BYJU'S Classes

Linear Programming

Introduction to Linear Programming Problems

Road Map

Illustration on Un-Bounded feasible solution

Illustration on Bounded feasible solution

Linear Programming Problem

Linear Programming Problem

Definition :

It is a process of finding the optimal value (maximum or minimum) of a linear function (objective function) of some variables, subject to linear constraints (equalities / inequalities).

Linear programming is a technique in which a linear function is maximized or minimized subject to various constraints.

This technique is useful for quantitative decisions in business planning , industrial engineering etc.

Method to solving Linear Programming Problem

Method of solving linear programming problem is referred as Corner Point Method. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m , respectively denote the largest and smallest values of these points.

3. When the feasible region is bounded , M and m are the maximum and minimum values of Z.

4. In case, the feasible region is unbounded ,

4. (a) M is the maximum value of Z, if the open half plane determined by

$ax + by > M$ has no point in common with the feasible region.

Otherwise, Z has no maximum value

4. (b) Similarly, m is the minimum value of Z, if the open half plane determined by

$ax + by < M$ has no point in common with the feasible region.

Otherwise, Z has no minimum value.

Let's try to understand these points using some illustrations

Question:

Solve the linear programming problem graphically :

Maximize $Z = 4x + y$

Subject to constraints : $x \geq 0 , y \geq 0$

$x + y \leq 50$

$3x + y \leq 90$

Solution:

Maximize $Z = 4x + y$; where Z is the Objective function

Subject to constraints : $x \geq 0 , y \geq 0$

$x + y \leq 50$

$3x + y \leq 90$

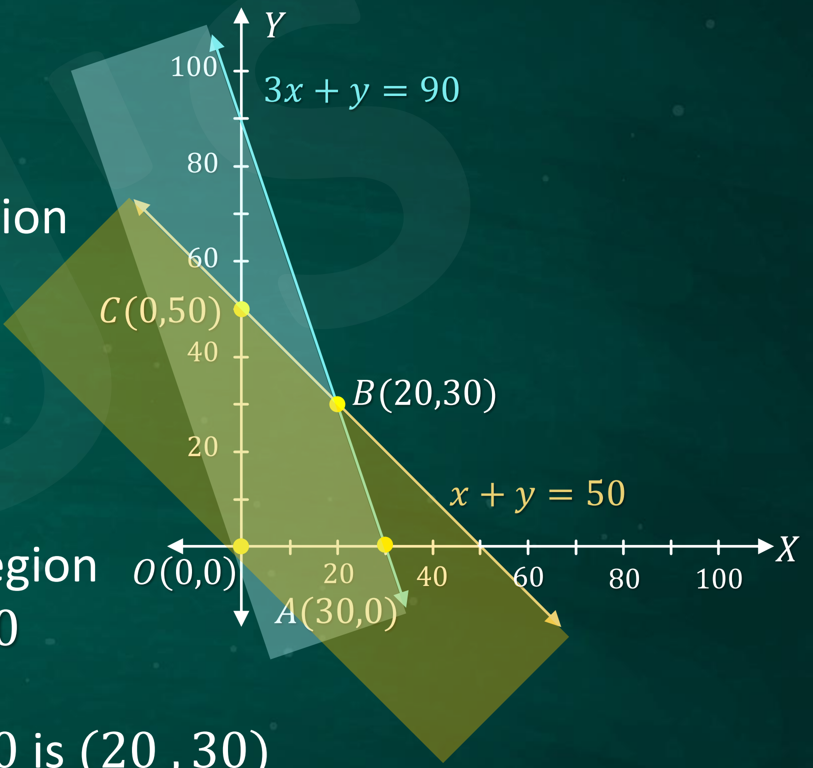
$x + y = 50$ is the line passing through the $(50, 0)$ and $(0, 50)$

Solution set for $x \geq 0, y \geq 0$ is first quadrant including co-ordinate axis

$(0, 0)$ satisfy $x + y \leq 50$ so solution set is the region below the line $x + y = 50$ including $x + y = 50$

$3x + y = 90$ is the line passing through the $(30, 0)$ and $(0, 90)$

$(0, 0)$ satisfy $3x + y \leq 90$ so solution set is the region below the line $3x + y = 90$ including $3x + y = 90$



Point of intersection of $3x + y = 90$ & $x + y = 50$ is $(20, 30)$

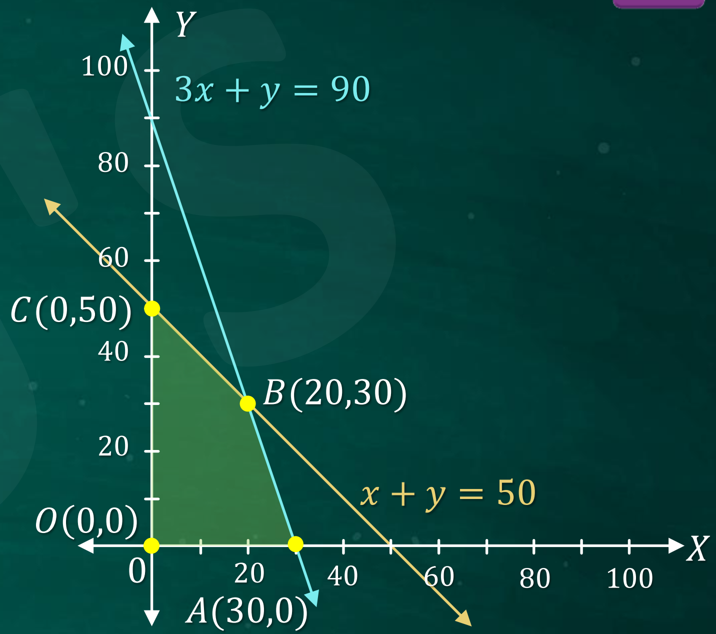
Solution set of all the constraints is shown in figure:

Shaded region is known as Feasible region

Points inside feasible region is Feasible solution

Points outside feasible region is Infeasible solution

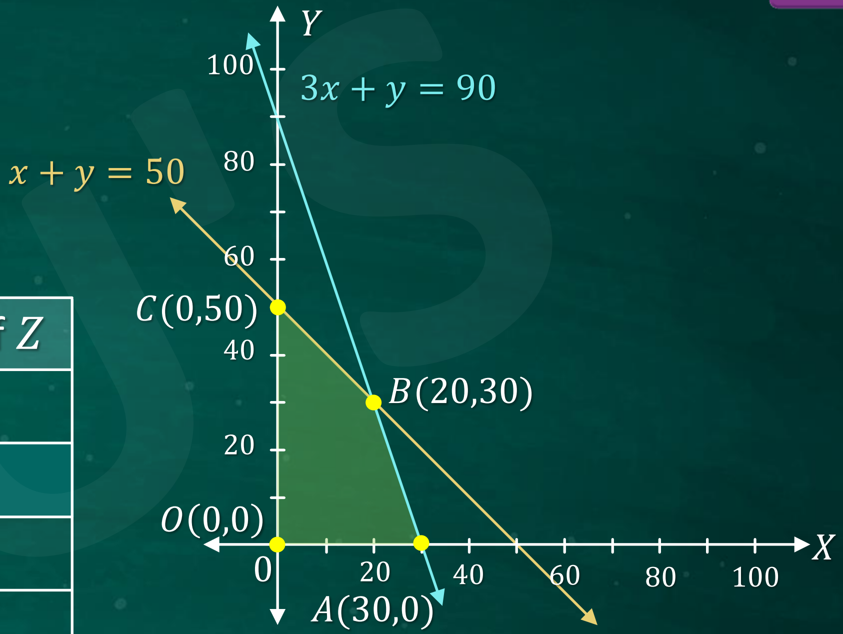
Points in the feasible region that gives the optimal value of Z : Optimal (feasible) solution



Here feasible region is bounded so optimal values occur at its corner points :

From graph, corner points are :
$(0,0)$, $(30,0)$, $(20,30)$, $(0,50)$

Vertex of feasible region	Corresponding value of Z
$O(0,0)$	0
$A(30,0)$	120 (Max)
$B(20,30)$	110
$C(0,50)$	50



Objective function Z is maximum at $A(30,0)$

Hence, optimal solution is $x = 30$, $y = 0$

Question:

Solve the linear programming problem graphically :

Minimize $Z = 200x + 500y$

Subject to constraints : $x \geq 0\,, y \geq 0$

$$x + 2y \geq 10$$
$$3x + 4y \leq 24$$

Solution:

Minimize $Z = 200x + 500y$; where Z is the Objective function

Subject to constraints : $x \geq 0\,, y \geq 0$

$$x + 2y \geq 10$$
$$3x + 4y \leq 24$$

$x + 2y = 10$ is the line passing through $(10, 0)$ and $(0, 5)$

Solution set for $x \geq 0, y \geq 0$ is first quadrant including co-ordinate axis

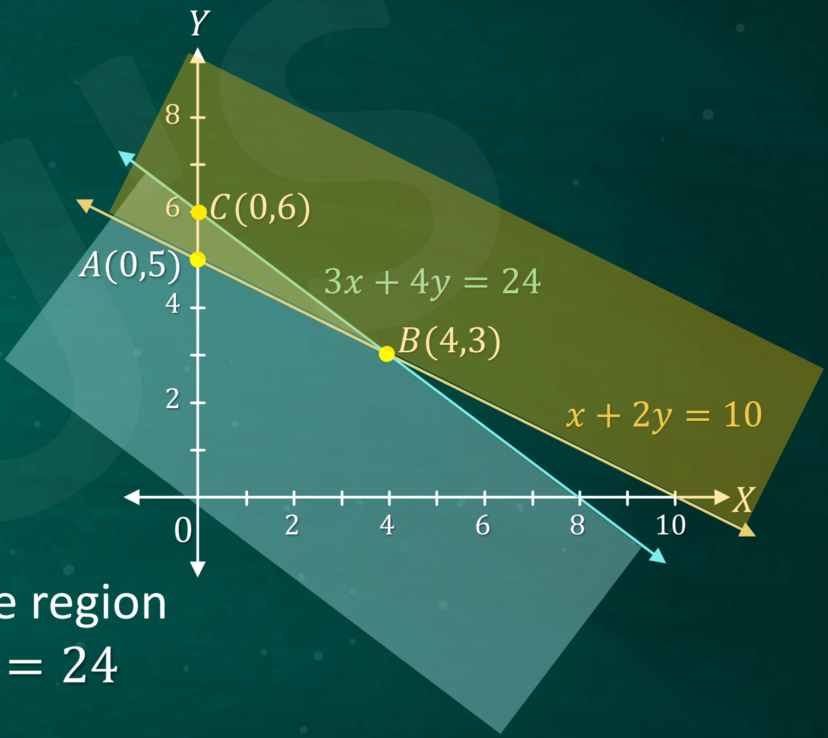
$(0, 0)$ does not satisfy $x + 2y \geq 10$ so solution set is the region above the line $x + 2y = 10$ including $x + 2y = 10$

$3x + 4y \leq 24$ is the line passing through $(8, 0)$ and $(0, 6)$

$(0, 0)$ satisfy $3x + 4y \leq 24$ so solution set is the region below the line $3x + 4y = 24$ including $3x + 4y = 24$

Point of intersection of $x + 2y = 10$ & $3x + 4y = 24$ is $(4, 3)$

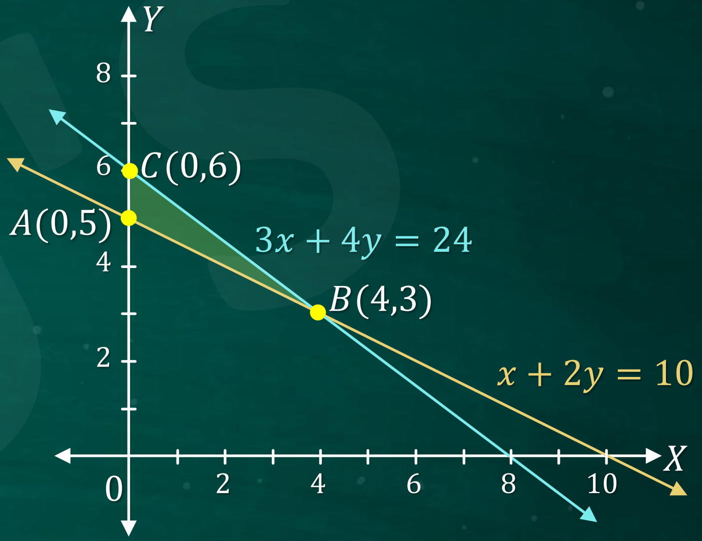
Solution set of all the constraints is shown in figure:



Here feasible region is bounded so optimal values occur at its corner points :

From graph, corner points are :
$(0,5)$, $(4,3)$, $(0,6)$

Vertex of feasible region	Corresponding value of Z
$A(0,5)$	2500
$B(4,3)$	2300 (Min)
$C(0,6)$	3000



Objective function Z is minimum at $A(4,3)$.

Hence, optimal solution is $x = 4$, $y = 3$

Question:

Solve the linear programming problem graphically :

Minimize $Z = 50x + 70y$

Subject to constraints : $x \geq 0 , y \geq 0$

$2x + y \geq 8$

$x + 2y \geq 10$

Solution:

Minimize $Z = 50x + 70y$; where Z is the Objective function

Subject to constraints : $x \geq 0 , y \geq 0$

$2x + y \geq 8$

$x + 2y \geq 10$

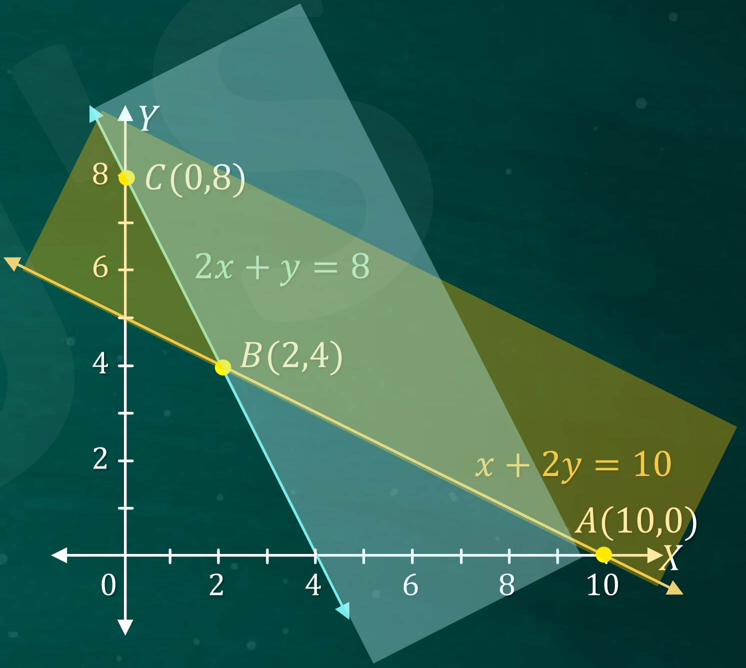
$2x + y = 8$ is the line passing through $(4, 0)$ and $(0, 8)$

Solution set for $x \geq 0, y \geq 0$ is first quadrant including co-ordinate axis

$(0, 0)$ does not satisfy $2x + y \geq 8$ so solution set is the region above the line $2x + y = 8$ including $2x + y = 8$

$x + 2y \geq 10$ is the line passing through $(10, 0)$ and $(0, 5)$

$(0, 0)$ does not satisfy $x + 2y \geq 10$ so solution set is the region above the line $x + 2y = 10$ including $x + 2y = 10$



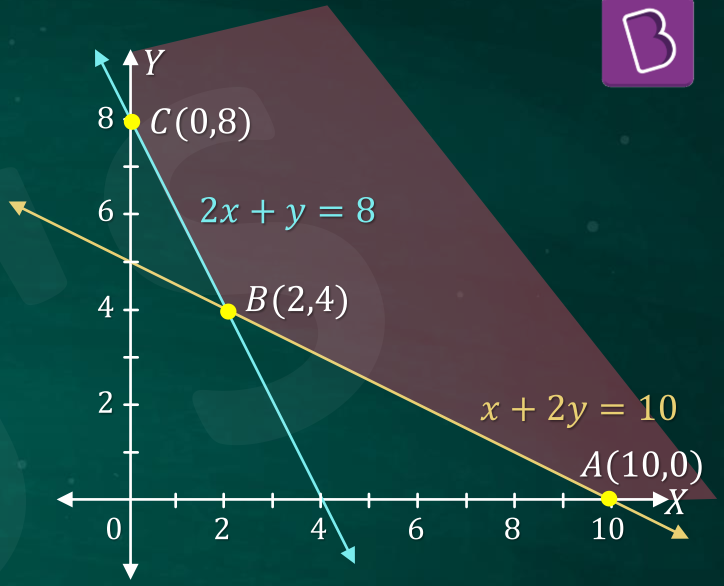
Point of intersection of $2x + y = 8$ & $x + 2y = 10$ is $(2, 4)$

Solution set of all the constraints is shown in figure:

Here feasible region is unbounded.

From graph , corner points are :

$(10, 0), (2,4), (0,8)$



Corner point	Corresponding value of Z
$A(10,0)$	500
$B(2,4)$	380
$C(0,8)$	560

Since, the region is unbounded, it's not necessary that minimum value of Z is 380.

Now, we will consider, the region $50x + 70y < 380$

$(0, 0)$ satisfy $50x + 70y < 380$ so solution set is towards the origin excluding $50x + 70y = 380$

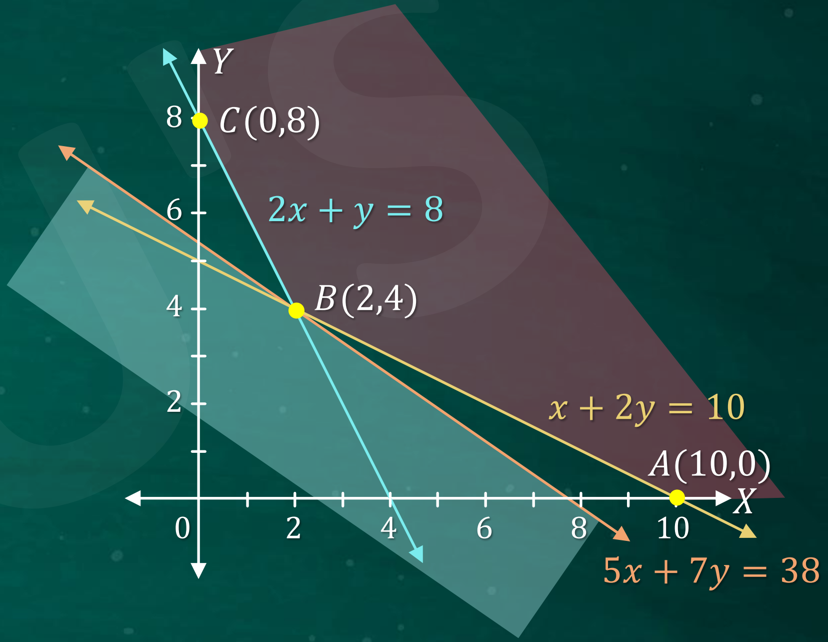
Since no point is common

So , no such value of (x, y) exist for which value of objective function Z is less than 380

At $B(2, 4)$ objective function $Z = 380$

So, minimum value of $\boxed{Z = 380}$

Hence , optimal solution is $x = 2, y = 4$



Question:

Solve the linear programming problem graphically :

Maximize $Z = 3x + 9y$

Subject to constraints : $x \geq 0\,, y \geq 0$

$$x \leq y$$

$$x + y \geq 10$$

$$x + 3y \leq 60$$

Solution:

Maximize $Z = 3x + 9y$; where Z is the Objective function

Subject to constraints : $x \geq 0\,, y \geq 0$

$$x \leq y$$

$$x + y \geq 10$$

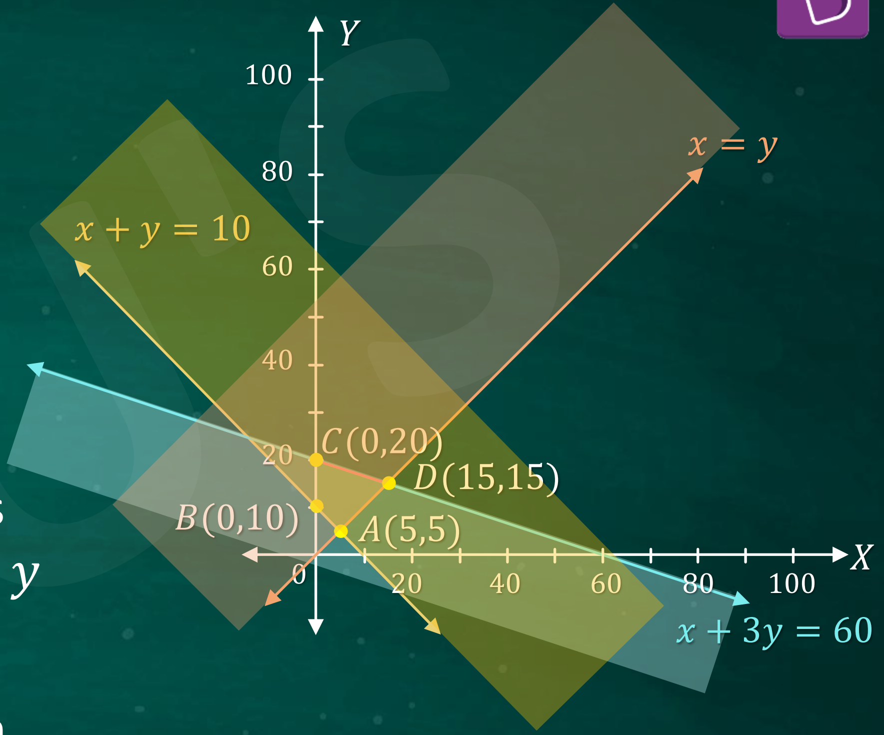
$$x + 3y \leq 60$$

$x + y = 10$ is the line passing through $(10\,,0)$ and $(0, 10)$

Solution set for $x \geq 0\,, y \geq 0$ is first quadrant including co-ordinate axis

$(1\,,0)$ does not satisfy $x \leq y$ so solution set is the region above the line $x = y$ including $x = y$

$(0\,,0)$ does not satisfy $x + y \geq 10$ so solution set is the region above the line $x + y = 10$ including $x + y = 10$

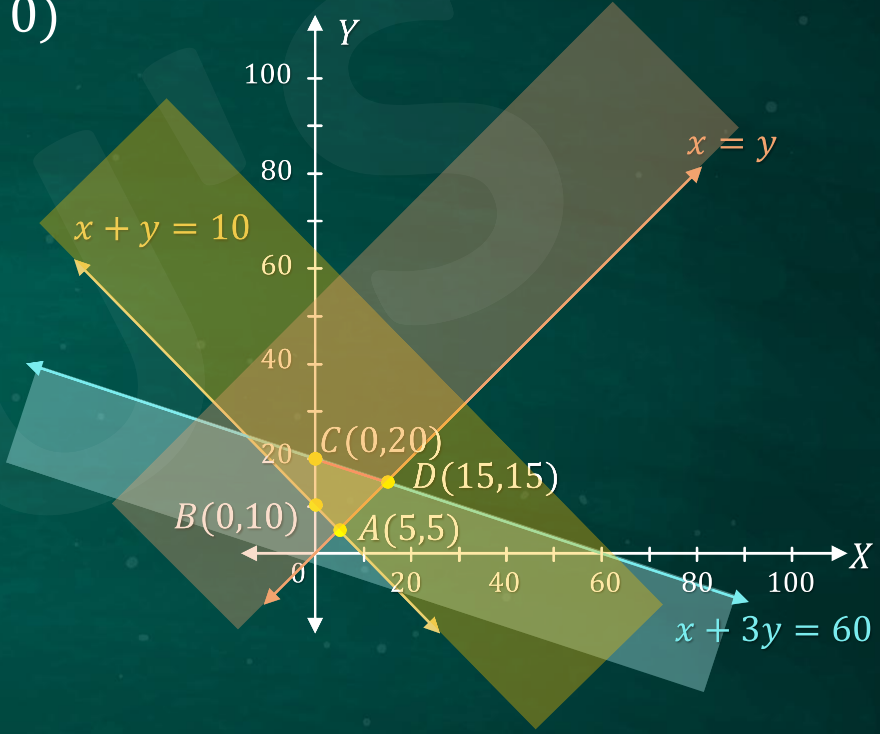


$x + 3y = 60$ is the line passing through $(60\,,0)$ and $(0, 20)$

$(0\,,0)$ satisfy $x + 3y \leq 60$ so solution set is the region below the line $x + 3y = 60$ including $x + 3y = 60$

Point of intersection of $x + 3y = 60$ & $y = x$ is $(15\,,15)$

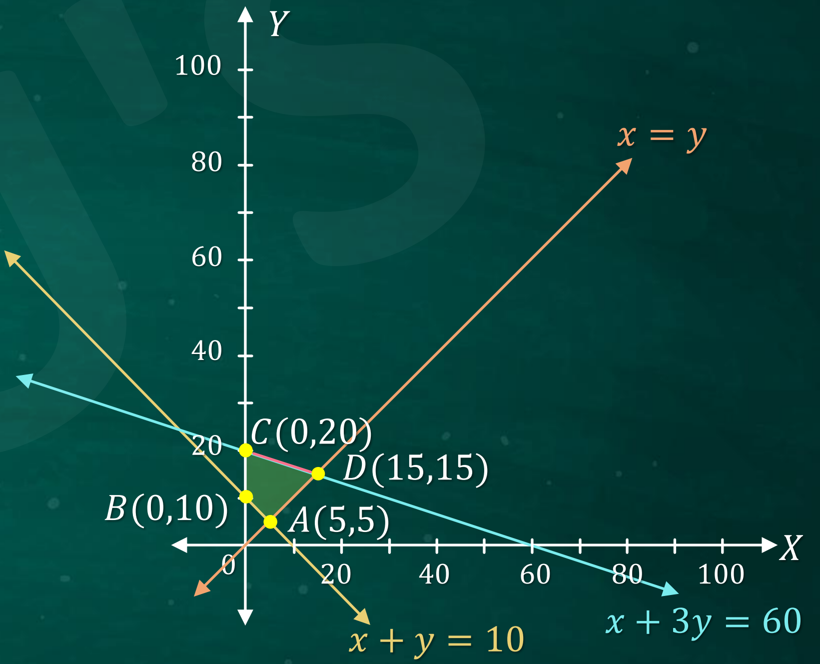
Point of intersection of $x + y = 10$ & $y = x$ is $(5\,,5)$



Here feasible region is bounded so optimal values occur at its corner points :

From graph , corner points are :
$(5, 5), (0, 10), (0, 20), (15, 15)$

Corner Point	Corresponding value of Z
$A(5,5)$	60
$B(0,10)$	90
$C(0,20)$	180 (Max)
$D(15,15)$	180 (Max)



Thus, For every point on entire line segment CD, Z will have maximum value.

Hence, this linear programming problem has multiple optimal solutions.

Question:

A manufacturer has three machines I, II & III installed in his factory. Machines I & II are capable of being operated for at most 12 hours whereas machine III is being operated for at least 5 hours a day. He produces only two items M & N each requiring the use of all three machines. The number of hours required for producing 1 unit of each of M & N on the three machines are given in the table:

Items	Number of hours required on the machines		
	I	II	III
M	1	2	1
N	2	1	1.25

He makes a profit of Rs. 600 and Rs. 400 on items M & N respectively. How many of each items should he produce so as to maximize the profit assuming that he can sell all the items that he produced. What will be the maximum profit ?

Solution:

Let x & y be the number of items of M & N produced by manufacturer respectively

There is a profit of Rs. 600 and Rs. 400 on each item of M & N respectively

Total profit on production = Rs ($600x + 400y$)

Maximize $Z = 600x + 400y$; where Z is the Objective function

Items	Number of hours required on the machines		
	I	II	III
$M \rightarrow x$	1	2	1
$N \rightarrow y$	2	1	1.25

Number of items can not be negative $\Rightarrow x \geq 0 , y \geq 0$

Machines I & II are capable of being operated for at most 12 hours so from table ,

$x + 2y \leq 12 , 2x + y \leq 12$

Machine III is being operated for at least 5 hours a day so from table ,

$x + \frac{5}{4}y \geq 5$

constraints :
$$\begin{aligned} x \geq 0 , y \geq 0 \\ x + 2y \leq 12 \\ 2x + y \leq 12 \\ x + \frac{5}{4}y \geq 5 \end{aligned}$$

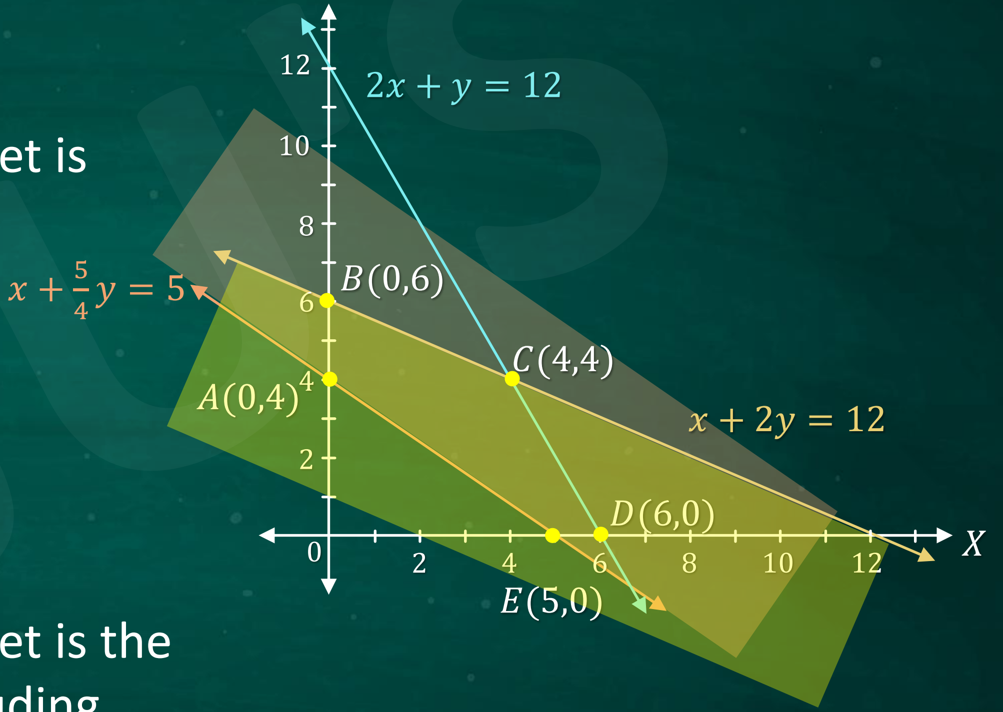
$x + 2y = 12$ is the line passing through the $(12 , 0)$ and $(0, 6)$

Solution set for $x \geq 0\,, y \geq 0$ is first quadrant including co-ordinate axis

$(0\,,0)$ satisfy $x + 2y \leq 12$ so solution set is the region below the line $x + 2y = 12$ including $x + 2y = 12$

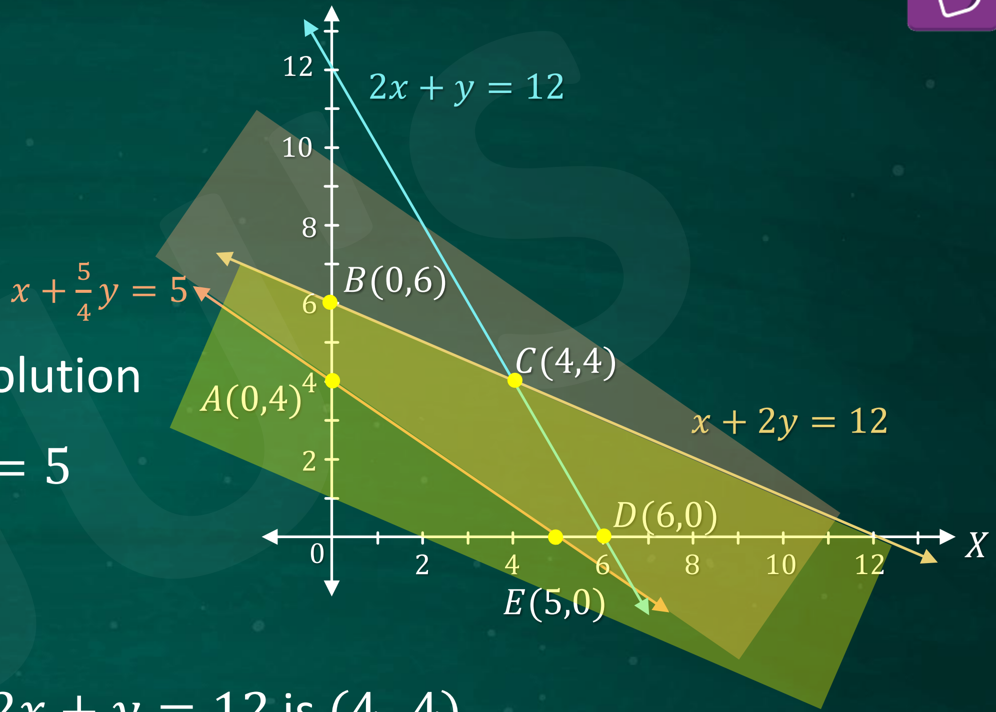
$2x + y \leq 12$ is the line passing through the $(6\,,0)$ and $(0, 12)$

$(0\,,0)$ satisfy $2x + y \leq 12$ so solution set is the region below the line $2x + y = 12$ including $2x + y = 12$



$x + \frac{5}{4}y \geq 5$ is the line passing through the $(5, 0)$ and $(0, 4)$

$(0, 0)$ does not satisfy $x + \frac{5}{4}y \geq 5$ so solution set is the region above the line $x + \frac{5}{4}y = 5$ including $x + \frac{5}{4}y = 5$

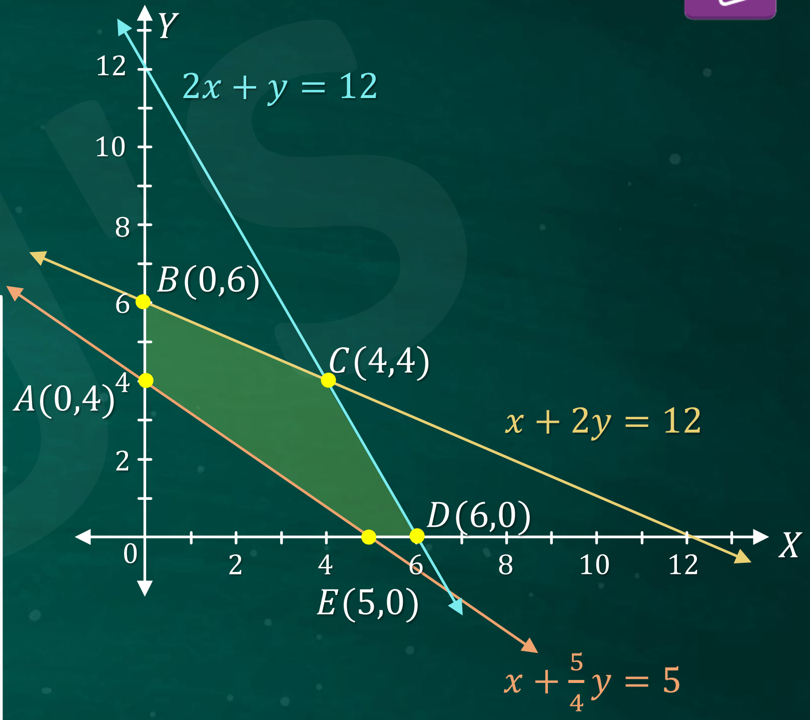


Point of intersection of $x + 2y = 12$ & $2x + y = 12$ is $(4, 4)$

Here feasible region is bounded so optimal values occur at its corner points :

From graph , corner points are :
$(0,4)$, $(0,6)$, $(4,4)$, $(6,0)$, $(5,0)$

Vertex of feasible region	Corresponding value of Z
$A(0,4)$	1600
$B(0,6)$	2400
$C(4,4)$	4000 (Max)
$D(6,0)$	3600
$E(5,0)$	3000



Objective function Z is maximum at $A(4,4)$

Hence , manufacturer should produce 4 units of each item M & N

Summary Sheet

- Linear programming is a technique in which a linear function is maximized or minimized subject to various constraints.
- This technique is useful for quantitative decisions in business planning , industrial engineering etc.
- Shaded region : Feasible region
- Points inside feasible region : Feasible solution
- Points outside feasible region: Infeasible solution
- Points in the feasible region that gives the optimal value of Z : Optimal (feasible) solution