TRIGONOMETRY PART-I

TRIGONOMETRIC FUNCTIONS

1. Relation between system of measurement of angles:

1Radian = $\frac{180}{\pi}$ degree $\approx$ 57°17'15" (approximately)

1degree = $\frac{\pi}{180}$ radian $\approx$ 0.0175 radian

2. Basic trigonometric identities:

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

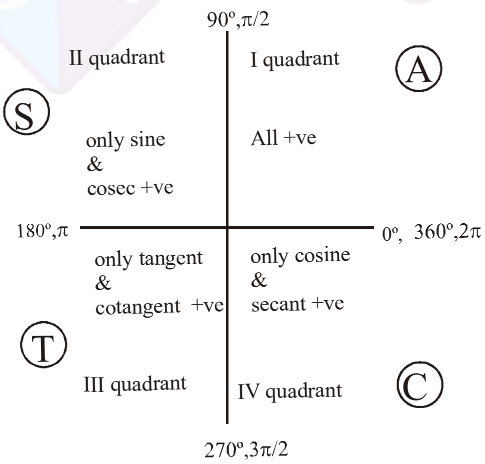
(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(c) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(d) $\text{coses}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(e) If $\text{coses}\theta + \cot\theta = k \Rightarrow \text{cosec}\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\text{cosec}\,\theta = k + \frac{1}{k}$

3. Signs of trigonometric functions in different quadrants:



4. Trigonometric functions of allied angles:

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi+\theta) = \cos\theta$, where $n \in I$

(b) $\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$\sin(90°-\theta) = \cos\theta$ $\cos(90°-\theta) = \sin\theta$

$\sin(90°+\theta) = \cos\theta$ $\cos(90°+\theta) = -\sin\theta$

$\sin(180°-\theta) = \sin\theta$ $\cos(180°-\theta) = -\cos\theta$

$\sin(180°+\theta) = -\sin\theta$ $\cos(180°+\theta) = -\cos\theta$

$\sin(270°-\theta) = -\cos\theta$ $\cos(270°-\theta) = -\sin\theta$

$\sin(270°+\theta) = -\cos\theta$ $\cos(270°+\theta) = \sin\theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$, $\tan n\pi = 0$ where $n \in I$

(ii) $\sin(2n + 1)\frac{\pi}{2} = (-1)^n$: $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

5. Important trignometric formulae

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A-B) = \dfrac{\cot B\cot A + 1}{\cot B - \cot A}$

(ix) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.

(x) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$.

(xi) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$.

(xii) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

(xiii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xiv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xv) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xvi) $\cos D - \cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xvii) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(xviii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(xix) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\dfrac{1+\cos 2\theta}{2}}$

(xx) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\dfrac{1-\cos 2\theta}{2}}$

(xxi) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$

(xxii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(xxiii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(xxiv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(xxv) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$

(xxviii) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

$= \sum \sin A\cos B\cos C - \prod \sin A$

(xxix) $\cos(A+B+C)$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$\prod \cos A - \sum \sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(xxx) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

$= \dfrac{S_1 - S_3}{1 - S_2}$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots.\ \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6. Values of some T-Ratios for angles 18°, 36°, 15°, 22.5°, 67.5, ... etc.

(a) $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72° = \sin\frac{\pi}{10}$

(b) $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54° = \cos\frac{\pi}{5}$

(c) $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \sin\frac{\pi}{12}$

(d) $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(f) $\tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(g) $\tan(22.5°) = \sqrt{2} - 1 = \cot(67.5°) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(h) $\tan(67.5°) = \sqrt{2} + 1 = \cot(22.5°)$

7. Maximum & minimum values of trigonometric expressions:

(a) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$

(c) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(d) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or $a^2 + b^2$, if for some real θ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a,b,>0\}$

(e) Minimum value of $a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta$ is either $2ab$ or $a^2 + b^2$, if for some real θ equation $a\sin\theta = b\operatorname{cosec}\theta$ is true or not true $\{a,b,> 0\}$

8. Important results:

(a) $\sin\theta \sin(60° - \theta) \sin(60°+\theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta \cos(60° - \theta) \cos(60°+\theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta \tan(60° - \theta) \tan(60°+\theta) = \tan 3\theta$

(d) $\cot\theta \cot(60° - \theta) \cot(60°+\theta) = \cot 3\theta$

(e) (i) $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \frac{3}{2}$

(ii) $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \frac{3}{2}$

(f) (i) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi$, $n \in I$

(ii) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n+1)\frac{\pi}{2}$, $n \in I$

(g) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$, if $\theta \neq n\pi$

(h) $\cot A - \tan A = 2\cot 2A$

9. Conditional Identities:

If $A + B + C = 180°$, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(c) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(d) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(e) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(f) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(g) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

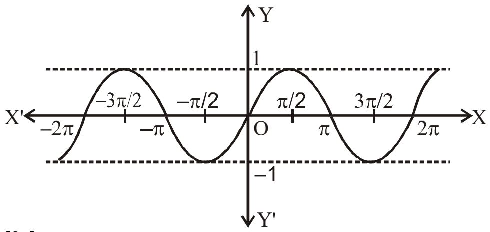
(h) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. Domains, Ranges and Periodicitity of trigonometric functions:

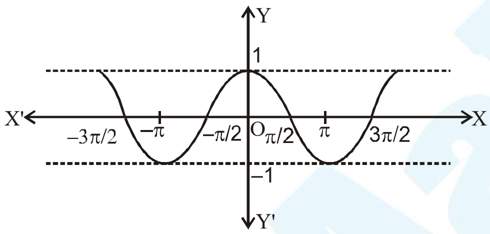
T-Ratio	Domain	Range	Period
sin x	R	$[-1,1]$	2π
cos x	R	$[-1,1]$	2π
tan x	$R-\{(2n+1)\pi/2\ ;n\in I\}$	R	π
cot x	$R-\{n\pi\ ;n\in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2\ ;n\in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
cosec x	$R-\{n\pi\ :n\in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

11. Graph of trigonometric functions:

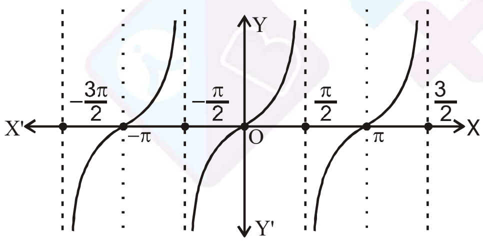
(a) y = sinx



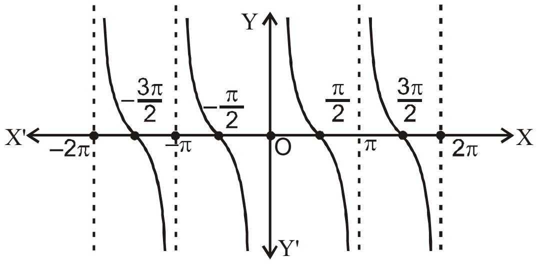
(b) y = cosx



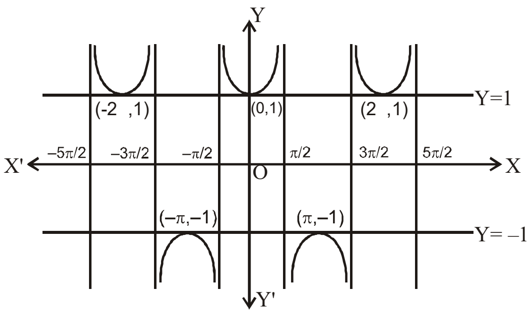
(c) y = tanx



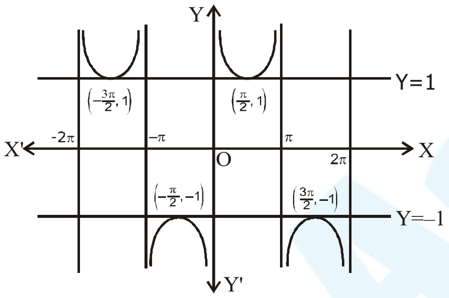
(iv) **y = cotx**



(v) **y = secx**



(vi) **y = cosecx**



12. Important note :

(a) The sum of interior angles of a polygon of n-sides

$= (n-2)\times 180° = (n-2)\pi$

(b) Each interior angle of a regular polygon of n sides

$$= \frac{(n-2)}{n}\times 180° = \frac{(n-2)}{n}\pi$$

(c) Sum of exterior angles of a polygon of any number of sides = 360° = 2 π

TRIGONOMETRY PART-II

TRIGONOMETRIC EQUATIONS

1. Trigonometric Equation:

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. Solution of trigonometric equation:

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) Principal solution :- The solution of the trigonometric equation lying in the interval $[0,2\pi]$.

(b) General solution :- Since all the trigonometric functions are many to one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solutions of trigono-metric equation.

3. General solutions of some trigonometric equations (To be remembered)

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi, n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha, n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi, n\in I$
(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi\pm\alpha, n\in I$,
(j) For $n \in I$, $\sin n\pi = 0$

$\sin(n\pi+\theta) = (-1)^n \sin\theta$

$\cos(n\pi+\theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n, n\in I$

(l) If n is an odd integer then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, $\cos\frac{n\pi}{2} = 0$

(m) $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$, $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$

(where n is odd integer)

4. General solutions of equation $a\cos\theta + b\sin\theta = c$:

Consider, $a\sin\theta + b\cos\theta = c$.......... (i)

$$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has the solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to $\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$

Now this equation can be solved easiliy.

5. General solution of equation of form:

$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + \ldots\ldots + a_n\cos^n x = 0$
Where a_0, a_1 .... a_n are real numbers.
Such an equation is solved by dividing equation by $\cos^n x$.

6. Important tips:

(a) For equation of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.
(b) Avold squaring the equations, if possible, because it may lead to extraneous solutions.
(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.
(d) The answer should not contain such values of θ which make any of the terms undefined or infinite.
(e) Check that denominator is not zero at any stage while solving equations.

(f) (i) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$

(ii) If $\cot\theta$ or $\mathrm{cosec}\,\theta$ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as tan θ and sec θ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.

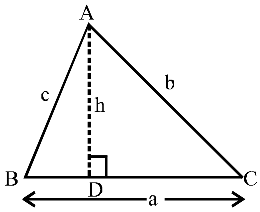
SOLUTIONS OF TRIANGLE

1. Sine formulae:

In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\lambda=\frac{abc}{2\Delta}=2R$$

where R is circumaradius and Δ is area of triangle



2. Cosine Formulae:

(a) $\cos A=\frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A$

(b) $\cos B=\frac{c^2+a^2-b^2}{2ac}$

(c) $\cos C=\frac{a^2+b^2-c^2}{2ab}$

3. Projection formulae:

(a) $b \cos C + c \cos B = a$
(b) $c \cos A + a \cos C = b$
(c) $a \cos B + b \cos A = c$

4. Napier's analogy (Tangent Rule):

(a) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$

(b) $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$

(c) $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

5. Half Angle formulae:

$s=\frac{a+b+c}{2}$ = semi-permeter of triangle

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

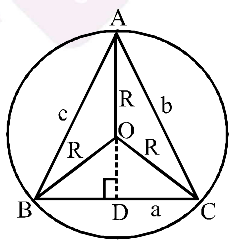
$= \frac{1}{2}\,bc\sin A = \frac{1}{2}\,ca\,\sin B = \frac{1}{2}\,ab\sin C$

$= \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}$

6. Radius of the circumcircle ‘R’:

Circumcentre is the point of concurrence of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$.



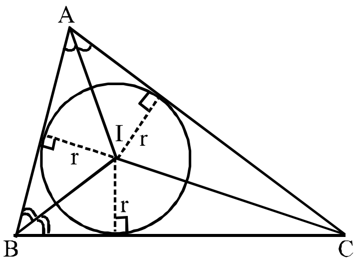
7. Radius of the incircle ‘r’:

Point of intersection of internal angle bisectors is incentre and perpen dicular distance of incentre from any side is called inradius 'r'

$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$

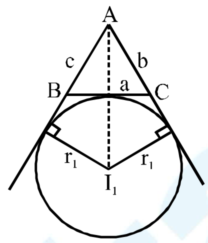
$= (s - c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= a\dfrac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\dfrac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\dfrac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$



8. Radii of the Ex-Circles:

Point of intersection of two external angle and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius.If r_1 is the radius of in-scribed circle opposite to angle A of ΔABC and so on then :



(a) $r_1 = \dfrac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(b) $r_2 = \dfrac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \dfrac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

(c) $r_3 = \dfrac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

9. Length of angle bisector, Medians & Altitude:

If m_a, β_a & h_a are the lengths of a median, an angle bisector & altiude from the angle A then,

$\frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A} = m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ and $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$, $h_a = \dfrac{a}{\cot B + \cot C}$

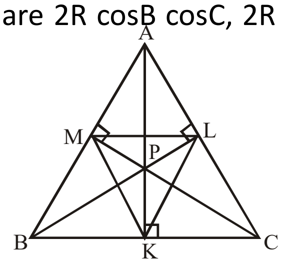
Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

10. Orthocentre and Pedal triangle:

(a) The point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the edal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB & 2R cosC.

(c) The distance of orthocentre from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.



(d) The sides of the pedal triangle are a cos A (=R sin 2A) bcosB = (R sin2B) and c cos C (=R sin 2C) and its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$

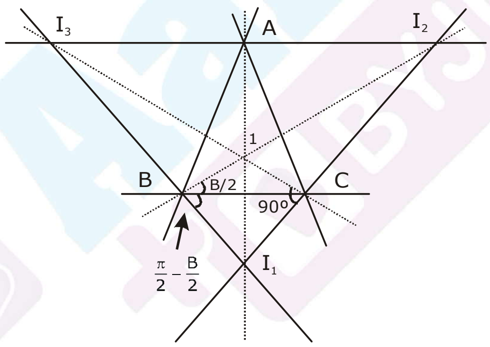
(e) Circumradii of the triangles PBC, PCA, PAB and ABC are equal

(f) **A**rea of pedal triangle = $2\Delta \cos A \cos B \cos C$

$$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$$

(g) Circumradii of pedal triangle = R/2

11. Ex-Central Tariangle:



(a) The trianlge formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(b) Incentre I of Δ ABC is the orthocentre of the excentral $\Delta\, I_1I_2I_3$.

(c) Δ ABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(d) The sides of the excentral triangle are

$$4R\cos\frac{A}{2}, 4R\cos\frac{B}{2} \text{ and } 4R\cos\frac{C}{2} \text{ and its angles are } \frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2} \text{ and } \frac{\pi}{2}-\frac{C}{2}$$

(e) $II_1 = 4R \sin\frac{A}{2}; II_2 = 4R\sin\frac{B}{2}; II_3 = 4R\sin\frac{C}{2}$

12. The distance between the special points:

(a) The distance between circumcentre and orthocentre is

$$= R\sqrt{1-8\cos A\cos B\cos C}$$

(b) The distance between circumcentre and incentre is

$$=\sqrt{R^2-2Rr}$$

(c) The distance between incentre and orthocentre is

$$=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$$

(d) The distances between circumcentre & excentres are

$$OI_1=R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\sqrt{R^2+2Rr_1}$$ & so on.

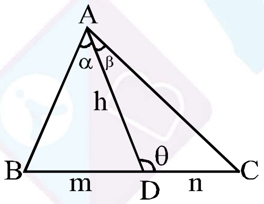
(e) Distance between circumcentre and centroid

$$OG = \sqrt{R^2-\frac{1}{9}(a^2+b^2+c^2)}$$

13. m-n Theorem:

$(m + n) \cot \theta = m \cot \alpha - n \cot \beta$

$(m + n) \cot \theta = n \cot \beta - m \cot \alpha.$



14. Important Points:

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

(b) In right angled triangle

(i) $a^2 + b^2 + c^2 = 8R^2$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$ (iii) $r : R : r_1 = 1 : 2 : 3$

(iv) area $=\frac{\sqrt{3}a^2}{4}$ (v) $R=\frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies

(1) inside an acute angled triangle

(2) outside an obtuse angled triangle &

(3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle all these centres coincide

15. Regular Polygon:

Consider a 'n' sided regular polygon of side length 'a'

(a) Radius of incircle of this polygon of side length r = $\frac{a}{2}\cot\frac{\pi}{n}$

(b) Radius of circumcircle of this polygon R = $\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

(c) Perimeter & area of regular polygon perimeter = na = 2nr tan $\frac{\pi}{n}$ = 2n R sin $\frac{\pi}{n}$

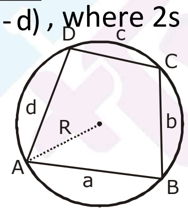
Area = $\frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{4}na^2\cot\frac{\pi}{n}$

16. Cyclic Quadrilateral:

(a) Quadriateral ABCD is cyclic if $\angle A + \angle C = \pi = \angle B + \angle D$

(opposite angle are supplementary angles)

(b) Area = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where 2s = a + b + c + d,



(c) cos B = $\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$ & similarly other angles

(d) Ptolemy's theorem : If ABCD is cyclic quadrilateral, then AC·BD = AB·CD + BC·AD

17. Solution of Triangles:

Case-I : If three sides are given then to find out three angles use

cosA = $\frac{b^2+c^2-a^2}{2bc}$, cos B = $\frac{c^2+a^2-b^2}{2ac}$ & cos C = $\frac{a^2+b^2-c^2}{2ab}$

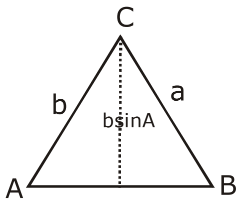
Case-II : Two sides & included angle are given :

Let sides a, b & angle C are given then use tan $\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$ and find value of A – B .... (i)

& $\frac{A+B}{2} = 90^\circ - \frac{C}{2}$ .... (ii) $c = \frac{a \sin C}{\sin A}$ .... (iii)

Case - III:

Two sides a, b & angle A opposite to one of then is given



(a) If a < b sin A No triangle exist

(b) If a = bsin A & A is acute, then one triangle exist which is right angle.

(c) a > b sin A, a < b & A is acute, then two triangle exist

(d) a > b sinA, a > b & A is acute,then one triangle exist

(e) a > b sin A & A is obtuse, then there is one triangle if a > b & no triangle if a < b.

HEIGHTS AND DISTANCES

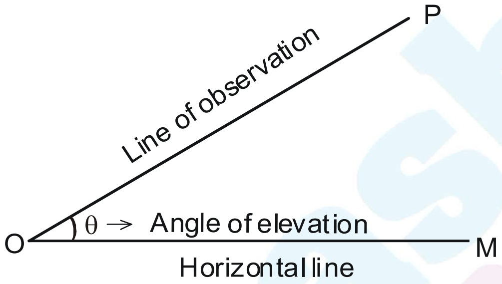
Introduction

One of the important applications of trigonometry is to find the height and distance of the point which are not directly measurable. This is done with the help of trigonometric ratios.

DEFINITIONS

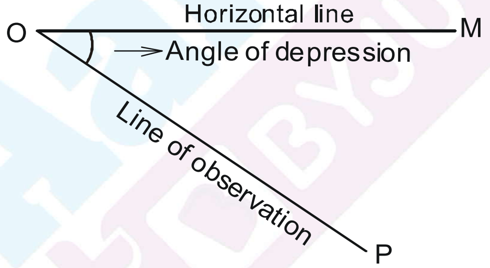
Angle of elevation :

Let O and P be two points where P is at a higher level than O. Let O be the position of the observer and P be the position of the object. Draw a horizontal line OM through the point O. OP is called the line of observation or line of sight. Then $\angle POM = \theta$ is called the angle of elevation of P as observed from O.



Angle of depression :

In the figure, if P be at a lower level than O, then $\angle MOP = \theta$ is called the angle of depression.



Note :

(1) The angle of elevation or depression is the angle between the line of observation and the horizontal line according as the object is at a higher or lower level than the observer.

(2) The angle of elevation or depression is always measured from horizontal line through the point of observation.

Some useful Results

In a triangle ABC,

$\sin\theta = \frac{p}{h}$, $\cos\theta = \frac{b}{h}$

$\tan\theta = p/b$