

METHODS OF DIFFERENTIATION

THEOREMS OF DIFFERENTIATION



What you already know

- Limits
- Continuity and differentiability



What you will learn

- Definition of $\frac{dy}{dx}$
- Derivative of standard function
- Theorem of differentiation

Definition of $\frac{dy}{dx}$

$\frac{dy}{dx}$ is the instantaneous change in y with respect to x.

Change in y with respect to x = $\frac{\Delta y}{\Delta x}$

When $\Delta x \rightarrow 0$, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Let us consider the graph of function $y = f(x)$. The two points on the graph are $P(x, f(x))$ and $Q(x + h, f(x + h))$.

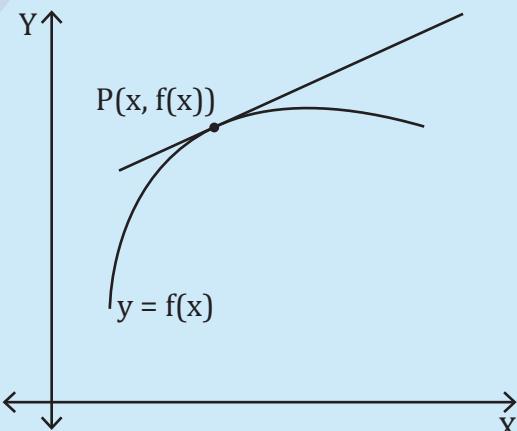
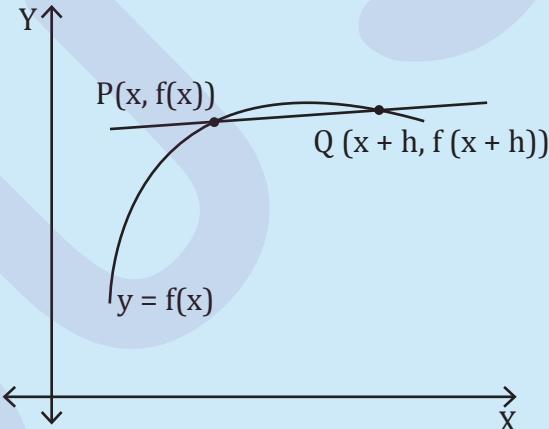
We can see that PQ is a secant to the curve.

When $h \rightarrow 0$, then point Q will reach to point P, i.e., $Q \rightarrow P$, and the secant will become the tangent to the curve.

$$\text{Hence, slope of the tangent} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This is known as the first principle of differentiation/ab-initio method.





Solve

Find the derivatives of the following functions by using the first principle of differentiation.

$$(i) f(x) = e^x \quad (ii) f(x) = \tan x$$

Solution

$$(i) \text{ Given, } f(x) = e^x$$

By using the first principle of differentiation, we get,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ &= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\ \Rightarrow f'(x) &= e^x \end{aligned}$$

$$(ii) \text{ Given, } f(x) = \tan x$$

By using the first principle of differentiation, we get,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\tan x + \tan h - \tan x}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\tan x + \tan h - \tan x(1 - \tan x \tan h)}{h(1 - \tan x \tan h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)} \\ &= \lim_{h \rightarrow 0} \frac{(1 + \tan^2 x)}{(1 - \tan x \tan h)} \times \lim_{h \rightarrow 0} \frac{\tan h}{h} \\ \Rightarrow f'(x) &= \sec^2 x \quad \left(\because \lim_{h \rightarrow 0} \frac{\tan h}{h} = 1 \right) \end{aligned}$$

Derivative of Standard Function

$$\frac{d(\text{Constant})}{dx} = 0$$

$$\frac{d(x^n)}{dx} = n x^{n-1}$$

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\tan x)}{dx} = \sec^2 x$$



$$\frac{d(a^x)}{dx} = a^x(\ln a)$$

$$\frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{x(\ln a)}$$

$$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{d(\sec x)}{dx} = \sec x \tan x$$

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$$

Theorem of Differentiation

(1) If $y = k f(x)$, then $\frac{dy}{dx} = k \frac{d}{dx}(f(x)) = k f'(x)$

Example:

$$\frac{d(2x^2)}{dx} = 2 \frac{d(x^2)}{dx} = 4x$$

(2) If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)) = f'(x) \pm g'(x)$

To prove $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

Proof

Let $p(x) = f(x) + g(x)$

$$\begin{aligned}\Rightarrow p'(x) &= \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} = \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ \Rightarrow p'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ \Rightarrow p'(x) &= f'(x) + g'(x)\end{aligned}$$

Example:

$$\frac{d}{dx}(\sin x + x) = \cos x + 1$$

(3) If $y = f(x). g(x)$, then

$$\frac{dy}{dx} = g(x) \frac{d}{dx}(f(x)) + f(x) \frac{d}{dx}(g(x)) = f'(x)g(x) + f(x)g'(x) \quad (\text{Product rule})$$



Proof

Let $p(x) = f(x) \times g(x)$

$$\Rightarrow p'(x) = \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} = \lim_{h \rightarrow 0} \frac{(f(x+h)g(x+h)) - (f(x)g(x))}{h}$$

$$\Rightarrow p'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - (f(x)g(x))}{h}$$

(By adding and subtracting $f(x)g(x+h)$ in the numerator)

$$= \lim_{h \rightarrow 0} g(x+h) \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \rightarrow 0} f(x) \left(\frac{g(x+h) - g(x)}{h} \right)$$

$$\Rightarrow p'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(4) \text{ If } y = \frac{f(x)}{g(x)}, \text{ then } \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \text{ (Quotient rule)}$$

$$(5) \text{ If } y = f(g(h(x))), \text{ then } \frac{dy}{dx} = f'(g(h(x))). g'(h(x)). h'(x) \text{ (Chain rule)}$$

Example:

(i) For $y = \sin(x^2)$, we get,

$$\frac{dy}{dx} = \frac{d(\sin x^2)}{dx} = \cos x^2 \frac{dx^2}{dx} = 2x \cos x^2$$

(ii) For $y = \ln(\tan x)$, we get,

$$\frac{dy}{dx} = \frac{d(\ln(\tan x))}{dx} = \frac{1}{\tan x} \times \frac{d(\tan x)}{dx} = \frac{\sec^2 x}{\tan x}$$



Solve

Find the derivatives/differentials of the following with respect to x.

$$(i) \quad y = \sqrt{x} + x^{\frac{3}{4}} - 3x^4$$

$$(ii) \quad y = e^x(x^2 + 1)$$

$$(iii) \quad y = \frac{x}{1+x^2}$$



Solution

(i) $y = \sqrt{x} + x^{\frac{3}{4}} - 3x^4$ (Given)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(\sqrt{x})}{dx} + \frac{d(x^{\frac{3}{4}})}{dx} - \frac{d(3x^4)}{dx} \\ &= \frac{1}{2}x^{\frac{1}{2}-1} + \frac{3}{4}x^{\frac{3}{4}-1} - 12x^3 \\ &= \frac{1}{2\sqrt{x}} + \frac{3}{4x^{\frac{1}{4}}} - 12x^3\end{aligned}$$

(ii) $y = e^x(x^2 + 1)$ (Given)

$$\begin{aligned}\frac{dy}{dx} &= e^x \frac{d}{dx}(x^2 + 1) + (x^2 + 1) \frac{d}{dx}e^x \quad (\text{Product rule}) \\ &= e^x(2x) + (x^2 + 1)e^x \\ &\Rightarrow \frac{dy}{dx} = e^x(x^2 + 2x + 1) = e^x(x+1)^2\end{aligned}$$

(iii) $y = \frac{x}{1+x^2}$ (Given)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x^2)\frac{dx}{dx} - x\frac{d(1+x^2)}{dx}}{(1+x^2)^2} \quad (\text{Quotient rule}) \\ &\Rightarrow \frac{dy}{dx} = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{(1-x^2)}{(1+x^2)^2}\end{aligned}$$



Solve

If $f(1) = 1$ and $f'(1) = 3$, then find the derivative of $f(f(f(f(x)))) + (f(x))^2$ at $x = 1$

- (a) 33 (b) 12 (c) 9 (d) 15

Solution

Step 1

Let $y = f(f(f(f(x)))) + (f(x))^2$

$$\Rightarrow \frac{dy}{dx} = f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) + 2f(x) \cdot f'(x)$$

Step 2

$$\begin{aligned}\text{At } x = 1, \frac{dy}{dx} &= f'(f(f(f(1)))) \cdot f'(f(f(1))) \cdot f'(1) + 2f(1) \cdot f'(1) \\ &= f'(f(1)) \cdot f'(1) \cdot f'(1) + 2f(1) \cdot f'(1) \\ &= f'(1) \cdot f'(1) \cdot f'(1) + 2f(1) \cdot f'(1) \\ &= 3 \times 3 \times 3 + 6 = 33\end{aligned}$$

\therefore Option (a) is the correct answer.



Concept Check

1. Find the derivative/differential of the following with respect to x: $y = \sqrt{\sin(2x+3)} + x^2 \ln x$

2. If $y(\alpha) = \sqrt{2\left(\frac{\tan \alpha + \cot \alpha}{1 + \tan^2 \alpha}\right) + \frac{1}{\sin^2 \alpha}}$, $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$, then at $\alpha = \frac{5\pi}{6}$, find $\frac{dy}{d\alpha}$.

 (a) 4 (b) $-\frac{1}{4}$ (c) -4 (d) $\frac{4}{3}$



Summary Sheet

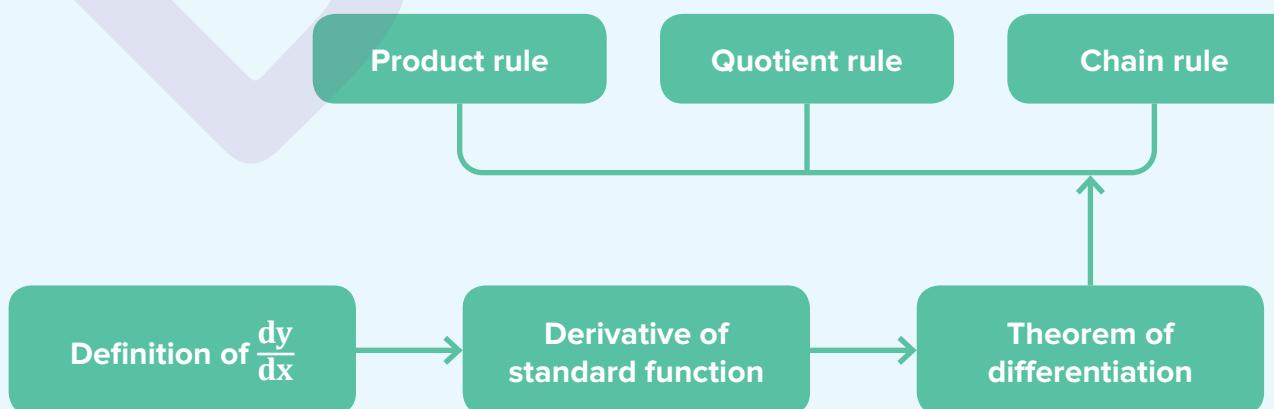


Key Takeaways

- At a point, the tangent is the limiting case of the secant passing through that point.
- If $y = k f(x)$, then $\frac{dy}{dx} = k \frac{d}{dx}(f(x)) = k f'(x)$
- If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)) = f'(x) + g'(x)$
- If $y = f(x) \cdot g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f(x)) g(x) + f(x) \frac{d}{dx}(g(x)) = f'(x) g(x) + f(x) g'(x)$ (Product rule)
- If $y = \frac{f(x)}{g(x)}$, then $\frac{dy}{dx} = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$ (Quotient rule)
- If $y = f(g(h(x)))$, then $\frac{dy}{dx} = f'(g(h(x))). g'(h(x)). h'(x)$ (Chain rule)



Mind Map





Self-Assessment

If $y = \sec(\tan^{-1}x)$, then find $\frac{dy}{dx}$ at $x = 1$



Answers

Concept Check

$$1. y = \sqrt{\sin(2x+3)} + x^2 \ln x \quad (\text{Given})$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{\sin(2x+3)} \right) + \frac{d}{dx} (x^2 \ln x) \\ &= \frac{1}{2\sqrt{\sin(2x+3)}} \frac{d}{dx} \sin(2x+3) + x^2 \frac{d}{dx} (\ln x) + \ln x \frac{d}{dx} (x^2) \\ &= \frac{2\cos(2x+3)}{2\sqrt{\sin(2x+3)}} + \frac{x^2}{x} + 2x(\ln x) \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos(2x+3)}{\sqrt{\sin(2x+3)}} + x + 2x(\ln x)\end{aligned}$$

2. Step 1

$$\begin{aligned}y(\alpha) &= \sqrt{2 \left(\frac{\tan \alpha + \cot \alpha}{1 + \tan^2 \alpha} \right) + \frac{1}{\sin^2 \alpha}} \\ &= \sqrt{\frac{2 \left(\tan \alpha + \frac{1}{\tan \alpha} \right)}{(1 + \tan^2 \alpha)} + \cosec^2 \alpha} \\ &= \sqrt{\frac{2(1 + \tan^2 \alpha)}{\tan \alpha (1 + \tan^2 \alpha)} + \cosec^2 \alpha} \\ &= \sqrt{2 \cot \alpha + \cosec^2 \alpha} \\ &= \sqrt{2 \cot \alpha + 1 + \cot^2 \alpha} \\ &= \sqrt{(1 + \cot \alpha)^2} = |1 + \cot \alpha|\end{aligned}$$

Step 2

For $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$, we get,

$$1 + \cot \alpha < 0$$

$$\Rightarrow y(\alpha) = -(1 + \cot \alpha)$$

$$\Rightarrow \frac{dy}{d\alpha} = \cosec^2 \alpha$$

$$\text{At } \alpha = \frac{5\pi}{6}, \frac{dy}{d\alpha} = 4$$

∴ Option (a) is the correct answer.

**Self-Assessment**

$$y = \sec(\tan^{-1}x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec(\tan^{-1}x))$$

$$= \sec(\tan^{-1}x) \times \tan(\tan^{-1}x) \times \frac{d}{dx}(\tan^{-1}x) \quad (\text{Chain rule})$$

$$= \sec(\tan^{-1}x) \times x \times \frac{1}{1+x^2}$$

$$\text{At } x=1, \frac{dy}{dx} = \sec \frac{\pi}{4} \times \frac{1}{2} = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}$$

METHODS OF DIFFERENTIATION

DIFFERENTIATION OF FUNCTIONS INVOLVING INVERSE



What you already know

- Definition of $\frac{dy}{dx}$
- Derivative of standard function
- Theorems of differentiation



What you will learn

- Differentiation of inverse trigonometric functions
- Standard substitutions
- Inverse functions

Differentiation of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

$$\frac{d}{dx}(\cosec^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

To prove $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

Proof:

Let $y = \sin^{-1} x, -1 < x < 1$

$\Rightarrow x = \sin y$

On differentiating with respect to y , we get,

$$\begin{aligned} \frac{dx}{dy} &= \cos y \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\cos y} \quad \left(\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \right) \end{aligned}$$



$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

To prove $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$

Proof:

We know that,

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{d}{dx}(\cos^{-1} x) &= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(\sin^{-1} x) \\ \Rightarrow \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1 - x^2}} \quad \left(\because \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0, \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \right) \end{aligned}$$

Example:

Let us find $\frac{dy}{dx}$ when $y = \sin^{-1}(2x + 3)$, $x \in (-2, -1)$

By applying the chain rule, we get,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x+3)^2}} \cdot 2 = \frac{2}{\sqrt{1 - (2x+3)^2}}$$



If $f(x) = 2\sqrt{x+1} \sin^{-1} x + 4\sqrt{1-x}$, then find $f'(x)$.

a) $2x \sin^{-1} x$

b) $\sqrt{1-x} \sin^{-1} x$

c) $\frac{\sin^{-1} x}{\sqrt{x+1}}$

d) $\sqrt{1-x^2}$

Solution

If $y = f(g(h(x)))$, $\frac{dy}{dx} = f'(g(h(x))) g'(h(x)) h'(x)$



$$\begin{aligned}
 \frac{d}{dx} f(x) &= 2 \times \frac{d}{dx} \left((\sqrt{x+1}) (\sin^{-1} x) \right) + 4 \times \frac{d}{dx} \sqrt{1-x} \\
 &= 2 \left(\frac{1}{2\sqrt{x+1}} \cdot \sin^{-1} x + \sqrt{x+1} \cdot \frac{1}{\sqrt{1-x^2}} \right) + \frac{4}{2\sqrt{1-x}} \cdot (-1) \\
 &= \frac{\sin^{-1} x}{\sqrt{x+1}} + \frac{2}{\sqrt{1-x}} - \frac{2}{\sqrt{1-x}} \\
 &= \frac{\sin^{-1} x}{\sqrt{x+1}}
 \end{aligned}$$

So, option (c) is the correct answer.

 If $y = \cos^{-1} \left(\frac{3}{5} \cos x - \frac{4}{5} \sin x \right)$, where $x \in \left(0, \frac{\pi}{2} \right)$, then $\frac{dy}{dx}$ is

- a) 0 b) 1 c) $\frac{3}{5}$ d) $\sin x$

Solution

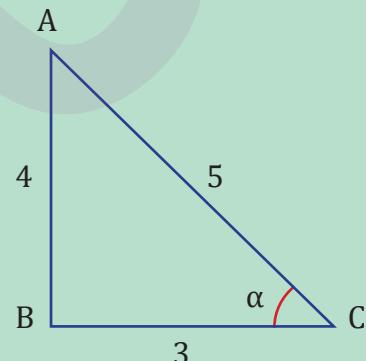
Step 1:

$$y = \cos^{-1} \left(\frac{3}{5} \cos x - \frac{4}{5} \sin x \right)$$

$$\text{Let } \cos \alpha = \frac{3}{5} \Rightarrow \sin \alpha = \frac{4}{5}$$

$$\Rightarrow y = \cos^{-1}(\cos \alpha \cos x - \sin \alpha \sin x)$$

$$\Rightarrow y = \cos^{-1}(\cos((x + \alpha)))$$

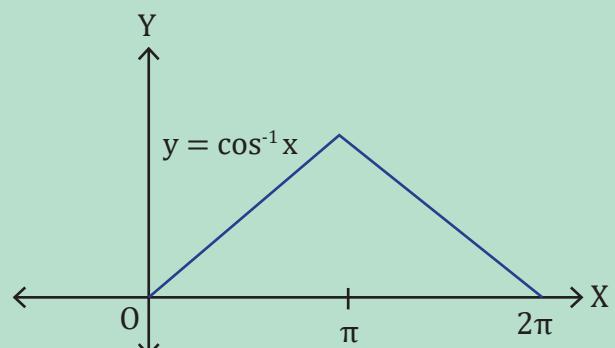


Step 2:

$$\text{Also, } \sin \alpha > 0, \cos \alpha > 0 \Rightarrow \alpha \in \left(0, \frac{\pi}{2} \right)$$

$$\text{So, } x + \alpha \in (0, \pi) \text{ as } x \in \left(0, \frac{\pi}{2} \right)$$

$$\cos^{-1}(\cos(x + \alpha)) = x + \alpha$$



Step 3:

$$\Rightarrow y = x + \alpha$$

$$\Rightarrow \frac{dy}{dx} = 1$$

So, option (b) is the correct answer.



If $y = \tan^{-1}\left(\frac{2-3x}{3+2x}\right) + \tan^{-1}\left(\frac{3x-1}{3x+1}\right)$, $x > 0$, then the value of $y'(1)$ is

- a) 0
- b) $\frac{2}{3}$
- c) $-\frac{1}{5}$
- d) $-\frac{1}{2}$

Solution

Step 1:

$$\begin{aligned} \tan^{-1}\left(\frac{2-3x}{3+2x}\right) &= \tan^{-1}\left(\frac{\frac{2}{3}-x}{1+\frac{2}{3}x}\right) \\ &= \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}x \quad \left(\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y \right) \end{aligned}$$

Similarly,

$$\tan^{-1}\left(\frac{3x-1}{1+3x}\right) = \tan^{-1}(3x) - \tan^{-1}(1)$$

Step 2:

$$\begin{aligned} y &= \tan^{-1}\left(\frac{2-3x}{3+2x}\right) + \tan^{-1}\left(\frac{3x-1}{3x+1}\right) \\ &= \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}x + \tan^{-1}(3x) - \tan^{-1}(1) \end{aligned}$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2} + \frac{3}{1+(3x)^2}$$

$$\frac{dy}{dx} (\text{at } x=1) = -\frac{1}{2} + \frac{3}{10} = -\frac{1}{5}$$

So, option (c) is the correct answer.



Note

Some standard substitutions are given below:

Form	Substitutions
$a^2 - x^2, \sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $a \cos \theta$
$x^2 + a^2, \sqrt{x^2 + a^2}$	$x = a \tan \theta$ or $a \cot \theta$
$x^2 - a^2, \sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \cosec \theta$
$\frac{a+x}{a-x}, \sqrt{\frac{a+x}{a-x}}$	$x = a \cos \theta$



Let us see how these substitutions work.

By substituting $x = a \sin \theta$ in $y = \sqrt{a^2 - x^2}$, we get,

$$y = \sqrt{a^2 - (a \sin \theta)^2} = \sqrt{a^2 \cos^2 \theta} = |a \cos \theta|$$

Similarly, let us substitute $x = a \cos \theta$ in $y = \sqrt{\frac{a+x}{a-x}}$

$$\begin{aligned} y &= \sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{a+a \cos \theta}{a-a \cos \theta}} \\ &= \sqrt{\frac{1+\cos \theta}{1-\cos \theta}} \\ &= \sqrt{\frac{2\cos^2 \frac{\theta}{2}}{2\sin^2 \frac{\theta}{2}}} = \left| \cot \frac{\theta}{2} \right| \end{aligned}$$



Find $\frac{dy}{dx}$ when $y = \cos^{-1}\left(\sqrt{1-x^2}\right)$, $x \in (0, 1)$

Solution

Step 1:

Let $x = \sin \theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$ as $x \in (0, 1)$

$$\begin{aligned} \Rightarrow y &= \cos^{-1}\left(\sqrt{1-(\sin \theta)^2}\right) \\ &= \cos^{-1}(|\cos \theta|) \end{aligned}$$

Step 2:

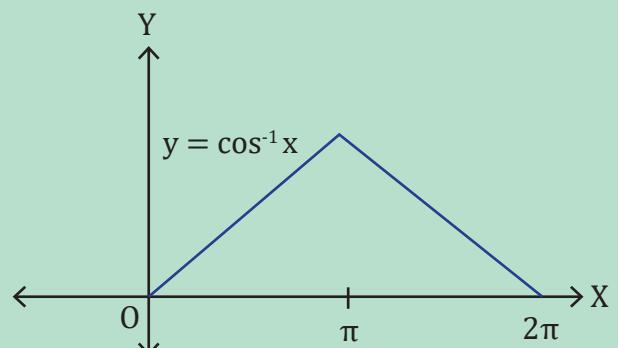
For $\theta \in \left(0, \frac{\pi}{2}\right)$, $|\cos \theta| = \cos \theta$

$$\Rightarrow y = \cos^{-1}(\cos \theta)$$

Also, for $x \in (0, \pi)$, $\cos^{-1}(\cos x) = x$

$$\Rightarrow y = \theta = \sin^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$





If $y = \tan^{-1} \left(\frac{\sqrt{1+(x)^2} - 1}{x} \right)$, then find $\frac{dy}{dx}$ when $x \in [-1, 1] - \{0\}$

a) $\frac{1}{(1+x^2)}$

b) $\frac{2}{1+4x^2}$

c) $\frac{1}{2(1+x^2)}$

d) $\frac{x}{1+x^2}$

Solution

Step 1:

Let $x = \tan \theta$, $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$ as $x \in [-1, 1]$

$$y = \tan^{-1} \left(\frac{\sqrt{1+(\tan \theta)^2} - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{|\sec \theta| - 1}{\tan \theta} \right)$$

Step 2:

As $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$, $|\sec \theta| = \sec \theta$

$$\begin{aligned} y &= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) \\ &= \tan^{-1} \left(\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right) \\ &= \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) = \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)} \right) \\ &= \tan^{-1} \left(\tan \left(\frac{\theta}{2} \right) \right) \end{aligned}$$

Step 3:

$\frac{\theta}{2} \in \left[-\frac{\pi}{8}, \frac{\pi}{8} \right]$ as $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$

And $\tan^{-1}(\tan x) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

$$\Rightarrow y = \tan^{-1} \left(\tan \left(\frac{\theta}{2} \right) \right) = \frac{\theta}{2} = \frac{\tan^{-1} x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

So, option (c) is the correct answer.



Inverse Functions

If $f(x)$ and $g(x)$ are differentiable functions such that $f^{-1}(x) = g(x)$, then,

$$(i) f'(g(x)) = \frac{1}{g'(x)}$$

$$(ii) g'(f(x)) = \frac{1}{f'(x)}$$

Proof:

i)

When $f^{-1}(x) = g(x)$, we have $f(g(x)) = x$

On differentiating with respect to x , we get,

$$f'(g(x)) g'(x) = 1 \Rightarrow f'(g(x)) = \frac{1}{g'(x)}$$

ii)

When $f^{-1}(x) = g(x)$, we have $g(f(x)) = x$

On differentiating with respect to x , we get,

$$g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

IIT JEE 2009



If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is 1:

Solution

Step 1:

Given, $f^{-1}(x) = g(x)$

$$g'(f(x)) = \frac{1}{f'(x)}$$

In order to calculate $g'(1)$, we have to find the value of x for which $f(x) = 1$

$$\text{Let } f(a) = a^3 + e^{\frac{a}{2}}$$

One can see that $a = 0$ satisfies the given equation as $f(0) = 0^3 + e^{\frac{0}{2}} = 1$

$$\text{Now, } g'(f(0)) = \frac{1}{f'(0)} \Rightarrow g'(1) = \frac{1}{f'(0)}$$

Step 2:

$$\begin{aligned} f(x) &= x^3 + e^{\frac{x}{2}} & \Rightarrow f'(0) &= \frac{1}{2} \\ \Rightarrow f'(x) &= 3x^2 + \frac{e^{\frac{x}{2}}}{2} & \Rightarrow g'(1) &= 2 \end{aligned}$$



Concept Check

1. Find $\frac{dy}{dx}$ when $y = \cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)$, $x \in (-1, 1)$

2. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then the value of $g'(x)$ is: ★ JEE MAIN 2014

- (a) $5x^4$ (b) $\frac{1}{1+(g(x))^5}$ (c) $1 + (g(x))^5$ (d) $1 + x^5$



Summary Sheet



Key Takeaways

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

- Some standard substitutions are given below:

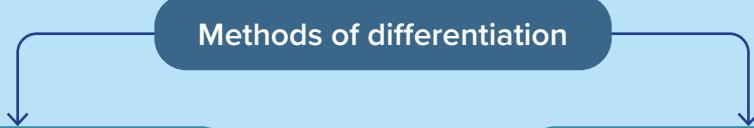
Form	Substitutions
$a^2 - x^2, \sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $a \cos \theta$
$x^2 + a^2, \sqrt{x^2 + a^2}$	$x = a \tan \theta$ or $a \cot \theta$
$x^2 - a^2, \sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \cosec \theta$
$\frac{a+x}{a-x}, \sqrt{\frac{a+x}{a-x}}$	$x = a \cos \theta$

- If $f(x)$ and $g(x)$ are differentiable functions such that $f^{-1}(x) = g(x)$, then,

(i) $f'(g(x)) = \frac{1}{g'(x)}$ (ii) $g'(f(x)) = \frac{1}{f'(x)}$



Mind Map



Differentiation using standard substitutions

Differentiation of inverse trigonometric functions



Self-Assessment

If $y = \sin^{-1}(2x\sqrt{1-x^2})$, $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then find $\frac{dy}{dx}$



Answers

Concept Check

1.

Step 1:

Let $x = \cos \theta$, $\theta \in (0, \pi)$ as $x \in (-1, 1)$

$$\begin{aligned} \Rightarrow y &= \cot^{-1}\left(\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right) \\ &= \cot^{-1}\left(\sqrt{\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}}\right) \\ &= \cot^{-1}\left(\left|\cot\frac{\theta}{2}\right|\right) \end{aligned}$$

Step 2:

$$\text{Since } \theta \in (0, \pi), \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow y = \cot^{-1}\left|\cot\frac{\theta}{2}\right| = \cot^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$\text{Also, } \cot^{-1}(\cot x) = x, x \in \mathbb{R}$$

$$\begin{aligned} \Rightarrow y &= \frac{\theta}{2} \\ \Rightarrow y &= \frac{\cos^{-1} x}{2} \\ \Rightarrow \frac{dy}{dx} &= \frac{-1}{2\sqrt{1-x^2}} \end{aligned}$$

**2.**

Given, $f^{-1}(x) = g(x)$

$$\Rightarrow f'(g(x)) = \frac{1}{g'(x)}, \text{ i.e., } g'(x) = \frac{1}{f'(g(x))}$$

$$\Rightarrow g'(x) = \frac{1}{1 + (g(x))^5}$$

So, option (b) is the correct answer

Self-Assessment

Step 1:

$$y = \sin^{-1}(2x\sqrt{1-x^2})$$

$$\text{Let } x = \sin \theta, \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \text{ as } x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow y = \sin^{-1}\left(2 \sin \theta \sqrt{1 - (\sin \theta)^2}\right)$$

$$= \sin^{-1}(2 \sin \theta |\cos \theta|)$$

Step 2:

Since $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, we have,

$$y = \sin^{-1}(2 \sin \theta \cos \theta) = \sin^{-1}(\sin 2\theta), 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Also, $\sin^{-1}(\sin(x)) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

METHODS OF DIFFERENTIATION

DIFFERENTIATION USING DIFFERENT METHODS



What you already know

- Differentiation of inverse trigonometric functions
- Standard substitutions



What you will learn

- Logarithmic differentiation
- Implicit differentiation
- Parametric differentiation

Logarithmic Differentiation

Logarithmic differentiation is used when functions are multiplied, divided, or raised to another function.

Let us see how logarithm makes the process of differentiation easy.

Example:

$$(a) \text{ Let } f(x) = \frac{(x-2)^2 (x-3)^3}{(x-1)^4 (x+4)^5} \dots (i)$$

$$\Rightarrow \ln(f(x)) = \ln(x-2)^2 + \ln(x-3)^3 - \ln(x-1)^4 - \ln(x+4)^5 \\ = 2\ln(x-2) + 3\ln(x-3) - 4\ln(x-1) - 5\ln(x+4) \dots (ii)$$

Clearly, differentiation of (ii) is easier than differentiation of (i).

(b) Functions of the form $f(x)^{g(x)}$ are always differentiated by taking \ln on both sides.

Example:

$$f(x) = x^{\sin x}$$

$$\ln(f(x)) = \sin x \cdot \ln(x)$$

Now, the differentiation can be done easily.



Differentiate w.r.t x

$$(i) y = x^x$$

$$(ii) y = \frac{(1+x)^{\frac{2}{3}} (3x-2)^{\frac{7}{4}}}{(5-2x)^{\frac{5}{7}} (6-5x)^{\frac{1}{3}}}$$



Solution

(i)

Step 1

Taking \ln on both sides, we get,

$$\ln y = \ln(x^x) = x \ln(x)$$

Step 2

Differentiating both sides w.r.t x , we get,

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \cdot \ln x) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \left(x \cdot \frac{1}{x} + 1 \cdot \ln x \right) = (1 + \ln x) \\ \Rightarrow \frac{dy}{dx} &= y(1 + \ln x) = x^x(1 + \ln x)\end{aligned}$$

(ii)

Step 1

Taking \ln on both sides, we get,

$$\begin{aligned}\ln y &= \ln \left(\frac{(1+x)^{\frac{2}{3}} (3x-2)^{\frac{7}{4}}}{(5-2x)^{\frac{5}{7}} (6-5x)^{\frac{1}{3}}} \right) \\ &= \ln \left((1+x)^{\frac{2}{3}} \right) + \ln \left((3x-2)^{\frac{7}{4}} \right) - \ln \left((5-2x)^{\frac{5}{7}} \right) - \ln \left((6-5x)^{\frac{1}{3}} \right) \\ &= \frac{2}{3} \ln(1+x) + \frac{7}{4} \ln(3x-2) - \frac{5}{7} \ln(5-2x) - \frac{1}{3} \ln(6-5x)\end{aligned}$$

Step 2

Differentiating both sides w.r.t x , we get,

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{2}{3} \left(\frac{1}{1+x} \right) + \frac{7}{4} \left(\frac{3}{3x-2} \right) - \frac{5}{7} \left(\frac{-2}{5-2x} \right) - \frac{1}{3} \left(\frac{-5}{6-5x} \right) \\ \frac{dy}{dx} &= y \left(\frac{2}{3(1+x)} + \frac{21}{4(3x-2)} + \frac{10}{7(5-2x)} + \frac{5}{3(6-5x)} \right) \\ \frac{dy}{dx} &= \left(\frac{(1+x)^{\frac{2}{3}} (3x-2)^{\frac{7}{4}}}{(5-2x)^{\frac{5}{7}} (6-5x)^{\frac{1}{3}}} \right) \left(\frac{2}{3(1+x)} + \frac{21}{4(3x-2)} + \frac{10}{7(5-2x)} + \frac{5}{3(6-5x)} \right)\end{aligned}$$



For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, find the value of $(1 + \ln 2x)^2 \frac{dy}{dx}$.

JEE MAIN JAN 2019

(a) $\ln 2x$

(b) $x \ln 2x$

(c) $\frac{x \ln 2x + \ln 2}{x}$

(d) $\frac{x \ln 2x - \ln 2}{x}$

Solution

Step 1

$$(2x)^{2y} = 4e^{2x-2y}$$

Taking \ln on both sides,

$$2y \ln 2x = (2x - 2y) \ln e + \ln 4$$

$$\Rightarrow 2y \ln 2x = (2x - 2y) + \ln 4 \quad (\because \ln e = 1)$$

$$\Rightarrow y = \frac{x + \ln 2}{1 + \ln 2x}$$

Step 2

Differentiating both sides w.r.t x , we get,

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + \ln 2x) \frac{d}{dx}(x + \ln 2) - (x + \ln 2) \frac{d}{dx}(1 + \ln 2x)}{(1 + \ln 2x)^2} \\ &= \frac{(1 + \ln 2x)(1) - (x + \ln 2) \left(\frac{2}{2x} \right)}{(1 + \ln 2x)^2} = \frac{1 + \ln 2x - 1 - \frac{\ln 2}{x}}{(1 + \ln 2x)^2} \\ &\Rightarrow \frac{dy}{dx} (1 + \ln 2x)^2 = \frac{x \ln 2x - \ln 2}{x} \end{aligned}$$

So, option (d) is the correct answer.

Implicit Differentiation

Explicit function

If, in an equation, y is expressed solely in terms of x , then it's known as an explicit function.

Example:

$2y - x^3 + 3x - 5 = 0$ can be written as $2y = x^3 + 3x - 5$

Implicit function

If the relation between x and y is an equation in which y can't be expressed solely in terms of x , then it's known as an implicit function.

Example:

$$x^3 + y^3 = 3xy$$

Steps to get $\frac{dy}{dx}$ for an implicit function

(1) Differentiate the equation w.r.t x .

(2) Collect the terms of $\frac{dy}{dx}$.



Find $\frac{dy}{dx}$ when $x(\sin y) + y = 3$

Solution

Differentiating both sides w.r.t x, we get,

$$\begin{aligned} \frac{d}{dx}(x \sin y) + \frac{dy}{dx} &= \frac{d(3)}{dx} \\ \Rightarrow x \cos y \frac{dy}{dx} + \sin y + \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx}(1 + x \cos y) &= -\sin y \\ \Rightarrow \frac{dy}{dx} &= \frac{-\sin y}{1 + x \cos y} \end{aligned}$$



Let $x^k + y^k = a^k$, ($a, k > 0$) and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then what is the value of k?

JEE MAIN 2020

(a) $\frac{3}{2}$

(b) $\frac{4}{3}$

(c) $\frac{1}{3}$

(d) $\frac{2}{3}$

Solution

Step 1

$$x^k + y^k = a^k$$

Differentiating both sides w.r.t x, we get,

$$\begin{aligned} \frac{d}{dx}(x^k) + \frac{d}{dx}(y^k) &= \frac{d}{dx}(a^k) \\ \Rightarrow k x^{k-1} + k y^{k-1} \frac{dy}{dx} &= 0 \\ \Rightarrow \left(\frac{x^{k-1}}{y^{k-1}}\right) + \frac{dy}{dx} &= 0 \dots (i) \end{aligned}$$

Step 2

On comparing (i) with $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}}$, we get,

$$1 - k = \frac{1}{3} \Rightarrow k = \frac{2}{3}$$

So, option (d) is the correct answer.



If $ax^2 + 2hxy + by^2 = 0$, then find $\frac{dy}{dx}$.

- (a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $-\frac{x}{y}$ (d) $-\frac{y}{x}$

Solution

Step 1

$$ax^2 + 2hxy + by^2 = 0$$

Differentiating both sides w.r.t x, we get,

$$\begin{aligned} 2ax + 2h\left(x \frac{dy}{dx} + y\right) + 2by \frac{dy}{dx} &= 0 \\ \Rightarrow ax + h\left(x \frac{dy}{dx} + y\right) + by \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} = -\frac{ax + hy}{hx + by} &\dots(\text{i}) \end{aligned}$$

So, option (b) is the correct answer.

Alternate Method

$$ax^2 + 2hxy + by^2 = 0$$

Homogeneous equation in x and y of degree n, represents n straight lines passing through the origin.

Here, n = 2.

The general equation of a line passing through origin is given by $y = mx$.

$$m = \text{slope} = \frac{dy}{dx} = \frac{y}{x}$$

Step 2

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow ax^2 + hxy + hxy + by^2 = 0$$

$$\Rightarrow x(ax + hy) + y(hx + by) = 0$$

$$\Rightarrow \frac{ax + hy}{hx + by} = -\frac{y}{x} \dots(\text{ii})$$

From (i) and (ii), we have $\frac{dy}{dx} = \frac{y}{x}$



If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$, where $\sin x > 0$, find $\frac{dy}{dx}$.

- (a) $\frac{\cos x}{2y - 1}$ (b) $\frac{\sin x}{y + 1}$ (c) $\frac{2 \cos x}{y - 2}$ (d) $\frac{\sin x}{y + 2}$

Solution

We can write $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$ as $y = \sqrt{\sin x + y}$. Squaring on both sides, we get,
 $y^2 = \sin x + y$



Differentiating both sides w.r.t x, we get,

$$2y \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y - 1}$$

So, option (a) is the correct answer.

Parametric Differentiation

Sometimes the equation of a curve is not given in cartesian form, i.e, $y = f(x)$, but in parametric form: $x = h(t)$, $y = g(t)$. In this section we will see how to calculate the derivative $\frac{dy}{dx}$ from a knowledge of the parametric derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We then extend this to the determination of the second derivative $\frac{d^2y}{dx^2}$.

We have, $\frac{dy}{dt} = g'(t)$ and $\frac{dx}{dt} = h'(t)$, where $x = h(t)$, $y = g(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{h'(t)}$$



If $x = 2e^{-t}$, $y = 4e^t$, then find $\frac{dy}{dx}$.

(a) e^{2t}

(b) $-2e^{2t}$

(c) e^{-2t}

(d) e^t

Solution

$$x = 2e^{-t} \Rightarrow \frac{dx}{dt} = -2e^{-t}$$

$$y = 4e^t \Rightarrow \frac{dy}{dt} = 4e^t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4e^t}{-2e^{-t}} = -2e^{2t}$$

So, option (b) is the correct answer.



If $x = 3 \cos \theta - \cos^3 \theta$, $y = 3 \sin \theta - \sin^3 \theta$, then $\frac{dy}{dx}$ is:

(a) $\tan^3 \theta$

(b) $\cot^3 \theta$

(c) $-\cot^3 \theta$

(d) $-\tan^3 \theta$

Solution

Step 1

$$x = 3 \cos \theta - \cos^3 \theta$$

$$\frac{dx}{d\theta} = -3 \sin \theta + 3 \cos^2 \theta \sin \theta$$

$$\frac{dx}{d\theta} = -3 \sin \theta (1 - \cos^2 \theta) = -3 \sin^3 \theta$$

$$y = 3 \sin \theta - \sin^3 \theta$$

$$\frac{dy}{d\theta} = 3 \cos \theta - 3 \sin^2 \theta \cos \theta$$

$$\frac{dy}{d\theta} = 3 \cos \theta (1 - \sin^2 \theta) = 3 \cos^3 \theta$$

Step 2

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \cos^3 \theta}{-3 \sin^3 \theta} = -\cot^3 \theta$$

So, option (c) is the correct answer.



Concept Check

1. Find $\frac{dy}{dx}$ where $y = x^{\tan x} + (\sin x)^{\cos x}$

2. If $x^2 + xy - y^3 = 1$, then find $\frac{dy}{dx}$.

3. If $x = a \left(\cos t + \ln \left(\tan \left(\frac{t}{2} \right) \right) \right)$, $y = a(\sin t)$, then, $\frac{dy}{dx}$ is:



Summary Sheet



Key Takeaways

- Logarithmic differentiation is used when functions are multiplied, divided, or raised to another function.
- **Explicit function**-If in an equation y is expressed solely in terms of x , then it is known as an explicit function.



• **Implicit function-**If the relation between x and y is an equation in which y can't be expressed solely in terms of x, then it is known as an implicit function.

• Steps to get $\frac{dy}{dx}$ for an implicit function

(1) Differentiate the equation w.r.t x.

(2) Collect the terms of $\frac{dy}{dx}$.

• **Parametric Differentiation-** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$



Mind Map

Implicit Differentiation

Methods of Differentiation

Logarithmic Differentiation

Parametric Differentiation



Self-Assessment

If $x = \frac{1+t}{t^3}$, $y = \frac{3}{2t^2} + \frac{2}{t}$ then, $x\left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx}$ is



Answers

Concept Check

1. Step 1

Let $u = x^{\tan x}$, $v = (\sin x)^{\cos x}$

$\Rightarrow y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$



Step 2

$$u = x^{\tan x}$$

Taking \ln on both sides, we get,

$$\ln u = \tan x \cdot \ln x$$

Differentiate the equation w.r.t x .

$$\frac{1}{u} \frac{du}{dx} = \left(\frac{\tan x}{x} + \sec^2 x \ln x \right)$$

$$\Rightarrow \frac{du}{dx} = x^{\tan x} \left(\frac{\tan x}{x} + \sec^2 x \ln x \right) \dots (i)$$

From (i) and (ii),

$$\Rightarrow \frac{dy}{dx} = x^{\tan x} \left(\frac{\tan x}{x} + \ln x \sec^2 x \right) + (\sin x)^{\cos x} (\cos x \cot x - \sin x (\ln \sin x))$$

2. $x^2 + xy - y^3 = 1$

Differentiate the equation w.r.t x .

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(1)$$

$$\Rightarrow 2x + x \frac{dy}{dx} + y - 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - x \frac{dy}{dx} = 2x + y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y}{3y^2 - x}$$

3. Step 1

$$y = a(\sin t)$$

$$\Rightarrow \frac{dy}{dt} = a(\cos t)$$

$$x = a \left(\cos t + \ln \left(\tan \left(\frac{t}{2} \right) \right) \right) = a \cos t + a \ln \left(\tan \left(\frac{t}{2} \right) \right)$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + \frac{a}{2} \frac{\sec^2 \left(\frac{t}{2} \right)}{\tan \left(\frac{t}{2} \right)} = -a \sin t + \frac{a}{2} \frac{1}{\sin \left(\frac{t}{2} \right) \cos \left(\frac{t}{2} \right)}$$

$$= -a \sin t + \frac{a}{\sin t}$$

Step 3

$$v = (\sin x)^{\cos x}$$

Taking \ln on both sides, we get,

$$\ln v = \cos x \cdot \ln \sin x$$

Differentiate the equation w.r.t x .

$$\frac{1}{v} \frac{dv}{dx} = \left(\frac{\cos x}{\sin x} \cdot \cos x - \sin x (\ln \sin x) \right)$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left(\frac{\cos^2 x}{\sin x} - \sin x (\ln \sin x) \right) \dots (ii)$$



Step 2

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{a \cos t}{a \left(\frac{1 - \sin^2 t}{\sin t} \right)} = \frac{a \cos t \sin t}{a(\cos^2 t)} = \tan t$$

So, $\frac{dy}{dx} = \tan t$

Self-Assessment

Step 1

$$\frac{dy}{dt} = \frac{3}{2}(-2t^{-3}) - \frac{2}{t^2} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{1+t}{t^3} \right) = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = t$$

Step 2

$$x \left(\frac{dy}{dx} \right)^3 - \frac{dy}{dx} = \frac{1+t}{t^3} (t^3) - t = 1$$

METHODS OF DIFFERENTIATION

HIGHER ORDER DERIVATIVES



What you already know

- Differentiation of inverse trigonometric functions
- Standard substitutions



What you will learn

- Derivative of one function with respect to another
- Higher order derivatives

Derivative of One Function with respect to Another Function

Let us consider two functions, $f(x)$ and $g(x)$.

Let $u = f(x)$, $v = g(x)$

Hence, $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

Therefore, the derivative of $f(x)$ with respect to $g(x)$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{g'(x)}$$



Find the derivative of

(i) $\sin x$ with respect to $\cos x$ (ii) $y = \ln x$ with respect to $z = e^x$

Solution

(i)

Let $u = \sin x$ and $v = \cos x$

Now, the derivative of u with respect to v ,

$$\begin{aligned}\frac{du}{dv} &= \frac{\frac{du}{dx}}{\frac{dv}{dx}} \\ &= \frac{\frac{d(\sin x)}{dx}}{\frac{d(\cos x)}{dx}} = \frac{\cos x}{-\sin x} = -\cot x\end{aligned}$$

(ii)

$y = \ln x$ and $z = e^x$

Now, the derivative of y with respect to z

$$\begin{aligned}\frac{dy}{dz} &= \frac{\frac{dy}{dx}}{\frac{dz}{dx}} \\ &= \frac{\frac{d(\ln x)}{dx}}{\frac{d(e^x)}{dx}} = \frac{\frac{1}{x}}{e^x} = \frac{1}{xe^x}\end{aligned}$$



Differentiate $\left(\frac{x^4 + x^2 + 1}{x^2 - x + 1} \right)$ with respect to $x^2 + 1$

Solution

Step 1:

$$\begin{aligned} \text{Let } y &= \left(\frac{x^4 + x^2 + 1}{x^2 - x + 1} \right) \\ &= \frac{(x^2 + x + 1)(x^2 - x + 1)}{(x^2 - x + 1)} \\ &= x^2 + x + 1 \end{aligned}$$

$$\text{and } z = x^2 + 1$$

Step 2:

Now, the derivative of y with respect to z

$$\begin{aligned} \frac{dy}{dz} &= \frac{\frac{dy}{dx}}{\frac{dz}{dx}} \\ \Rightarrow \frac{dy}{dz} &= \frac{\frac{d(x^2 + x + 1)}{dx}}{\frac{d(x^2 + 1)}{dx}} = \frac{2x + 1}{2x} = 1 + \frac{1}{2x} \end{aligned}$$

Higher Order Derivatives

If $y = f(x)$,

Then, $\frac{dy}{dx} = y' = y_1 = f' \rightarrow$ First order derivative

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = y'' = y_2 = f'' \rightarrow$ Second order derivative

.....

.....

$\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right) = y''^{\text{n times}} = y_n = f''^{\text{n times}} \rightarrow n^{\text{th}}$ order derivative



If $y = \cos x$, then find $y_1^2 + y_2^2$

Solution

$$y_1 = \frac{dy}{dx} = \frac{d(\cos x)}{dx} = -\sin x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx}(-\sin x) = -\cos x$$

$$\therefore y_1^2 + y_2^2 = (-\sin x)^2 + (-\cos x)^2 = \sin^2 x + \cos^2 x = 1$$



If $y = (x + \sqrt{x^2 - 1})^m$, then show that $(x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = m^2 y$

Solution

Step 1:

$$y = (x + \sqrt{x^2 - 1})^m$$

Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}(x + \sqrt{x^2 - 1})^m$$

$$= m(x + \sqrt{x^2 - 1})^{m-1} \left(1 + \frac{2x}{2\sqrt{x^2 - 1}} \right)$$

$$= m(x + \sqrt{x^2 - 1})^{m-1} \frac{(x + \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}}$$

$$= \frac{m(x + \sqrt{x^2 - 1})^m}{\sqrt{x^2 - 1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{my}{\sqrt{x^2 - 1}}$$

Step 2:

$$\frac{dy}{dx} = \frac{my}{\sqrt{x^2 - 1}}$$

$$\Rightarrow (\sqrt{x^2 - 1})y' = my \dots\dots\dots\dots\dots (i)$$

Differentiating both sides with respect to x,

$$\left(\frac{2x}{2\sqrt{x^2 - 1}} \right)y' + (\sqrt{x^2 - 1})y'' = my'$$

$$\Rightarrow \frac{x}{\sqrt{x^2 - 1}} \times y' + \sqrt{x^2 - 1} \times y'' = my'$$

$$\Rightarrow xy' + (x^2 - 1)y'' = my'\sqrt{x^2 - 1}$$

From equation (i)

$$x \frac{dy}{dx} + (x^2 - 1) \frac{d^2y}{dx^2} = m^2 y$$

Hence, proved.

Higher order derivative - Parametric form

Let two variables x and y be functions of a parameter t.

i.e., $y = f(t)$, $x = g(t)$

Hence, $\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) = \frac{f'(t)}{g'(t)}$ and,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$



Note

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}, \text{ but } \frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$



JEE MAIN JAN 2019



If $x = 3 \tan t$ and $y = 3 \sec t$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

- (a) $\frac{1}{6\sqrt{2}}$ (b) $\frac{1}{3\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{6}$

Solution

Step 1:

$$\begin{aligned}x &= 3 \tan t \Rightarrow \frac{dx}{dt} = 3 \sec^2 t \\y &= 3 \sec t \Rightarrow \frac{dy}{dt} = 3 \sec t \times \tan t \\&\Rightarrow \frac{dy}{dx} = \frac{3 \sec t \times \tan t}{3 \sec^2 t} = \sin t \\&\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} (\sin t) \\&\quad \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \right) \\&= \frac{\cos t}{3 \sec^2 t}\end{aligned}$$

Step 2:

$$\begin{aligned}&\text{At } t = \frac{\pi}{4} \\&\frac{d^2y}{dx^2} = \frac{\cos\left(\frac{\pi}{4}\right)}{3\sec^2\left(\frac{\pi}{4}\right)} \\&= \frac{1}{3} \times \left(\cos \frac{\pi}{4} \right)^3 \\&= \frac{1}{6\sqrt{2}}\end{aligned}$$

∴ Option (a) is the correct answer



If $x = 2\sin \theta - \sin 2\theta$ and $y = 2\cos \theta - \cos 2\theta$, $\theta \in [0, 2\pi]$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \pi$.

- (a) $\frac{3}{8}$ (b) $\frac{3}{2}$ (c) $\frac{3}{4}$ (d) $-\frac{3}{4}$

JEE MAIN JAN 2020

Solution

Step 1:

$$\begin{aligned}x &= 2 \sin \theta - \sin 2\theta \\&\Rightarrow \frac{dx}{d\theta} = 2 \cos \theta - 2 \cos 2\theta \\y &= 2 \cos \theta - \cos 2\theta \\&\Rightarrow \frac{dy}{d\theta} = -2 \sin \theta + 2 \sin 2\theta\end{aligned}$$

$$\begin{aligned}&\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} = \frac{2\sin 2\theta - 2\sin \theta}{2\cos \theta - 2\cos 2\theta} \\&= \frac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} \\&= \frac{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} \\&\Rightarrow \frac{dy}{dx} = \cot\left(\frac{3\theta}{2}\right)\end{aligned}$$


Step 2:

$$\frac{dy}{dx} = \cot\left(\frac{3\theta}{2}\right)$$

Differentiating both sides with respect to x

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{d\theta}\left(\cot\frac{3\theta}{2}\right) \\ &= \frac{-\csc^2\left(\frac{3\theta}{2}\right) \times 3}{2(\cos\theta - \cos 2\theta)} \\ &= \frac{-\csc^2\left(\frac{3\theta}{2}\right) \times 3}{2(\cos\theta - \cos 2\theta)}\end{aligned}$$

At $\theta = \pi$

$$\frac{d^2y}{dx^2} = \frac{-\csc^2\left(\frac{3\pi}{2}\right) \times 3}{2(\cos\pi - \cos 2\pi)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-3}{2}}{-4} = \frac{3}{8}$$

∴ Option (a) is the correct answer


Concept Check

1. Find the derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$, where $x \in \left(0, \frac{\pi}{2}\right)$.

- (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) 2 (d) 1

JEE MAIN APR 2019

2. If $y^2 + \ln(\cos^2 x) = y$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then which of the following is true?

- (a) $y'(0) + y''(0) = 3$ (b) $y''(0) = 2$ (c) $y'(0) + y''(0) = 1$ (d) $y''(0) = 0$

JEE MAIN SEPT 2019

3. Find the second derivative of $\ln x$ with respect to $\sin x$.


Summary Sheet

Key Takeaways

- Let $u = f(x)$, $v = g(x)$ then, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{g'(x)}$
- $\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right) = y'' \dots \dots \dots \text{n times} = y_n = f'' \dots \dots \dots \text{n times} \rightarrow n^{\text{th}} \text{ order derivative}$



- Let two variables x and y be functions of a parameter t i.e., $y = f(t)$, $x = g(t)$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$



Mind Map

Derivative of one function
with respect to another

Higher order derivative

Parametric form



Self-Assessment

If $y = \sec(\tan^{-1} x)$, then find $\frac{dy}{dx}$ at $x = 1$



Answers

Concept Check

1.

Step 1:

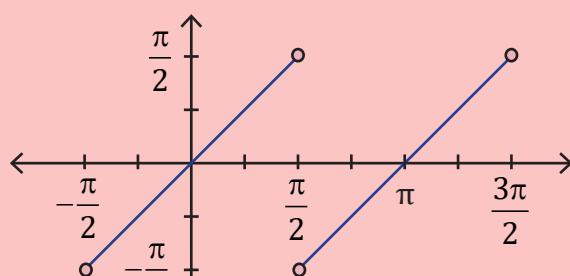
$$\begin{aligned} \text{Let } y &= \tan^{-1} \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right) \\ &= \tan^{-1} \left(\frac{\frac{\sin x}{\cos x} - 1}{\frac{\sin x}{\cos x} + 1} \right) \end{aligned}$$

(Dividing numerator and denominator by $\cos x$)

$$= \tan^{-1} \left(\frac{\tan x - \tan \frac{\pi}{4}}{1 + \tan x \cdot \tan \frac{\pi}{4}} \right)$$

$$\begin{aligned} y &= \tan^{-1} \left(\tan \left(x - \frac{\pi}{4} \right) \right) \\ &= x - \frac{\pi}{4} \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right) \end{aligned}$$

From the graph of $\tan^{-1}(\tan x)$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
 $\tan^{-1}(\tan t) = t$



$$\text{and } z = \frac{x}{2}$$

**Step 2:**

Now, derivative of y with respect to z,

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$= \frac{d\left(x - \frac{\pi}{4}\right)}{dx} = \frac{1}{\frac{d\left(\frac{x}{2}\right)}{dx}} = \frac{1}{\frac{1}{2}} = 2$$

\therefore Option (c) is the correct answer

2.

Step 1:

$$\text{Given, } y^2 + \ln(\cos^2 x) = y$$

By substituting $x = 0$, we get,

$$y^2 + \ln 1 = y$$

$$\Rightarrow y^2 - y = 0 \Rightarrow y = 0, 1$$

Differentiating both sides,

$$2yy' + \frac{1}{\cos^2 x} \times 2 \cos x \times (-\sin x) = y'$$

$$2yy' - 2 \tan x = y' \dots\dots\dots(1)$$

By substituting $x = 0$, we get,

$$2y(0)y'(0) - 2 \tan 0 = y'(0)$$

$$\text{If } y(0) = 0, \text{ then } y'(0) = 0$$

$$\text{If } y(0) = 1, \text{ then } y'(0) = 0$$

3.

Step 1:

$$\text{Let } y = \ln x, z = \sin x$$

$$\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{\frac{1}{x}}{\cos x} = \frac{1}{x \cos x}$$

Step 2:

$$\begin{aligned} \frac{d^2y}{dz^2} &= \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(\frac{1}{x \cos x}\right) \\ &= \frac{x \sin x - \cos x}{x^2 \cos^3 x} \end{aligned}$$

Step 2:

Differentiating (1) on both sides,

$$2yy'' + 2y'y' - 2 \sec^2 x = y''$$

$$\Rightarrow 2yy'' + 2(y')^2 - 2 \sec^2 x = y''$$

By substituting $x = 0$, we get

$$2y(0)y''(0) + 2(y'(0))^2 - 2 \sec^2 0 = y''(0)$$

$$\text{If } y(0) = 0 \text{ and } y'(0) = 0, \text{ then } y''(0) = -2$$

$$\text{If } y(0) = 1 \text{ and } y'(0) = 0, \text{ then } y''(0) = 2$$

$$\Rightarrow |y''(0)| = 2$$

\therefore Option (b) is the correct answer.

Self-Assessment

$$y = \sec(\tan^{-1} x)$$

$$\therefore \frac{dy}{dx} = \sec(\tan^{-1} x) \cdot \tan(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

At $x = 1$,

$$\begin{aligned} \frac{dy}{dx} &= \sec(\tan^{-1} 1) \cdot \tan(\tan^{-1} 1) \cdot \frac{1}{1+1} \\ &= \sec \frac{\pi}{4} \times 1 \times \frac{1}{2} = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

METHODS OF DIFFERENTIATION

DIFFERENTIATION OF DETERMINANTS



What you already know

- Logarithmic differentiation
- Differentiation of inverse trigonometric functions



What you will learn

- Differentiation of determinants

Differentiation of Determinants

To differentiate a determinant, we differentiate one row at a time, keeping others unchanged. That is,

$$\text{If } \Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}, \text{ then}$$

Differentiating w.r.t x, we get,

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

The same operation can be performed column wise.



If $\begin{vmatrix} e^x & \sin x \\ \cos x & \ln(1+x) \end{vmatrix} = A + Bx + Cx^2 + Dx^3 + \dots$, then the ordered point (A, B) is :

- (a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)

Solution

Step 1:

$$\begin{vmatrix} e^x & \sin x \\ \cos x & \ln(1+x) \end{vmatrix} = A + Bx + Cx^2 + Dx^3 + \dots, \text{ is an identity in } x, \text{ so it must hold true for all values of } x \in \mathbb{R}.$$

Step 2:

Value of A can be obtained by substituting $x = 0$

$$\begin{aligned} A &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \quad \left(\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$



Step 3:

To obtain the value of B, let's differentiate the given equation and then substitute $x = 0$

Differentiating w.r.t x, we get,

$$\begin{vmatrix} e^x & \cos x \\ \cos x & \ln(1+x) \end{vmatrix} + \begin{vmatrix} e^x & \sin x \\ -\sin x & \frac{1}{1+x} \end{vmatrix} = B + 2Cx + 3Dx^2 + \dots$$

Substituting $x = 0$, we get,

$$\begin{aligned} B &= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

The ordered pair (A, B) is (0, 0)

So, option (a) is the correct answer.



If $xe^{xy} = y + \sin^2 x$, then find the value of $\frac{dy}{dx}$ at $x = 0$.

Solution

Step 1:

$$xe^{xy} = y + \sin^2 x \quad (\text{i})$$

Differentiating both sides w.r.t x, we get,

$$e^{xy} + xe^{xy} \left(x \frac{dy}{dx} + y \right) = \frac{dy}{dx} + 2 \sin x \cos x \quad (\text{ii})$$

Step 2:

To find out the value of $\frac{dy}{dx}$ at $x = 0$, we need the value of y at $x = 0$

Substituting $x = 0$ in (i), we get,

$$0 = y + 0 \Rightarrow y = 0$$

Step 3:

Substituting $x = 0, y = 0$ in equation (ii), we get,

$$\frac{dy}{dx} = 1$$



IIT JEE 2012



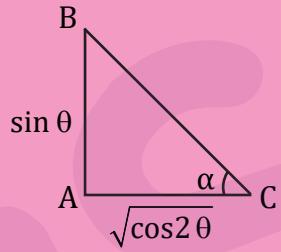
Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is:

Solution

Step 1:

$$\text{Let } \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \alpha$$

$$\Rightarrow \tan\alpha = \frac{\sin\theta}{\sqrt{\cos 2\theta}}, f(\theta) = \sin\alpha$$



Step 2:

Now,

$$BC = \sqrt{\sin^2 \theta + \cos 2\theta} = \sqrt{\sin^2 \theta + 1 - 2\sin^2 \theta}$$

$$= \sqrt{1 - \sin^2 \theta} = |\cos \theta| = \cos \theta \left(\because -\frac{\pi}{4} < \theta < \frac{\pi}{4} \right)$$

$$\Rightarrow \sin \alpha = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\frac{d}{d(\tan\theta)}(f(\theta)) = \frac{d}{d(\tan\theta)}(\tan\theta) = 1$$

JEE MAIN 2013



If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is:

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

Solution

Step 1:

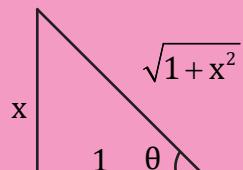
Given $y = \sec(\tan^{-1}x)$

Let $\tan^{-1}x = \theta \Rightarrow \tan \theta = x$ where $\theta \in \left(0, \frac{\pi}{2}\right)$ as $x = 1$
 $\Rightarrow y = \sec \theta$

Step 2:

$$\cos \theta = \frac{1}{\sqrt{1+x^2}}, y = \frac{1}{\cos \theta}$$

$$\Rightarrow y = \sqrt{1+x^2}$$



Step 3:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \times (2x) = \frac{x}{\sqrt{1+x^2}}$$



Substituting $x = 1$, we get,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

So, option (a) is the correct answer.

Alternate solution

$$y = \sec(\tan^{-1}x)$$

$$\frac{dy}{dx} = \sec(\tan^{-1}x) \cdot \tan(\tan^{-1}x) \cdot \frac{1}{1+x^2}$$

Substituting $x = 1$, we get,

$$\begin{aligned}\frac{dy}{dx} &= \sec(\tan^{-1}1) \cdot \tan(\tan^{-1}1) \cdot \frac{1}{2} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

IIT JEE 2007



$\frac{d^2x}{dy^2}$ equals to:

- (a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Solution

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d\left(\frac{dx}{dy}\right)}{dy}$$

$$= \frac{\frac{d}{dx}\left(\frac{dx}{dy}\right)}{\frac{dy}{dx}} = \frac{\frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)}{\frac{dy}{dx}}$$

$$\frac{d^2x}{dy^2} = \frac{\left(\frac{-1}{\left(\frac{dy}{dx}\right)^2}\right) \times \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{-\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^3}$$

So, option (d) is the correct answer.



-  If $f''(x) = -f(x)$ where $f(x)$ is continuous, twice differentiable function and $g(x) = f'(x)$. If $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and $F(5) = 5$, then $F(10)$ is :
- (a) 0 (b) 5 (c) 10 (d) 25

Solution

Step 1:

$$g(x) = f'(x) \Rightarrow g'(x) = f''(x)$$

$$\text{Also, } f''(x) = -f(x)$$

$$\Rightarrow g'(x) = -f(x) \dots\dots (i)$$

Step 2:

$$\begin{aligned} F(x) &= \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2 \\ \Rightarrow F'(x) &= 2f\left(\frac{x}{2}\right) \cdot f'\left(\frac{x}{2}\right) \cdot \frac{1}{2} + 2g\left(\frac{x}{2}\right) \cdot g'\left(\frac{x}{2}\right) \cdot \frac{1}{2} \\ &= f\left(\frac{x}{2}\right) \cdot f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right) \cdot g'\left(\frac{x}{2}\right) \\ &= f\left(\frac{x}{2}\right) \cdot f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right) \cdot \left(-f\left(\frac{x}{2}\right)\right) \quad (\because g'(x) = -f(x), g(x) = f'(x)) \\ &= 0 \end{aligned}$$

Step 3:

$$F'(x) = 0 \Rightarrow F(x) = \text{constant}$$

$$\text{Also, } F(5) = 5 \Rightarrow F(10) = 5$$

So, option (b) is the correct answer.



Concept Check

1. If $f(x) = \begin{vmatrix} x & x^2 \\ x^3 & 2 \end{vmatrix}$, then the value of $f'(1)$ is :

- (a) 0 (b) 2 (c) -3 (d) 1

2. Let $f(x)$ be a quadratic expression which is positive for all real values of x .

If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x :

- (a) $g(x) > 0$ (b) $g(x) < 0$ (c) $g(x) = 0$ (d) $g(x) \geq 0$

3. If $y = \sin^{-1} \left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2} \right)$, where $x+x^2 \leq 1$ and $\frac{dy}{dx} = \frac{1}{2\sqrt{x(1-x)}} + p$, then p is :

- (a) 0 (b) $\frac{1}{\sqrt{1-x}}$ (c) $\sin^{-1} \sqrt{x}$ (d) $\frac{1}{1-x^2}$



Summary Sheet



Key Takeaways

- **Differentiation of determinants**

To differentiate a determinant, we differentiate one row at a time, keeping others unchanged.

If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then

$$\Delta'(\mathbf{x}) = \begin{vmatrix} f_1'(\mathbf{x}) & f_2'(\mathbf{x}) & f_3'(\mathbf{x}) \\ g_1(\mathbf{x}) & g_2(\mathbf{x}) & g_3(\mathbf{x}) \\ h_1(\mathbf{x}) & h_2(\mathbf{x}) & h_3(\mathbf{x}) \end{vmatrix} + \begin{vmatrix} f_1(\mathbf{x}) & f_2(\mathbf{x}) & f_3(\mathbf{x}) \\ g_1'(\mathbf{x}) & g_2'(\mathbf{x}) & g_3'(\mathbf{x}) \\ h_1(\mathbf{x}) & h_2(\mathbf{x}) & h_3(\mathbf{x}) \end{vmatrix} + \begin{vmatrix} f_1(\mathbf{x}) & f_2(\mathbf{x}) & f_3(\mathbf{x}) \\ g_1(\mathbf{x}) & g_2(\mathbf{x}) & g_3(\mathbf{x}) \\ h_1'(\mathbf{x}) & h_2'(\mathbf{x}) & h_3'(\mathbf{x}) \end{vmatrix}$$

Note: The same operation can be performed column wise.



Mind Map

Methods of Differentiation

Differentiation of Determinants



Self-Assessment

(1) If $y = a \cos(\ln x) + b \sin(\ln x)$, then $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx}$ is:



Answers

Concept Check

(1) Given, $f(x) = \begin{vmatrix} x & x^2 \\ x^3 & 2 \end{vmatrix}$

Differentiating both sides w.r.t x, we get,

$$f'(x) = \begin{vmatrix} 1 & 2x \\ x^3 & 2 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ 3x^2 & 0 \end{vmatrix}$$

$$f'(1) = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$



$$= (2-2) + (0-3) \quad \left(\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right)$$

$$= -3$$

So, option (c) is the correct answer.

2.

Solution

Step 1:

Let $f(x) = ax^2 + bx + c$

Since, $f(x)$ is positive for all real values of x . So, $b^2 - 4ac < 0$

Let $a = 1$.

$f(x) = x^2 + bx + c$ and $b^2 - 4c < 0$

Step 2:

$$g(x) = f(x) + f'(x) + f''(x)$$

$$g(x) = x^2 + x(b+2) + (b+2+c)$$

Discriminant (D) of $g(x) = (b+2)^2 - 4(b+2+c) = b^2 - 4c - 4 < 0$ as $b^2 - 4c < 0$

For $g(x)$, coefficient of x^2 is positive and D is negative.

$g(x) > 0$ for all real values of x .

So, option (a) is the correct answer.

3.

Step 1:

Given $y = \sin^{-1}(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2})$, where $x+x^2 \leq 1 \dots (i)$

Also, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$, $x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \dots (ii)$

Comparing (i) and (ii),

$$\begin{aligned} y &= \sin^{-1}(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}) \\ &= \sin^{-1}\left(x\sqrt{1-\left(\sqrt{x}\right)^2} + \sqrt{x}\sqrt{1-x^2}\right) \\ &= \sin^{-1}(x) + \sin^{-1}(\sqrt{x}) \end{aligned}$$

Step 2:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} \dots (iii) \end{aligned}$$

It is given that,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}(1-x)} + p$$

Comparing it with (iii), we get,

$$p = \frac{1}{\sqrt{1-x^2}}$$

So, option (d) is the correct answer.

**Self-Assessment**

$$y = a \cos(\ln x) + b \sin(\ln x)$$

$$\frac{dy}{dx} = -\frac{a}{x} \sin(\ln x) + \frac{b}{x} \cos(\ln x)$$

$$x \frac{dy}{dx} = -a \sin(\ln x) + b \cos(\ln x)$$

Differentiating w.r.t x, we get,

$$xy'' + y' = -\frac{a}{x} \cos(\ln x) - \frac{b}{x} \sin(\ln x)$$

$$x^2 y'' + xy' = -a \cos(\ln x) - b \sin(\ln x)$$

$$x^2 y'' + xy' = -y$$

So, option (c) is the correct answer.