SEQUENCE & SERIES

1. Arithmetic Progression (AP) :

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If 'a' is the **first term** & 'd' is the **common difference**, then AP can be written as a, a + d, a + 2d,........ a + (n–1)d, ...

(a) n^{th} term of this AP is $T_n = a + (n-1)d$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$ where l is the last term

(c) Also n^{th} term $[T_n = S_n - S_{n-1}]$

Note :

(i) If sum of first n terms of an AP is of the form An^2+Bn i.e.a quadratic expression in n, then in such case the common difference is twice the coefficient of n^2.i.e. 2A

(ii) If n^{th} term of an AP is of the form An + B i.e. a linear expression in n, then in such case the coefficient of n is the common difference of the AP i.e. A

(iii) Three numbers in AP can be taken as a – d, a, a+d:
four numbers in AP can be taken as a – 3d, a – d, a + d, a +3d
five numbers in AP are a – 2d, a – d, a, a + d, a + 2d &
six terms in AP are a –5d, a – 3d, a – d, a + d, a + 3d, a+5d etc.

(iv) If for an AP p^{th} term is q,q^{th} term is p, then r^{th} term is = p +q – r & $(p + q)^{th}$ term is 0.

(v) If a_1, a_2, a_3 ...... and b_1, b_2, b_3 ...... are two AP s, then, $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$........ are also in AP

(vi) (a) If each term of an AP is increased or decreased by the same number then the resulting sequence is also an AP having the same common difference.

(b) If each term of an AP is muitiplied or divided by the same non zero number ‘k’ then the resulting sequence is also an AP whose common difference is kd or d/k respectively, where d is common difference of original AP.

(vii) Any term of an AP (except the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2}, \quad k < r$$

2. Geometric Progression (GP):

GP is a sequence of non zero numbers whose succeeding terms is equal to the preceeding terms multiplied by a constant.Thus in a GP the ratio of successive terms taken in order is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2,ar^3,ar^4,......... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term $\quad T_n = a r^{n-1}$

(b) Sum of the first n terms $S_n = \dfrac{a(r^n - 1)}{r - 1}$ **if** $r \neq 1$

(C) Sum of infinite GP when $|r| < 1$ & $n \to \infty$, $r^n \to 0$

$$S_\infty = \frac{a}{1-r}; |r| < 1$$

(d) If a,b, c are in GP$\Rightarrow b^2 = ac$ and loga, logb, logc, are in AP.

(e) In a GP, product of k^{th} term from begining and k^{th} term from the last is always constant which equal to product of first term and last term.

(f) Three numbers in **GP** $\quad$ **a/r, a ar**

Four numbers in **GP** $\quad$ $a/r^3, a/r, ar, ar^3$,

Five numbers in **GP** $\quad$ $a/r^2, a/r, a, ar, ar^2$

Six numbers in **GP** $\quad$ $a/r^5, a/r^3, a/r, ar, ar^3, ar^5$

(g) If each term of a **GP** is raised to the same power, then resulting series also a **GP.**

(h) If each term of a GP is multiplied or divided by the same non-zero quantity, then the resulting sequence is also a GP.

(i) If a_1, a_2, a_3...... and b_1, b_2, b_3,..... be two GP.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2......... and $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}$ .... will also form a GP & common ratio will be $r_1 \cdot r_2$ and $\dfrac{r_1}{r_2}$ respectively.

(j) In a positive GP every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e. $T_r = \sqrt{T_{r-k}T_{r+k}}$, $k < r$

(k) If a_1, a_2, a_3......... a_n, is a GP of non zero, non negative terms, then **log a_1, log a_2,.....log a_n** is an **AP** and **vice-versa.**

3. Harmonic progtession (HP) :

A non zero sequence is said to HP if the reciprocals of its terms are in AP. If the sequence a_1, a_2, a_3,...... a_n is a HP then $1/a_1, 1/a_2$,.......... $1/a_n$. is an AP. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\dfrac{1}{a}, \dfrac{1}{a+d}, \dfrac{1}{a+2d},$ .... $\dfrac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero. If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

4. Means:

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c, are in AP then b is AM of a &c.

n-arithmetic means between two numbers ;

If a, b, are any two given numbers & a, A_1, A_2,.....A_n, b are in AP then A_1, A_2....A_n are the n AM's between a & b, then

$A_1 = a + d$,$A_2 = a + 2d$,........, $A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

(b) Geometric mean (GM):

If a, b, c are in GP, b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$

n-geometric means between two numbers :

If a, b are two given positive numbers & a, G_1,G_2, ...G_n, b are in GP then G_1, G_2, G_3, ...G_n, are n GMs between a&b. $G_1 = ar$,$G_2 = ar^2$, ... $G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to n^{th} power of the single GM between a&b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a &

(C) Harnonic mean (HM) :

If a, b, c are in HP, then b is HM between a & c, then $b = \frac{2ac}{a+c}$

Important Note :

(i) If A, G, H, are respectively AM, GM, HM between two positive number a & b then

(a) $G^2 = AH$ (A,G, H constitute a GP) (b)$A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let a_1, a_2, ... a_n. be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonir mean (H) as A

mean (H) as $A = \frac{a_1 + a_2 + ... + a_n}{n}$

$G = (a_1 a_2 .... a_n)^{1/n}$ and H $\dfrac{n}{\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}+......\dfrac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$.

5. Arithmetico - Geometric series:

Sum of First n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n-1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]r^n}{1-r}, r \neq 1$

Sum to infinity :

If $|r| < 1$ & $n \to \infty$ then $= 0 \Rightarrow S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

6. Sigma Notations:

Therorems :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(b) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k$ = nk : where k is a constant.

7. Results:

(a) $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \dfrac{n}{30}$ $(n + 1)(2n + 1)(3n^2 + 3n - 1)$

BINOMIAL THEOREM

1. $(x+y)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1}y \,.....+ {}^nC_r x^{n-r}y^r \,...+ {}^nC_n y^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$, **where n ∈ N.**

Important Terms in the Binomial Expansion are :

(a) General term: The general term or the $(r + 1)^{th}$ term in the expansion of $(x+y)^n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r} \cdot y^r$.

(b) Middle term :

The middle term (s) is the expansion of $(x+y)^n$ is (are):

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term independent of x :

Term independent of x contains no x: Hence find the value of r for which the exponent of x is zero.

2. If $(\sqrt{A} + B)^n = I + f$, where I & n are positive integers & $0 \le f < I$, then

(a) $(I + f)\, f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I + f)\,(I - f) = K^n$ if n is even & $\sqrt{A} - B < I$

3. Some Results on Binomial Coefficients :

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} .... + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(f) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!\,n!}$

(h) $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + ... + C_{n-r}\, C_n = \frac{(2n)!}{(n+r)!\,(n-r)!}$

Note :- $(2n)! = 2^n \times n!\, [1.3.5.....(2n-1)]$

4. Greatest Coefficient & Greatest Term in Expansion of $(x+a)^n$:

(a) If n is given, greatest coefficient is ${}^{n}C_{n/2}$.

If n is odd, greatest coefficient is ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \& \ T_{p+1} & \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is an integer} \\ T_{q+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is noninteger and} \in (q, q+1),\ q \in I \end{cases}$$

6. Binomial Theorem for Negative or Fractional Indices :

If $n \in Q$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \infty$ provided $|x| < 1$.

Note :

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots \infty$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots \infty$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \infty$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots \infty$

6. Exponential Series :

(a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \infty$; where x may be any real or complex number &

$e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(b) $a^x = 1 + \frac{x}{1!}\ln a + \frac{x^2}{2!}\ln^2 a + \frac{x^3}{3!}\ln^3 a + \ldots\ldots \infty$, where $a > 0$.

7. Logarithmic Series :

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \infty$, where $-1 < x \le 1$

(b) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots \infty$, where $-1 \le x < 1$

(c) $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right)$ $|x| < 1$

PERMUTATIONS AND COMBINATIONS

1. Fundamental Principle of Counting :

If the first job can be completed in ‘m’ ways, after completion of first job, secound job can be completed in ‘n’ ways, then the two jobs in succession can be completed in ‘m x n’ ways. This can be extendedz to more jobs too, Also known as Fundamental principal of multiplication.

if the first job can be completed in ‘m’ ways, after completion of first job, secound job can be completed in ‘n’ ways, then the two jobs independently can be completed in ‘m+n’ ways. This can be extended to more jobs too. Also known as Fundamental principal of addition.

2. Factorial :

A Useful Notation : $n! = n(n-1)(n-2) \dots\dots\dots 3,2,1$:

$n! = n.(n-1)!$ where $n \in W$

$0! = 1! = 1$

$(2n)! = 2^n . n!$

Note: factorials of negative integers are not defined.

3. Permutation :

(a) ^nP_r denotes the number of permutations of n different things taken r at a time ($n \in N, r \in W, n \geq r$) $^nP_r = n(n-1)(n-2) \dots\dots\dots (n-r+1) = \frac{n!}{(n-r)!}$ (repetition is not allowed)

(b) The number of permutations of n things taken all at a time wher p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining n – (p +q + r) are all different is : $\frac{n!}{p!q!r!}$

(c) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r! \cdot {}^{n-1}C_{r-1}$.

(d) The number of permutation of n different objects taken r at a time when repetition is allowed any number of times is n.n.n........ r times = n^r.

(e) (i) The number of circular permutations of n different things taken all at a time is : $(n-1)! = \frac{^nP_n}{n}$

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$

(ii) The number of circular permutation of n different things taking r at a time distin guishing clockwise & anticlockwise arrangement is $\frac{^nP_r}{r}$

4. Combination

(a) ^nC_r denotes the number of combinations of n different things taken r at a time, and

$^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$ where $r \le n$; $n \in N$ and $r \in W$. ^nC_r is also denoted by $\binom{n}{r}$ or C (n,r). (if not repeated).

(b) The number of combination of n different things taking r at a time

(i) when p particular things are always to be included = $^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded = $^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded = $^{n-p-q}C_{r-p}$

(c) Given n different objects, the number of ways of selecting atleast one of them is , $^nC_1 + {^nC_2} + {^nC_3} + \ldots\ldots + {^nC_n} = 2^n - 1$.

This can also be stated as the total number of combinations of n distinct things.

(d) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r + ... things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : (p + 1) (q + 1) (r + 1) ...... –1.

(iii) The total number of ways of selecting one or more thing from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things $(p + 1)(q + 1)(r + 1)2^n - 1$

5. Divisors :

Let $N = p^a.q^b.r^c$ .... where p,q,r...... are distinct primes & a, b, c...... are natural numbers then :

(a) The total numbers of divisors of N including 1& N is = (a +1) (b + 1) (c + 1).....

(b) The sum of these divisors is = $(p^0 + p^1 + p^2 + \ldots + p^a)$ $(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)$....

(c) Number of ways in which N can be resolved as a product of the factor is $= \frac{1}{2}(a + 1)$ (b + 1) (c + 1) ...... if N is not a perfect square.

$= \frac{1}{2}[(a + 1)(b + 1)(c + 1) \ldots\ldots + 1]$ if N is perfect square.

(d) Number of ways in which a composite number N can be resolve into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

6. Divison and Distribution :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups containing m & n things respectively is : $\frac{(m+n)!}{m!n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of sub-division is $\frac{(2n)!}{n!n!2!}$: for any one way it is possible to inter change the two groups without obtaining a new distribution.

(iii) if 2n things are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!n!(2!)} \times 2!$.

(b) (i) Number of ways in which (m+n+p) different things can be divided into three groups containing m,n, & p things respectively is $\frac{(m+n+p)!}{m!n!p!}$ $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!n!3!n!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into *l* groups containing p objects each, m groups containing q objects each is equal to $\frac{n!(l+m)!}{(p!)^l(q!)^m l!m!}$

Here lp + mq = n

(d) Number of ways in which n identical things can be distributed to p persons if there is no restriction to the number of things received by them = p^n.

(e) Number of ways in which n distinct things may be distributed among p persons if each persons may receive none, one or more things is : ${}^{n+p-1}C_n$.

7 Dearrangement :

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\ldots\ldots+(-1)^n\frac{1}{n!}\right]$$

8 Important Result :

(a) Number of rectangles of any size in a square of size $n\times n$ is $^nC_2 \cdot {}^nC_2$ & number of square of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangle of any size in a rentangle of size $n\times p$ $(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ & number of squares of any size is $\sum_{r=1}^{n}(n+1-r)\,.\,(p+1-r)$.

(c) If there are n points in a plane of which m (< n) are collinear :

(i) Total number of lines obtained by joining these points is $^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangle $^nC_3 - {}^mC_3$

(d) Maximum number of point of intersection of n circles is $^nP_2$ & n lines is $^nC_2$.