = 1$
∴ The point of contact of the tangent is (1, 1).
Hence, the equation of tangent is $y - 1 = 2(x - 1)$ (by point-slope form)
$\Rightarrow y - 1 = 2x - 2$
$\Rightarrow 2x - y = 1$

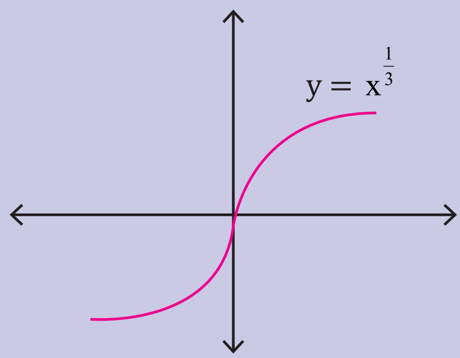
(ii)
$f(x) = x^{\frac{1}{3}}$
At $x = 0, y = f(x) = 0$
∴ The point of contact of the tangent is (0, 0).

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h}$$

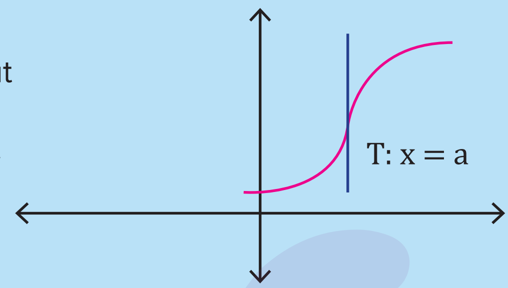
$$=\lim_{h\to 0}\left(\frac{1}{h^2}\right)^{\frac{1}{3}}$$

$f'(0) \to \infty$
⇒The tangent at point (0, 0) is a vertical tangent, i.e., $x = 0$



Note

If L.H.D. =R.H.D.→ ∞ or -∞, f(x) is non-differentiable, but there is a vertical tangent at point P.
The equation of tangent is $x = a$, where a is a constant.
(in the adjacent diagram)



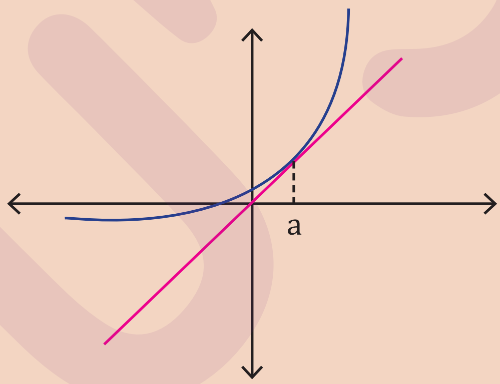
Relation Between Continuity and Differentiability

Case I

If a function is differentiable, then it will be continuous as well.

Example:

The given graph is continuous as well as differentiable at point $x = a$

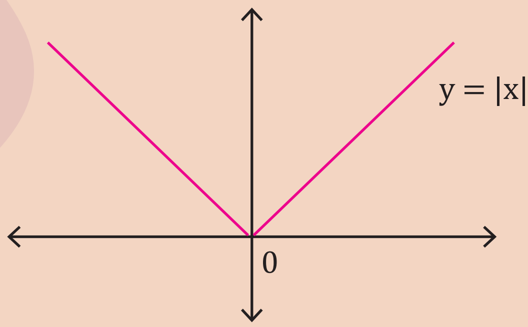


Case II

If a function is continuous, then it may or may not be differentiable.

Example:

In the given graph, we can observe a sharp edge at the point $x = 0$
Hence, the graph of the function $y = |x|$ is continuous but not differentiable at $x = 0$

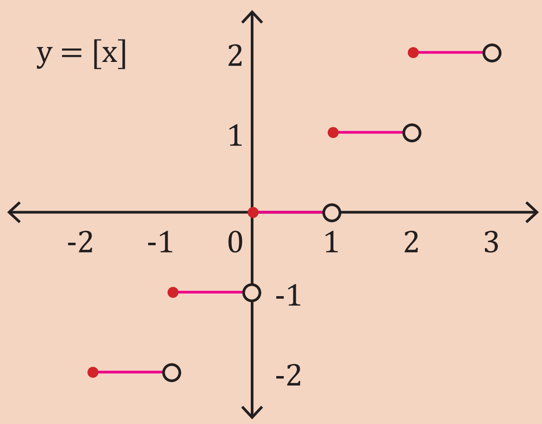


Case III

If a function is discontinuous, then it will be non-differentiable.

Example:

In the given graph, we can observe the breakage.
Hence, the graph of the function $y = [x]$ is neither continuous nor differentiable on $\mathbb{Z}$.

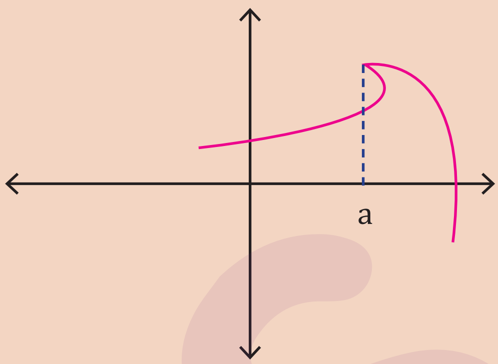


Case IV

If a function is non-differentiable, then it may or may not be discontinuous.

Example:

Because of the sharp edge at the point $x = a$, the given function is not differentiable, but it is continuous.



If the function $f(x) = \begin{cases} A + Bx^2, x < 1 \\ 3Ax - B + 2, x \geq 1 \end{cases}$ is differentiable at $x = 1$, then find the ordered pair (A, B).

(a) $(2, 3)$ **(b)** $(0, 1)$ **(c)** $(4, 2)$ **(d)** $(2, 4)$

Solution

Step 1:

Since $f(x)$ is derivable, it will be continuous.

At $x = 1$, L.H.L. = R.H.L. = $f(1)$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} A + Bx^2 = A + B$

$f(1) = 3A(1) - B + 2 = 3A - B + 2$

L.H.L = $f(1) \Rightarrow A + B = 3A - B + 2 \Rightarrow B = A + 1$ ......(i)

Step 2:

The function is derivable at $x = 1$

$\Rightarrow$ R.H.D. = L.H.D.

$$\Rightarrow \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h}$$

$$\Rightarrow \lim_{h \to 0^+} \frac{(3A(1+h) - B + 2) - (3A - B + 2)}{h} = \lim_{h \to 0^+} \frac{A + B(1-h)^2 - (3A - B + 2)}{-h}$$

$$\Rightarrow \lim_{h \to 0^+} \frac{3Ah}{h} = \lim_{h \to 0^+} \frac{A + B + Bh^2 - 2Bh - 3A + B - 2}{-h}$$

$$\Rightarrow 3A = \lim_{h \to 0^+} \frac{Bh^2 - 2Bh + (-2A + 2B - 2)}{-h}$$

$$\Rightarrow 3A = \lim_{h \to 0^+} \frac{Bh^2 - 2Bh}{-h} \qquad (-2A + 2B - 2 = 0 \text{ from equation (i)})$$

$3A = 2B$ . . . (ii)

By solving (i) and (ii), we get,

$A=2, B=3$

$\therefore$ **Option (a) is the correct answer.**

Alternate method

Step 1:

Since $f(x)$ is derivable, it will be continuous.

At $x = 1$, L.H.L. = R.H.L. = $f(1)$

L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} A + Bx^2 = A + B$

$f(1) = 3A(1) - B + 2 = 3A - B + 2$

L.H.L = $f(1) \Rightarrow A + B = 3A - B + 2 \Rightarrow B = A + 1$ ......(i)

Step 2:

$f(x)$ is derivable at $x = 1$

$\Rightarrow$ L.H.D. = R.H.D.

$\Rightarrow f'(1^-) = f'(1^+)$

$\Rightarrow 2B(1) = 3A(1)$

$\Rightarrow 2B = 3A$ . . . (ii)

By solving (i) and (ii), we get,

$A = 2, B = 3$

∴ **Option (a) is the correct answer.**

Check the differentiability of the following function:

$$f(x) = \begin{cases} e^{2x}, & x \le 0 \\ 2\sin x, & x > 0 \end{cases} \quad \text{at } x = 0$$

Solution

For a function to be differentiable, it must be continuous at the given point.

L.H.L. = $\lim_{x\to 0^-} e^{2x} = 1$ and R.H.L. = $\lim_{x\to 0^+} 2\sin x = 0$

Here, L.H.L. ≠ R.H.L

The function is discontinuous.

As the function is discontinuous at $x = 0$, it will not be differentiable.

Note

Continuity is the necessary condition for differentiability of a function. So, to check the differentiability of a function, first check its continuity at the given point.

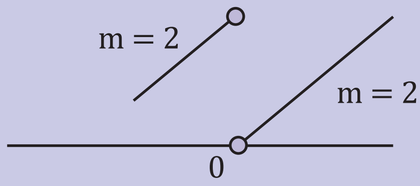
Let us consider the previous illustration. $f(x) = \begin{cases} e^{2x}, & x \le 0 \\ 2\sin x, & x > 0 \end{cases}$

L.H.D = $f'(0^-) = 2e^{2\times 0} = 2$

and R.H.D. = $f'(0^+) = 2\cos 0^0 = 2$

Here, L.H.D = R.H.D but the function is not continuous at $x = 0$ and because of that, the function is not differentiable.

From the graph of $f(x)$, we can see that at $x = 0^+$ and $x = 0^-$, the graph has the same slope. Hence,
L.H.D = R.H.D
However, the graph has a breakage at $x = 0$. Hence, it does not satisfy the basic condition of differentiability, which is continuity.



Concept Check

1. Check differentiability of function $f(x) = \begin{cases} x + [2x], & x < 1 \\ \{x\} + 1, & x \geq 1 \end{cases}$ at $x = 1$, where [.] denotes G.I.F and {.} denotes fractional part function.

2. Comment on derivative of $f(x)$ at $x = 0$, where $f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

3. If $f(x) = \begin{cases} \dfrac{x\left(3e^{\frac{1}{x}} + 4\right)}{2 - e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is:

   (a) Continuous as well as differentiable at $x = 0$
   (b) Continuous but not differentiable at $x = 0$
   (c) Neither differentiable at $x = 0$ nor continuous at $x = 0$
   (d) None of these

Summary Sheet

Key Takeaways

- Tangent is the limiting case of secant.
- A function is said to be differentiable at a point P, if it has a unique tangent (of finite slope) at point P.
- A function is differentiable if it does not have a jump (break) or a sharp corner.
- If a function is differentiable then it will be continuous as well.
- If a function is continuous then it may or may not be differentiable.
- If a function is discontinuous then it will be non-differentiable.
- If a function is non-differentiable then it may or may not be discontinuous.

Mind Map

- Concept of tangent — Differentiability
  - Condition for Differentiability
  - Tangent at a point
  - Relation between continuity and differentiability

Self-Assessment

The number of values of $x \in [0, 2]$ at which $f(x) = \left|x - \frac{1}{2}\right| + |x - 1| + \tan x$ is not differentiable is:

(a) 0 (b) 1 (c) 3 (d) None of these

Answers

Concept Check

1.

First, let's check continuity of the function $f(x) = \begin{cases} x + [2x], x < 1 \\ \{x\} + 1, x \geq 1 \end{cases}$

$$\text{L.H.L.} = \lim_{x \to 1^-} x + [2x]$$
$$= 1 + [2^-] = 1 + 1 = 2$$
$f(1) = \{1\} + 1 = 1$

L.H.L $\neq f(1)$

$\Rightarrow$The function $f(x)$ is discontinuous.

Since the function is discontinuous, it will not be differentiable.

2.

Step 1:

First, let's check continuity of the function $f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases}$

$$\text{L.H.L.} = \lim_{x \to 0^-} x \tan^{-1}\left(\frac{1}{x}\right)$$
$$= 0 \qquad \left(\because -\frac{\pi}{2} < \tan^{-1}\left(\frac{1}{x}\right) < \frac{\pi}{2}\right)$$

$$\text{R.H.L.} = \lim_{x\to 0^+} x\tan^{-1}\left(\frac{1}{x}\right)$$

$$= 0 \qquad \left(\because -\frac{\pi}{2} < \tan^{-1}\left(\frac{1}{x}\right) < \frac{\pi}{2}\right)$$

$f(0) = 0$

We have, L.H.L. = R.H.L. = f(0)

$\Rightarrow$ f(x) is continuous at x = 0

Step 2:

$$\text{L.H.D.} = \lim_{h\to 0^+} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0^+} \frac{-h\tan^{-1}\left(\frac{1}{-h}\right)}{-h} = -\frac{\pi}{2}$$

Now,

$$\text{R.H.D.} = \lim_{h\to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h\tan^{-1}\left(\frac{1}{h}\right)}{h} = \frac{\pi}{2}$$

We have

L.H.D. $\neq$ R.H.D.

$\Rightarrow$ f(x) is non-differentiable.

3.

Step 1:

For continuity,

$$\text{L.H.L.} = \lim_{x\to 0^-} \frac{x\left(3e^{\frac{1}{x}} + 4\right)}{2 - e^{\frac{1}{x}}}$$

$$x \to 0^- \Rightarrow \frac{1}{x} \to -\infty \Rightarrow e^{\frac{1}{x}} \to 0$$

$\therefore$ L.H.L. = 0

$$\text{R.H.L.} = \lim_{x\to 0^+} \frac{x\left(3e^{\frac{1}{x}} + 4\right)}{2 - e^{\frac{1}{x}}}$$

$$\text{R.H.L.} = \lim_{x\to 0^+} \frac{xe^{\frac{1}{x}}\left(3 + \frac{4}{e^{\frac{1}{x}}}\right)}{e^{\frac{1}{x}}\left(\frac{2}{e^{\frac{1}{x}}} - 1\right)}$$

$$\text{R.H.L.} = \lim_{x\to 0^+} \frac{x\left(3 + \frac{4}{e^{\frac{1}{x}}}\right)}{\left(\frac{2}{e^{\frac{1}{x}}} - 1\right)}$$

$x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty \Rightarrow e^{\frac{1}{x}} \to \infty$ and

$$\frac{1}{e^{\frac{1}{x}}} \to 0$$

R.H.L. = 0

f(0) = 0

We have, L.H.L. = R.H.L = f(0)

$\therefore$ f(x) is continuous.

Step 2:

For differentiability,

$$\text{L.H.D.} = \lim_{h \to 0^+} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h \to 0^+} \frac{\frac{(-h)\left(3e^{\frac{-1}{h}}+4\right)}{2-e^{-\frac{1}{h}}}}{-h}$$

$$h \to 0^+ \Rightarrow -\frac{1}{h} \to -\infty \Rightarrow e^{-\frac{1}{h}} \to 0$$

$\therefore$ L.H.D. = 2

$$\text{R.H.D.} = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h\left(\frac{3e^{\frac{1}{h}}+4}{2-e^{\frac{1}{h}}}\right)}{h}$$

$$= \lim_{h \to 0^+} \left(\frac{3+\frac{4}{e^{\frac{1}{h}}}}{\frac{2}{e^{\frac{1}{h}}}-1}\right) = -3$$

$$h \to 0^+ \Rightarrow \frac{1}{h} \to +\infty \Rightarrow e^{\frac{1}{h}} \to \infty$$

$$\frac{1}{e^{\frac{1}{h}}} \to 0$$

We have, L.H.L. $\neq$ R.H.L

$\therefore$ f(x) is non-differentiable.

$\therefore$ **Option (b) is the correct answer.**

Self Assessment

- $\left|x-\frac{1}{2}\right|$ is continuous everywhere but not differentiable at $\frac{1}{2}$.
- $|x - 1|$ is continuous everywhere but not differentiable at 1.
- tan x is continuous in [0, 2], except at $x = \frac{\pi}{2}$.

Hence, f(x) is not differentiable at $x = \frac{1}{2}, 1, \frac{\pi}{2}$.

MATHEMATICS

CONTINUITY AND DIFFERENTIABILITY

DIFFERENTIABILITY FOR SOME SPECIAL FUNCTIONS

NOTES

What you already know

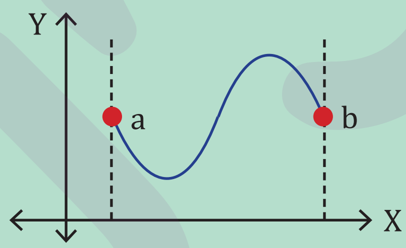
- Continuity and discontinuity
- Basics of differentiability
- Some standard functions

What you will learn

- Differentiability in an interval
- Theorems on differentiability

Differentiability in an Interval

A function $f(x)$ is differentiable or derivable in the interval $[a,b]$ if it is:
- Differentiable at each point of (a, b)
- Differentiable at the endpoints



Also,

R.H.D. $= f'(a^+) = \lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ and L.H.D. $= f'(b^-) = \lim\limits_{h\to 0^+} \dfrac{f(b-h)-f(b)}{-h}$ exist finitely.

Polynomial, trigonometric, exponential, and logarithmic functions are derivable at their respective domains.

Differentiability in an interval has to be checked for the cases:
- At boundary points, where function changes expression
- For $[f(x)]$, $\{f(x)\}$, $[.]$ denotes the G.I.F. and $\{.\}$ denotes fractional part function, where $f(x)$ becomes an integer.
- For $\text{sgn}(f(x))$, where $f(x) = 0$
- For $|f(x)|$, where $f(x) = 0$

Comment on the differentiability of $f(x) = |x - 1|$

Solution

Method 1

Step 1:

Given, $f(x) = |x - 1|$

$$f(x) = \begin{cases} (x - 1), & x - 1 \geq 0 \\ -(x - 1), & x - 1 < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} (x - 1), & x \geq 1 \\ (1 - x), & x < 1 \end{cases}$$

Step 2:

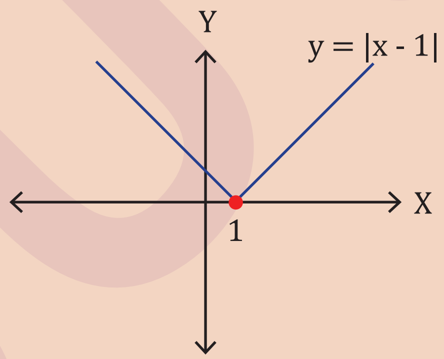
At $x = 1$,

R.H.D. $= \lim\limits_{h \to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0^+} \dfrac{(1+h)-1-0}{h} = 1$, and

L.H.D. $= \lim\limits_{h \to 0^+} \dfrac{f(1-h)-f(1)}{-h} = \lim\limits_{h \to 0^+} \dfrac{1-(1-h)-0}{-h} = -1$

Clearly, L.H.D. $\neq$ R.H.D.

$\Rightarrow$ At $x = 1$, function $f(x)$ is non-differentiable



Method 2

We can see that function $f(x)$ has sharp corner at $x = 1$

$\Rightarrow$ At $x = 1$, function $f(x)$ is non-differentiable

Note

Special functions

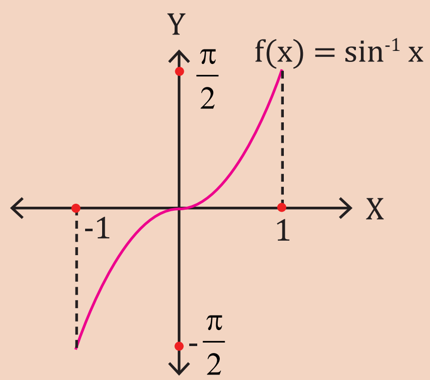
- For inverse trigonometric functions, check the differentiability at their boundary points of the domain.

Comment on the differentiability of $f(x) = \sin^{-1} x$; $x \in [-1, 1]$.

Solution

Given, $f(x) = \sin^{-1} x$

At $x = 1$,



$$\text{L.H.D.} = \lim_{h\to 0^+}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+}\frac{\sin^{-1}(1-h)-\frac{\pi}{2}}{-h} = \lim_{h\to 0^+}\frac{\cos^{-1}(1-h)}{h},$$

Let us consider $1 - h = \cos\theta$

$$\Rightarrow \text{L.H.D.} = \lim_{\theta\to 0^+}\frac{\theta}{1-\cos\theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

By applying L'Hospital's rule, we get

$$\Rightarrow \text{L.H.D.} = \lim_{\theta\to 0^+}\frac{1}{\sin\theta} = \frac{1}{0^+} = +\infty$$

Clearly L.H.D. does not exist finitely.

$\Rightarrow$ Function $f(x)$ is non-differentiable at $x = 1$

Also,

at $x = -1$, function $f(x)$ has a vertical tangent.

$\Rightarrow$ At $x = -1$, function f is non-differentiable.

Comment on the continuity and the differentiability of:

$$f(x) = \begin{cases} \left|x-\frac{1}{2}\right|, & 0 \le x \le 1 \\ x[x], & 1 < x \le 2 \end{cases}$$ **, Where [.] denotes the GIF**

Solution

Given,

$$f(x) = \begin{cases} \left|x-\frac{1}{2}\right|, & 0 \le x \le 1 \\ x[x], & 1 < x \le 2 \end{cases} = \begin{cases} -\left(x-\frac{1}{2}\right), & 0 \le x \le \frac{1}{2} \\ \left(x-\frac{1}{2}\right), & \frac{1}{2} < x \le 1 \\ x, & 1 < x < 2 \\ 2x, & x = 2 \end{cases}$$

The possible points of discontinuity are

$x = \frac{1}{2}, 1, 2$

At $x = 1$,

$$\text{L.H.L.} = \lim_{x\to 1^-}\left(x-\frac{1}{2}\right) = \frac{1}{2}$$

$$f(1) = \left|x-\frac{1}{2}\right| = \frac{1}{2}$$

$$\text{R.H.L.} = \lim_{x\to 1^+} x[x] = 1\times 1 = 1$$

L.H.L. $\neq$ R.H.L.

$\Rightarrow$ Function $f(x)$ is discontinuous at $x = 1$

At $x = 2$,

L.H.L. $= \lim_{x \to 2^-} x[x] = 2 \times 1 = 2$

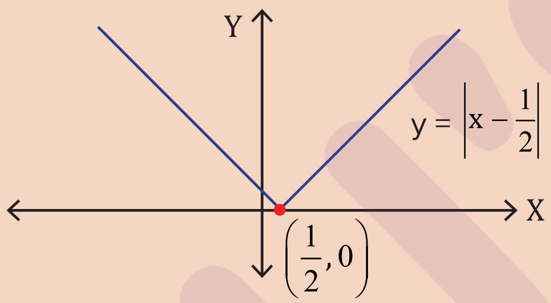
$f(2) = 2 \times 2 = 4$

L.H.L. $\neq f(2)$

$\Rightarrow$ Function $f(x)$ is discontinuous at $x = 2$

At $x = 0$, given modulus function is continuous and right differentiable.

But at $x = \frac{1}{2}$,given modulus function continuous but not differentiable (makes sharp corner at $x = \frac{1}{2}$).



Therfore,

Points (x)	0	$\frac{1}{2}$	1	2
Continuity	Continuous	Continuous	Discontinuous	Discontinuous
Differentiability	Differentiable	Non-Differentiable	Non-Differentiable	Non-Differentiable

Theorems on Differentiability

- If $f(x)$ and $g(x)$ both are differentiable at $x = a$, then at $x = a$, the functions $f(x) \pm g(x)$, $f(x) \times g(x)$, $\frac{f(x)}{g(x)}$; $(g(a) \neq 0)$ will also be differentiable.

  Example:
  Both $f(x) = x$ and $g(x) = \sin x$ are differentiable at $x = 0$.
  $\Rightarrow f(x) \pm g(x) = x \pm \sin x$ and $f(x) \times g(x) = x \cdot \sin x$ are differentiable at $x = 0$.
- If $f(x)$ is non-differentiable and $g(x)$ is differentiable at $x = a$, then the functions :
  1. $f(x) \pm g(x)$ will be non-differentiable at $x = a$
  2. $f(x) \times g(x)$, $\frac{f(x)}{g(x)}$ $(g(a) \neq 0)$ may or may not be differentiable at $x = a$

  Example:
  $f(x) = x$ and $g(x) = |x|$
  $f(x) \pm g(x) = x \pm |x|$ is non differentiable at $x = 0$ and $f(x) \times g(x) = x|x|$ is differentiable at $x = 0$

- If $f(x)$ and $g(x)$ both are non-differentiable at $x = a$, then we get,
  1. At most, one of $f(x) \pm g(x)$ may be differentiable at $x = a$.
  2. $f(x) \times g(x)$, $\frac{f(x)}{g(x)}$;$(g(a) \neq 0)$ may or may not be differentiable at $x = a$

Note

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \times g(x)$	$\frac{f(x)}{g(x)}, g(x) \neq 0$
Differentiable	Differentiable	Differentiable	Differentiable	Differentiable
Differentiable	Non-Differentiable	Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable
Non-Differentiable	Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable

If $f(x)$ is a differentiable function and $f(x) \times g(x)$ is differentiable at $x = a$, then which of the following condition/conditions is/are correct?
(a) $g(x)$ must be differentiable at $x = a$
(b) If $g(x)$ may be discontinuous, then $f(a) = 0$
(c) If $f(a) \neq 0$, then $g(x)$ must be differentiable
(d) $g(x)$ is continuous

Solution

Given, $f(x)$ is a differentiable function.
Let us consider $h(x) = f(x) \times g(x)$

$$\frac{d}{dx}\left[h(x)\right]_{x=a} = f'(a) \times g(a) + \lim_{h \to 0^+}\left[\frac{g(a+h)-g(a)}{h}\right] \times f(a) \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now, we will consider the following two cases:

Case 1: $f(a) \neq 0$
According to equation (i), $g(x)$ must be a differentiable function.

Case 2: $f(a) = 0$
According to equation (i), $g(x)$ can also be a discontinuous function.
Hence, options (b) and (c) are the correct answers.

Concept Check

1. Comment on the differentiability of the following functions in their respective domains:
   (i) $f(x) = |x^2 - 3x + 2|$
   (ii) $f(x) = |\ln x|$

2. If function $f(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$ is differentiable, then find the value of $k + m$. **JEE MAIN 2015**

   (a) 2 (b) $\frac{16}{5}$ (c) $\frac{10}{3}$ (d) 4

3. Find the set of points, where $f(x) = \frac{x}{1+|x|}$ is differentiable. **JEE MAIN 2015**

   (a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$ (c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

Summary Sheet

Key Takeaways

- A function $f(x)$ is differentiable or derivable in the interval $[a, b]$ in the following conditions:

1. If it is differentiable at each point of (a, b)
2. If it is differentiable at the endpoints

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \times g(x)$	$\frac{f(x)}{g(x)}, g(x) \neq 0$
Differentiable	Differentiable	Differentiable	Differentiable	Differentiable
Differentiable	Non-Differentiable	Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable
Non-Differentiable	Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable	Differentiable / Non-Differentiable

Mind Map

Differentiability in an inerval ← Continuity and differentiability → Theorems on differentiability

Self-Assessment

A function f(x) such that $f\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}-|x|$, $\forall$ x. Find $f'\left(\frac{\pi}{2}\right)$, if it exists.

Answers

Concept Check

1.

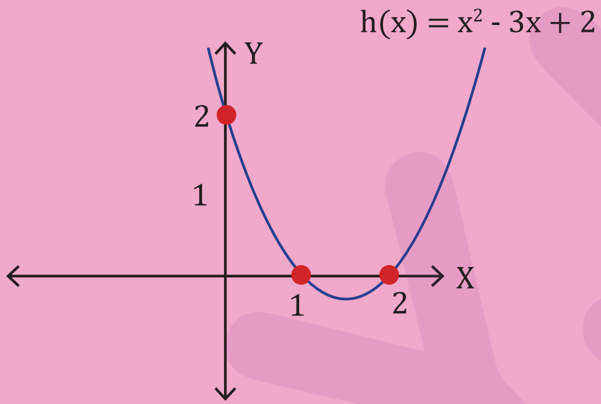
(i) Given, $f(x) = |x^2 - 3x + 2|$

Let $h(x) = x^2 - 3x + 2$

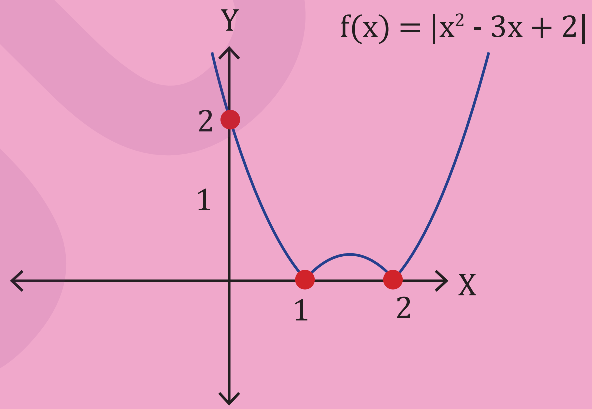
We will draw the graph of the function.

$f(x) = |h(x)|$

$f(x) = |(x - 1)(x - 2)|$



Now, at $x = 1, 2$, we can see that function f(x) is non-differentiable.
(there is sharp edge at points $x = 1, 2$)

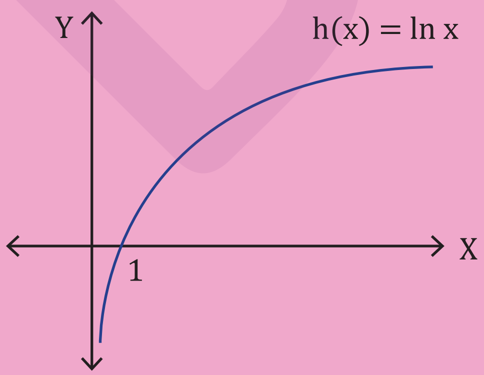


(ii) Given, $f(x) = |\ln x|$

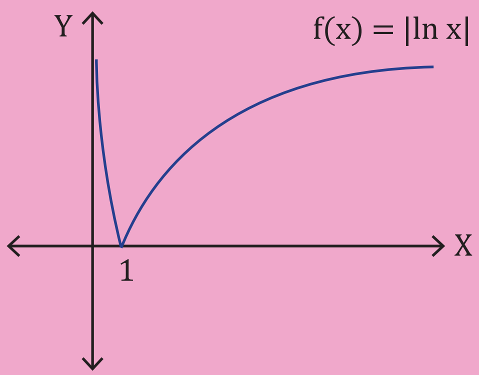
Let $h(x) = \ln x$

We will draw the graph of the function.

$f(x) = |h(x)|$



Now, at $x = 1$, we can see that function f(x) is non-differentiable.
(there is sharp edge at point $x = 1$)



2.

Given $f(x) = \begin{cases} k\sqrt{x+1}, 0 \le x \le 3 \\ mx + 2, 3 < x \le 5 \end{cases}$ is differentiable

$\Rightarrow$ L.H.L. = R.H.L. = f(a) and L.H.D. = R.H.D.

At $x = 3$, function f is continuous

$\Rightarrow \lim_{x \to 3^+} (mx + 2) = f(3)$

$\Rightarrow 3m + 2 = 2k$.........................(i)

Also, L.H.D. = R.H.D. at $x = 3$

$\Rightarrow \frac{k}{2\sqrt{x+1}} = m \Rightarrow \frac{k}{4} = m$................(ii)

After solving equations (i) and (ii), we get,

$m = \frac{2}{5}$ and $k = \frac{8}{5}$

$\Rightarrow k + m = 2$

Hence, option (a) is the correct answer.

3.

Given, $f(x) = \frac{x}{1+|x|}$

Clearly, the possible point of non-differentiability is $x = 0$

At $x = 0$,

$$\text{L.H.D.} = \lim_{h \to 0^+} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h \to 0^+} \frac{\frac{-h}{1+|h|}-0}{-h} = 1$$

$$\text{R.H.D.} = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{\frac{h}{1+h}-0}{h} = 1$$

Here, LHD = RHD

At $x = 0$, function f is differentiable.

Now, we can see that function f is differentiable in $(-\infty, \infty)$.

Therefore, option (b) is the correct answer.

Self-Assessment

Given $f\left(x+\frac{\pi}{2}\right) = \frac{\pi}{2} - |x|$

At $x = \frac{\pi}{2}$,

$$\text{R.H.D.} = \lim_{h \to 0^+} \frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0^+} \frac{\frac{\pi}{2}-|h|-\frac{\pi}{2}}{h} = -1$$

$$\text{L.H.D.} = \lim_{h \to 0^+} \frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h \to 0^+} \frac{\frac{\pi}{2}-|-h|-\frac{\pi}{2}}{-h} = 1$$

$\Rightarrow f'\left(\frac{\pi}{2}\right)$ does not exist.

MATHEMATICS

CONTINUITY AND DIFFERENTIABILITY

MAXIMUM OR MINIMUM OF TWO FUNCTIONS

What you already know

- Differentiability in an interval
- Theorems on differentiability

What you will learn

- Max {f(x), g(x)} or Min {f(x), g(x)}
- Functional equations

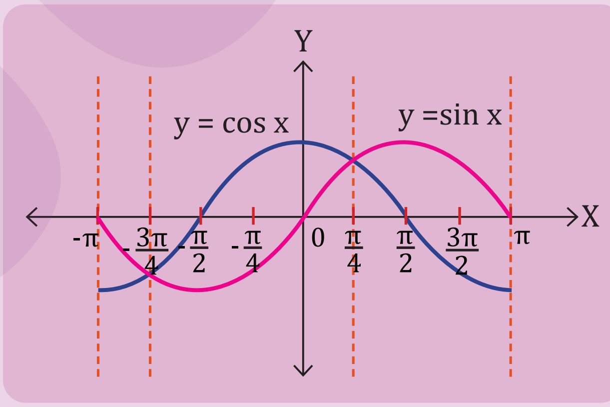
Max {f(x), g(x)} or Min {f(x), g(x)}

Draw the graph of $f(x) = \min\{\sin x, \cos x\}, x \in [-\pi, \pi]$

Solution

Step 1:

Draw the graphs of $\sin x$ and $\cos x$ between the same coordinate axis.



Step 2:

Check which graph has the lower value in the particular interval.

Between $-\pi$ to $\frac{-3\pi}{4}$, $\cos x$ is smaller than $\sin x$

$\Rightarrow f(x) = \min\{\sin x, \cos x\} = \cos x, x \in [-\pi, \frac{-3\pi}{4})$

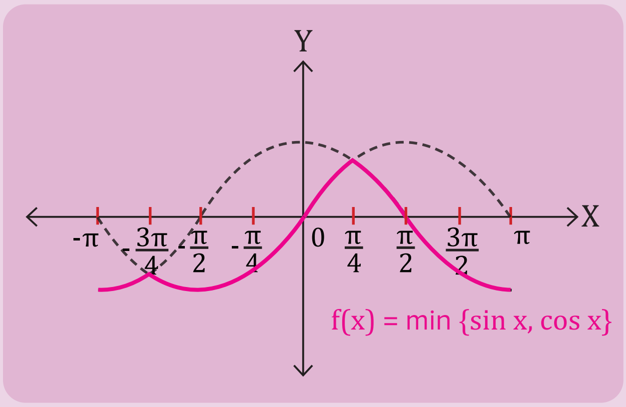
Between $\frac{-3\pi}{4}$ to $\frac{\pi}{4}$, $\sin x$ is smaller than $\cos x$

$\Rightarrow f(x) = \min\{\sin x, \cos x\} = \sin x, x \in [\frac{-3\pi}{4}, \frac{\pi}{4})$

Between $\frac{\pi}{4}$ to π, $\cos x$ is smaller than $\sin x$

$\Rightarrow f(x) = \min\{\sin x, \cos x\} = \cos x, x \in [\frac{\pi}{4}, \pi]$

The final graph of f(x) is shown in the diagram.



JEE Main 2019

Let S be the set of points in the interval $(-\pi, \pi)$ at which the function $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following?

(a) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right\}$

(b) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

(c) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$

(d) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

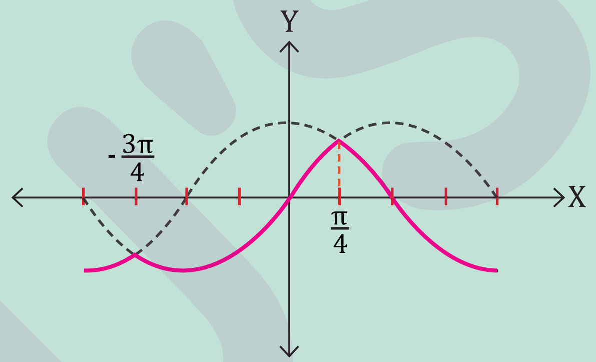
Solution

Step 1:

$f(x) = \min\{\sin x, \cos x\}$ is shown in figure.

Clearly, from the figure, at $x = -\frac{3\pi}{4}$ and $x = \frac{\pi}{4}$, $f(x)$ is not differentiable.

$$S = \left\{-\frac{3\pi}{4}, \frac{\pi}{4}\right\} \subset \left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right\}$$



Hence, option (a) is the correct answer.

AIEEE 2007

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Which of the following is true?

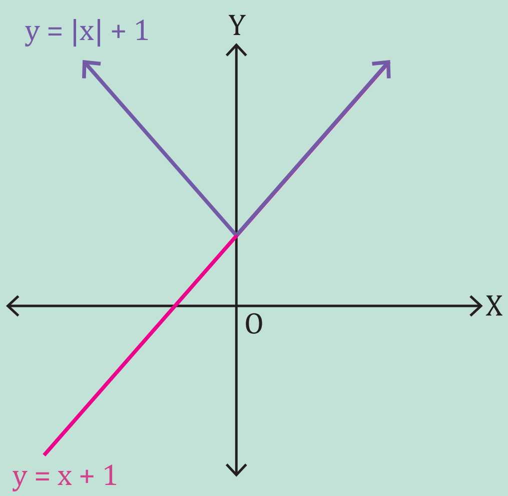
(a) $f(x) \geq 1, \forall x \in \mathbb{R}$

(b) $f(x)$ is not differentiable at $x = 1$

(c) $f(x)$ is differentiable everywhere.

(d) $f(x)$ is not differentiable at $x = 0$

Solution



Step 1:

Draw the graphs of $y = x + 1$ and $y = |x| + 1$ between the same coordinate axis.

Step 2:

We can see from the graph, for $x < 0$, $y = x + 1$ has the lower value.

For $x > 0$, $y = x + 1$ and $y = |x| + 1$ have the same value.

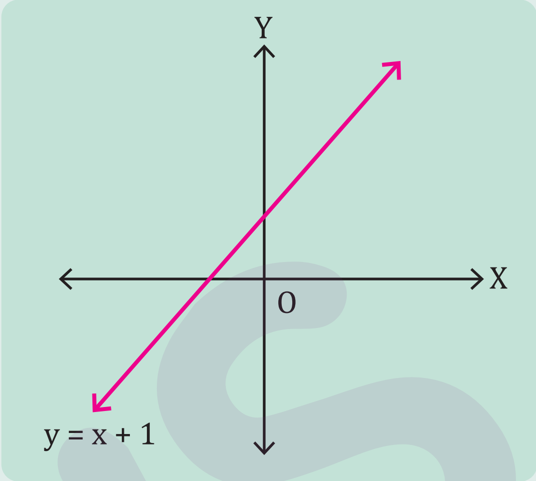
Step 3:

So, the final graph of $f(x) = \min\{x + 1, |x| + 1\} = x + 1$ is shown in the figure.

$f(x)$ is continuous everywhere and there are no sharp edges in the graph.

$\Rightarrow f(x)$ is differentiable everywhere.

Hence, option (c) is the correct answer.



JEE Main 2019

Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let S be the set of points in the interval $(-4, 4)$ at which f is not differentiable. What is S?

(a) $\{-2, -1, 0, 1, 2\}$ **(b)** $\{-2, 2\}$ **(c)** $\{-2, -1, 1, 2\}$ **(d)** An empty set

Solution

Step 1:

Plot the graph of $\max\{|x|, x^2\}, |x| \le 2$

To get this, draw the graphs of $|x|$ and x^2 on the same coordinate plane.

The upper portion of the graph will give the max of $|x|$ and x^2

Between -2 to -1, x^2 has the larger value.

$\Rightarrow \max\{|x|, x^2\} = x^2, x \in (-2, -1)$

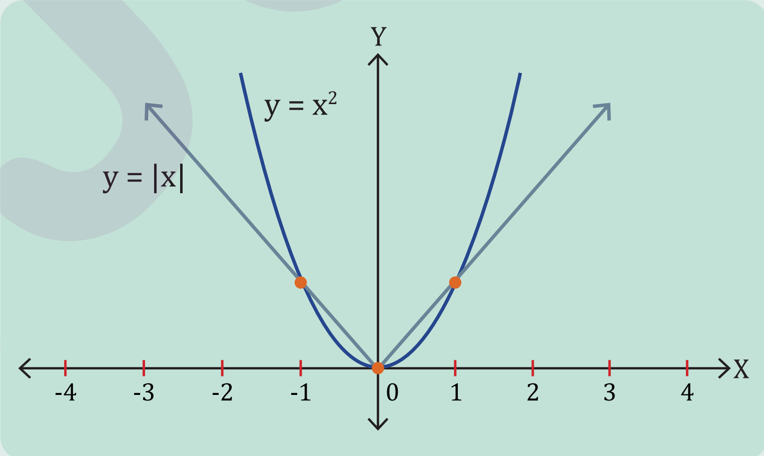
Between -1 to 1, $|x|$ has the larger value.

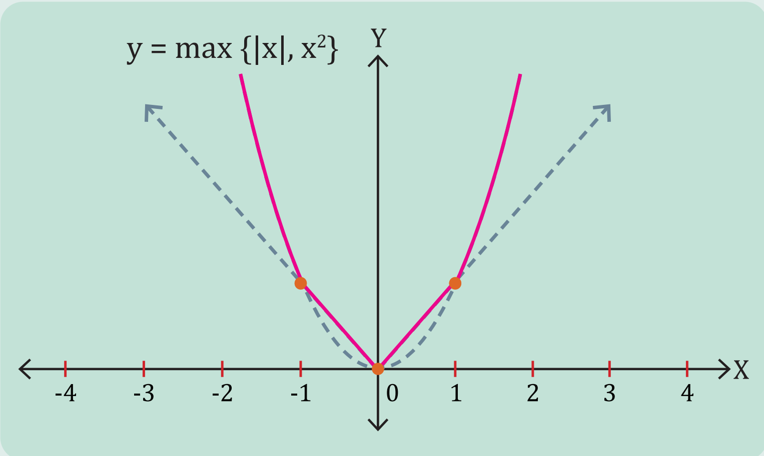
$\Rightarrow \max\{|x|, x^2\} = |x|, x \in (-1, 1)$

Between 1 to 2, x^2 has the larger value.

$\Rightarrow \max\{|x|, x^2\} = x^2, x \in (1, 2)$

So, the graph of $\max\{|x|, x^2\}, |x| \le 2$ is shown in the figure.

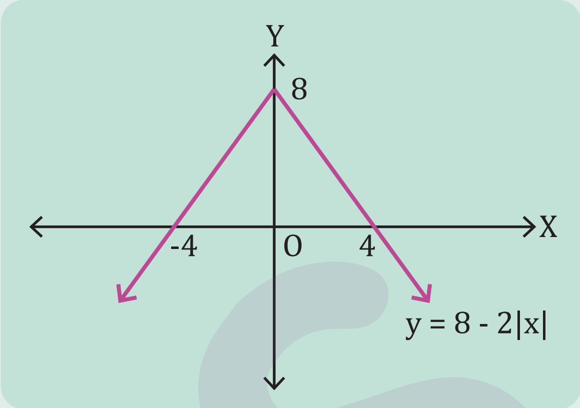




Step 2:

Plot the graph of $y = 8 - 2|x|$

To get this, draw the graph of $-2|x|$, then lift it up by 8 units.



Step 3:

Plot the graphs of $y = 8 - 2|x|$ for $2 < |x| \le 4$ and max $\{|x|, x^2\}$, $|x| \le 2$ on the same coordinate axis.

So, the final graph of f(x) is shown in the figure.

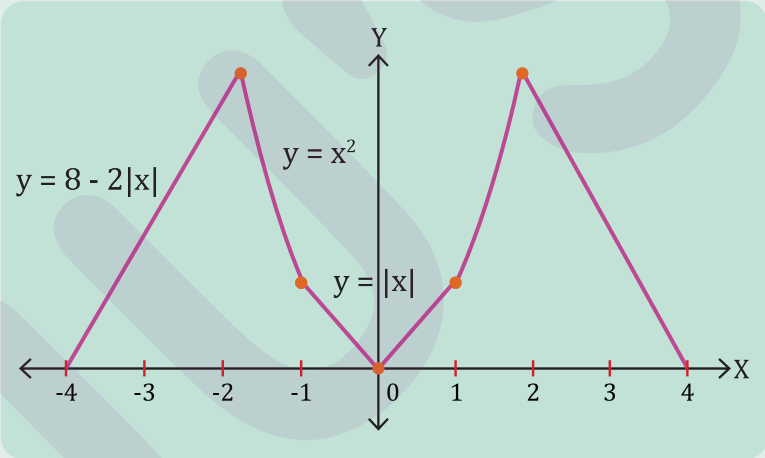
We can see from the diagram that the sharp edges are present at

$x = \{-2, -1, 0, 1, 2\}$

$\Rightarrow$ f(x) is non-differentiable at

$x = \{-2, -1, 0, 1, 2\}$

Hence, option (a) is the correct answer.



JEE Main 2019

Let $f(x) = \begin{cases} -1, -2 \le x < 0 \\ x^2 - 1, 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then in the interval $(-2, 2)$, 'g' is?

(a) Not differentiable at two points

(b) Not continuous

(c) Not differentiable at one point

(d) Differentiable at one point

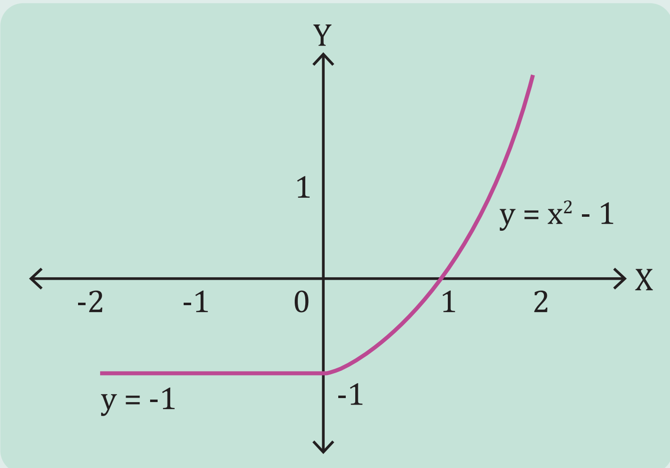
Solution

Step 1:

Plot the graph of $f(x) = \begin{cases} -1, -2 \le x < 0 \\ x^2 - 1, 0 \le x \le 2 \end{cases}$

$y = -1$ is the straight line from -2 to 0.

To draw the graph of $y = x^2 - 1$, pull the graph of $y = x^2$ by 1 unit below the X-axis.



Graph of $y = f(x)$

Step 2:

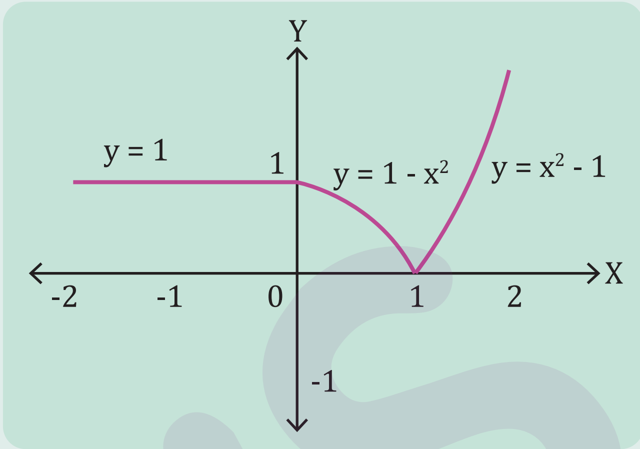
Plot the graph of $|f(x)|$

$$|f(x)| = \begin{cases} |-1|, & -2 \le x < 0 \\ |x^2 - 1|, & 0 \le x \le 2 \end{cases}$$

$$|f(x)| = \begin{cases} 1, & -2 \le x < 0 \\ 1 - x^2, & 0 \le x \le 1 \\ x^2 - 1, & 1 < x \le 2 \end{cases}$$

We learnt that the graph of $|f(x)|$ is the graph of $f(x)$ drawn on positive Y-axis and mirror image of $f(x)$ drawn on negative Y-axis along the X-axis.

So, the graph of $|f(x)|$ is shown in the figure.



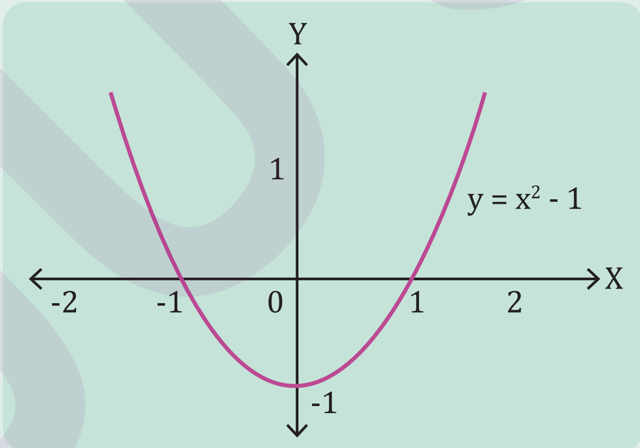
Graph of $y = |f(x)|$

Step 3:

Plot the graph of $f(|x|)$

$f(|x|) = x^2 - 1, -2 \le x \le 2$

We learnt that the graph of $f(|x|)$ is the graph of $f(x)$ drawn on the positive X-axis and its mirror image is along the Y-axis.



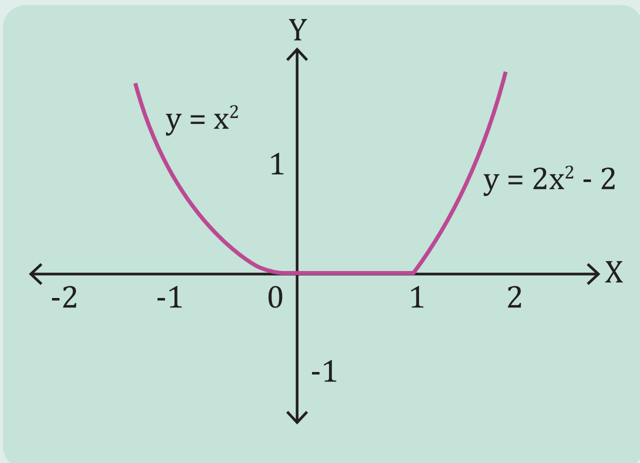
Graph of $y = f(|x|)$

Step 4:

$$g(x) = |f(x)| + f(|x|) = \begin{cases} 1 + (x^2 - 1), & -2 \le x < 0 \\ 1 - x^2 + (x^2 - 1), & 0 \le x \le 1 \\ (x^2 - 1) + (x^2 - 1), & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} x^2, & -2 \le x < 0 \\ 0, & 0 \le x \le 1 \\ 2(x^2 - 1), & 1 < x \le 2 \end{cases}$$

Plot the graph of $g(x)$



Graph of $y = g(x)$

We can see from the diagram that the sharp corner in the graph is at $x = 1$

$\Rightarrow g(x)$ is not differentiable at $x = 1$

Check differentiability at $x = 0$

At $x < 0$, $g(x) = x^2 \Rightarrow g'(x) = 2x$

$$\text{L.H.D.} = \lim_{x \to 0^-} 2x = 0$$

At $x > 0$, $g(x) = 0 \Rightarrow g'(x) = 0$

$$\text{R.H.D.} = \lim_{x \to 0^+} 0 = 0$$

$\Rightarrow$ L.H.D. = R.H.D.

$\Rightarrow$ g(x) is differentiable at x = 0

Hence, option (c) is the correct answer.

Functional Equations

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}$, if $f(x)$ is differentiable at $x = 0$

(a) $f(x)$ is differentiable only in a finite interval containing zero.
(b) $f(x)$ is continuous $\forall x \in \mathbb{R}$
(c) $f'(x)$ is constant $\forall x \in \mathbb{R}$
(d) $f(x)$ is differentiable except at two points.

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Solution

Step 1:

$f(x+y) = f(x) + f(y), \forall x \in \mathbb{R} \dots (1)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(h)}{h}$$

By substituting $x = 0$ and $y = 0$ in equation (1), we get,

$f(0) = f(0) + f(0) \Rightarrow f(0) = 2f(0) \Rightarrow f(0) = 0$

Step 2:

$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(h)}{h}$ is in the form of $\frac{0}{0}$

By using L' hospital Rule, we get,

$f'(x) = \lim_{h \to 0} f'(h) = f'(0) =$ constant

$\Rightarrow f'(x) = k$

By taking integral on both sides, we get,

$\int f'(x)dx = \int k\,dx \Rightarrow f(x) = kx + c$

By substituting $x = 0$, we get,

$c = f(0) = 0$

$\Rightarrow f(x) = kx$

So, $f(x) = kx$ is continuous $\forall x \in \mathbb{R}$ and $f'(x) = k$ is constant $\forall x \in \mathbb{R}$

Hence, options (b) and (c) are the correct answers.

Alternative solution

$f(x+y)=f(x)+f(y), \forall\ x, y \in \mathbb{R}$

By using partial derivative with respect to x, we get,

$f'(x+y)=f'(x)+0$

By substituting $x=0$, we get,

$f'(y)=f'(0) \Rightarrow f'(x)=f'(0)$

By taking integral on both the sides, we get,

$\int f'(x)dx=\int f'(0)dx \Rightarrow f(x)=f'(0)x+c$

When $x=0,\ f(0)=0 \Rightarrow c=0$

$f(x)=kx$

So, $f(x) = kx$ is continuous $\forall\ x \in \mathbb{R}$ and $f'(x) = k$ is constant $\forall\ x \in \mathbb{R}$

Hence, options (b) and (c) are the correct answers.

If f is a real valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, $\forall\ x, y \in R$ and $f(0) = 0$, then find $f(1)$.

(a) 1 (b) 2 (c) 0 (d) -1

AIEEE 2005

Solution

$|f(x)-f(y)| \le (x-y)^2$

By replacing $x=x+h$ and $y=x$, we get,

$\Rightarrow |f(x+h)-f(x)| \le (x+h-x)^2$

$\Rightarrow |f(x+h)-f(x)| \le |h|^2$

$\Rightarrow \lim\limits_{h\to 0}\left|\dfrac{f(x+h)-f(x)}{h}\right| \le \lim\limits_{h\to 0}|h|$

$\Rightarrow |f'(x)| \le 0$, but in general $|f'(x)| \ge 0$

$\Rightarrow |f'(x)|=0 \Rightarrow f(x)=c$

$\Rightarrow$ Given $f(0)=0 \Rightarrow c=0$

$\Rightarrow f(x)=f(1)=0$

Hence, option (c) is the correct answer.

Concept Check

1. $f\left(\dfrac{x+y}{2}\right)=\dfrac{f(x)+f(y)}{2}, \forall\ x, y \in \mathbb{R}$ and $f(0)=1,\ f'(0)=-1$

   If $f(x)$ is differentiable $\forall\ x \in \mathbb{R}$, then find $f(x)$.

   (a) x (b) $1 - x$ (c) $2x$ (d) $2x + 1$

Summary Sheet

- To find the max {f(x), g(x)} or the min {f(x), g(x)}, plot the graphs of f(x) and g(x) on the same coordinate plane. Then figure out their maximum and minimum.
- To solve the functional equations, use the standard definition of f'(x) and substitute the condition given in the question.

Mind Map

Max {f(x), g(x)} or Min {f(x), g(x)} → **Functional equations**

Self-Assessment

If a differentiable function f is satisfying a relation

$$f(x+y) = f(x) + f(y) + 2xy(x+y) - \frac{1}{3}, \forall\ x, y \in R \text{ and } \lim_{h\to 0}\frac{3f(h)-1}{6h} = \frac{2}{3}$$

then find the value of [f(2)] (where [x] represents the greatest integer function).

Answers

Concept Check

1. $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \forall\ x, y \in \mathbb{R}$ and $f(0) = 1$, $f'(0) = -1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right) - f\left(\frac{2x+2(0)}{2}\right)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\left(\frac{f(2x)+f(2h)}{2}\right) - \left(\frac{f(2x)+f(0)}{2}\right)}{h} = \lim_{h\to 0}\frac{f(2h)-f(0)}{2h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(2h+0)-f(0)}{2h} = f'(0)$$

Given $f'(0) = -1$

$\Rightarrow f'(x) = -1$

By taking integral on both sides, we get,

$\int f'(x)dx = \int -1\,dx$

$\Rightarrow f(x) = -x + c$

Given $f(0) = 1 \Rightarrow f(0) = c = 1$

$\Rightarrow f(x) = 1 - x$

Hence, option (b) is the correct answer.

Self-Assessment

Step 1:

Given, $f(x+y) = f(x) + f(y) + 2xy(x+y) - \frac{1}{3}, \forall\ x, y \in \mathbb{R}$ ...(1)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x+0)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\left(f(x)+f(h)+2xh(x+h)-\frac{1}{3}\right)-\left(f(x)+f(0)-\frac{1}{3}\right)}{h} \quad \{\text{Using (1)}\}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} + 2x^2 = f'(0) + 2x^2$$

Step 2:

$$\lim_{h\to 0}\frac{3f(h)-1}{6h} = \lim_{h\to 0}\frac{f(h)-\frac{1}{3}}{2h} = \lim_{h\to 0}\frac{f(h)-f(0)}{2h} = \frac{f'(0)}{2} = \frac{2}{3}$$

$\left\{\text{By replacing } x = 0,\ y = 0 \text{ in equation 1, we get, } f(0) = \frac{1}{3}\right\}$

$$\Rightarrow f'(0) = \frac{4}{3}$$

$$\therefore f'(x) = \frac{4}{3} + 2x^2$$

$$\Rightarrow f(x) = \frac{2x^3}{3} + \frac{4}{3}x + k \quad \Rightarrow f(0) = k = \frac{1}{3}$$

$$\therefore f(x) = \frac{2x^3}{3} + \frac{4}{3}x + \frac{1}{3} \Rightarrow f(2) = \frac{25}{3}$$

$$\Rightarrow [f(2)] = 8$$