BYJU'S Classes

MATHEMATICS

INVERSE TRIGONOMETRIC FUNCTIONS

DOMAIN, RANGE AND GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

What you already know

- Function and classification of functions
- Basic graph of functions

What you will learn

- Condition for existence of inverse of a function
- Domain, range, and graph of inverse trigonometric functions

Inverse function recap

If $f : A \to B$ is a one-one and an onto function, then $f^{-1} : B \to A$ is its inverse.
If f and f^{-1} are the inverse of each other, then we get the following:

(i) Domain of f = Range of f^{-1}
(ii) Range of f = Domain of f^{-1}
(iii) $(f^{-1} of)(x) = f^{-1}(f(x)) = x$

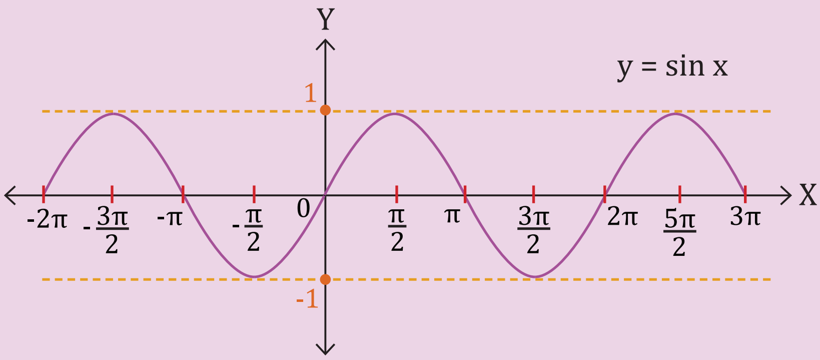
Condition for Existence of Inverse of a Function

An inverse of a function exists only if the function is one-one and onto, i.e., bijective.
If a function is not bijective in the given domain, then its domain is modified to get the inverse.
The modified domain is known as a restricted domain.

Example

Let us consider the following function: $f : \mathbb{R} \to \mathbb{R} : f(x) = \sin x$

From the given graph, we can see that the given function is neither one-one nor onto as a horizontal line will cut the graph at more than one point and codomain ($\mathbb{R}$) is not equal to the range of the function.

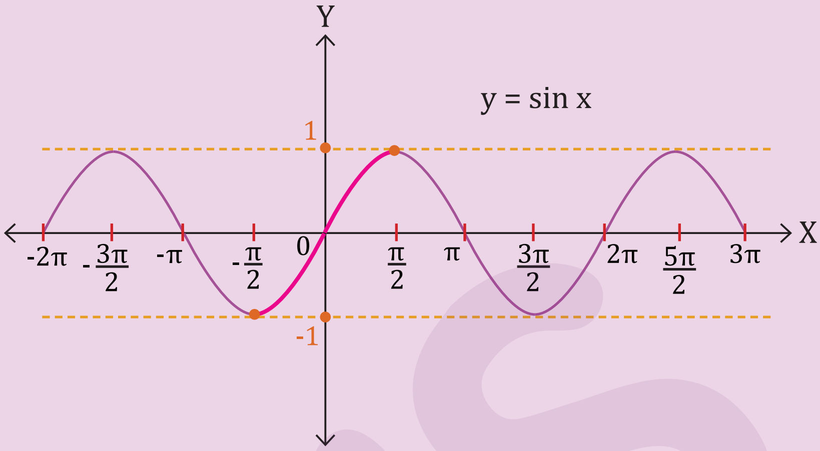


To make the function bijective domain and codomain of the given function is modified to

$f:\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ as

shown in the figure.

Now, we can see that in the modified domain and codomain, the given function $f(x) = \sin x$ is both one-one and onto function, i.e., bijective.



Note

- Inverse trigonometric function gives angle in radians. For example, $\sin^{-1}x$ is the measure of the angle in radian.
- There is a difference between $\sin^{-1}x$ (or) (arcsin x) and $(\sin x)^{-1}$
- $(\sin x)^{-1} = \frac{1}{\sin x}$ (wherever it exists)

Domain, Range, and Graph of Inverse Trigonometric Functions

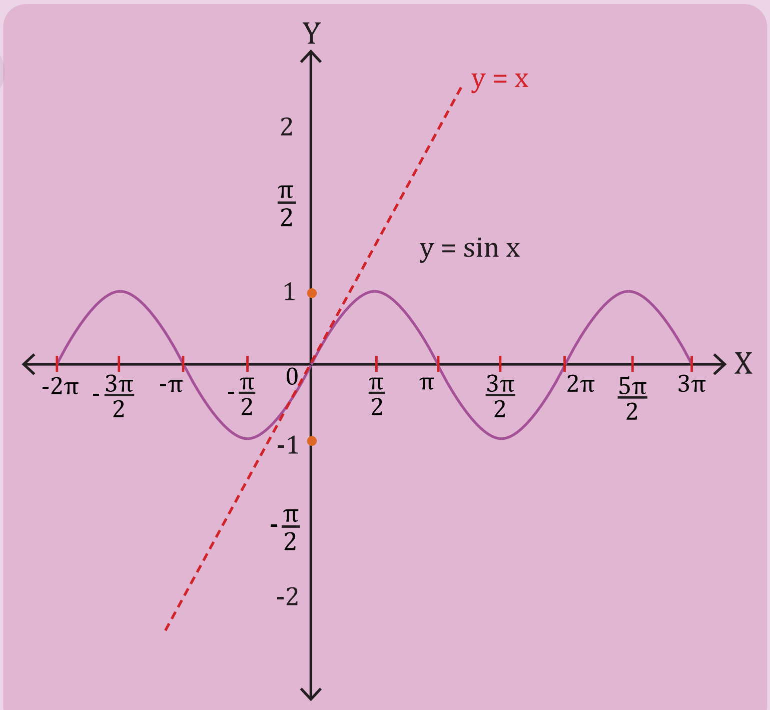
Domain, range, and graph of $f(x) = \sin^{-1}x$

For the function $f(x) = \sin x$,

Restricted domain $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range = [-1, 1]

Here, we can see that the nature of the graph of $f(x)$ is strictly increasing.

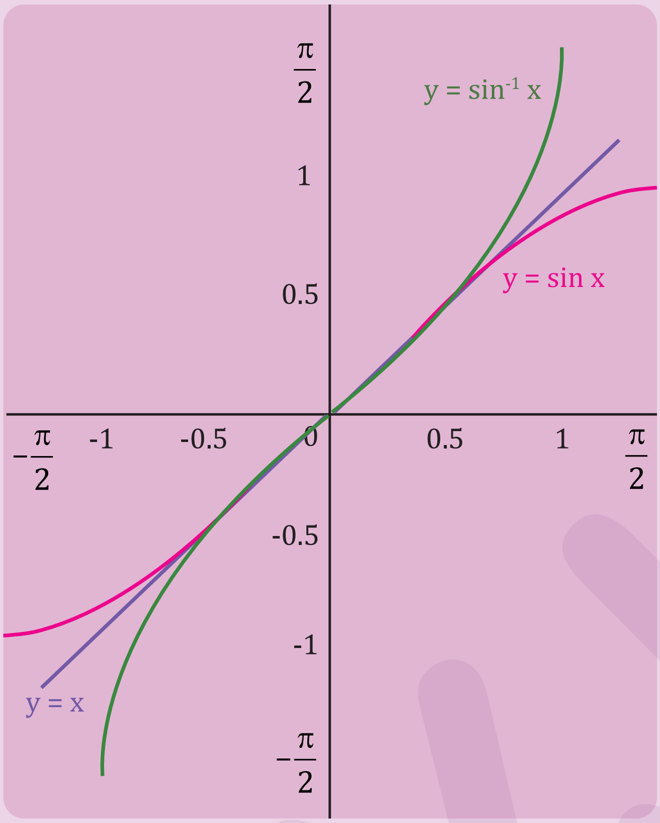


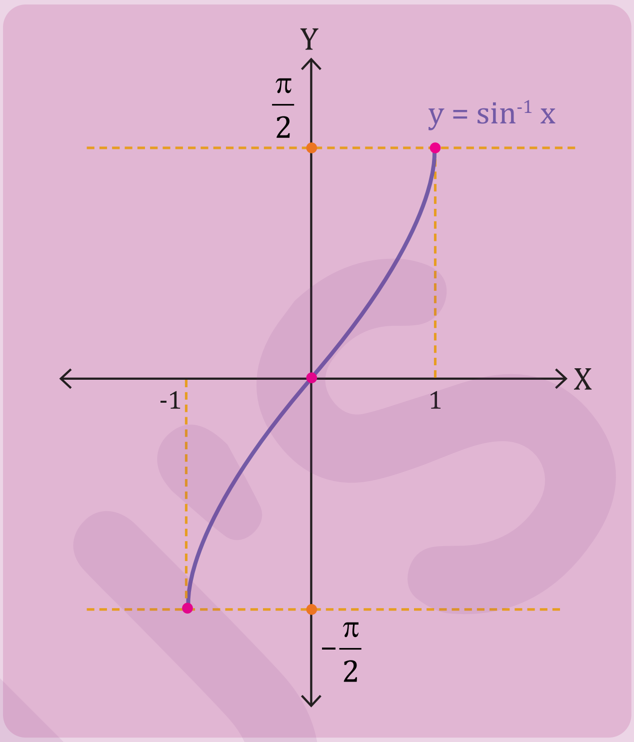
By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of inverse of $f(x)$, i.e., $\sin^{-1}x$.

For the inverse trigonometric function $g(x) = \sin^{-1} x$, we get,

Domain = $[-1, 1]$

Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





Note

For the inverse trigonometric function $g(x) = \sin^{-1} x$, if x is positive, then angle $g(x)$ lies in the first quadrant, and if x is negative, then angle $g(x)$ lies in the fourth quadrant.

Example

(i) $\sin^{-1} 1 = \theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

(ii) $\sin^{-1}\left(-\frac{1}{2}\right) = \alpha \Rightarrow \sin\alpha = -\frac{1}{2} \Rightarrow \alpha = -\frac{\pi}{6}$

(iii) $\sin^{-1}(\pi) =$ Not defined as $\pi \notin [-1, 1]$

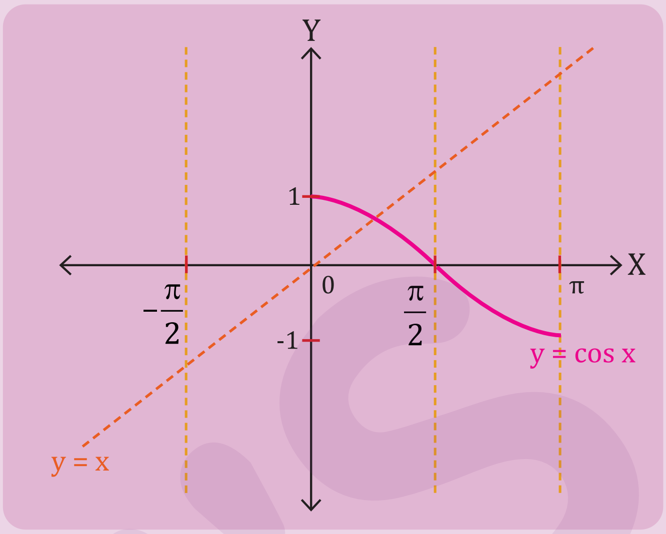
Domain, range, and graph of $f(x) = \cos^{-1} x$

For the cosine function $f(x) = \cos x$,

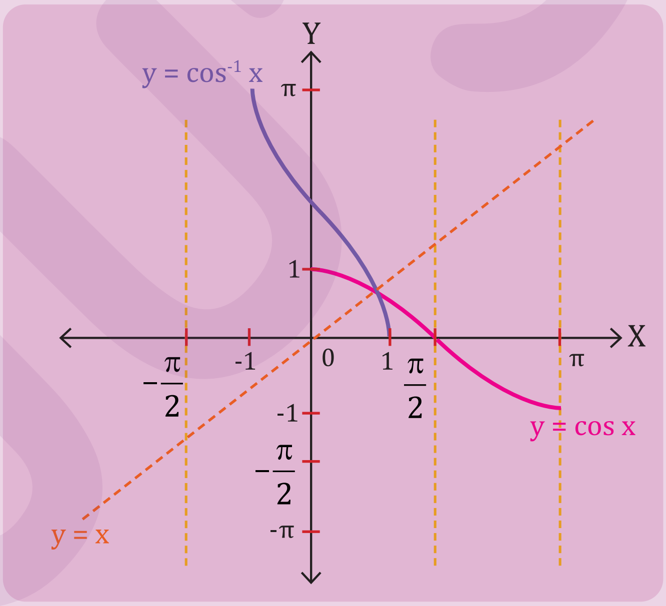
Restricted domain = $[0, \pi]$

Range = $[-1, 1]$

Here, we can see that the nature of the graph of $f(x)$ is strictly decreasing.



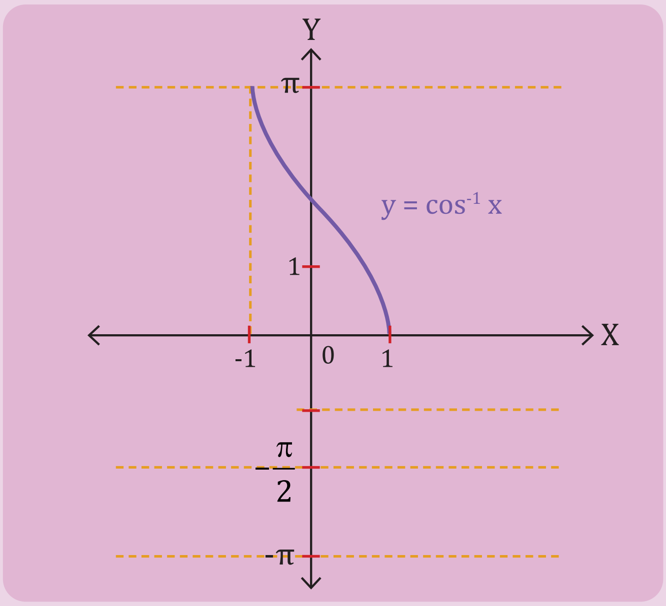
By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of inverse of $f(x)$, i.e., $\cos^{-1} x$.



For the inverse trigonometric function, $g(x) = \cos^{-1} x$, we get,

Domain = $[-1, 1]$

Range = $[0, \pi]$



Note

For the inverse trigonometric function $g(x) = \cos^{-1} x$, if x is positive, then angle $g(x)$ lies in the first quadrant, and if x is negative, then angle $g(x)$ lies in the second quadrant.

Example

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta \Rightarrow \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}$$

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then find the value of $x^{50} + y^{50} + z^{50}$.

Solution

We know, $0 \le \cos^{-1}x \le \pi$

Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

It is only possible if

$\cos^{-1}x = \cos^{-1}y = \cos^{-1}z = \pi$

$\Rightarrow x = y = z = \cos\pi = -1$

$\therefore x^{50} + y^{50} + z^{50} = (-1)^{50} + (-1)^{50} + (-1)^{50}$

$= 1 + 1 + 1 = 3$

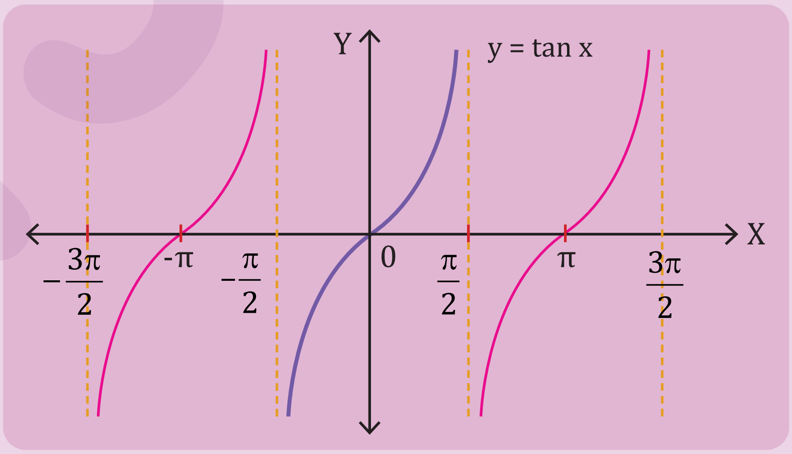
Domain, range, and graph of $f(x) = \tan^{-1}x$

For the tangent function $f(x) = \tan x$,

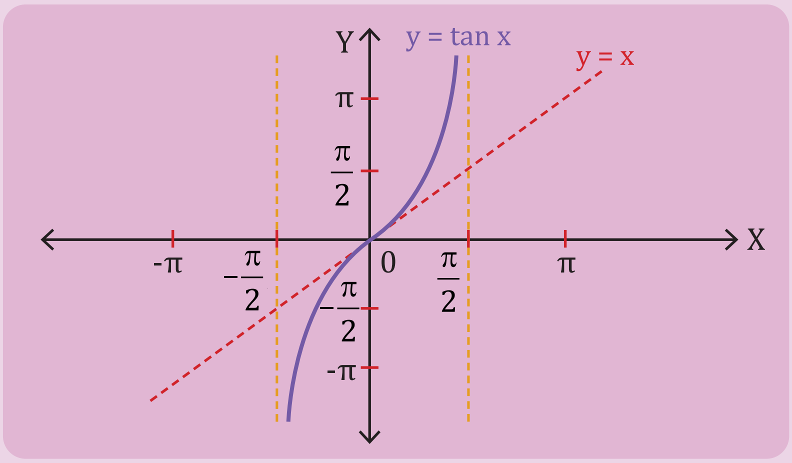
Restricted domain = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

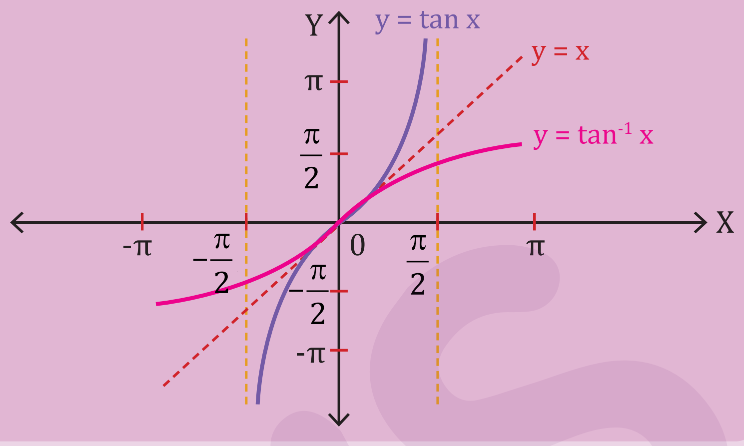
Range = $\mathbb{R}$

Here, we can see that the nature of the graph of $f(x)$ is strictly increasing.



By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of the inverse of $f(x)$, i.e., $\tan^{-1}x$.

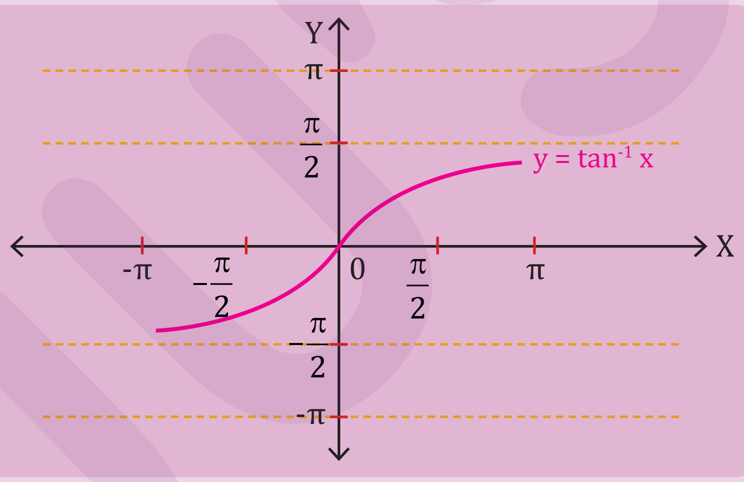




For the inverse trigonometric function $g(x) = \tan^{-1} x$, we get,

Domain = $\mathbb{R}$

Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Note

For the inverse trigonometric function $g(x) = \tan^{-1} x$, if x is positive, then $g(x)$ lies in the first quadrant, and if x is negative, then $g(x)$ lies in the fourth quadrant.

What is the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$?

Solution

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} + \left(-\frac{\pi}{6}\right) = \frac{3\pi}{4}$$

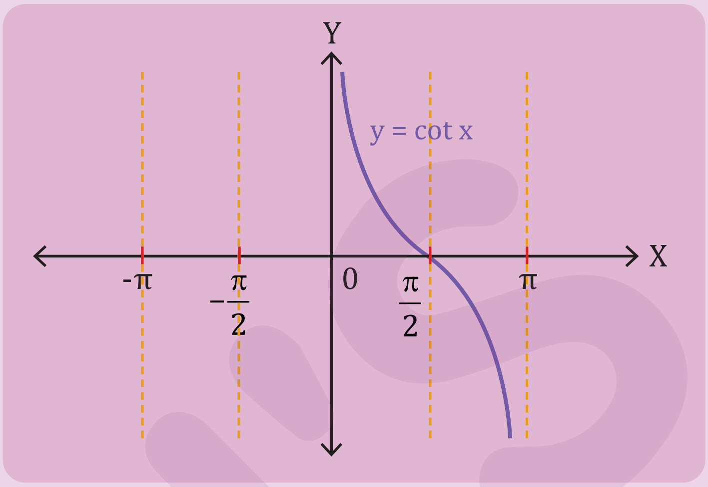
Domain, range, and graph of $f(x) = \cot^{-1} x$

For the cotangent function $f(x) = \cot x$,

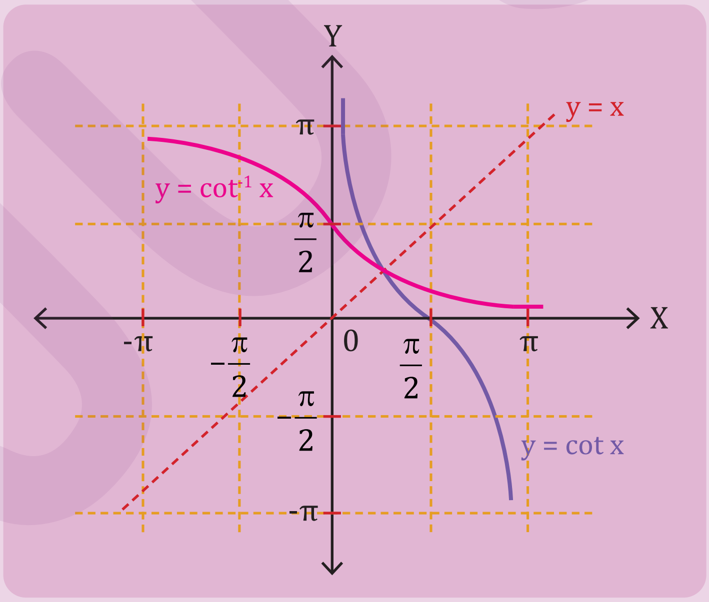
Restricted domain = $(0, \pi)$

Range = $\mathbb{R}$

Here, we can see that the nature of the graph of $f(x)$ is strictly decreasing.



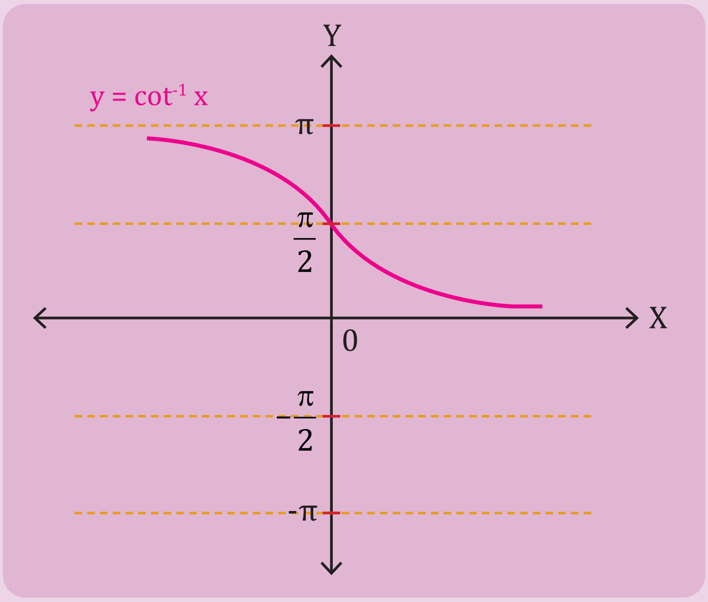
By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of the inverse of $f(x)$, i.e., $\cot^{-1} x$.



For the inverse trigonometric function $g(x) = \cot^{-1} x$, we get,

Domain = $\mathbb{R}$

Range = $(0, \pi)$



Note

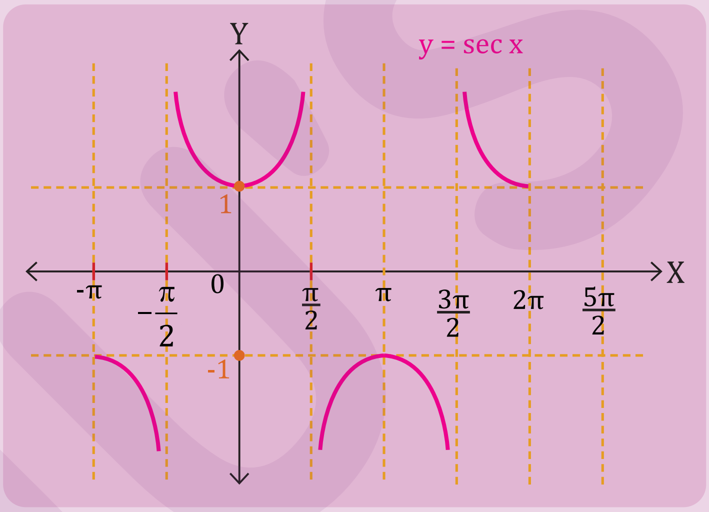
For the inverse trigonometric function $g(x) = \cot^{-1} x$, if x is positive, then $g(x)$ lies in the first quadrant, and if x is negative, then $g(x)$ lies in the second quadrant.

Domain, range, and graph of $f(x) = \sec^{-1} x$

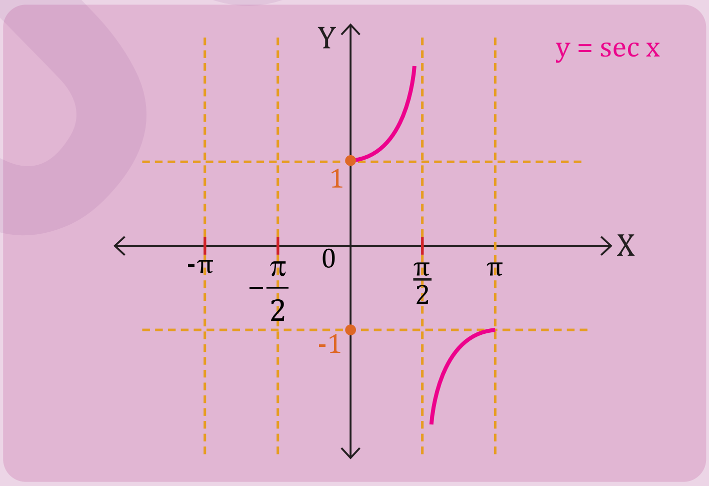
For the secant function $f(x) = \sec x$,

Restricted domain = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

Range = $(-\infty, -1] \cup [1, \infty)$



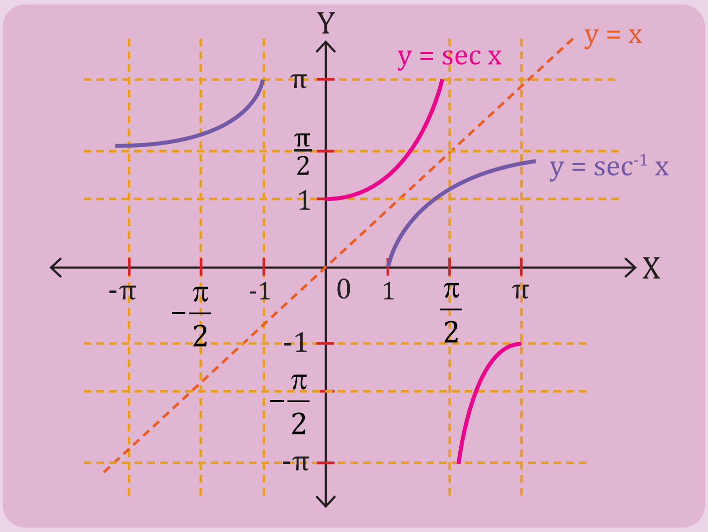
By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of the inverse of $f(x)$, i.e., $\sec^{-1} x$.

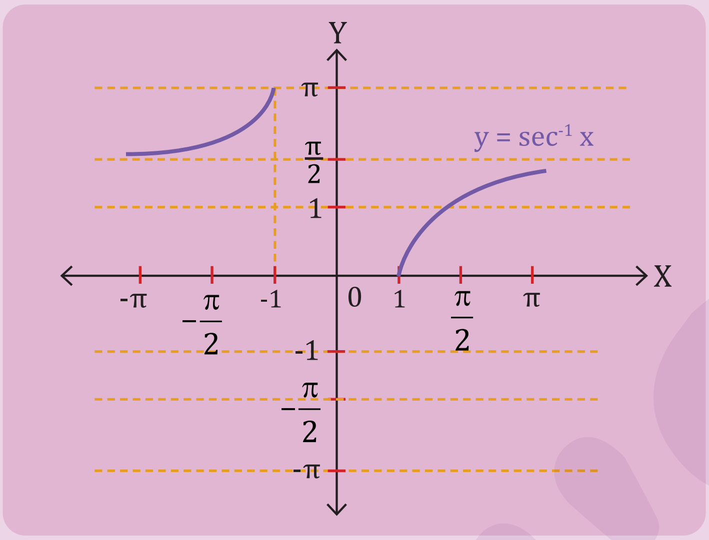


For the inverse trigonometric function $g(x) = \sec^{-1} x$, we get,

Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$





Note

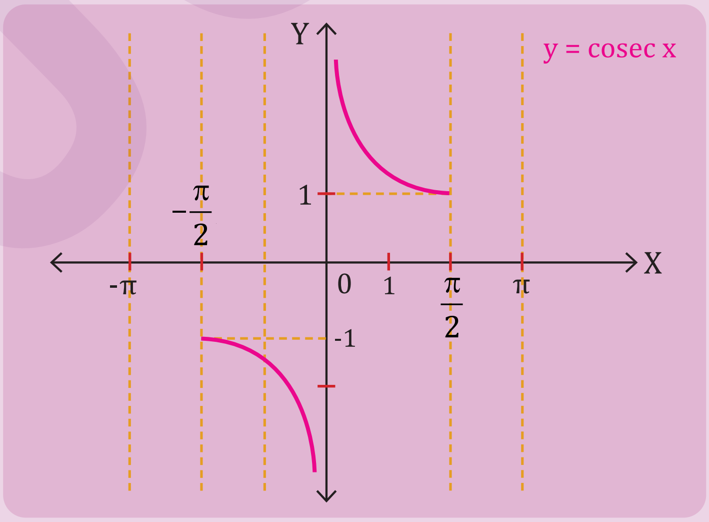
For the inverse trigonometric function $g(x) = \sec^{-1}x$, if x is positive, then $g(x)$ lies in the first quadrant, and if x is negative, then $g(x)$ lies in the second quadrant.

Domain, range, and graph of $f(x) = \operatorname{cosec}^{-1} x$

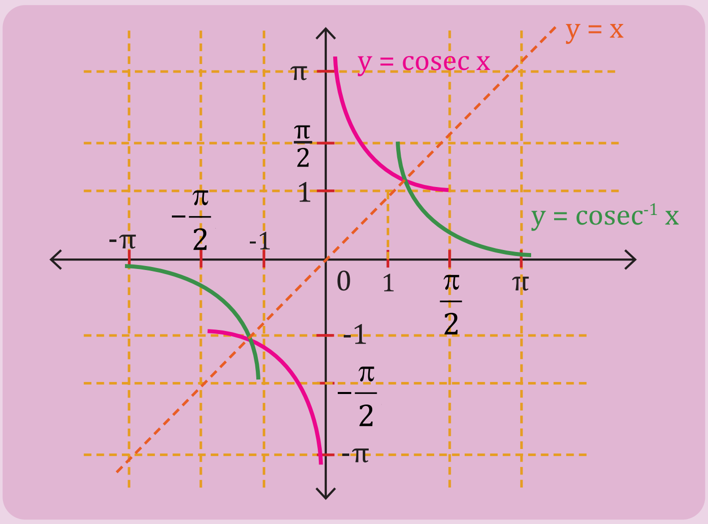
For the cosecant function $f(x) = \operatorname{cosec} x$,

Restricted domain = $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) - \{0\}$

Range = $(-\infty, -1] \cup [1, \infty)$



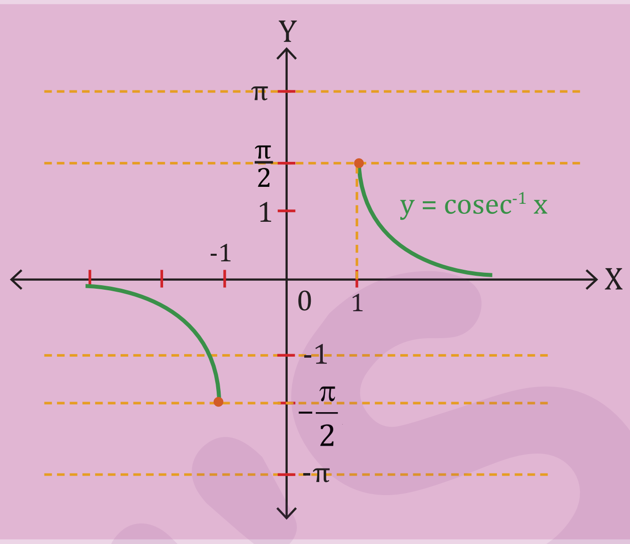
By taking the mirror image of $f(x)$ about the line $y = x$, we get the graph of the inverse of $f(x)$, i.e., $\operatorname{cosec}^{-1} x$.



For the inverse trigonometric function $g(x) = \text{cosec}^{-1} x$,

Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$



Note

For the inverse trigonometric function $g(x) = \text{cosec}^{-1} x$, if x is positive, then $g(x)$ lies in the first quadrant, and if x is negative, then $g(x)$ lies in the fourth quadrant.

Example

(i) $\text{cosec}^{-1}(1) = \dfrac{\pi}{2}$

(ii) $\text{cosec}^{-1}\left(\dfrac{1}{2}\right) =$ Not defined as $\dfrac{1}{2} \notin (-\infty, -1] \cup [1, \infty)$

Find domain and range of $\cos^{-1}[x]$, where [.] represents the greatest integer function.

Solution

Step 1

Given function, $y = \cos^{-1}[x]$

for the function $y = \cos^{-1}x$, $-1 \le x \le 1$

$\Rightarrow -1 \le [x] \le 1$

$\Rightarrow -1 \le x < 2$

Step 2

$-1 \le [x] \le 1 \Rightarrow [x] = -1, 0, 1$

$[x] = -1 \Rightarrow \cos^{-1}(-1) = \pi$

$[x] = 0 \Rightarrow \cos^{-1}(0) = \dfrac{\pi}{2}$

$[x] = 1 \Rightarrow \cos^{-1}(1) = 0$

The range of the function is $\{0, \dfrac{\pi}{2}, \pi\}$.

Concept Check

1. Find the domain and range of $y = \sin^{-1}(e^x)$
2. Find the domain of $f(x) = \sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ JEE ADVANCED 2003

Summary Sheet

Key Takeaways

- An inverse of a function exists only if the function is one-one and onto, i.e., bijective.
- If a function is not bijective in the given domain, then its domain is modified to get the inverse. Thus, a modified domain is known as a restricted domain.
- For $g(x) = \sin^{-1} x$,
  Domain = $[-1, 1]$
  Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- For $g(x) = \cos^{-1} x$,
  Domain = $[-1, 1]$
  Range = $[0, \pi]$
- For $g(x) = \tan^{-1} x$,
  Domain = $\mathbb{R}$
  Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- For $g(x) = \cot^{-1} x$,
  Domain = $\mathbb{R}$
  Range = $(0, \pi)$
- For $g(x) = \sec^{-1} x$,
  Domain = $(-\infty, -1] \cup [1, \infty)$
  Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$
- For $g(x) = \operatorname{cosec}^{-1} x$,
  Domain = $(-\infty, -1] \cup [1, \infty)$
  Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Mind Map

Condition for existence of inverse → Domain, range and graph of inverse trigonometric functions →

- $f(x) = \sin^{-1}x$
- $f(x) = \cos^{-1}x$
- $f(x) = \tan^{-1}x$
- $f(x) = \cot^{-1}x$
- $f(x) = \sec^{-1}x$
- $f(x) = \mathrm{cosec}^{-1}x$

Self-Assessment

Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

Answers

Concept Check

1.

Step 1:

The domain of $\sin^{-1}x$ is $-1 \le x \le 1$

$\Rightarrow$ For $\sin^{-1}e^x$, we have $-1 \le e^x \le 1$

$\Rightarrow 0 < e^x \le 1$ (Since exponential function cannot be negative)

$\Rightarrow x \le 0 \rightarrow$ Domain of the function

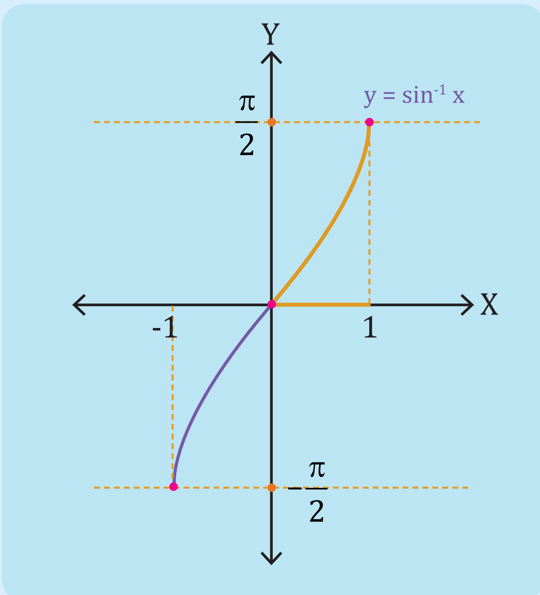
Step 2:

$0 < e^x \le 1$

$\Rightarrow \sin^{-1}(0) < \sin^{-1}(e^x)\ \ \sin^{-1}1$

$\Rightarrow 0 < \sin^{-1}(e^x) \le \frac{\pi}{2}$

$\Rightarrow$ Range = $(0, \frac{\pi}{2}]$



2.

Given, $y = f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$

For the real value of y, we get,

$\sin^{-1}(2x) + \frac{\pi}{6} \geq 0$

$\Rightarrow \sin^{-1}(2x) \geq -\frac{\pi}{6}$

Step 2:

We know that the maximum value of

$\sin^{-1}x$ is $\frac{\pi}{2}$.

So,

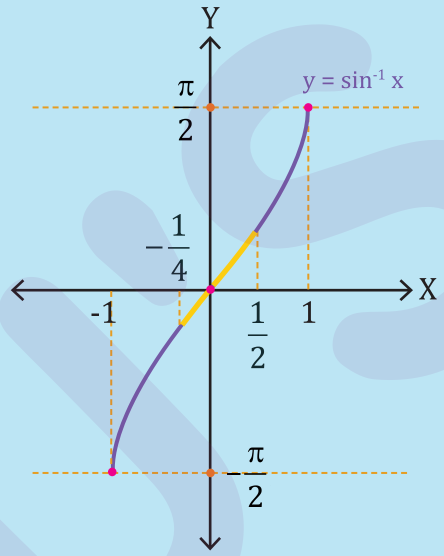
$\Rightarrow \frac{\pi}{2} \geq \sin^{-1}(2x) \geq -\frac{\pi}{6}$

$\Rightarrow \sin\frac{\pi}{2} \geq 2x \geq \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow 1 \geq 2x \geq -\frac{1}{2}$

$\Rightarrow \frac{1}{2} \geq x \geq -\frac{1}{4}$

$\Rightarrow x \in \left[-\frac{1}{4}, \frac{1}{2}\right]$



Self-Assessment

$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

MATHEMATICS

INVERSE TRIGONOMETRIC FUNCTIONS

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

What you already know

- Conditions for the existence of inverse functions
- Domain, range, and graph of ITF

What you will learn

- Properties of inverse function $f^{-1}(-x)$
- Properties of inverse function $f(f^{-1}(x))$
- Graphs of inverse function $f(f^{-1}(x))$
- Properties and graphs of inverse function $f^{-1}(f(x))$

Property 1: Properties of Inverse Trigonometric Function $f^{-1}(-x)$

1. $\sin^{-1}(-x) = -\sin^{-1}x;\ |x| \le 1$
2. $\tan^{-1}(-x) = -\tan^{-1}x;\ x \in \mathbb{R}$
3. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x;\ |x| \ge 1$
4. $\cos^{-1}(-x) = \pi - \cos^{-1}x;\ |x| \le 1$
5. $\cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in \mathbb{R}$
6. $\sec^{-1}(-x) = \pi - \sec^{-1}x;\ |x| \ge 1$

To prove $\sin^{-1}(-x) = -\sin^{-1}x;\ |x| \le 1$

Proof

Let $\sin^{-1}(-x) = \theta$, where $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow -x = \sin\theta$ or $x = -\sin\theta$

We know that $\sin(-\theta) = -\sin\theta$

Hence, $x = -\sin\theta$ can be written as $x = \sin(-\theta)$

$\Rightarrow \sin^{-1}x = -\theta$

$\Rightarrow \sin^{-1}x = -\sin^{-1}(-x)$ or $\sin^{-1}(-x) = -\sin^{-1}(x)$

Hence proved.

To prove $\cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in \mathbb{R}$

Proof

Let $\cot^{-1}(-x) = \theta,\ \theta \in (0, \pi)$

$\Rightarrow -x = \cot\theta$ or $x = -\cot\theta$

We know that $\cot(\pi - \theta) = -\cot\theta$

Hence, $x = -\cot\theta$ can be written as $x = \cot(\pi - \theta)$

$\Rightarrow \cot^{-1}x = (\pi - \theta)$ or $\theta = \pi - \cot^{-1}x$

$\Rightarrow \cot^{-1}(-x) = \pi - \cot^{-1}x$

Hence proved.

Note

1. $\sin^{-1}(x)$, $\tan^{-1}(x)$, $\text{cosec}^{-1}(x)$ are **odd functions** $(f(-x) = -f(x))$
2. $\cos^{-1}(x)$, $\cot^{-1}(x)$, $\sec^{-1}(x)$ are **neither even nor odd functions.**

Evaluate: $\sin^{-1}\left(\frac{-1}{2}\right)$

Solution

$$\sin^{-1}\left(\frac{-1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$$

Property 2: Properties of Inverse Function $f(f^{-1}(x))$

1. $\sin(\sin^{-1}(x)) = x;\ x \in [-1,1]$
2. $\cos(\cos^{-1}(x)) = x;\ x \in [-1,1]$
3. $\tan(\tan^{-1}(x)) = x;\ x \in \mathbb{R}$
4. $\cot(\cot^{-1}(x)) = x;\ x \in \mathbb{R}$
5. $\text{cosec}(\text{cosec}^{-1}(x)) = x; |x| \geq 1$
6. $\sec(\sec^{-1}(x)) = x; |x| \geq 1$

Proof

To prove $\sin(\sin^{-1}(x)) = x;\ x \in [-1,1]$

Let $y = \sin(\sin^{-1}(x))$ and $\sin^{-1}(x) = \theta$, $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow x = \sin\theta$.......(1)

Substituting $\sin^{-1}(x) = \theta$ in y, we get,

$y = \sin\theta$ ...(2)

From (1) and (2), we get,

$y = x$

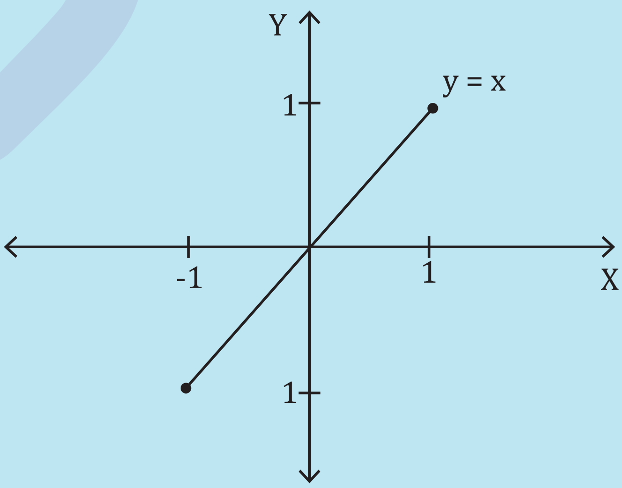
$\therefore \sin(\sin^{-1}(x)) = x$

Graphs of Inverse Function $f(f^{-1}(x))$

1. $y = \sin(\sin^{-1}(x))$ and $y = \cos(\cos^{-1}(x))$ for $x \in [-1, 1]$

We know that $y = \sin(\sin^{-1}(x)) = x;\ x \in [-1, 1]$ and $y = \cos(\cos^{-1}(x)) = x;\ x \in [-1, 1]$

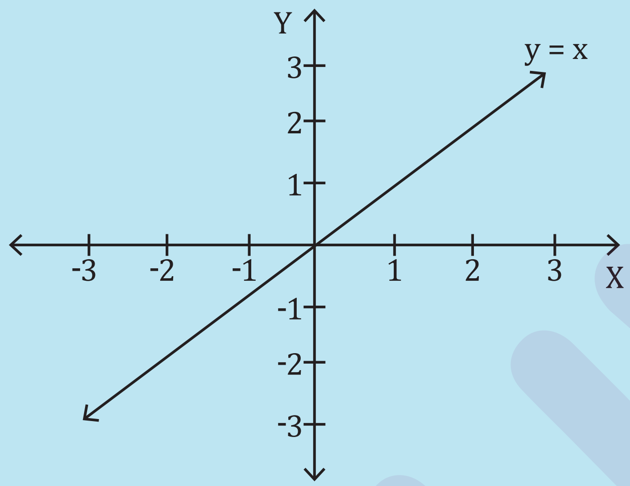
So, the graph will be the line $y = x;\ x \in [-1, 1]$



2. $y = \tan(\tan^{-1}(x)$ and $y = \cot(\cot^{-1}(x))$ for $x \in \mathbb{R}$

We know that $y = \tan(\tan^{-1}(x)) = x \,;\, x \in \mathbb{R}$ and $y = \cot(\cot^{-1}(x)) = x \,;\, x \in \mathbb{R}$

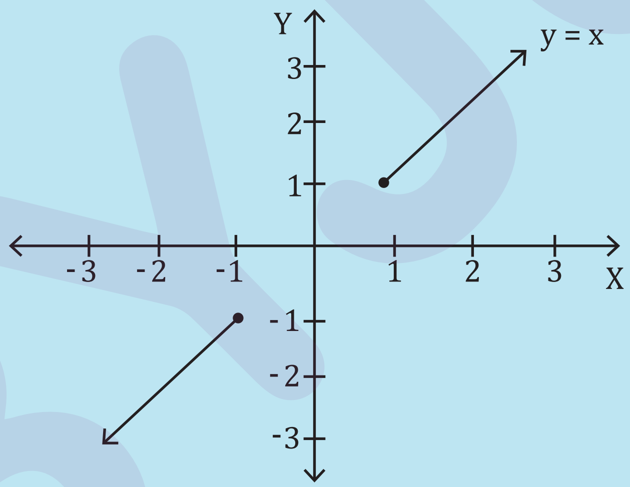
So, the graph will be the line $y = x \,;\, x \in \mathbb{R}$



3. $y = \operatorname{cosec}(\operatorname{cosec}^{-1}(x)$ and $y = \sec(\sec^{-1}(x))$ for $|x| \geq 1$

We know that $y = \operatorname{cosec}(\operatorname{cosec}^{-1}(x)) = x \,;\, |x| \geq 1$ and $y = \sec(\sec^{-1}(x)) = x \,;\, |x| \geq 1$

So, the graph will be the line $y = x \,;\, |x| \geq 1$



Evaluate: $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Solution

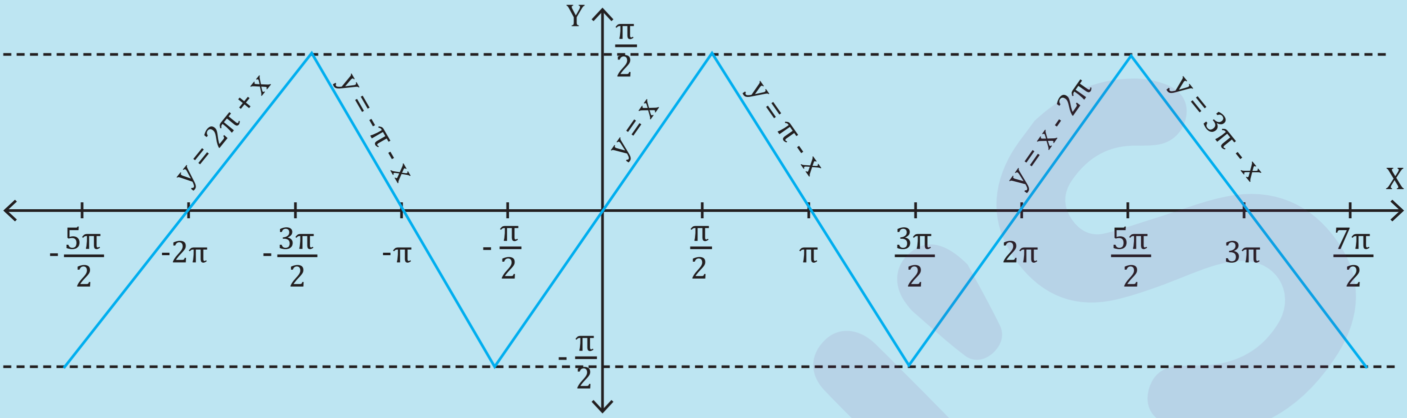
$\cos(\cos^{-1}(x)) = x \,;\, x \in [-1, 1]$

$\Rightarrow \cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = -\frac{\sqrt{3}}{2}$

Property 3: Properties of Inverse Function $f^{-1}(f(x))$

1. $\sin^{-1}(\sin(x))$

Graph of $y = \sin^{-1}(\sin(x))$



Proof

Consider $y = \sin^{-1}(\sin(x))$

We know that $\sin^{-1}(\sin(x)) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Also, $y = \sin^{-1}(\sin(x)) \Rightarrow \sin y = \sin x$

That is, $y = n\pi + (-1)^n x, n \in \mathbb{Z}$ ...(i)

$\sin^{-1}(\sin(x)) = n\pi + (-1)^n x, n \in \mathbb{Z}$

The graph of $\sin^{-1}(\sin(x))$ will be a straight line.

We have to ensure that $(n\pi + (-1)^n x) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ as $\sin^{-1}(\sin(x)) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Case 1: $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

If $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then $n = 0$ in (i)

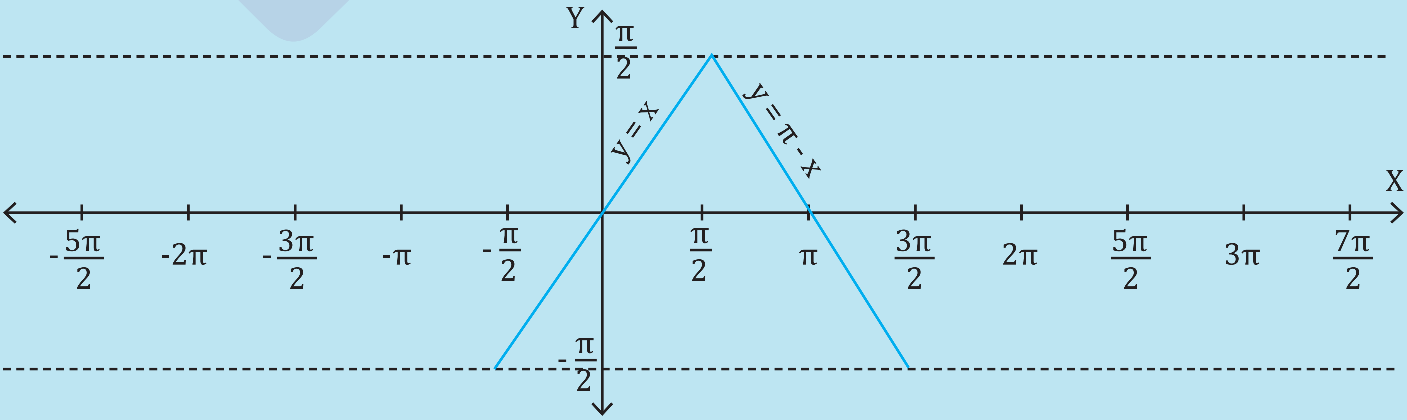
$\Rightarrow y = x$

Case 2: $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

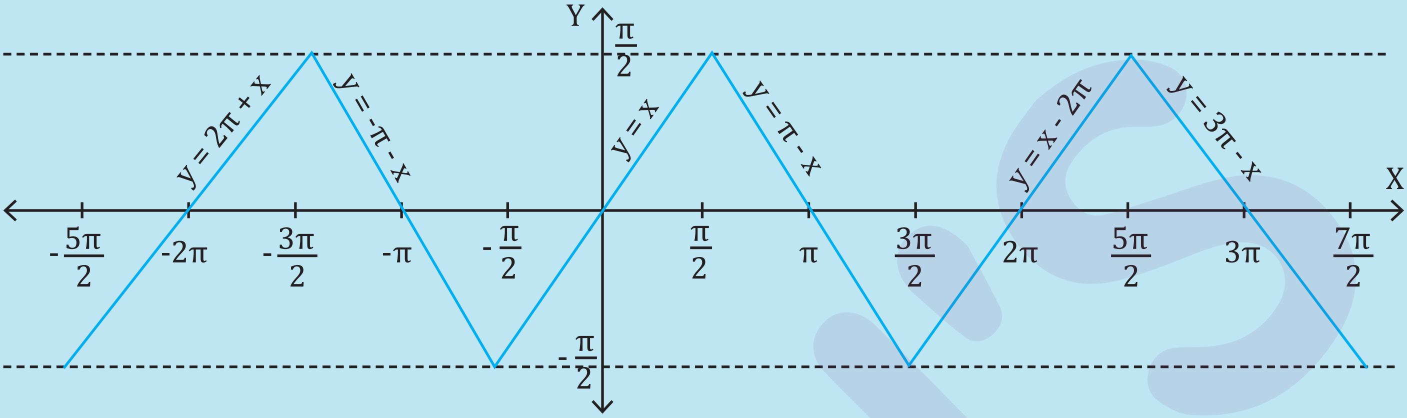
If $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, then $n = 1$ in (i)

$\Rightarrow y = \pi - x$

Let us plot the graph of $y = \sin^{-1}(\sin(x))$ for $x \in \left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$



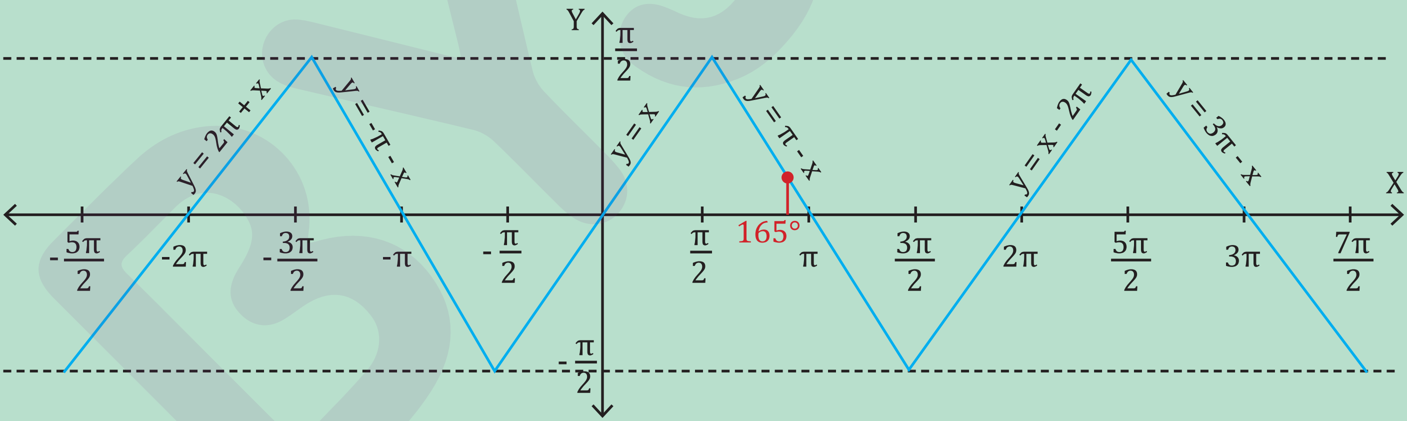
We also know that if $g(x)$ is periodic with period T, then $f(g(x))$ is also periodic with period T. That means the period of $\sin^{-1}(\sin(x))$ is 2π, and we have already plotted the graph for 2π length. So, the graph will simply repeat itself after every 2π interval.
The graph of $y = \sin^{-1}(\sin(x))$ is as follows:



Evaluate: $\sin^{-1}(\sin 165°)$.

Solution

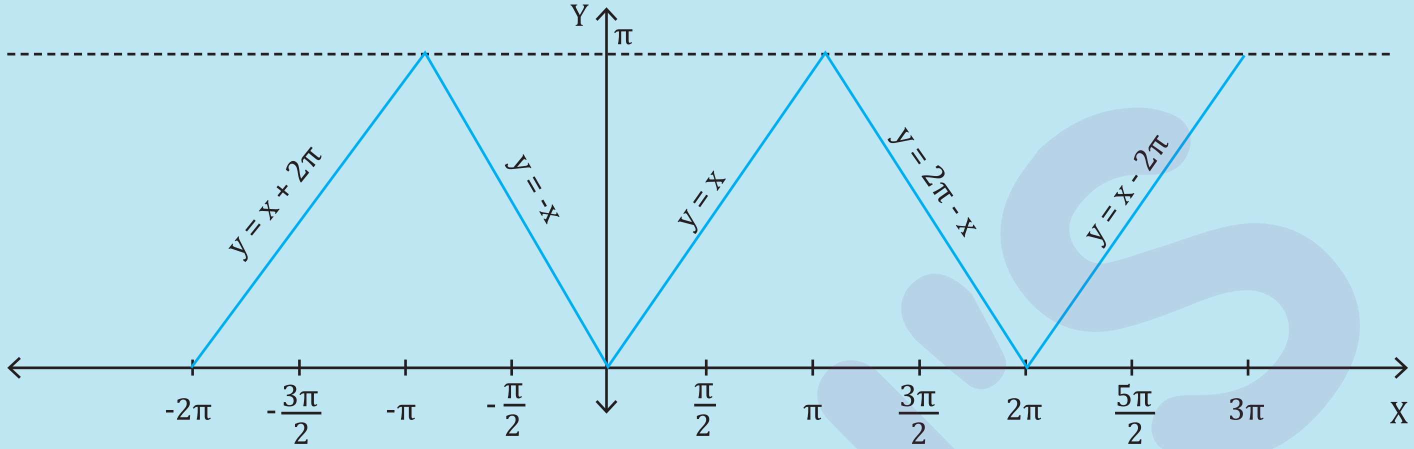
Step 1: 165° lies between 90° and 180°.
From the graph, it is clear that $y = \sin^{-1}(\sin(x))$ will follow $y = \pi - x$



Step 2:
$\sin^{-1}(\sin 165°) = 180° - 165° = 15°$

2. $\cos^{-1}(\cos(x))$

Graph of $y = \cos^{-1}(\cos(x))$



Proof

Consider $y = \cos^{-1}(\cos(x))$

We know that $\cos^{-1}(\cos(x)) \in [0, \pi]$

$\Rightarrow y \in [0, \pi]$

Also, $y = \cos^{-1}(\cos(x)) \Rightarrow \cos y = \cos x$

That is, $y = 2n\pi \pm x, n \in \mathbb{Z}$ ...(i)

$\cos^{-1}(\cos(x)) = 2n\pi \pm x, n \in \mathbb{Z}$

$\Rightarrow$ The graph of $\cos^{-1}(\cos(x))$ will be a straight line.

We have to ensure that $(2n\pi \pm x) \in [0, \pi]$ as $\cos^{-1}(\cos(x)) \in [0, \pi]$

Case 1: $x \in [0, \pi]$

If $x \in [0, \pi]$, then $n = 0$ in (i)

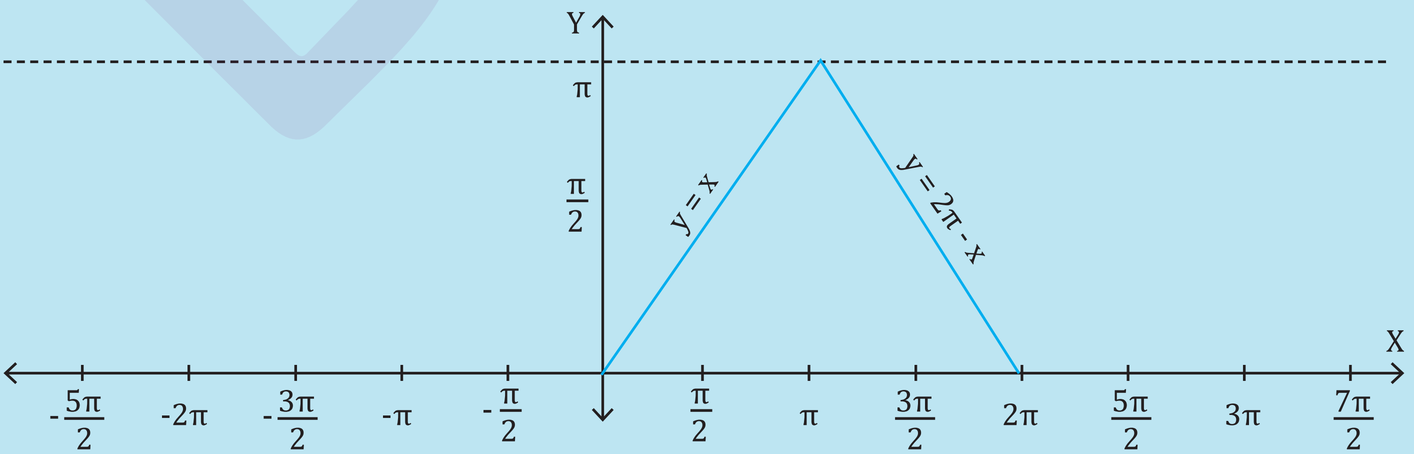
$\Rightarrow y = x$

Case 2: $x \in [\pi, 2\pi]$

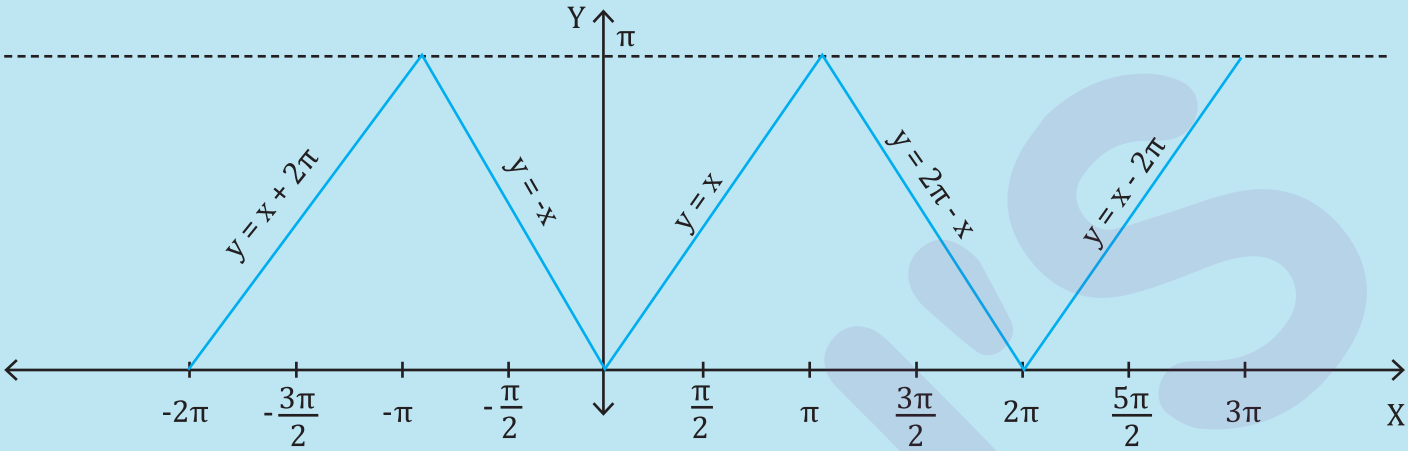
If $x \in [\pi, 2\pi]$, then $n = 1$ in (i)

$\Rightarrow y = 2\pi - x$

Let us plot the graph of $y = \cos^{-1}(\cos(x))$ for $x \in [0, 2\pi]$



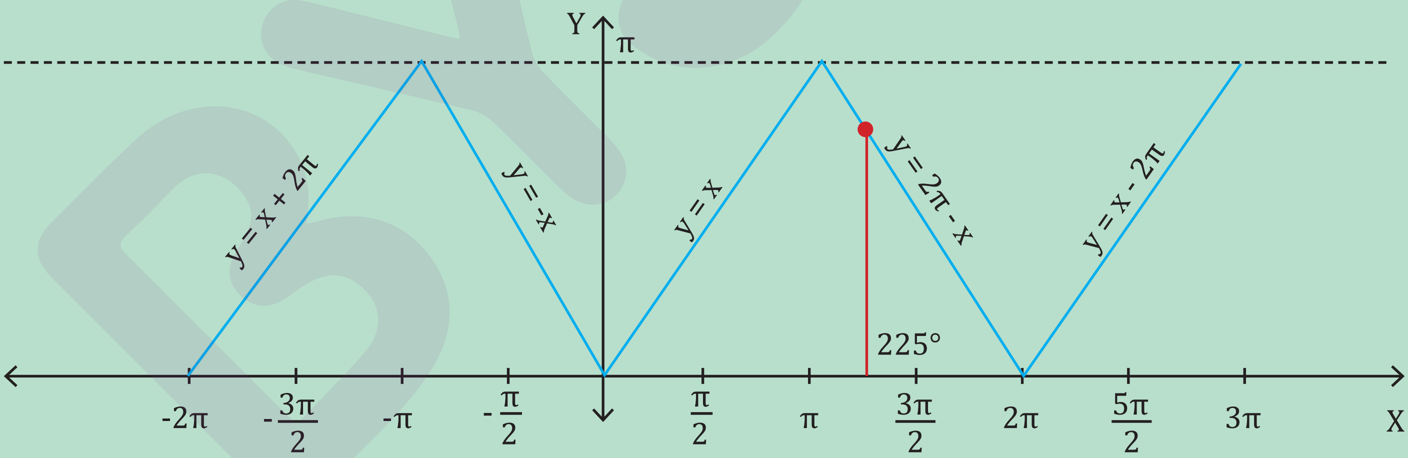
We also know that if g(x) is periodic with period T, then f(g(x)) is also periodic with period T. That means the period of $\cos^{-1}(\cos(x)$ is 2π, and we have already plotted the graph for 2π length. So, the graph will simply repeat itself after every 2π interval.
The graph of $y = \cos^{-1}(\cos(x))$ is as follows:



Evaluate: $\cos^{-1}(\cos 225°)$.

Solution

Step 1: 225° lies between 180° and 360°.
From the graph, it is clear that $y = \cos^{-1}(\cos(x))$ will follow $y = 2\pi - x$



Step 2:
$\cos^{-1}(\cos 225°) = 360° - 225° = 135°$

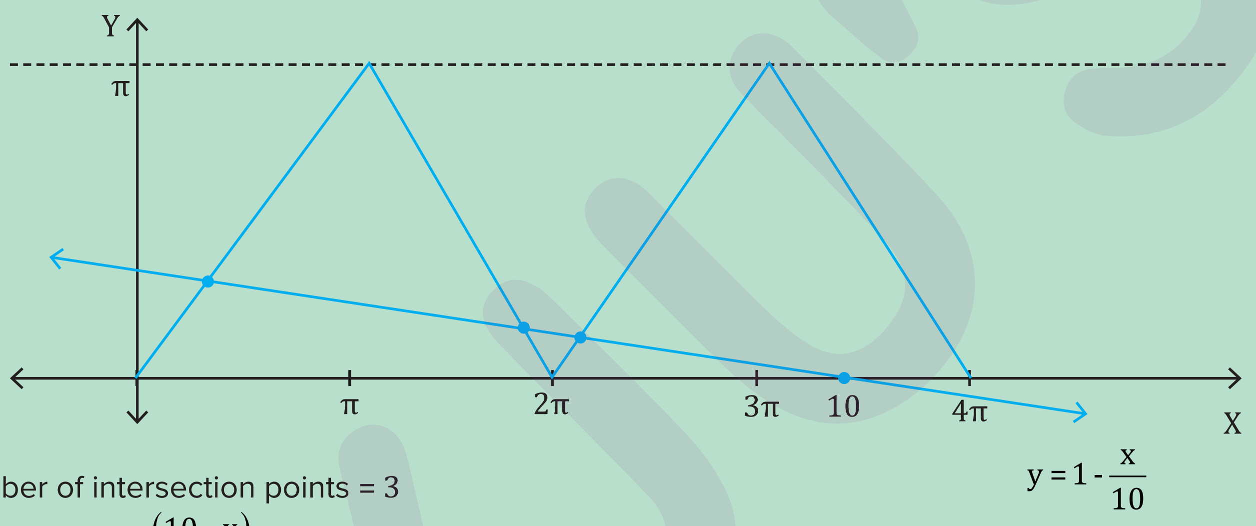
JEE ADVANCED 2014

Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. Find the number of points for $x \in [0, 4\pi]$ satisfying the equation $f(x) = \dfrac{(10 - x)}{10}$

Solution

Step 1: Let us plot $y = \cos^{-1}(\cos x)$ and $y = \dfrac{(10 - x)}{10}$ for $x \in [0, 4\pi]$. The number of solutions will be equal to the number of intersection points.

Step 2:



Number of intersection points = 3

Hence, $f(x) = \dfrac{(10 - x)}{10}$ has 3 solutions in $x \in [0, 4\pi]$.

Concept Check

1. Evaluate the following:

   $(a)\ 2\cot^{-1}(-\sqrt{3})$ $\qquad$ $(b)\ \sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right)$

2. Evaluate the following:

   $(a)\ \operatorname{cosec}\left(\operatorname{cosec}^{-1}\left(\dfrac{1}{2}\right)\right)$ $\qquad$ $(b)\ \sin(\sin^{-1}(2))$

3. Evaluate $\sin^{-1}(\sin 7)$.
4. Evaluate $\cos^{-1}(\cos (7))$.

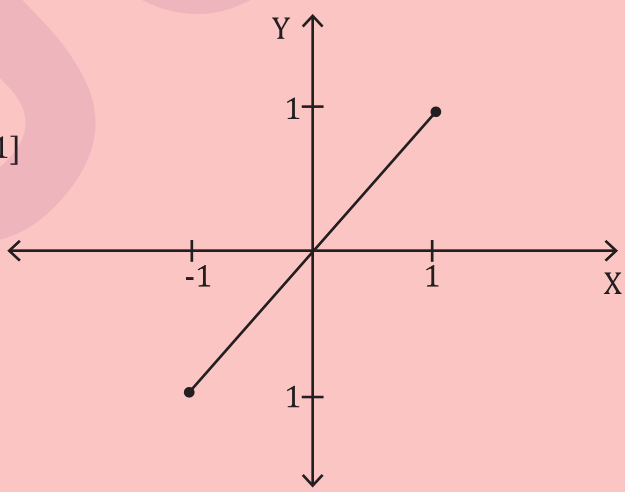
Summary Sheet

Key formulae

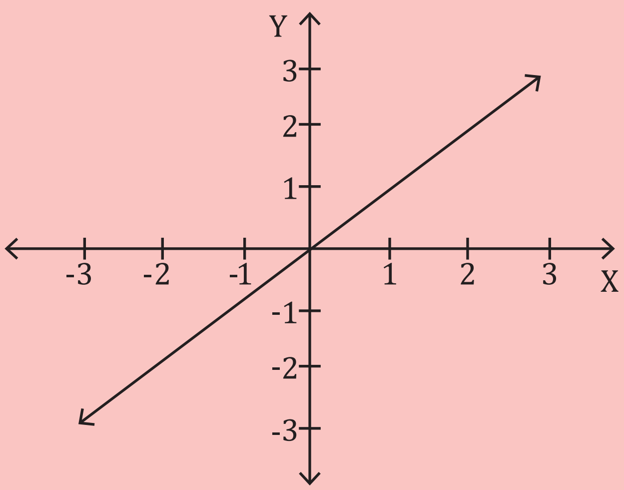
- **Properties of inverse function $f^{-1}(-x)$**
  1. $\sin^{-1}(-x) = -\sin^{-1}x;\ |x| \le 1$
  2. $\tan^{-1}(-x) = -\tan^{-1}x;\ x \in \mathbb{R}$
  3. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x;\ |x| \ge 1$
  4. $\cos^{-1}(-x) = \pi - \cos^{-1}x;\ |x| \le 1$
  5. $\cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in \mathbb{R}$
  6. $\sec^{-1}(-x) = \pi - \sec^{-1}x;\ |x| \ge 1$
- **Properties of inverse function $f(f^{-1}(x))$**
  1. $\sin(\sin^{-1}(x)) = x;\ x \in [-1,1]$
  2. $\cos(\cos^{-1}(x)) = x;\ x \in [-1,1]$
  3. $\tan(\tan^{-1}(x)) = x;\ x \in \mathbb{R}$
  4. $\cot(\cot^{-1}(x)) = x;\ x \in \mathbb{R}$
  5. $\operatorname{cosec}(\operatorname{cosec}^{-1}(x)) = x; |x| \ge 1$
  6. $\sec(\sec^{-1}(x)) = x; |x| \ge 1$

Graphs

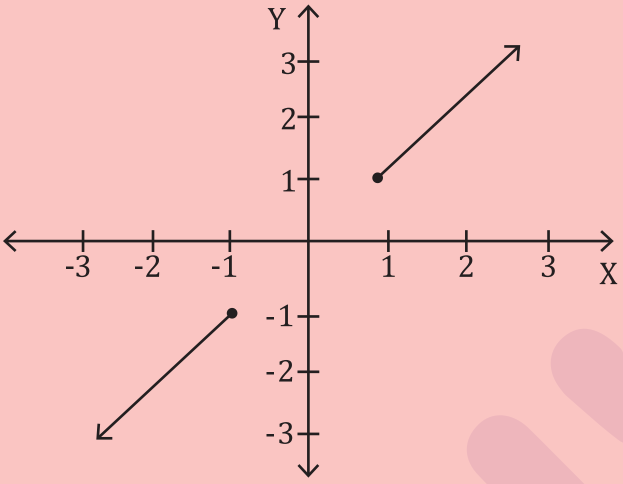
1. $y = \sin(\sin^{-1}(x))$ and $y = \cos(\cos^{-1}(x))$ for $x \in [-1, 1]$



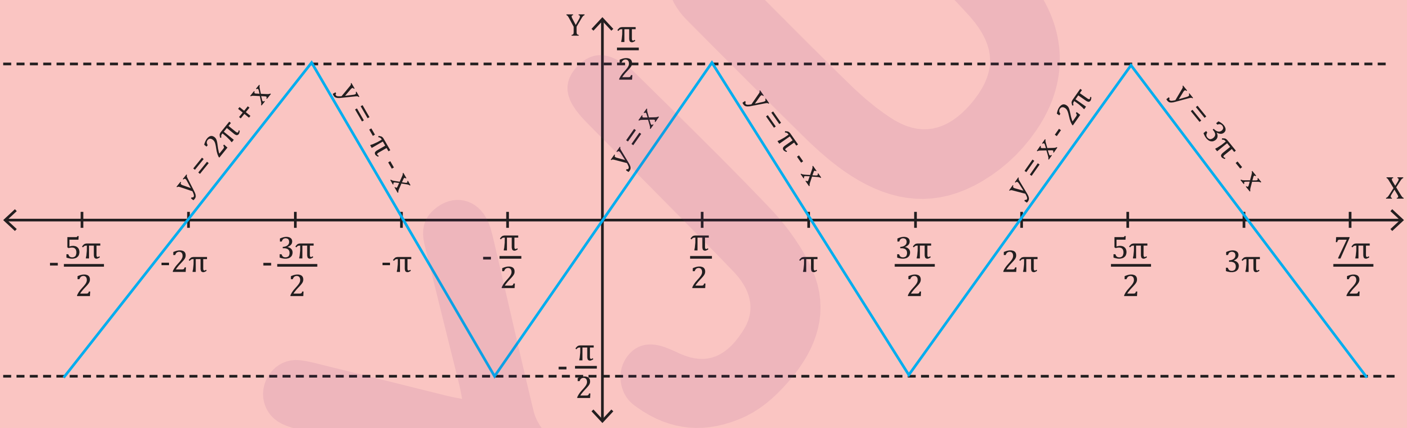
2. $y = \tan(\tan^{-1}(x))$ and $y = \cot(\cot^{-1}(x))$ for $x \in \mathbb{R}$



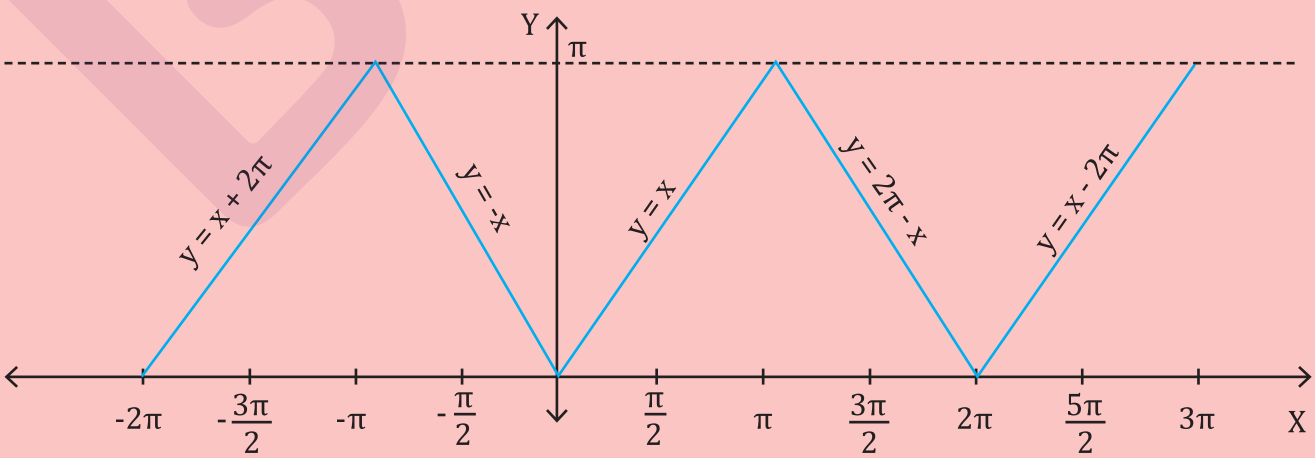
3. $y = \text{cosec}(\text{cosec}^{-1}(x))$ and $y = \sec(\sec^{-1}(x))$ for $|x| \geq 1$



4. $y = \sin^{-1}(\sin(x))$



5. $y = \cos^{-1}(\cos(x))$



Mind Map

Properties of ITF

- Properties of $f^{-1}(-x)$
- Properties of $f^{-1}(f(x))$
- Properties of $f(f^{-1}(x))$

Self-Assessment

Evaluate $\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$.

Answers

Concept Check

1.

(a) Step 1:

$$2\cot^{-1}\left(-\sqrt{3}\right) = 2\left(\pi - \cot^{-1}\sqrt{3}\right)$$

Step 2:

$$= 2\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{3}$$

(b) Step 1:

$$\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$$

Step 2:

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

2.

(a)

Domain of $\operatorname{cosec}^{-1} x$ is $|x| \geq 1$

$\therefore \operatorname{cosec}\left(\operatorname{cosec}^{-1}\left(\frac{1}{2}\right)\right)$ is not defined.

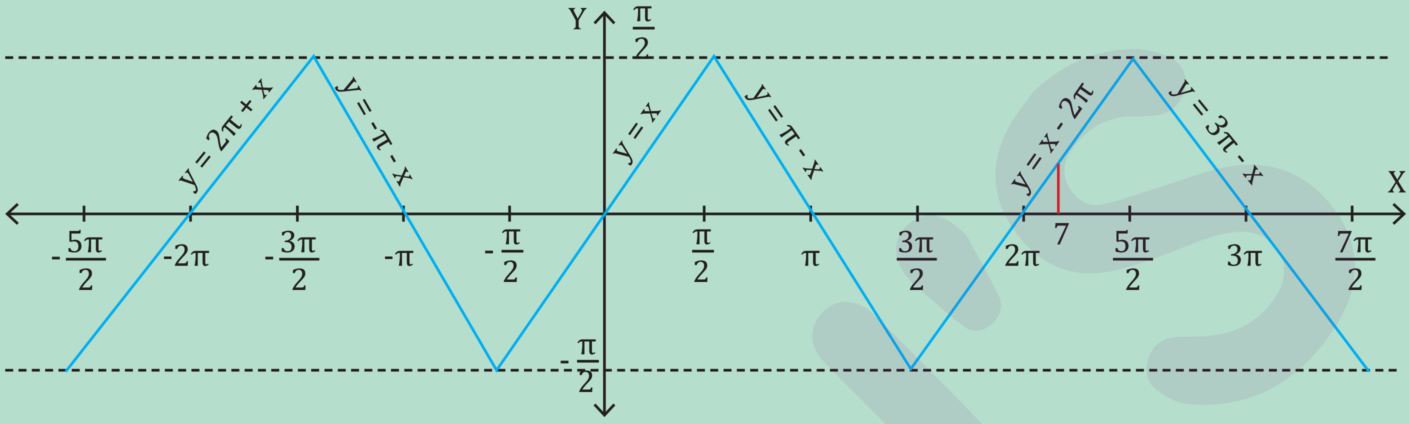
(b)

Domain of $\sin^{-1} x$ is $x \in [-1, 1]$

$\therefore \sin\left(\sin^{-1}(2)\right)$ is not defined.

3.

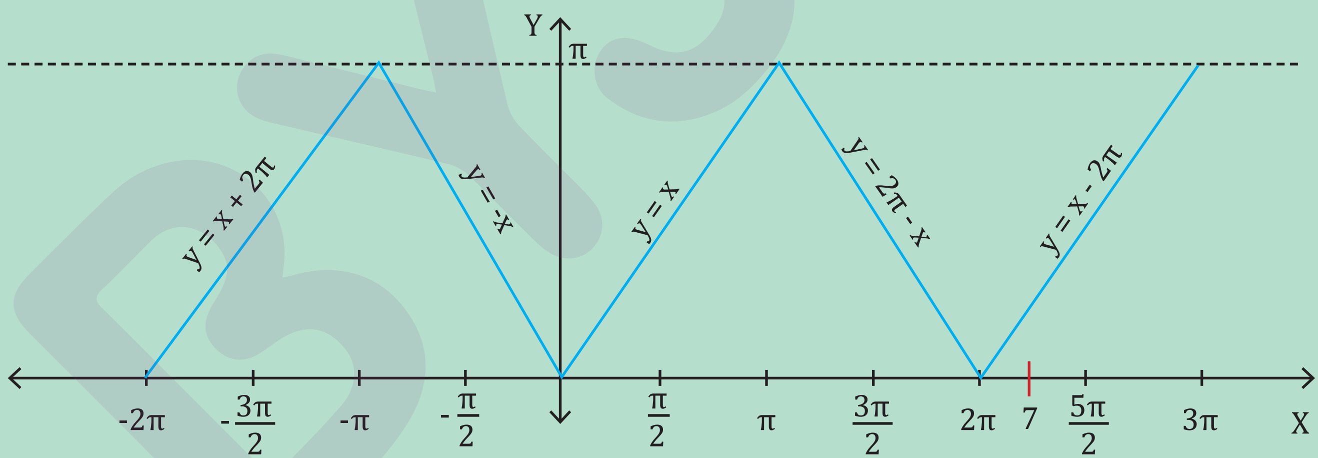
Step 1: 7 lies between 2π and $\frac{5\pi}{2}$.
From the graph, it is clear that
$y = \sin^{-1}(\sin(x))$ will follow
$y = x - 2\pi$



Step 2: $\sin^{-1}(\sin 7) = 7 - 2\pi$

4.

Step 1: 7 lies between 2π and 3π.
From the graph, it is clear that
$y = \cos^{-1}(\cos(x))$ will follow
$y = x - 2\pi$

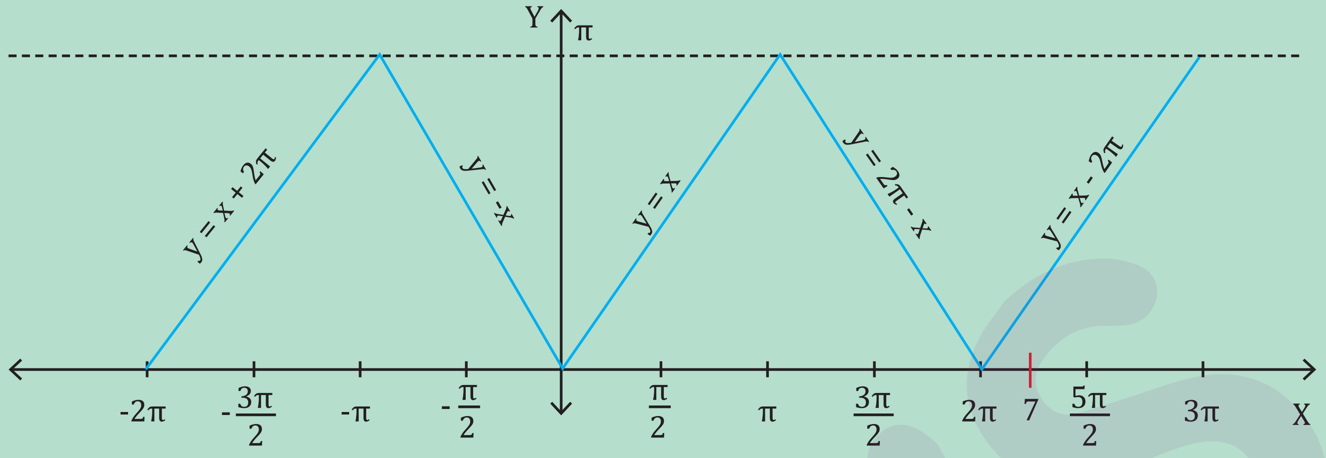


Step 2: $\cos^{-1}(\cos 7) = 7 - 2\pi$

Self-Assessment

Step 1: $\frac{7\pi}{6}$ lies between π and 2π.

From the graph, it is clear that $y = \cos^{-1}(\cos(x))$ will follow
$y = 2\pi - x$



Step 2: $\cos^{-1}\cos\left(\frac{7\pi}{6}\right) = 2\pi - \frac{7\pi}{6} = \frac{5\pi}{6}$

BYJU'S Classes

MATHEMATICS

INVERSE TRIGONOMETRIC FUNCTIONS

MORE ON PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

What you already know

- Function and classification of functions
- Basic graphs of functions
- Domain, range, and graph of inverse trigonometric functions

What you will learn

- Function of the form $f^{-1}(f(x))$
- Properties of inverse trigonometric function $f^{-1}\left(\frac{1}{x}\right)$

Properties of Inverse Function $f^{-1}(f(x))$ (cont.)

Let us consider function $y = \tan^{-1}(\tan x)$.
Given function is an inverse tangent function

$\therefore$ Range of the function is $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Also, x is an argument of the tangent function

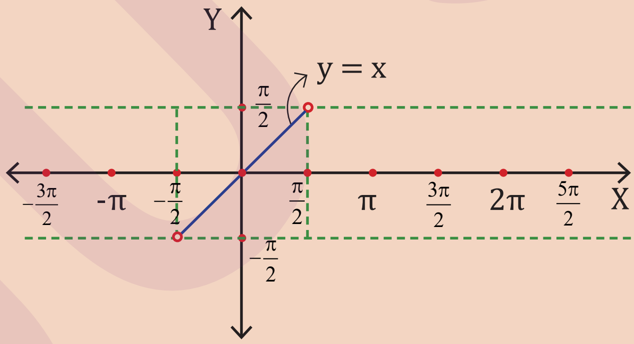
$\therefore$ Domain of the function is $x \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$

As the value of the function oscillates in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function is periodic and depends on the period of $\tan x$.
We know that the period of $\tan x$ is π .
$\therefore$ Period of $y = \tan^{-1}(\tan x)$ is also π
So, let's plot the graph of $y = \tan^{-1}(\tan x)$ for the length π.

$y = \tan^{-1}(\tan x)$
$\Rightarrow \tan y = \tan x$
$\Rightarrow y = n\pi + x \ldots$ (i)
$\Rightarrow$ Graph of $y = \tan^{-1}(\tan x)$ will be a straight line with slope 1.

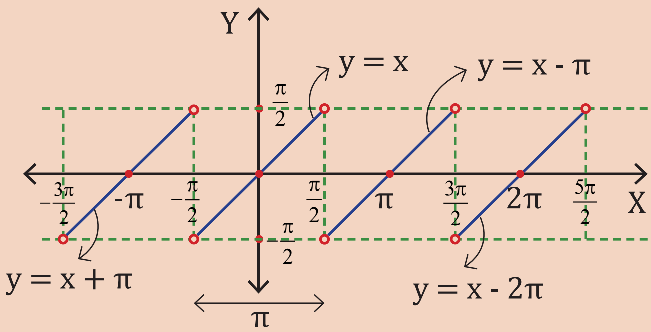
Now, by substituting $n = 0$, we get the relation between x and y for the first interval,
$y = x$

So, for the first interval, the function $y = \tan^{-1}(\tan x)$ can be plotted as shown in the figure above.

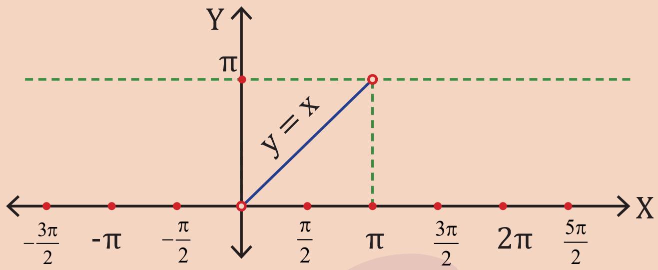


As the function is periodic with the period π, the slope of the graph is always 1.

Hence, for all the intervals graph of the function $y = \tan^{-1}(\tan x)$ can be plotted as shown in the figure below.



Let us consider the function $y = \cot^{-1}(\cot x)$.
Given function is an inverse cotangent function
∴ Range of the function is $y \in (0, \pi)$
Also, x is the argument of the tangent function
∴ Domain of the function is $x \in \mathbb{R} - \{n\pi\}; n \in \mathbb{Z}$
As the value of the function oscillates in the range $(0, \pi)$, the function is periodic and depends on the period of cot x.
We know that the period of cot x is π.
∴ Period of $y = \cot^{-1}(\cot x)$ is also π
So, let's plot the graph of $y = \cot^{-1}(\cot x)$ for the length π.



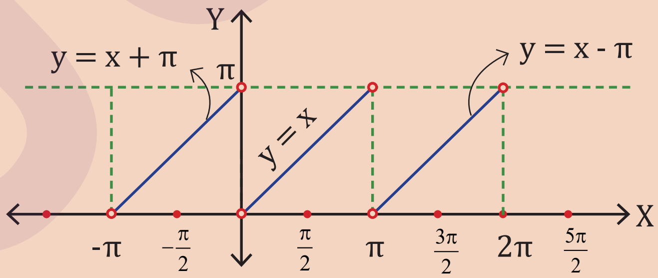
$y = \cot^{-1}(\cot x)$
$\Rightarrow \cot y = \cot x$
$\Rightarrow \tan x = \tan y$
$\Rightarrow y = n\pi + x$ . . . (i)
⇒ Graph of $y = \cot^{-1}(\cot x)$ will be a straight line with slope 1.

Now, by substituting $n = 0$, we get the relation between x and y for the first interval,
$y = x$

So, the first interval the function $y = \cot^{-1}(\cot x)$ can be plotted as shown in the figure above.

As the function is periodic with the period π, the slope of the graph is always 1.

Hence, all the interval graphs of the function $y = \cot^{-1}(\cot x)$ can be plotted as shown in the adjacent figure.

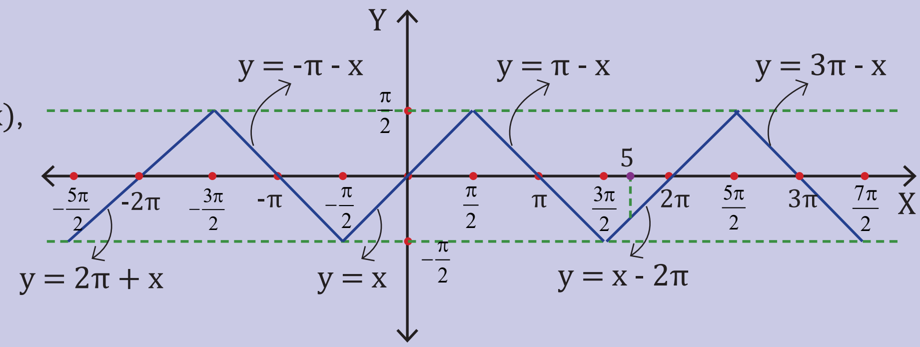


Find $\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$.

Solution

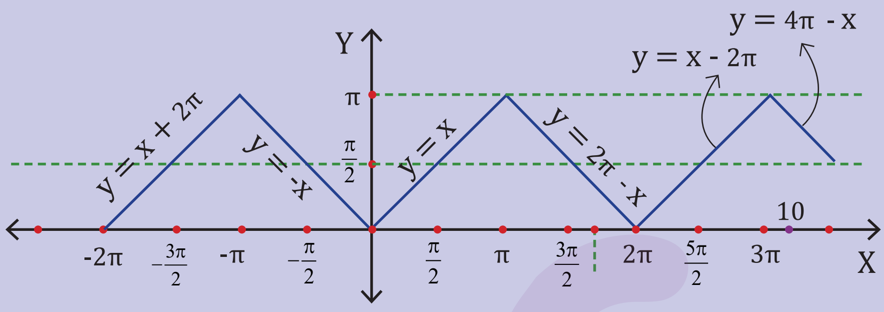
Step 1:

From the graph of $y = \sin^{-1}(\sin x)$, we get,
For $x = 5$, $y = x - 2\pi$
$\Rightarrow y = 5 - 2\pi$
$\Rightarrow \sin^{-1}(\sin 5) = 5 - 2\pi$ . . . . (i)



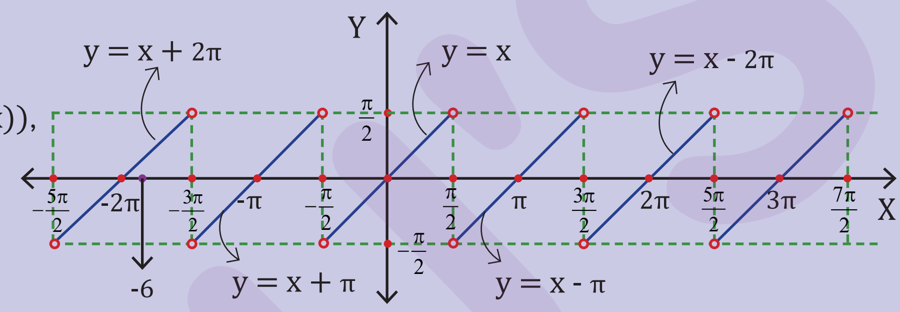
Step 2:

From the graph of $y = \cos^{-1}(\cos x)$,
we get,
For $x = 10$, $y = 4\pi - x$
$\Rightarrow y = 4\pi - 10$
$\Rightarrow \cos^{-1}(\cos 10) = 4\pi - 10 \ldots .$ (ii)



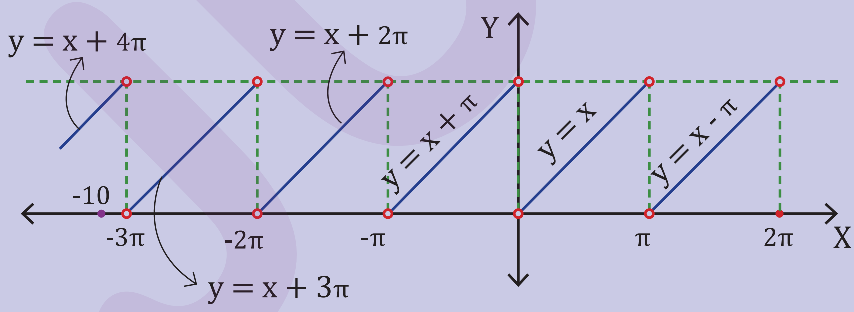
Step 3:

From the graph of $y = \tan^{-1}(\tan (x))$,
For $x = -6$, $y = x + 2\pi$, we get,
$\Rightarrow y = 2\pi - 6$
$\Rightarrow \tan^{-1}(\tan (-6)) = 2\pi - 6 \ldots .$ (iii)



Step 4:

From the graph of $y = \cot^{-1}(\cot (x))$,
For $x = -10$, $y = x + 4\pi$, we get,
$\Rightarrow y = 4\pi - 10$
$\Rightarrow \cot^{-1}(\cot (-10)) = 4\pi - 10 \ldots .$ (iv)



Step 5:

$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan (-6)) + \cot^{-1}(\cot (-10))$
$= 5 - 2\pi + 4\pi - 10 + 2\pi - 6 + 4\pi - 10$ (from (i), (ii), (iii), (iv))
$= 8\pi - 21$

Let us consider the function $y = \sec^{-1}(\sec x)$.
Given function is an inverse secant function
$\therefore$ Range of the function is $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Also, x is the argument of the secant function
$\therefore$ Domain of the function is $x \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$

We know that the period of $\sec x$ is 2π.
$\therefore$ Period of $y = \sec^{-1}(\sec x)$ is also 2π
So, let's plot the graph of $y = \sec^{-1}(\sec x)$ for the length 2π.

$\Rightarrow y = \sec^{-1}(\sec x)$
$\Rightarrow \sec y = \sec x$
$\Rightarrow \cos x = \cos y$
$\Rightarrow y = 2n\pi \pm x \ldots$ (i)

Hence, the graph of the function $y = \sec^{-1}(\sec x)$ will be a replica of the graph of the function $y = \cos^{-1}(\cos x)$, except that $y = \sec^{-1}(\sec x)$ is not defined for
$x = \left\{(2n+1)\frac{\pi}{2}\right\}$; $n \in \mathbb{Z}$

Now, let us consider the function
$y = \text{cosec}^{-1}(\text{cosec } x)$.
Given function is an inverse cosecant function

$\therefore$ Range of the function is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

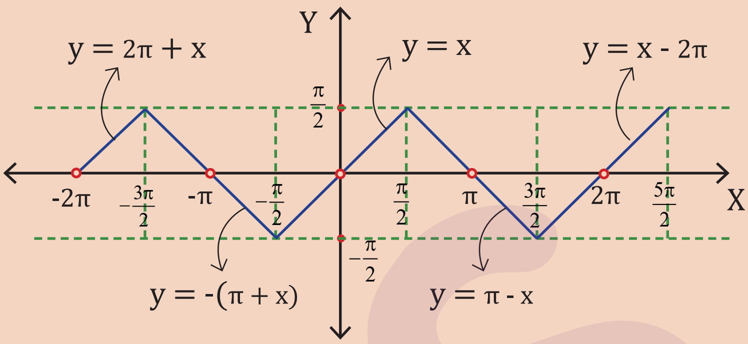
Also, x is the argument of the cosecant function

$\therefore$ Domain of the function is $x \in \mathbb{R} - \{n\pi\}$

We know that the period of cosec x is 2π

$\therefore$ Period of $y = \text{cosec}^{-1}(\text{cosec } x)$ is also 2π

So, let's plot the graph of $y = \text{cosec}^{-1}(\text{cosec } x)$ for the length 2π.



$\Rightarrow y = \text{cosec}^{-1}(\text{cosec } x)$

$\Rightarrow \text{cosec } y = \text{cosec } x$

$\Rightarrow \sin y = \sin x$

$\Rightarrow y = n\pi + (-1)^n x \ldots$ (i)

Hence, the graph of the function $y = \text{cosec}^{-1}(\text{cosec } x)$ will be a replica of the graph of the function $y = \sin^{-1}(\sin x)$, except $y = \text{cosec}^{-1}(\text{cosec } x)$ is not defined for $x = n\pi;\ n \in \mathbb{Z}$

Property 3: Function of the Form $f^{-1}(f(x))$ (For the Principal Values of x Only)

- $\sin^{-1}(\sin(x)) = x;\ \forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1}(\cos(x)) = x;\ \forall\ x \in [0, \pi]$
- $\tan^{-1}(\tan(x)) = x;\ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot^{-1}(\cot(x)) = x;\ \forall\ x \in (0, \pi)$
- $\text{cosec}^{-1}(\text{cosec } x) = x;\ \forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sec^{-1}(\sec(x)) = x;\ \forall\ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

If $x^2 + 2x + n > 10 + \sin^{-1}(\sin 9) + \tan^{-1}(\tan 9)$ for all real numbers x, then the possible value of n can be:

(a) 11 (b) 12 (c) 13 (d) 14

Solution

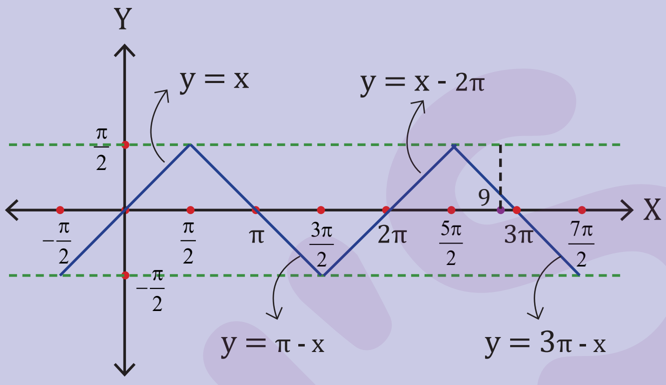
Step 1:

From the graph of $y = \sin^{-1}(\sin x)$, we get,

For $x = 9$, $y = 3\pi - x$

$\Rightarrow y = 3\pi - 9$

$\Rightarrow \sin^{-1}(\sin 9) = 3\pi - 9$ . . . . (i)



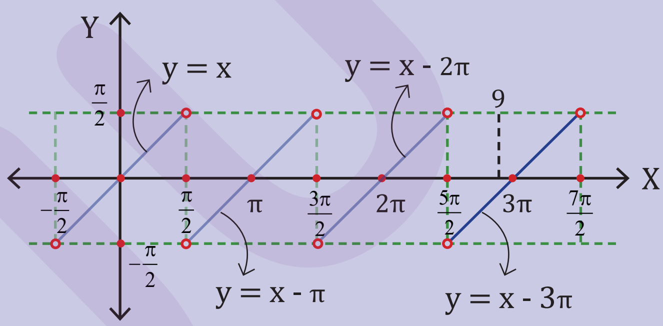
Step 2:

From the graph of $y = \tan^{-1}(\tan (x))$

For $x = 9$, $y = x - 3\pi$, we get,

$\Rightarrow y = 9 - 3\pi$

$\Rightarrow \tan^{-1}(\tan 9) = 9 - 3\pi$ . . . . (ii)



Step 3:

$x^2 + 2x + n > 10 + \sin^{-1}(\sin 9) + \tan^{-1}(\tan 9)$

$x^2 + 2x + n > 10 + 3\pi - 9 + 9 - 3\pi$ (from (i) and (ii))

$x^2 + 2x + n > 10$

$x^2 + 2x + (n - 10) > 0$

Here, $a > 0$

We know that for $a > 0$, $y = ax^2 + bx + c > 0 \ \forall\, x \in \mathbb{R}$ only if $D < 0$

$2^2 - 4(n - 10) < 0$

$4 - 4n + 40 < 0$

$n > 11$

∴ **Options (b), (c), and (d) are the correct answers.**

Property 4: Properties of Inverse Function $f^{-1}\left(\frac{1}{x}\right)$

- $\operatorname{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$; $|x| \geq 1$

Proof

Let $\operatorname{cosec}^{-1}(x) = \theta\ ;\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\operatorname{cosec}\theta = x$

$\Rightarrow \frac{1}{\sin\theta} = x$

$\Rightarrow \sin\theta = \left(\frac{1}{x}\right)$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \operatorname{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

As $\frac{1}{x} \neq 0$, so $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Similarly,

- $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right);\ |x| \geq 1$
- $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right); x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right); x < 0 \end{cases}$

Justification:

Range of $\cot^{-1}x$ is $(0, \pi)$, but the range of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, we can not equate L.H.S. and R.H.S. directly.

For $x > 0$, both L.H.S. and R.H.S. lie in the interval $\left(0, \frac{\pi}{2}\right)$.

For $x < 0$, L.H.S. lies in the interval $\left(\frac{\pi}{2}, \pi\right)$, and R.H.S. lies in the interval $\left(-\frac{\pi}{2}, 0\right)$. So, π is added to the R.H.S. to make the range of L.H.S. and R.H.S. equal

Note

$\sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ → Not identical, because L.H.S. $\sin^{-1}x$ is defined for $x = 0$, but R.H.S. $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ is not defined for $x = 0$.

Due to the same reason, $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$ → Not identical

Find the value of $\sec^{-1}\left(\sqrt{2}\right) + \cot^{-1}\left(-\sqrt{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution

Step 1:

Given,

$$\sec^{-1}\left(\sqrt{2}\right) + \cot^{-1}\left(-\sqrt{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \sec^{-1}\left(\sqrt{2}\right) + \pi + \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \left(\cot^{-1}(x) = \pi + \tan^{1}\left(\frac{1}{x}\right); x < 0\right)$$

$$= \sec^{-1}\left(\sqrt{2}\right) + \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad (\tan^{-1}(-x) = -\tan^{-1}(x))$$

$$= \sec^{-1}\left(\sqrt{2}\right) + \pi$$

Step 2:

Let $\sec^{-1}\left(\sqrt{2}\right) = \alpha$

$\Rightarrow \sec\alpha = \sqrt{2}$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \alpha = \frac{\pi}{4}$

$\therefore \sec^{-1}\left(\sqrt{2}\right) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

Property 5: Properties of some particular Inverse Functions

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}; x \in [-1, 1]$
- $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}; |x| \geq 1$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}; x \in \mathbb{R}$

Proof

Let $\sin^{-1}x = \theta \, ; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin\theta = x$

$\Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = x$

$\because -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} \geq -\theta \geq -\frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} \geq -\theta + \frac{\pi}{2} \geq -\frac{\pi}{2} + \frac{\pi}{2}$

$\Rightarrow \pi \geq \frac{\pi}{2} - \theta \geq 0$

Which is the range of the inverse cosine function

$\Rightarrow \frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Similarly, the other two results can also be proved.

Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Solution

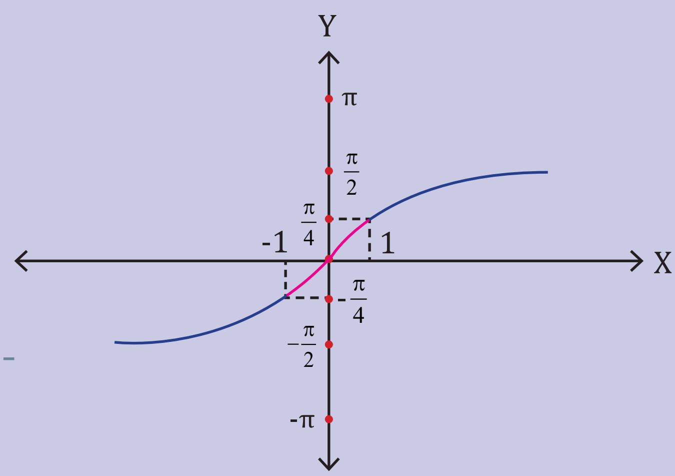
Step 1:

Domain of $\sin^{-1}x$ is $x \in [-1, 1]$

Domain of $\cos^{-1}x$ is $x \in [-1, 1]$

Domain of $\tan^{-1}x$ is $x \in \mathbb{R}$

$\therefore$ Domain of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is $x \in [-1, 1]$



Step 2:

We know that for $x \in [-1, 1]$,

$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$

From the graph of the inverse tangent function,

for $x \in [-1, 1]$, $-\frac{\pi}{4} \leq \tan^{-1}(x) \leq \frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} + \frac{\pi}{2} \leq \sin^{-1}(x) + \cos^{-1}(x) + \tan^{-1}(x) \leq \frac{\pi}{4} + \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{4} \leq f(x) \leq \frac{3\pi}{4}$

$\therefore$ Range of the function $f(x)$ is $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

The greatest and least value of $(\sin^{-1}(x))^2 + (\cos^{-1}(x))^2$ are _____ and _____, respectively.

(a) $\frac{5\pi^2}{4}, \frac{\pi^2}{8}$ (b) $\frac{\pi}{2}, -\frac{\pi}{2}$ (c) $\frac{\pi^2}{4}, -\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{4}, 0$

Solution

Step 1:

Given,

$y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$

$\Rightarrow y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2 + 2\sin^{-1}x\cos^{-1}x - 2\sin^{-1}x\cos^{-1}x$

$\Rightarrow y = (\sin^{-1}x + \cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x$

$$\Rightarrow y = \left(\frac{\pi}{2}\right)^2 - 2\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$\Rightarrow y = \frac{\pi^2}{4} - \pi\sin^{-1}x + 2\left(\sin^{-1}x\right)^2$$

$$\Rightarrow y = 2\left\{\left(\sin^{-1}x\right)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{8}\right\}$$

$$\Rightarrow y = 2\left[\left(\sin^{-1}x\right)^2 - 2\left(\sin^{-1}x\right)\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 - \left(\frac{\pi}{4}\right)^2 + \frac{\pi^2}{8}\right]$$

$$\Rightarrow y = 2\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$$

Step 2:

$$\therefore y_{min} = 2\left[0 + \frac{\pi^2}{16}\right] = \frac{\pi^2}{8}$$

and $$y_{max} = 2\left[\left(-\frac{\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$$

$$= 2\left(\frac{9\pi^2}{16} + \frac{\pi^2}{16}\right) = \frac{5\pi^2}{4}$$

$\therefore$ **Option (a) is the correct option.**

Concept Check

AIEEE 2007

1. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is:

(a) 1 (b) 3 (c) 4 (d) 5

2. Solve $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Summary Sheet

Key Takeaways

- $\sin^{-1}(\sin(x)) = x;\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1}(\cos(x)) = x;\ \forall\, x \in [0, \pi]$
- $\tan^{-1}(\tan(x)) = x;\ \forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot^{-1}(\cot(x)) = x;\ \forall\, x \in (0, \pi)$
- $\mathrm{cosec}^{-1}(\mathrm{cosec}\, x) = x;\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sec^{-1}(\sec(x)) = x;\ \forall\, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $\mathrm{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right);\ |x| \geq 1$
- $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right);\ |x| \geq 1$

- $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right); & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right); & x < 0 \end{cases}$
- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};\ x \in [-1, 1]$
- $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2};\ |x| \geq 1$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2};\ x \in \mathbb{R}$

Mind Map

Properties of inverse function

- Properties of inverse function $f^{-1}(x) + g^{-1}(x) = \frac{\pi}{2}$
- Properties of inverse function $f^{-1}\left(\frac{1}{x}\right)$
- Function of the form $f^{-1}(f(x))$

Self-Assessment

A value of x satisfying the equation $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$ is:

(a) $\frac{1}{2}$ (b) -1 (c) 0 (d) $-\frac{1}{2}$

Answers

Concept Check

1.

Step 1:

Given,

$$\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \operatorname{cosec}^{-1}\left(\frac{5}{4}\right)$$

$$\sin^{-1}\left(\frac{x}{5}\right) = \sec^{-1}\left(\frac{5}{4}\right) \qquad \left(\because \sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}, |x| \geq 1\right)$$

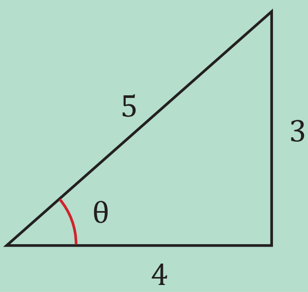
$$\sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right) \qquad \left(\because \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right); |x| \geq 1\right)$$

Step 2:

Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta, \theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \cos\theta = \frac{4}{5}$

Here, θ is an acute angle in the right angle triangle, as shown in the figure.



$\sin^{-1}\left(\frac{x}{5}\right) = \theta$

$\Rightarrow \frac{x}{5} = \sin\theta$

$\Rightarrow \frac{x}{5} = \frac{3}{5}$ (From the triangle)

$\Rightarrow x = 3$

∴ **Option (b) is the correct answer.**

2.

$-1 \leq x^2 - 2x + 1 \leq 1 \ldots$ (i)

$-1 \leq x^2 - x \leq 1 \ldots$ (ii)

Given, $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

$\Rightarrow x^2 - 2x + 1 = x^2 - x \quad [\sin^{-1}(A) + \cos^{-1}(A) = \frac{\pi}{2}]$

$\Rightarrow 2x - x = 1$

$\Rightarrow x = 1$

$x = 1$ satisfies both (i) and (ii)

Hence, $x = 1$ is the solution of $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Self Assessment

Step 1:

Given equation is $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1}x]$

Let $\cot^{-1}(1 + x) = a$

$\Rightarrow \cot a = 1 + x$

We know,

$\operatorname{cosec} a = \sqrt{1+\cot^2 a} = \sqrt{1+(1+x)^2} = \sqrt{x^2+2x+2}$

Also, $\sin a = \frac{1}{\operatorname{cosec} a}$

$\Rightarrow \sin a = \frac{1}{\sqrt{x^2+2x+2}}$

$\Rightarrow a = \sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)$

Let $\tan^{-1}x = b$

$\Rightarrow x = \tan b$

We know that $\sec b = \sqrt{1+\tan^2 b} = \sqrt{1+x^2}$

Also, $\cos b = \frac{1}{\sec b}$

$\Rightarrow \cos b = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow b = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

Step 2:

Given equation is $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$

$\Rightarrow \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right) = \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$

$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \sqrt{1+x^2} = \sqrt{x^2+2x+2}$

$\Rightarrow 1 + x^2 = x^2 + 2x + 2$ (Squaring on both sides)

$\Rightarrow 2x = -1$

$\Rightarrow x = -\frac{1}{2}$

BYJU'S Classes

MATHEMATICS

INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

INTERCONVERSION OF INVERSE TRIGONOMETRIC FUNCTIONS

What you already know

- Domain, range, and graph of ITF
- Properties and graphs of the inverse function $f^{-1}(f(x))$

What you will learn

- Interconversion of different ITF
- Sum of angles in terms of $\tan^{-1} x$
- Difference of angles in terms of $\tan^{-1} x$

Interconversion of Different ITF

Consider a △ ABC in which △ BAC = θ

We know,

$$\sin\theta = \frac{p}{h} \Rightarrow \theta = \sin^{-1}\left(\frac{p}{h}\right)$$

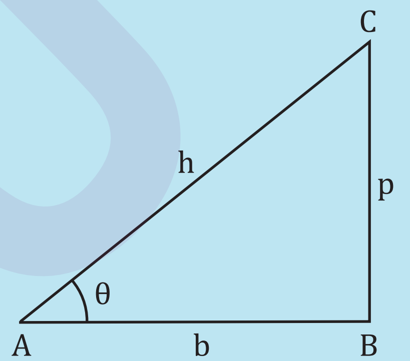
$$\cos\theta = \frac{b}{h} \Rightarrow \theta = \cos^{-1}\left(\frac{b}{h}\right)$$

$$\tan\theta = \frac{p}{b} \Rightarrow \theta = \tan^{-1}\left(\frac{p}{b}\right)$$

$$\operatorname{cosec}\theta = \frac{h}{p} \Rightarrow \theta = \operatorname{cosec}^{-1}\left(\frac{h}{p}\right)$$

$$\sec\theta = \frac{h}{b} \Rightarrow \theta = \sec^{-1}\left(\frac{h}{b}\right)$$

$$\cot\theta = \frac{b}{p} \Rightarrow \theta = \cot^{-1}\left(\frac{b}{p}\right)$$



For $x > 0$,

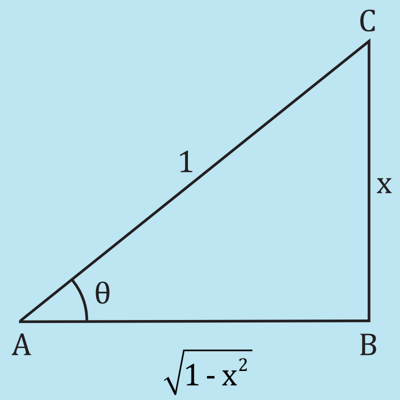
$$\sin\theta = \frac{x}{1} \Rightarrow \theta = \sin^{-1} x$$

$$\cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\operatorname{cosec}\theta = \frac{1}{x} \Rightarrow \theta = \sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \Rightarrow \theta = \sin^{-1} x = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$



$$\cot\theta = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = \sin^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Find the value of the expression $\sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right)$

Solution

Step 1:

We know that, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

$$\Rightarrow \sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right) = \sin\left(\frac{1}{2}\left(\pi - \cot^{-1}\left(\frac{3}{4}\right)\right)\right)$$

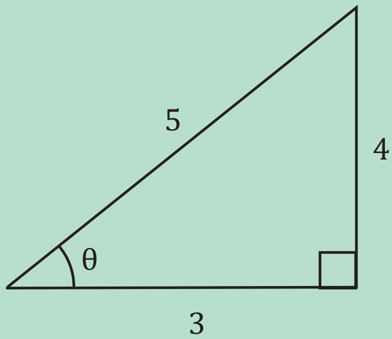
$$= \sin\left(\frac{\pi}{2} - \frac{\cot^{-1}\left(\frac{3}{4}\right)}{2}\right)$$

$$= \cos\left(\frac{\cot^{-1}\left(\frac{3}{4}\right)}{2}\right)$$

Step 2:

Let $\cot^{-1}\frac{3}{4} = \theta \Rightarrow \cot\theta = \frac{3}{4}$

$$\Rightarrow \cos\left(\frac{\cot^{-1}\left(\frac{3}{4}\right)}{2}\right) = \cos\left(\frac{\theta}{2}\right)$$



Since, $\cos\theta = \frac{3}{5}$

$$\Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{3}{5}$$

$$\Rightarrow \cos\frac{\theta}{2} = \pm\frac{2}{\sqrt{5}}$$

Also, $\cot^{-1}\frac{3}{4} = \theta \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ or $\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)$

So, $\cos\left(\frac{\theta}{2}\right) > 0 \Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{2}{\sqrt{5}}$

JEE MAIN 2008

Find the value of $\cot\left(\operatorname{cosec}^{-1}\left(\frac{5}{3}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$ (c) $\frac{3}{17}$ (d) $\frac{4}{17}$

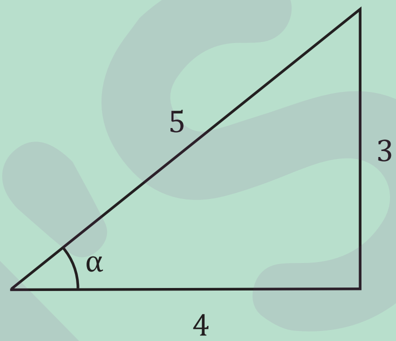
Solution

Step 1:

Let $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right)=\alpha, \alpha \in\left(0, \frac{\pi}{2}\right)$ and $\tan^{-1}\left(\frac{2}{3}\right)=\beta, \beta \in\left(0, \frac{\pi}{2}\right)$

$\Rightarrow \operatorname{cosec} \alpha=\frac{5}{3}$ and $\tan \beta=\frac{2}{3}$ or $\cot \beta=\frac{3}{2}$

Also, $\operatorname{cosec} \alpha=\frac{5}{3} \Rightarrow \cot \alpha=\frac{4}{3}$



Step 2:

$$\cot(\alpha+\beta)=\frac{\cot \alpha \cdot \cot \beta - 1}{\cot \alpha + \cot \beta}=\frac{\frac{4}{3} \cdot \frac{3}{2}-1}{\frac{4}{3}+\frac{3}{2}}=\frac{6}{17}$$

So, option (b) is the correct choice.

JEE ADVANCED 2019

If $0<x<1$, then $\sqrt{1+x^2}\left[\left\{x \cos\left(\cot^{-1} x\right)+\sin\left(\cot^{-1} x\right)\right\}^2-1\right]^{\frac{1}{2}}$ is equal to

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x (c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

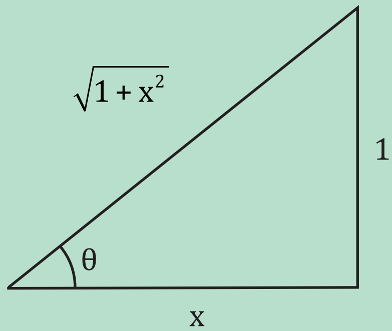
Solution

Step 1:

Let $\cot^{-1} x=\theta, 0<\theta<\frac{\pi}{2}$

$\Rightarrow \cot \theta=x$

$\therefore \sin \theta=\frac{1}{\sqrt{1+x^2}}$ and $\cos \theta=\frac{x}{\sqrt{1+x^2}}$



Step 2:

The expression reduces to $\sqrt{1+x^2}\left[\{x \cos \theta+\sin \theta\}^2-1\right]^{\frac{1}{2}}$

Substituting $\sin \theta=\frac{1}{\sqrt{1+x^2}}$ and $\cos \theta=\frac{x}{\sqrt{1+x^2}}$, we get,

$$\sqrt{1+x^2}\left[\left\{x \times \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left\{\frac{1+x^2}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[x^2\right]^{\frac{1}{2}}$$

$$= |x|\sqrt{1+x^2}$$

$$= x\sqrt{1+x^2} \quad (|x| = x, x > 0)$$

So, option (c) is the correct answer.

Sum of Angles in Terms of arctan x

$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x \ge 0, y \ge 0, xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x \ge 0, y \ge 0, xy > 1 \end{cases}$$

Proof

Let $\tan^{-1} x = A, A \in \left[0, \frac{\pi}{2}\right)$

Let $\tan^{-1} y = B, B \in \left[0, \frac{\pi}{2}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = \tan^{-1}(\tan(A+B)), 0 \le A+B < \pi$$

Also, from the graph of $\tan^{-1}(\tan\theta)$, we know that,

$\tan^{-1}(\tan\theta) = \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$\tan^{-1}(\tan\theta) = \theta - \pi, \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(\tan(A+B)) = \begin{cases} A+B, & 0 \le A+B < \frac{\pi}{2} \\ A+B-\pi, & \frac{\pi}{2} < A+B < \pi \end{cases}$$

Case 1: $A + B < \frac{\pi}{2}$

$\Rightarrow\ A < \frac{\pi}{2} - B$

$\Rightarrow \tan A < \tan\left(\frac{\pi}{2} - B\right)$

$\Rightarrow \tan A < \cot\ B$

$\Rightarrow \tan A \tan B < 1$ or $xy < 1$

That means, $A + B < \frac{\pi}{2}$ is equivalent to $xy < 1$

Similarly, $A + B > \frac{\pi}{2}$ is equivalent to $xy > 1$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \begin{cases} \tan^{-1}x + \tan^{-1}y & x \ge 0, y \ge 0, xy < 1 \\ \tan^{-1}x + \tan^{-1}y - \pi & x \ge 0, y \ge 0, xy > 1 \end{cases}$$

$$\therefore\ \tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & x \ge 0, y \ge 0, xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & x \ge, y \ge 0, xy > 1 \end{cases}$$

Case 2: $A + B = \frac{\pi}{2}$

$\Rightarrow\ A = \frac{\pi}{2} - B$

$\Rightarrow \tan A = \tan\left(\frac{\pi}{2} - B\right)$

$\Rightarrow \tan A = \cot\ B$

$\Rightarrow \tan A \tan B = 1$ or $xy = 1$

$\therefore$ When $xy = 1$ we have, $A + B = \frac{\pi}{2}$ or $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2}$

Prove that $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

Solution

Step 1:
Let's first evaluate $\tan^{-1}(2) + \tan^{-1}(3)$
Here, $2 \times 3 = 6 > 1$

Step 2:
Also, we know that

$$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad x \ge 0, y \ge 0, xy > 1$$

$$\Rightarrow \tan^{-1}(2) + \tan^{-1}(3) = \pi + \tan^{-1}\left(\frac{2+3}{1-2\cdot 3}\right) = \pi + \tan^{-1}(-1) = \pi - \tan^{-1}(1)$$

Step 3:

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \tan^{-1}(1) + \pi - \tan^{-1}(1)$$

$$= \pi$$

Addition formulae for sum of angles in terms of arctan x

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \begin{cases} \tan^{-1}\dfrac{(x+y+z-xyz)}{\left(1-\sum xy\right)} & x \geq 0, y \geq 0, z \geq 0, \sum xy < 1 \\ \pi + \tan^{-1}\dfrac{(x+y+z-xyz)}{\left(1-\sum xy\right)} & x \geq 0, y \geq 0, z \geq 0, \sum xy > 1 \end{cases}$$

Note

1. If $xy + yz + zx = 1$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$

2 If $x + y + z = xyz$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

If $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) = k \cdot \pi$, then find the value of k.

Solution

Step 1:

Let $\sin^{-1}\left(\frac{12}{13}\right) = \alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \sin\alpha = \left(\frac{12}{13}\right)$ or $\tan\alpha = \left(\frac{12}{5}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right) = \beta, \beta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \cos\beta = \left(\frac{4}{5}\right)$ or $\tan\beta = \left(\frac{3}{4}\right)$

Step 2:

$$\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

Let's first evaluate $\tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right)$

Here, $\frac{12}{5} \times \frac{3}{4} = \frac{36}{20} > 1$

Step 3:
Also, we know that,

$$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad x \geq 0, y \geq 0, xy > 1$$

$$\Rightarrow \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) = \pi + \tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}}\right) = \pi + \tan^{-1}\left(-\frac{63}{16}\right)$$

Step 4:

$$\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$= \pi + \tan^{-1}\left(-\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \pi - \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \pi$$

$$\Rightarrow k = 1$$

Difference of Angles in Terms of arctan x

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \quad x \geq 0, y \geq 0$$

The value of $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(-\frac{8}{19}\right)$ is:

Solution

Step 1:
Let's first evaluate

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$$

Here,

$$\frac{3}{4} \times \frac{3}{5} = \frac{9}{20} < 1$$

Step 2:

Also, we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad x \geq 0, y \geq 0, xy < 1$$

$$\Rightarrow \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\cdot\frac{3}{5}}\right) = \tan^{-1}\left(\frac{27}{11}\right)$$

Step 3:

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(-\frac{8}{19}\right) = \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right)$$

Now, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy} \quad x \geq 0, y \geq 0$

$$\Rightarrow \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right)$$

$$= \tan^{-1}\left(\frac{425}{425}\right) = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

JEE ADVANCED 2019

If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to?

Solution

Step 1:

We have, $\cos\alpha = \dfrac{3}{5}$

$\Rightarrow \tan\alpha = \dfrac{4}{3}$

Also, $\tan\beta = \dfrac{1}{3}$

Step 2:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\cdot\frac{1}{3}} = \frac{9}{13}$$

$$\therefore \alpha - \beta = \tan^{-1}\left(\frac{9}{13}\right)$$

Concept Check

1. Solve: $\cos\left(\tan^{-1}x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$; $x > 0$

2. Prove that, $\tan^{-1}(1) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$

Summary Sheet

Key Formulae

- **Interconversion of different ITF**

  For $x > 0$,

  $$\sin\theta = \frac{x}{1} \Rightarrow \theta = \sin^{-1}x$$

  $$\cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$$

  $$\tan\theta = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

  $$\operatorname{cosec}\theta = \frac{1}{x} \Rightarrow \theta = \sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

  $$\sec\theta = \frac{1}{\sqrt{1-x^2}} \Rightarrow \theta = \sin^{-1}x = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

  $$\cot\theta = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = \sin^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

- **Sum of Angles in terms of arctan x**

  $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x \ge 0, y \ge 0, xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x \ge 0, y \ge 0, xy > 1 \end{cases}$$

  $$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \begin{cases} \tan^{-1}\dfrac{(x+y+z-xyz)}{\left(1-\sum xy\right)} & x \ge 0, y \ge 0, z \ge 0, \sum xy < 1 \\ \pi + \tan^{-1}\dfrac{(x+y+z-xyz)}{\left(1-\sum xy\right)} & x \ge 0, y \ge 0, z \ge 0, \sum xy > 1 \end{cases}$$

- **Difference of angles in terms of arctan x**

  $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \qquad x \ge 0, y \ge 0$$

Mind Map

Inverse Trigonomteric Functions

- Interconversion of different ITF
- ITF formulae
  - Sum of angles in terms of $\tan^{-1} x$
  - Difference of angles in terms of $\tan^{-1} x$

Self-Assessment

Prove that $4\left(2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\frac{1}{7}\right)=\pi$

Answers

Concept Check

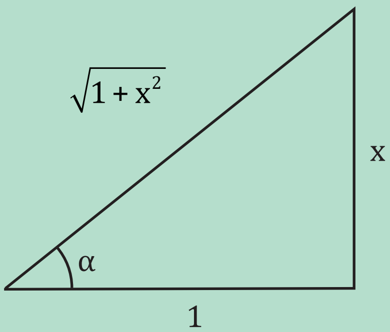
1.

Step 1:

Let $\tan^{-1}x=\alpha,\ 0<\alpha<\frac{\pi}{2}$

$\Rightarrow \tan\alpha = x$

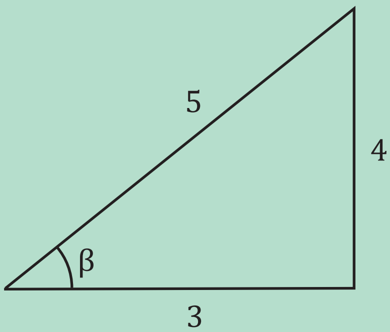
$\therefore \cos\alpha=\frac{1}{\sqrt{1+x^2}}$



Let $\cot^{-1}\left(\frac{3}{4}\right)=\beta,\ 0<\beta<\frac{\pi}{2}$

$\Rightarrow \cot\beta=\frac{3}{4}$

$\therefore \sin\beta=\frac{4}{5}$



Step 2:
Now, we have,
$\cos\alpha = \sin\beta$

$$\frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$$

Squaring both the sides and then solving, we get,

$$x = \pm\frac{3}{4}$$

However, it is given that $x > 0$

So, $x = \frac{3}{4}$ is the only solution.

2.

Step 1:

Let's first evaluate $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

Here,

$$\frac{1}{2}\times\frac{1}{3} = \frac{1}{6} < 1$$

Step 2:
Also, we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad x \geq 0, y \geq 0, xy < 1$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1}(1)$$

Step 3:

$$\tan^{-1}(1) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(1) + \tan^{-1}(1)$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

Self-Assessment

Step 1:

$$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

Let's first evaluate $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

Here,

$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9} < 1$

Step 2:

Also, we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad x \geq 0, y \geq 0, xy < 1$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

Step 3:

$$4\left(2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7}\right) = 4\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}\right)$$

$$= 4\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)\right)$$

$$= 4\left(\tan^{-1}(1)\right)$$

$$= 4 \times \frac{\pi}{4} = \pi$$

Hence proved

BYJU'S Classes

M A T H E M A T I C S

INVERSE TRIGONOMETRIC FUNCTIONS

SUM AND DIFFERENCE OF INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

What you already know

- Interconversion of different ITF
- Sum of angles in terms of $\tan^{-1} x$
- Difference of angles in terms of $\tan^{-1} x$

What you will learn

- Telescopic Series: Method of Difference
- Sum and Difference of angles in terms of $\sin^{-1} x$ and $\cos^{-1} x$

Telescopic Series: Method of Difference

Sum of series $\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^2} + \cdots$ n terms is

Solution

Step 1:

One can observe that the general term T_n is $\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$

Also, $\tan^{-1}\left(\frac{1}{1+n+n^2}\right) = \tan^{-1}\left(\frac{(n+1)-n}{1+(n)(n+1)}\right)$

Step 2:

For $x \geq 0, y \geq 0$, we have $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$

$\Rightarrow \tan^{-1}\left(\frac{(n+1)-(n)}{1+(n)(n+1)}\right) = \tan^{-1}(n+1) - \tan^{-1}(n)$

That means, $T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

Step 3:

Substituting $n = 1$, we get, $T_1 = \tan^{-1}(2) - \tan^{-1}(1)$

Substituting $n = 2$, we get, $T_2 = \tan^{-1}(3) - \tan^{-1}(2)$

Substituting $n = 3$, we get, $T_3 = \tan^{-1}(4) - \tan^{-1}(3)$

Similarly, substituting $n = n$, we get, $T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

Step 4:

$$\Rightarrow S_n = \sum T_n = \left(\tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2) + \cdots + \tan^{-1}(n+1) - \tan^{-1}(n)\right)$$

$$= \tan^{-1}(n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}(n+1) - \frac{\pi}{4}$$

JEE MAIN (JAN 2019)

The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is:

(a) $\frac{23}{22}$ (b) $\frac{22}{23}$ (c) $\frac{19}{21}$ (d) $\frac{21}{19}$

Solution

Step 1:

$$\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right) = \cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+2\sum_{p=1}^{n}p\right)\right)$$

$$= \cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+2\times\frac{n(n+1)}{2}\right)\right) \qquad \left(\sum_{p=1}^{n}p = \frac{n(n+1)}{2}\right)$$

$$= \cot\left(\sum_{n=1}^{19}\cot^{-1}\left(n^2+n+1\right)\right)$$

$$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{1}{\left(n^2+n+1\right)}\right)$$

Step 2:

$$\Rightarrow \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{1}{\left(n^2+n+1\right)}\right) = \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{(n+1)-(n)}{\left(1+(n)(n+1)\right)}\right)$$

$$= \cot\left(\sum_{n=1}^{19}\left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)\right) \left(\text{For } x \ge 0, y \ge 0, \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$$

$$= \cot\left(\tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2) + \cdots + \tan^{-1}(20) - \tan^{-1}(19)\right)$$

$$= \cot\left(\tan^{-1}(20) - \tan^{-1}(1)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{20-1}{1+20\times1}\right)\right) = \cot\left(\tan^{-1}\left(\frac{19}{21}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{21}{19}\right)\right) = \frac{21}{19}$$

So, option (d) is the correct answer.

Sum of series: $\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}\cdots n$ terms, where $x>0$ is

(a) $\tan^{-1}(x+n)-\tan^{-1}(x)$
(b) $\tan^{-1}(x+n-1)$
(c) $\tan^{-1}(x)$
(d) $\tan^{-1}(x+n+1)-\tan^{-1}(x)$

Solution

Step 1:
The given series can be re-written as

$$\tan^{-1}\frac{(x+1)-(x)}{1+(x)(x+1)}+\tan^{-1}\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}+\tan^{-1}\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}+\cdots n \text{ terms}$$

Step 2:

For $x \geq 0, y \geq 0, \tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y$

$$T_1=\tan^{-1}\frac{(x+1)-(x)}{1+(x+1)(x)}=\tan^{-1}(x+1)-\tan^{-1}(x)$$

$$T_2=\tan^{-1}\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}=\tan^{-1}(x+2)-\tan^{-1}(x+1)$$

$$T_3=\tan^{-1}\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}=\tan^{-1}(x+3)-\tan^{-1}(x+2)$$

$$T_4=\tan^{-1}\frac{(x+4)-(x+3)}{1+(x+4)(x+3)}=\tan^{-1}(x+4)-\tan^{-1}(x+3)$$

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.
.

$$T_n=\tan^{-1}\frac{(x+n)-(x+(n-1))}{1+(x+n)(x+(n-1))}=\tan^{-1}(x+n)-\tan^{-1}(x+(n-1))$$

$$\Rightarrow S_n=\sum T_n=\tan^{-1}(x+n)-\tan^{-1}(x)$$

So, option (a) is the correct answer.

Consider only the principal values of inverse function, the set
$A=\left\{x\geq 0:\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$

(a) Contains two elements
(b) Contains more than two elements
(c) It is an empty set
(d) It is a singleton set

Solution

Step 1:

Let $\tan^{-1}(2x) = \alpha \Rightarrow \tan\alpha = 2x$ and $\tan^{-1}(3x) = \beta \Rightarrow \tan\beta = 3x$

Given that $\alpha + \beta = \dfrac{\pi}{4}$

Step 2:

Taking tangent on both sides, we get,

$$\tan(\alpha+\beta) = \tan\frac{\pi}{4}$$

$$\Rightarrow 1 = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\Rightarrow \frac{2x+3x}{1-2x\cdot 3x} = \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow x = -1, \frac{1}{6}$$

Step 3:

But $x \ge 0$, so $x = -1$ is rejected.

$$\therefore x = \frac{1}{6}$$

So, option (d) is the correct answer.

Sum and Difference of Angles in terms of $\sin^{-1} x$ and $\cos^{-1} x$

Sum and Difference of Angles in terms of $\sin^{-1} x$

1. $$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \ge 0, y \ge 0, x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \ge 0, y \ge 0, x^2 + y^2 > 1 \end{cases}$$

2. $$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right); \ x \ge 0, y \ge 0$$

Sum and Difference of Angles in terms of $\cos^{-1} x$

1. $$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right); \ x \ge 0, y \ge 0$$

2. $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & x \ge 0, y \ge 0, x \le y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & x \ge 0, y \ge 0, x > y \end{cases}$$

JEE MAIN (APRIL 2019)

The value of $\sin^{-1}\left(\dfrac{12}{13}\right) - \sin^{-1}\left(\dfrac{3}{5}\right)$ is equal to

(a) $\dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{56}{65}\right)$

(b) $\pi - \sin^{-1}\left(\dfrac{63}{65}\right)$

(c) $\dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{9}{65}\right)$

(d) $\pi - \cos^{-1}\left(\dfrac{33}{65}\right)$

Solution

Step 1:

We know that, $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right); x \geq 0, y \geq 0$

$$\Rightarrow \sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{12}{13}\sqrt{1-\left(\frac{3}{5}\right)^2} - \frac{3}{5}\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\cdot\frac{4}{5} - \frac{3}{5}\cdot\frac{5}{13}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} - \frac{15}{65}\right)$$

$$= \sin^{-1}\left(\frac{33}{65}\right)$$

Step 2:

$$\Rightarrow \sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{33}{65}\right)$$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{33}{65}\right)$$

Step 3:

Also, $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}; x \geq 0$

$$\Rightarrow \sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{33}{65}\right)$$

$$= \frac{\pi}{2} - \sin^{-1}\sqrt{1-\left(\frac{33}{65}\right)^2}$$

$$= \frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$$

So, option (a) is the correct answer.

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution

Step 1:

Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

Let $\cos^{-1}x = \alpha,\ \cos^{-1}y = \beta,\ \cos^{-1}z = \gamma$

$\Rightarrow \cos\alpha = x, \cos\beta = y, \cos\gamma = z$

$\Rightarrow \alpha + \beta = \pi - \gamma$

Step 2:

Taking cosine on both sides, we get

$\cos(\alpha + \beta) = \cos(\pi - \gamma)$

$\Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos\gamma$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

Squaring both sides, we get,

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

Hence proved.

Concept Check

1. Find the sum of the series.

$$S = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{18}\right) + \tan^{-1}\left(\frac{1}{32}\right) + \cdots \infty$$

2. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \le x \le 1, -2 \le y \le 2, x \le \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to: **AIEEE 2005** **JEE MAIN (APRIL 2019)**

(a) $4\sin^2\alpha$ (b) $2\sin^2\alpha$

(c) $4\sin^2\alpha - 2x^2y^2$ (d) $4\cos^2\alpha + 2x^2y^2$

Summary Sheet

Key Formulae

Sum and Difference of Angles in terms of $\sin^{-1}x$

1. $$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \ge 0, y \ge 0, x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \ge 0, y \ge 0, x^2 + y^2 > 1 \end{cases}$$

2. $$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right); x \ge 0, y \ge 0$$

Sum and Difference of Angles in terms of $\cos^{-1}x$

1. $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right);\ x \geq 0, y \geq 0$

2. $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & x \geq 0, y \geq 0, x \leq y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & x \geq 0, y \geq 0, x > y \end{cases}$

Mind Map

Sum and difference of ITF

- Telescopic series: Method of difference
- Sum and difference of angles in terms of $\sin^{-1}x$ and $\cos^{-1}x$

Self-Assessment

Evaluate: $\sum_{n=1}^{100} \tan^{-1}\left(\frac{2n}{n^4 + n^2 + 2}\right)$

Answers

Concept Check

1.

Step 1:

We have, $T_n = \tan^{-1}\left(\frac{1}{2n^2}\right) = \tan^{-1}\left(\frac{2}{4n^2}\right)$

Step 2:

For $x \geq 0, y \geq 0$, we have $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$

$T_n = \tan^{-1}\left(\frac{2}{4n^2}\right)$

$= \tan^{-1}\left(\frac{2}{1 + (2n)^2 - 1}\right)$

$= \tan^{-1}\left(\frac{(2n + 1) - (2n - 1)}{1 + (2n + 1)(2n - 1)}\right) = \tan^{-1}(2n + 1) - \tan^{-1}(2n - 1)$

Step 3:

$T_1 = \tan^{-1}(3) - \tan^{-1}(1)$

$T_2 = \tan^{-1}(5) - \tan^{-1}(3)$

$T_3 = \tan^{-1}(7) - \tan^{-1}(5)$

.

.

.

$T_n = \tan^{-1}(2n + 1) - \tan^{-1}(2n - 1)$

Step 4:

$\Rightarrow S_n = \sum T_n = \tan^{-1}(2n + 1) - \tan^{-1}(1)$

$\Rightarrow S_\infty = \lim_{n \to \infty}(S_n) = \lim_{n \to \infty} \tan^{-1}(2n + 1) - \tan^{-1}(1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

2.

Step 1:

Let $\cos^{-1}x = A, \cos^{-1}\frac{y}{2} = B$

$\Rightarrow \cos A = x, \cos B = \frac{y}{2}$

$\Rightarrow A - B = \alpha$

Step 2:

Taking cosine on both the sides, we get,

$\cos(A - B) = \cos\alpha$

$\Rightarrow \cos A \cos B + \sin A \sin B = \cos\alpha$

$\Rightarrow x \cdot \frac{y}{2} + \sqrt{1 - x^2}\sqrt{1 - \frac{y^2}{4}} = \cos\alpha$

$\Rightarrow x \cdot \frac{y}{2} - \cos\alpha = -\sqrt{1 - x^2}\sqrt{1 - \frac{y^2}{4}}$

Squaring both sides, we get,

$\frac{x^2y^2}{4} - xy\cos\alpha + \cos^2\alpha = 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$

$\Rightarrow -4xy\cos\alpha + 4\cos^2\alpha = 4 - 4x^2 - y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4 - 4\cos^2\alpha = 4\sin^2\alpha$

So, option (a) is the correct answer.

Self-Assessment

Step 1:

$$\sum_{n=1}^{100} \tan^{-1}\left(\frac{2n}{n^4 + n^2 + 2}\right) = \sum_{n=1}^{100} \tan^{-1}\left(\frac{2n}{1 + n^4 + n^2 + 1}\right)$$

$$= \sum_{n=1}^{100} \tan^{-1}\left(\frac{\left(n^2 + n + 1\right) - \left(n^2 - n + 1\right)}{1 + \left(n^2 + n + 1\right)\left(n^2 - n + 1\right)}\right)$$

$$= \sum_{n=1}^{100} \left(\tan^{-1}\left(n^2 + n + 1\right) - \tan^{-1}\left(n^2 - n + 1\right)\right)$$

Step 2:

$$T_1 = \tan^{-1}(3) - \tan^{-1}(1)$$

$$T_2 = \tan^{-1}(7) - \tan^{-1}(3)$$

$$T_3 = \tan^{-1}(13) - \tan^{-1}(7)$$

$$T_4 = \tan^{-1}(21) - \tan^{-1}(13)$$

.

.

.

$$T_{100} = \tan^{-1}\left(100^2 + 100 + 1\right) - \tan^{-1}\left(100^2 - 100 + 1\right)$$

Step 3:

The required sum is $T_1 + T_2 + T_3 + T_4 + \cdots + T_{100}$

$$= \tan^{-1}\left(100^2 + 100 + 1\right) - \tan^{-1}(1)$$

$$= \tan^{-1}(10101) - \frac{\pi}{4}$$

MATHEMATICS

INVERSE TRIGONOMETRIC FUNCTIONS

MULTIPLE ANGLES OF INVERSE TRIGONOMETRIC FUNCTIONS

NOTES

What you already know

- Domain and range of inverse trigonometric functions
- Graphs of inverse trigonometric functions
- Graphs of $T^{-1}T$

What you will learn

- Multiple angles in terms of $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, and more
- Inequalities involving ITF

Multiple Angles in terms of $\sin^{-1}$

$$2\sin^{-1}x = \begin{cases} -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right); & -1 \le x < \frac{-1}{\sqrt{2}} \\ \sin^{-1}\left(2x\sqrt{1-x^2}\right); & \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right); & \frac{1}{\sqrt{2}} < x \le 1 \end{cases}$$

Proof

Step 1:

Let $\sin^{-1}x = \alpha, \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow x = \sin\alpha$

$\Rightarrow \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\alpha\sqrt{1-\sin^2\alpha}\right)$

$= \sin^{-1}\left(2\sin\alpha\left|\cos\alpha\right|\right)$

$= \sin^{-1}\left(2\sin\alpha\cos\alpha\right)\left(\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right)$

$= \sin^{-1}\left(\sin 2\alpha\right)$

Where $2\alpha \in [-\pi, \pi]$

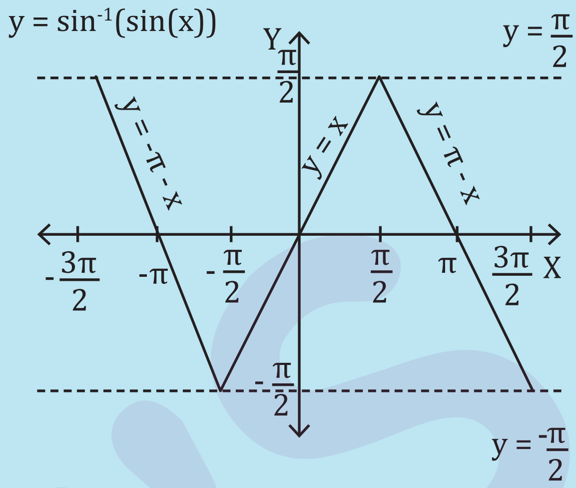
Step 2:

We have, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(\sin 2\alpha\right)$,

where $2\alpha \in [-\pi, \pi]$

Let us look at the graph of $\sin^{-1}(\sin x)$. Now, we can see that,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-\pi-2\alpha\ ;\ -\pi\le 2\alpha<\dfrac{-\pi}{2}\\[4pt] 2\alpha;\ \dfrac{-\pi}{2}\le 2\alpha\le\dfrac{\pi}{2}\\[4pt] \pi-2\alpha;\ \dfrac{\pi}{2}<2\alpha\le\pi\end{cases}$$



By substituting $\alpha=\sin^{-1}x$, we get,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-\pi-2\sin^{-1}x\ ;\ -\pi\le 2\sin^{-1}x<\dfrac{-\pi}{2}\\[4pt] 2\sin^{-1}x\ ;\dfrac{-\pi}{2}\le 2\sin^{-1}x\le\dfrac{\pi}{2}\\[4pt] \pi-2\sin^{-1}x\ ;\ \dfrac{\pi}{2}<2\sin^{-1}x\le\pi\end{cases}$$

This can also be written as follows :

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-\pi-2\sin^{-1}x\ ;\dfrac{-\pi}{2}\le\sin^{-1}x<\dfrac{-\pi}{4}\\[4pt] 2\sin^{-1}x\ ;\dfrac{-\pi}{4}\le\sin^{-1}x\le\dfrac{\pi}{4}\\[4pt] \pi-2\sin^{-1}x\ ;\ \dfrac{\pi}{4}<\sin^{-1}x\le\dfrac{\pi}{2}\end{cases}$$

Step 3:

The given piecewise function can be written as follows:

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-\pi-2\sin^{-1}x\ ;\ -1\le x<\dfrac{-1}{\sqrt{2}}\\[4pt] 2\sin^{-1}x\ ;\ \dfrac{-1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\[4pt] \pi-2\sin^{-1}x\ ;\ \dfrac{1}{\sqrt{2}}<x\le 1\end{cases}$$

Step 4:

$$2\sin^{-1}x=\begin{cases}-\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right)\ ;\ -1\le x<\dfrac{-1}{\sqrt{2}}\\[4pt] \sin^{-1}\left(2x\sqrt{1-x^2}\right)\ ;\ \dfrac{-1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\[4pt] \pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right)\ ;\ \dfrac{1}{\sqrt{2}}<x\le 1\end{cases}$$

AIEEE 2003

The trigonometric equation $\sin^{-1} x = 2\sin^{-1}\alpha$ has a solution for which of the following values of α ?

(a) All real values (b) $|\alpha| \leq \frac{1}{\sqrt{2}}$ (c) $|\alpha| \geq \frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}} < |\alpha| \leq \frac{1}{\sqrt{2}}$

Solution

Step 1:

We know that,

$\sin^{-1} x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow 2\sin^{-1}\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin^{-1}\alpha \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

Step 2:

$\frac{-\pi}{4} \leq \sin^{-1}\alpha \leq \frac{\pi}{4}$

$\Rightarrow \sin\left(\frac{-\pi}{4}\right) \leq \sin\left(\sin^{-1}\alpha\right) \leq \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \frac{-1}{\sqrt{2}} \leq \alpha \leq \frac{1}{\sqrt{2}}$

Therefore, $|\alpha| \leq \frac{1}{\sqrt{2}}$

So, option (b) is the correct answer.

$2\cos^{-1} x = 2\pi - \cos^{-1}(2x^2 - 1)$ is true for which values of x?

(a) $x \in [0, 1]$ (b) $x \in [-1, 0]$ (c) $x \in [-1, 1]$ (d) $x \in \left[\frac{1}{2}, 1\right]$

Solution

Step 1:

Let $\cos^{-1} x = \theta$, where $\theta \in [0, \pi]$ and $x \in [-1, 1]$

$\Rightarrow x = \cos\theta$

Step 2:

To solve $2\cos^{-1} x = 2\pi - \cos^{-1}\left(2x^2 - 1\right)$

It is similar to $\cos^{-1}\left(2x^2 - 1\right) = 2\pi - 2\cos^{-1} x = 2\pi - 2\theta$ ....(i)

By substituting $x = \cos\theta$ in $\cos^{-1}\left(2x^2 - 1\right)$, we get,

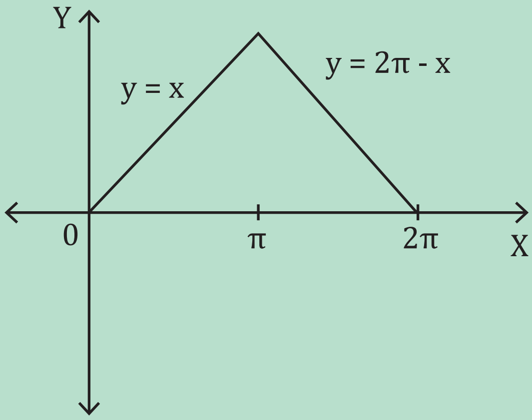
$\cos^{-1}\left(2x^2 - 1\right) = \cos^{-1}\left(2\cos^2\theta - 1\right)$

$= \cos^{-1}\left(\cos 2\theta\right)$

Step 3:

$\cos^{-1}\left(2x^2 - 1\right) = \cos^{-1}\left(\cos 2\theta\right)$, where $2\theta \in [0, 2\pi]$

Let us draw the graph of $\cos^{-1}(\cos x)$, where $x \in [0, 2\pi]$



We can see that for $2\theta \in [\pi, 2\pi]$, we get,

$\cos^{-1}\left(\cos 2\theta\right) = 2\pi - 2\theta$ which is similar to (i).

Step 4:

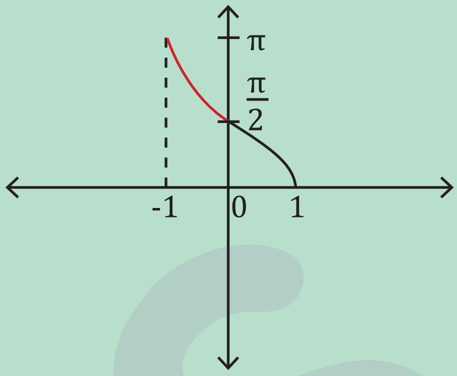
$\Rightarrow \pi \le 2\theta \le 2\pi$

$\Rightarrow \dfrac{\pi}{2} \le \theta \le \pi$

$\Rightarrow \dfrac{\pi}{2} \le \cos^{-1} x \le \pi$

Therefore, $x \in [-1, 0]$

So, option (b) is the correct answer.



If $f(x) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, $g(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, $h(x) = 2\tan^{-1}x$ and f, g, and h are identical functions, then x belongs to :

(a) $[0,1]$ (b) $[0,1)$ (c) $(-1,0]$ (d) $[-1,0)$

Solution

Step 1:

Let $\tan^{-1} x = \theta$, where $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and $x \in \mathbb{R}$.

$\Rightarrow x = \tan\theta$

Also, f, g, and h are identical functions.

$\Rightarrow f(x) = g(x) = h(x)$

Step 2:

Case 1: $h(x) = 2\tan^{-1} x = 2\theta, x \in \mathbb{R}$

Case 2:

$f(x) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

By substituting $x = \tan\theta$, we get,

$f(\theta) = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)$

$f(\theta) = \tan^{-1}(\tan 2\theta)$

If $\tan^{-1}(\tan 2\theta) = 2\theta$

$\Rightarrow 2\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ or $\theta \in \left(\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right)$

It means that $\tan^{-1}x \in \left(\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right)$ or $x \in (-1, 1)$.

Case 3:

$g(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

By substituting $x = \tan\theta$, we get,

$g(\theta) = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$g(\theta) = \cos^{-1}(\cos 2\theta)$

If $\cos^{-1}(\cos 2\theta) = 2\theta$

$\Rightarrow 2\theta \in [0, \pi)$ or $\theta \in \left[0, \dfrac{\pi}{2}\right)$

It means that $\tan^{-1}x \in \left[0, \dfrac{\pi}{2}\right)$ or $x \ge 0$.

Step 3:

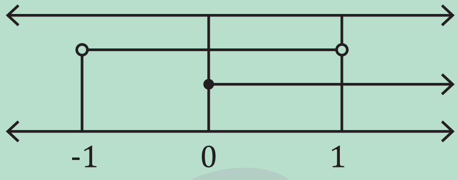
For f, g, and h to be identical, their domain must be identical.

So, the final domain is as follows:

$D_f \cap D_g \cap D_h$

$\Rightarrow x \in [0, 1)$

So, option (b) is the correct answer.



JEE ADVANCED 2015

If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then which of the following option(s) is / are correct?

(a) $\cos\beta > 0$ (b) $\sin\beta < 0$ (c) $\cos(\alpha+\beta) > 0$ (d) $\cos\alpha < 0$

Solution

We have to find the quadrants in which α and β lie.

Step 1:

Given, $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$

We know that, $\frac{6}{12} < \frac{6}{11} < \frac{6}{4\sqrt{3}}$

$\sin^{-1}x$ is an increasing function, so,

$\Rightarrow \sin^{-1}\frac{6}{12} < \sin^{-1}\frac{6}{11} < \sin^{-1}\frac{6}{4\sqrt{3}}$

$\Rightarrow \sin^{-1}\frac{1}{2} < \sin^{-1}\frac{6}{11} < \sin^{-1}\frac{\sqrt{3}}{2}$

$\Rightarrow \frac{\pi}{6} < \sin^{-1}\frac{6}{11} < \frac{\pi}{3}$

$\Rightarrow \frac{\pi}{2} < 3\sin^{-1}\frac{6}{11} < \pi$

Therefore, $\alpha \in \left(\frac{\pi}{2}, \pi\right)$

Step 2:

Given, $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$

We know that, $0 < \frac{4}{9} < \frac{4}{8}$

$\cos^{-1}x$ is a decreasing function, so,

$\Rightarrow \cos^{-1}0 > \cos^{-1}\frac{4}{9} > \cos^{-1}\frac{4}{8}$

$\Rightarrow \frac{\pi}{2} > \cos^{-1}\frac{4}{9} > \frac{\pi}{3}$

$\Rightarrow \frac{3\pi}{2} > 3\cos^{-1}\frac{4}{9} > \pi$

Therefore, $\beta \in \left(\pi, \frac{3\pi}{2}\right)$

Step 3:

Since α lies in the second quadrant, $\cos\alpha < 0$.

Since β lies in the third quadrant, $\cos\beta < 0$ and $\sin\beta < 0$

Also, $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\beta \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \alpha+\beta \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, which lies in the first or fourth quadrant.

Therefore, $\cos(\alpha+\beta) > 0$.

So, option (b),(c),(d) are the correct answers.

Inequalities Involving ITF

Solve: $\log_2(\tan^{-1}x) > 1$

Solution

Step 1:

$\log_2(\tan^{-1}x) > 1$

$\Rightarrow \tan^{-1}(x) > 2$

Step 2:

We know that, $\tan^{-1} x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

It means that $\tan^{-1}(x) > 2$ is not possible.

Therefore, $x \in \Phi$

JEE MAIN 2019

All the values of x satisfying the inequality $(\cot^{-1} x)^2 - 7\cot^{-1} x + 10 > 0$ lie in which of the following intervals?

(a) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$ (b) $(\cot 5, \cot 4)$

(c) $(-\infty, \cot 5) \cup (\cot 2, \infty)$ (d) $(\cot 2, \infty)$

Solution

Step 1:

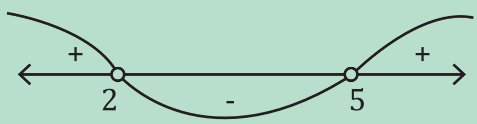
Let $\cot^{-1}x = t$, where $t \in (0, \pi)$

The inequality becomes,

$t^2 - 7t + 10 > 0$

$(t-5)(t-2) > 0$

$t \in (-\infty, 2) \cup (5, \infty)$



Step 2:

However, $t \in (0, \pi)$

By taking the intersection, we get,

$t \in (0, 2)$

$\cot^{-1}x \in (0, 2)$

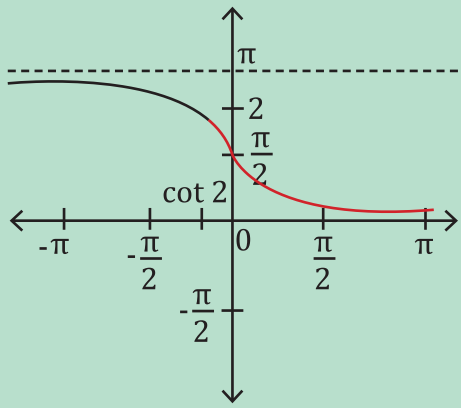
Step 3:

Let us see the graph of $\cot^{-1} x$.

For $\cot^{-1}x \in (0, 2)$, we get,

$x \in (\cot 2, \infty)$

So, option (d) is the correct answer.



Concept Check

1. If $\tan^{-1} y = \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$; $|x| < \dfrac{1}{\sqrt{3}}$, then what is the value of y? JEE MAIN 2015

(a) $\dfrac{3x+x^3}{1-3x^2}$ (b) $\dfrac{3x-x^3}{1+3x^2}$ (c) $\dfrac{3x+x^3}{1+3x^2}$ (d) $\dfrac{3x-x^3}{1-3x^2}$

2. Solve : $\sin^{-1}x > \dfrac{\pi}{6}$

3. For how many integral values of x, is the inequality $\cos^{-1}x > \cos^{-1}x^2$ true?

Summary Sheet

Key formulae

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -\pi - 2\sin^{-1}x\,; & -1 \le x < \dfrac{-1}{\sqrt{2}} \\ 2\sin^{-1}x\,; & \dfrac{-1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x\,; & \dfrac{1}{\sqrt{2}} < x \le 1 \end{cases}$$

Mind Map

Graph of Inverse Trigonomteric Ratios

Inequalities involving ITF

Graph of $f^{-1}(f(x))$

Multiple angles in terms of $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$

Self-Assessment

Solve the following inequality: $(\sec^{-1}x)^2 - 6\sec^{-1}x + 8 > 0$

Answers

Concept Check

1.

Step 1:

Let $\tan^{-1}x = \theta$, where $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $x \in \mathbb{R}$.

$\Rightarrow x = \tan\theta$

However, in the question, it is given that $|x| < \frac{1}{\sqrt{3}}$.

It means that $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}}$

Therefore, $\theta \in \left(\frac{-\pi}{6}, \frac{\pi}{6}\right)$ ....(i)

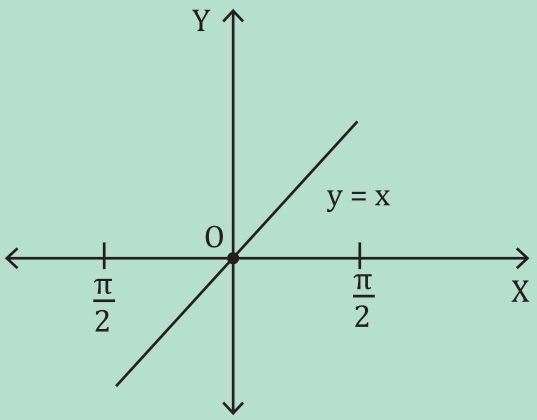
Step 2:

By substituting $x = \tan\theta$, we get

$\tan^{-1} y = \theta + \tan^{-1}\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right)$

$= \theta + \tan^{-1}(\tan 2\theta)$, where $2\theta \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ (from (i))

Let us examine the graph of $\tan^{-1}(\tan x)$ for $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$



For $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, $\tan^{-1}(\tan x) = x$

$\Rightarrow \tan^{-1} y = \theta + \tan^{-1}(\tan 2\theta) = \theta + 2\theta = 3\theta$

Step 3:

$$y = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3x - x^3}{1 - 3x^2}$$

Therefore, option (d) is the correct answer.

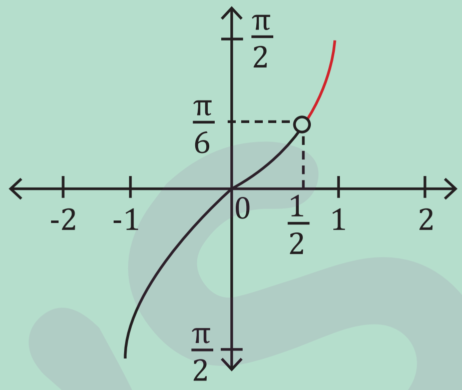
2.

Step 1:

From the graph,

We can see that $\sin^{-1}x$ is always less than or equal to $\frac{\pi}{2}$

So, $\frac{\pi}{2} \geq \sin^{-1}x > \frac{\pi}{6}$



Step 2:

Since $\sin^{-1}x$ is an increasing function, the given inequality can be modified as follows:

$$\sin\left(\frac{\pi}{2}\right) \geq x > \sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow 1 \geq x > \frac{1}{2}$$

Therefore, $x \in \left(\frac{1}{2}, 1\right]$

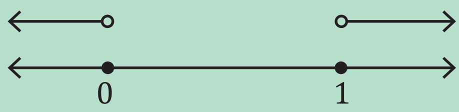
3.

Step 1:

Since $\cos^{-1}x$ is a decreasing function, the inequality $\cos^{-1}x > \cos^{-1}x^2$ can be reduced to $x^2 > x$

$\Rightarrow (x)(x - 1) > 0$

$\Rightarrow x \in (-\infty, 0) \cup (1, \infty)$



Step 2:

However, for $\cos^{-1}x$, $\cos^{-1}x^2$ is defined for $x \in [-1, 1]$

Step 3:

By taking the intersection, we get, $x \in [-1, 0)$

Therefore, the number of integral value possible is only 1.

Self-Assessment

$(\sec^{-1}x)^2 - 6\sec^{-1}x + 8 > 0$

Step 1:

Let $\sec^{-1}x = t$

The inequality becomes,

$t^2 - 6t + 8 > 0$

$(t - 4)(t - 2) > 0$

$t \in (-\infty, 2) \cup (4, \infty)$

$\Rightarrow \sec^{-1}x < 2$ or $\sec^{-1}x > 4$

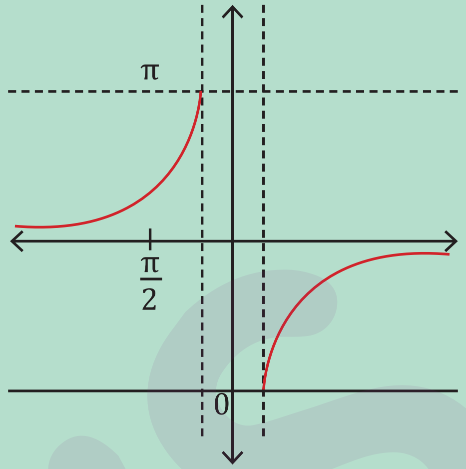
Step 2:

From the graph for $\sec^{-1}x$, we can see that,

$$\sec^{-1}x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

It means that $\sec^{-1}x > 4$ is not possible.

So, the inequality that needs to be solved is $\sec^{-1}x < 2$.



Step 3:

$\sec^{-1}x < 2$ is represented by the shaded region in the graph.

$\Rightarrow x \in (-\infty, \sec 2) \cup (1, \infty)$