# CIRCULAR MOTION

## 1. Kinematics of Circular Motion :

- Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} = \vec{a}_t + \vec{a}_c$
- Tangential acceleration $a_t = \frac{d|\vec{v}|}{dt} = \alpha r$

  [$\vec{a}_t$ = component of $\vec{a}$ along $\vec{v} = (\vec{a} \cdot \hat{v})\hat{v}$ ]
- Centripetal acceleration $a_c = \omega v = \frac{v^2}{r} = \omega^2 r$ or $\vec{a}_c = \omega^2 r(-\hat{r})$
- Magnitude of net acceleration $a = \sqrt{a_c^2 + a_1^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$
- The concept of radius of curvature : The normal on tangent at a point on the curve gives the direction of radius.

  i.e., $R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

## 2. Centripetal and Centrifugal Forces :

### 2.1 Centripetal Force :

Centripetal force can be expressed as

$$\vec{F} = -m\omega^2 \vec{r} = -m\omega^2 \hat{r} = -\left(\frac{mv^2}{r}\right)\hat{r}$$

(a) If the body comes to rest on a circular path i.e., $\vec{v} \to 0$, the body will move along the radius towards the centre and if $a_r$ vanishes, the body will fly off tangentially, so a tangential velocity and radial acceleration are necessary for uniform circular motion.

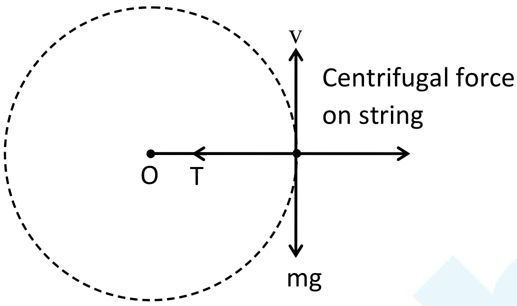
(b) As $F \neq 0$, so the body is not in equilibrium and linear momentum of the particle does not reamin conserved but angular momentum is conserved as the force is central i.e. $\tau = 0$

(c) In the case of circular motion, centripetal force changes only the direction of velocity of the particle.

**2.2 Centrifugal Force :**

(a) In a non-inertial frame Centrifugal force exist which is equal and opposite to centripetal force which exist in an inertial frame.

(b) Under centrifugal force body moves only along a sraight line. It appears when centripetal force ceases to exist. Since the body is viewed from a non-inertial frame.




(c) In an inertial frame, the centrifugal force does not act on the object.

(d) In non-inertial frames, centrifugal force arises as pseudo forces and need to be considered.

(e) Under centrifugal forces arises as pseudo force and need to be considered.

## 3. Maximum Speed of Vehicle on Circular Turning Roads

- **On Unbanked Roads (Friction Only) :**

$$v_{max} = \sqrt{\mu Rg}$$

- **On Frictionless Banked Road :**

$$v_{max} = \sqrt{Rg\tan\theta}$$

where $\theta \rightarrow$ Banking angle

- **On Frictional Banked Road :**

$$V_{min} = \sqrt{Rg\left(\frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right)} = \sqrt{Rg\tan(\theta-\phi)}$$

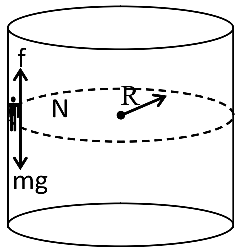
$$v_{max} = \sqrt{Rg\left(\frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}\right)} = \sqrt{Rg\tan(\theta+\phi)}$$

Where, $\phi$ = angle of friction = $\tan^{-1}(\mu)$
$\theta$ = angle of banking

- **Death Well :**

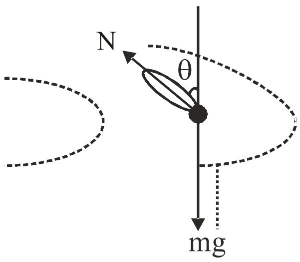
$$f = mg$$

$$N = \frac{mv^2}{R} = mR\omega^2$$



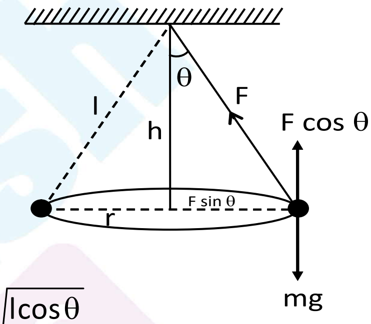
- **Bending of Cyclist :**

$$\tan\theta = \frac{v^2}{rg}$$




## 4. Conical Pendulum

If the bob of a simple pendulum is pulled to a side and whirled to move along a circle in horizontal plane, the string sweeps a cone and this arrangment is called conical pendulum. If *l* is length of the pendulum, F is tension in the string, r is radius of the horizontal circle and θ is the semivertical angle of the cone, then




$F \sin\theta = mr\omega^2$ and $F\cos\theta = mg$

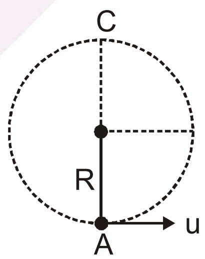
$$\tan\theta = \frac{r\omega^2}{g} \Rightarrow \omega = \sqrt{\frac{g\tan\theta}{r}}$$

$$\text{Time period } T = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

(from $r = l\sin\theta$)

## 5. Circular Motion in Vertical Plane

**A. Condition to Complete Vertical Circle :**




$$u \geq \sqrt{5gR}$$

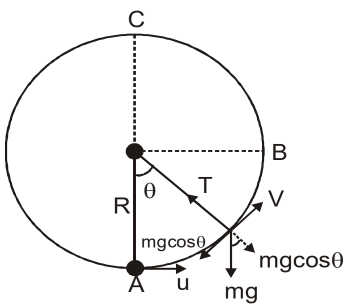
If $u \geq \sqrt{5gR}$ then Tension at C is equal to 0 and tension at A is equal to 6 mg

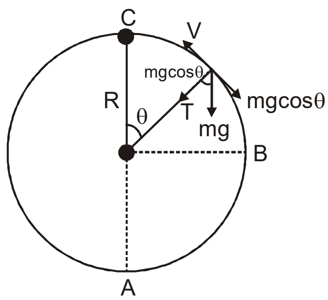
Velocity at B : $v_B = \sqrt{3gR}$

Velocity at C : $v_C = \sqrt{gR}$

From A to B : $T = mg\cos\theta + \dfrac{mv^2}{R}$

From B to C : $T = \dfrac{mv^2}{R} - mg\cos\theta$




**B. Condition to Pendulum Motion (Oscillation Condition)**

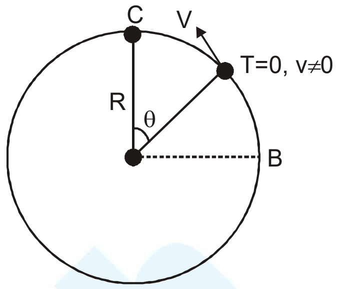
$$u \leq \sqrt{2gR} \text{ (in between A to B)}$$

Velocity can be zero but T never be zero between A & B.

Because T is given by $T = mg\cos\theta + \frac{mv^2}{R}$

**C. Condition for Leaving Path** :

$$\sqrt{2gR} < u < \sqrt{5gR}$$




Particle crosses the point B but not complete the vertical circle.
Tension will be zero in between B to C & the angle where T = 0

$$\cos\theta = \frac{u^2 - 2gR}{3gR}$$

θ is from vertical line.

**Note** : After leaving the circle the particle will follow a parabolic path.

- When a body is to move along a vertical circle it should have certain critical velocities at various points. Tension in string also changes from point to point. Tension will be maximum when the body is at the lower position.

- If the body has velocity less than the critical velocity, body cannot complete vertical circle. In such a case body may execute simple harmonic motion or it may leave the circle.

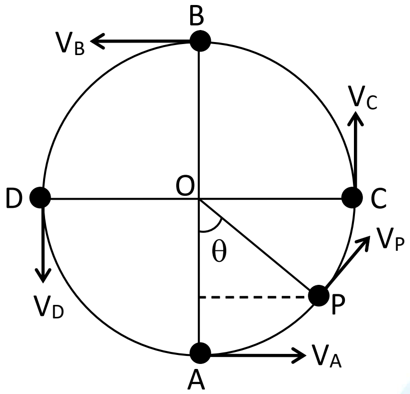
- Consider a body of m tied to one end of a string be whirled in a vertical circle of radius r in the vertical plane.

(a) For just completing vertical circle,

$$\text{Velocity of body}\begin{Bmatrix} \text{at the lowest point A is } V_A = \sqrt{5gr}\ \left(\text{i.e. } V_A \geq \sqrt{5gr}\right), \\ \text{at highest point B is } V_B = \sqrt{gr}\ \left(\text{i.e. } V_B \geq \sqrt{gr}\right), \\ \text{at horizontal point C or D is } \left(V_C \text{ or } V_D\right) = \sqrt{3gr}\ \left(\text{i.e. } V_C \text{ or } V_D \geq \sqrt{3gr}\right) \end{Bmatrix}$$

(b) For the body to complete the veritcal circle,

$$\text{Tension in the string} \begin{Bmatrix} \text{at the lowest point A is } T_A \geq 6\text{ mg}, \\ \text{at height point B is } T_B \geq 0, \\ \text{at horizontal point C or D is } (T_C \text{ or } T_D) \geq 3\text{mg} \end{Bmatrix}$$




(c) If the body at point P has critical velocity for just completing vertical circle

$V_P = \sqrt{gr(3+2\cos\theta)}$ and tension in the string is $T_P = 3\text{ mg }(1 + \cos\theta)$

$$\Rightarrow \quad T_P = \frac{M\{V_P^2 + g(r-h)\}}{r} \qquad \text{(where h is height of P above A)}$$

Here at A, $\theta = 0°$ and $V_P = V_A = \sqrt{5gr}$

at B, $\theta = 180°$ and $V_P = V_B = \sqrt{gr}$

at C, $\theta = 90°$ and $V_P = V_C = \sqrt{3gr}$

We can obtain tension also.

(d) $\sqrt{gr}$ is the least velocity at the top for the body to describe vertical circle if $V < \sqrt{gr}$ string slackens

(e) If the velocity of the body at the lowest point is less than $\sqrt{5gr}$, it cannot complete vertical cirlce.

(i) If $V_A < \sqrt{2gr}$, velocity of the body becomes zero at a certain point before tension of the string vanishes. So, body will oscillate about A.

(ii) If $V_A = \sqrt{2gr}$ here also body will oscilatles about A but along semcircle DAC.

(iii) If $\sqrt{2gr} < V_A < \sqrt{5gr}$, tension in the string vanishes before velocity of the body becomes zero. Then the body will leave the circle along the tangent at that point (This occurs at a point between C and B).

- When a body moves along a vertical circle by first maintaining its critical velocities at various points, then difference in the tension of the string when it is at the bottom most and top most points is equal to six times the weight of the body. Here tension in the string when the body is at the top most point is zero.
  If the body just complete vertical circle.
  $$T_A - T_B = 6\text{ mg}$$
  $$T_A - T_C = 3\text{ mg}$$

- A body can move along a vertical circle with uniform speed but the tension in the string should be adjusted from point to point.

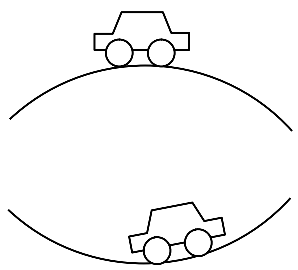
- When a body moves along a vertical circle with uniform speed, difference in the tension in the string when the body is at the lowest and top most positioin is equal to twice the weight of the body.

- When a body just moves in a vertical circle, its total energy only is constant. Its speed, linear velocity, linear momentum, angular momentum, angular velocity, P.E., K.E., centripetal force, tension in the string all the variable.

- When a body slides along an inclined plane of height 'h' and describes vertical cirlce of radius 'r' on reaching the bottom, then h = 5r/2.
  Here $mgh = \frac{1}{2} mv^2$ where $V = \sqrt{5gr}$ $\left(\because h = \frac{5r}{2}\right)$

- When a vehicle moving with certain speed is at the top of a convex shaped bridge or speed braker, the normal force on it is less than its weight. If that vehicle is at the lowest portion of a dip or concave shaped bridge, the normal force on it is greater than its weight.

- A car moving with speed V enters on a concave bridge of radius of curvature r. At the top most point of that bridge, normal reaction on the car is $N = mg - \frac{mV^2}{r}$.
  If the car moves on concave bridge and the bottom most point, $N = mg + \frac{mV^2}{r}$

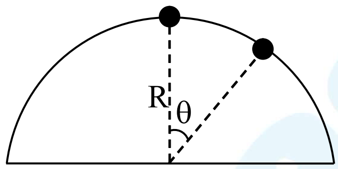


- If a bucket filled with water is whirled in a vertical circle at the end of a rope, water will not fall down when it is at the highest point if its velocity at the point is $V \geq \sqrt{gr}$ .

Here its period of revolution is $T \leq 2\pi\sqrt{\frac{r}{g}}$ , where r is the lenght of rope.

- A block of mass M hangs at the end of a string of length 'l'. A bullet of mass m flying horizontally hits the block and sticks to it. If the block now completes vertical circle, minimum velocity of the bullet is $V = \frac{M+m}{m}\sqrt{5gl}$

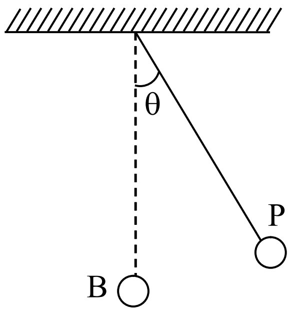
- A particle begins to slide without any friction from the top of a hemisphere of radius R as shown. If leaves the surface of hemisphere at height 'h' above the centre, such that h = 2r/3 and cos θ = 2/3.




If its velocity at the highest point is $\sqrt{gR}$ , it leaves the hemisphere along the tangent at that point without sliding down.

- A bob of mass m is suspended from point 'O' using an ideal string of length '*l*'. If the bob is pulled to a position P such that string makes an angle θ to the vertical and released, then velocity of the bob on reaching bottom most point B is $V = \sqrt{2gl(1-\cos\theta)}$ and in this position in the string is

$$T = \frac{mV^2}{l} + mg = mg\ (3 - 2\cos\theta)$$




- A shell of mass M hangs at the end of a string of length l. It explodes into two pieces, a piece of mass m flies of horizontally and the remaining fragment attached to the string just completes vertical circle.
mu = (M – m) V (numerically)

Where V = $\sqrt{5gl}$

$$\Rightarrow u = \frac{(M-m)\sqrt{5gl}}{m} = \left(\frac{M}{m}-1\right)\sqrt{5gl}$$

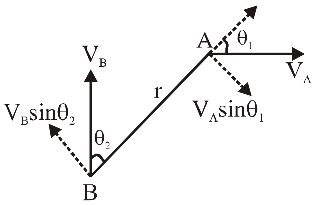
**KEY POINTS**

- Average angular velocity is a scalar physical quantity whereas instantaneous angular velocity is a vector physical quanity.
- Small Angular displacement $d\vec{\theta}$ is a vector quantity, but large angular displacement is scalar quantity.

  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ But $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$

**Relative Angular Velocity**

Relative angular velocity of a particle 'A' w.r.t. other moving particle 'B' is the angular velocity of the position vector of 'A' w.r.t. 'B'.




That means it is the rate at which position vector of 'A' w.r.t. 'B' rotates at that instant

$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{\text{Relative velocity of A w.r.t. B perpendicular to line AB}}{\text{seperation between A and B}}$$

here $(v_{AB})_\perp = v_A \sin\theta_1 + v_B \sin\theta_2$

$$\therefore \quad \omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$$

# WORK, ENERGY AND POWER

## 1. Work done :

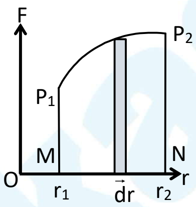
$W = \int dW = \int \vec{F}.d\vec{r} = \int Fdr\cos\theta$

For constant force W = $\vec{F}.\vec{d}$ = Fdcos$\theta$

For Unidirectional force

$W = \int dW = \int Fdx$ = Area between F – x curve and x-axis.

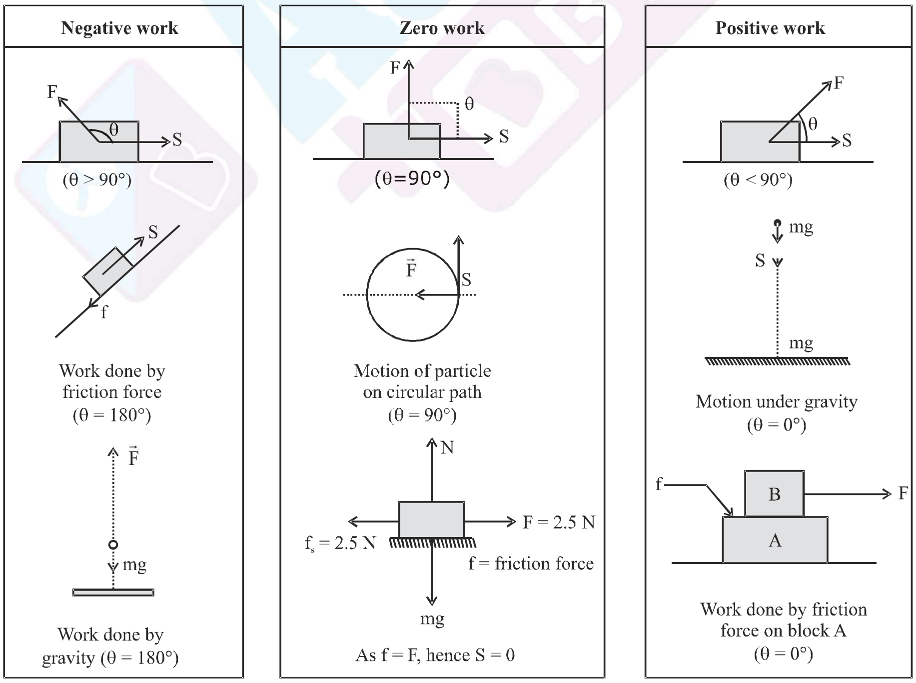
- **Calculation of work done from force-displacement graph :**




total work done, $W = \sum_{r_1}^{r_2} dW = \sum_{r_1}^{r_2} F.dr =$ area of $P_1P_2$ NM = $\int_{r_1}^{r_2} Fdr$

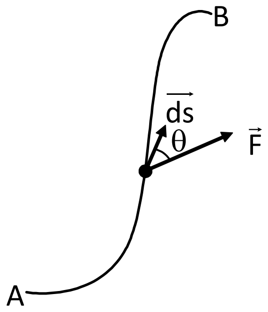
## 2. Nature of work done :

Although work done is a scalar quantity, yet its value may be positive,negative or even zero



## 3. Work done by variable force :

$$W_{AB} = \int_A^B \vec{F}.d\vec{s} = \int_A^B Fds\cos\theta$$




In term of ractangular components

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

then work done is

$$W_{AB} = \int_A^B (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

or $$W_{AB} = \int_{x_A}^{x_B} F_x d_x + \int_{y_B}^{y_B} F_y dy + \int_{Z_A}^{z_B} F_z dz$$

## 4. Conservative Forces :

Work done does not depend upon path.

- Work done in a round trip is zero.
- Central forces, spring forces etc. are conservative forces
- When only a conservative force acts within a system, the kinetic energy and potential energy can change into each other. However, their sum, the mechanical energy of the system, doesn't change.
- Work done is completely recoverable.
- If $\vec{F}$ is a conservative force then $\vec{\nabla} \times \vec{F} = 0$ (i.e. curl of $\vec{F}$ is zero)

## 5. Non-conservative Forces :

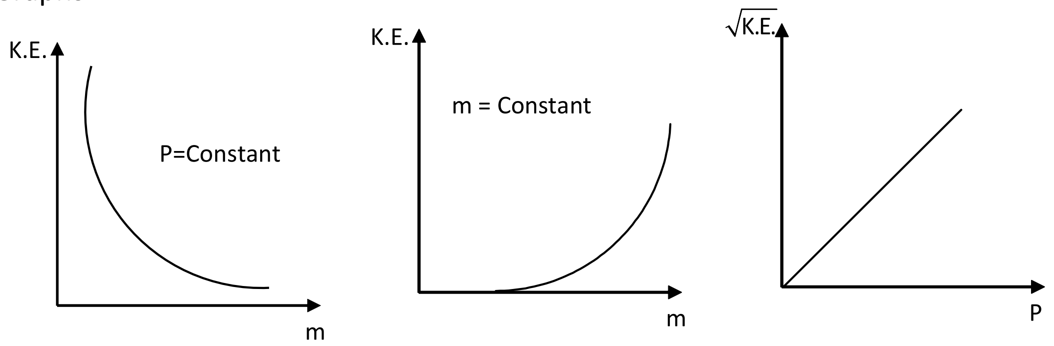
- Work done depends upon path.
- Work done in a round trip is not zero.
- Force are velocity- dependent & retarding in nature e.g. friction, viscous force etc.
- Work done against a non-conservative force may be dissipated as heat energy.
- Work done is not recoverable.

## 6. Kinetic energy :

- The energy possessed by a body, by the virtue of its motion is called kinetic energy.
  $$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}.\vec{v})$$
- Kinetic energy can never be negative, it is always positive.
- Relation between kinetic energy (K.E) and linear momentum (P) :

$$\therefore K.E. = \frac{P^2}{2m} \qquad P = \sqrt{2m(K.E.)}$$

Graphs -




## 7. Work-Energy Theorem :

According to this theorem work done by net force on a body is equal to change in its kinetic energy.

$\boxed{W = \Delta K.E.}$ or $\boxed{W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}$

Note -

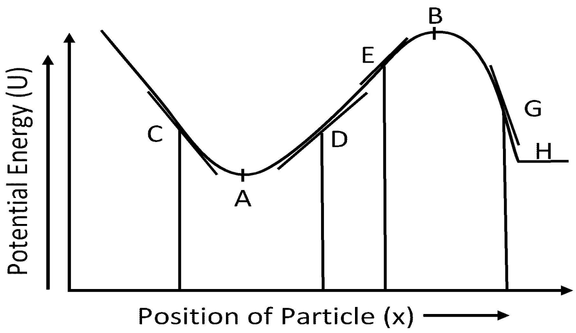
(i) If K.E. of the body decreases then work done is negative i.e. the force opposes the motion.

(ii) If K.E. of the body increases then work done is positive. i.e. the force supports the motion.

## 8. Potential energy :

- The energy which a body has by virtue of its position or configuration in a conservative force field.
- Potential energy is a relative quantity.
- Potential energy is defined only for conservative force field.
- Relationship between conservative force field and potential energy : $\vec{F} = -\text{grad}(U) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$
- If force varies only with one dimension (along x-axis) then $F = -\frac{dU}{dx} \Rightarrow U = -\int_{x_1}^{x_2} F dx$

## 9. Potential energy curve and equilibrium :




It is a curve which shows change in potential energy with position of a particle.

- **Nature of Forces :**

  At point **C :** slope $\frac{dU}{dx}$ is negative so F is positive.

  At point **D :** slope $\frac{dU}{dx}$ is positive so F is negative.

  At point **E :** slope $\frac{dU}{dx}$ is positive so F is negative.

  At point **G :** slope $\frac{dU}{dx}$ is negative so F is positive.

- **Stable Equilibrium :**
  When a particle is slightly displaced from equilibrium position and it tends to come back towards equilibrium then it is said to be in stable equilibrium.

  At point **A :** it is the point of stable equilibrium.

  At point **A :** $U = U_{min}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = positive

- **Unstable equilibrium :**
  When a particle is slightly displaced from equilibrium and it tends to move away from equilibrium position then it is said to be in unstable equilibrium.

  At point **B :** it is the point of unstable equilibrium.

  At point **B :** $U = U_{max}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = negative

- **Neutral equilibrium :**
  When a particle is slightly displaced from equilibrium position and no force acts on it then equilibrium is said to be neutral equilibrium. Point H is at neutral equilibrium $\Rightarrow$

  U = constant ; $\frac{dU}{dx} = 0, \frac{d^2U}{dx^2} = 0$

## 10. Law of conservation of Mechanical energy :

Total mechanical (kinetic + potential) energy of a system remains constant if only conservative forces are acting on the system of particles or the work done by all other forces is zero. From work energy theorem W = $\Delta$KE

**Proof :**

For internal conservative forces $W_{int} = -\Delta U$

So $W = W_{ext} + W_{int} = 0 + W_{int} = -\Delta U$

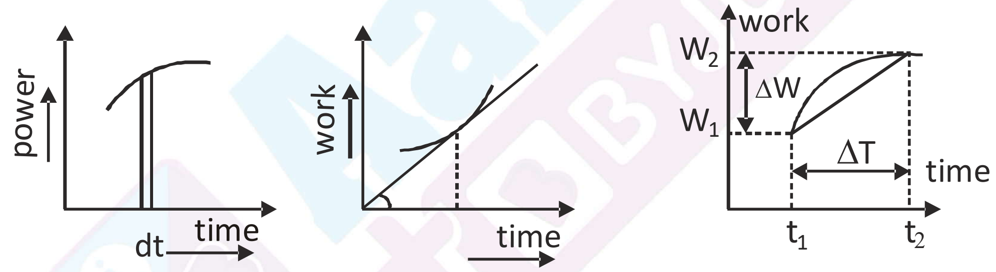
$\Rightarrow -\Delta U = \Delta KE \Rightarrow \Delta(KE + U) = 0 \Rightarrow KE + U = \text{(constant)}$

- Spring force F = –kx, Elastic potential energy stored in spring $U(x) = \frac{1}{2}kx^2$
- Mass and energy are equivalent and are related by $E = mc^2$

## 11. Power

- Power is a scalar quantity with dimension $M^1L^2T^{-3}$
- SI unit of power is J/s or watt
- 1 horsepower = 746 watt = 550 ft-lb/sec.

  **Average power $P_{av}$ = W/t**
- **Instantaneous power** $P = \frac{dW}{dt} = \vec{F}.\left(\frac{d\vec{r}}{dt}\right) = \vec{F}.\vec{v}$




$W = \int P dt$

fig. (a)

instantaneous power

$P = \frac{dW}{dt} = \tan\theta$

fig. (b)

average power

$\vec{P} = P_{avg.} = \frac{W_2 - W_1}{t_2 - t_1} = \frac{\Delta W}{\Delta t}$

fig. (c)

For a system of varying mass $\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$

If v = constant then $\vec{F} = \vec{v}\frac{dm}{dt}$ then $P = \vec{F}.\vec{v} = v^2\frac{dm}{dt}$

In rotatory motion : $P = \tau.\frac{d\theta}{dt} = \tau\omega$

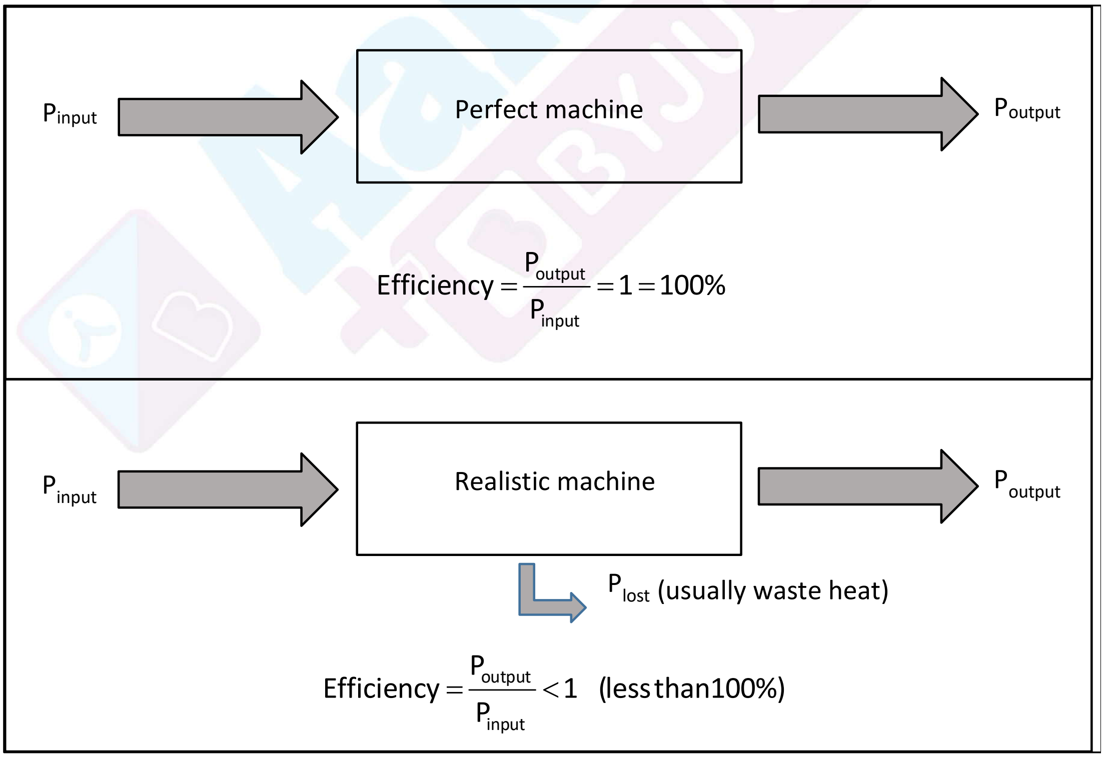
## 12. Key Points :

- A body may gain kinetic energy and potential energy simultaneously because principle of conservation of mechanical energy may not be valid every time.
- Comets move around the sun in elliptical orbits. The gravitational force on the comet due to sun is not normal to the comet's velocity but the work done by the gravitational force is zero in complete round trip because gravitational force is a conservative force.
- Work done by static friction may be positive because static friction may acts along the direction of motion of an object.

## 13. Efficiency :

Efficiency of a machine in % $\eta = \dfrac{\text{Work done}}{\text{energy input}} \times 100\%$

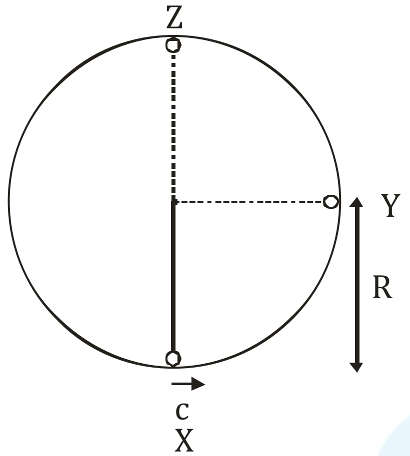
or Efficiency = $\dfrac{P_{output}}{P_{input}} \times 100\%$



## 14. Circular Motion in a Vertical Plane:

- **Vertical circular motion using a string:**

  Suppose a body is tied to a string and rotated in a vertical circle as shown




Between X and Y, tension will balance out weight and hence the string will always be taut. So the velocity required to reach Y can be found out by conserving mechanical energy.

$E_x$(Energy at X) = $\frac{1}{2}mu^2$

Since the particle just reaches point Y hence Velocity at Y is zero.

$E_y$=mgR

Equating both we get, u = $\sqrt{2gR}$

Now if want to find the minimum velocity to reach point Z, can I assume velocity to be zero at Z? The answer is no because if the velocity is zero at Z then weight will not be balanced and the string will become slack So at Z, velocity should be such that the weight is equal to the centripetal force making tension just to be zero.

$$\frac{mv^2}{R} = mg \qquad ...(1)$$

$$E_z = mg(2R) + \frac{mv^2}{2}$$

Substituting the value of v we get

$E_z$= 2.5mgR

Equating $E_x$ and $E_z$ we get,

u = $\sqrt{5gR}$

So now we have our critical values we can frame our cases, Case I : $u < \sqrt{2gR}$;

The ball will oscillate and never reach point Y.

Case II : $\sqrt{2gR} < u < \sqrt{5gR}$

The ball will lose contact somewhere between Y and Z and start projectile motion.

Case III: $u > \sqrt{5gR}$

The string will never become slack and complete the circle.

- **Vertical circular motion using a rod:**
  The situation is similar when the ball is tied to a rod and moved in vertical circle. The only difference is now the velocity at the top can be zero. As now the normal reaction of the rod can balance the weight at that point. Solving similarly as above we get the following cases for a rod:

  Case I : $u < \sqrt{2gR}$

  The body will oscillate and not reach point Y.

  Case II : $\sqrt{2gR} < u < \sqrt{4gR}$

  The ball will oscillate and cross point Y but not reach point Z.

  Case III : $u > \sqrt{4gR}$

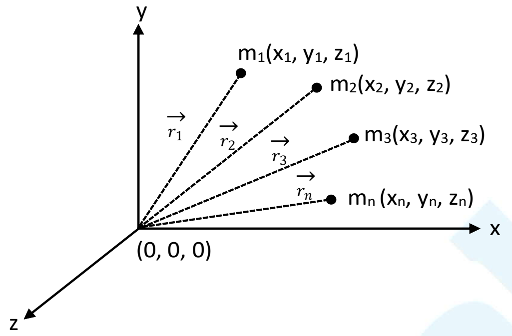
  The body will complete the circle.

# CENTRE OF MASS

## • Centre of mass :

For a system of particles, centre of mass is that point at which its total mass is supposed to be concentrated.

## • Centre of mass of system of discrete particles :




Total mass of the body : $M = m_1 + m_2 + ....... + m_n$ then

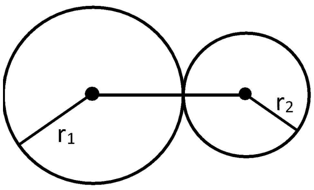
$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{1}{M}\sum_{i=1}^{i=n} m_i\vec{r}_i$$

co-ordinates of centre of mass :

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{i=n} m_i x_i,\ y_{cm} = \frac{1}{M}\sum_{i=1}^{i=n} m_i y_i \text{ and } z_{cm} = \frac{1}{M}\sum_{i=1}^{i=n} m_i z_i$$

• For a two particle system, distances of particles from centre of mass are in the reverse ratio of the masses i.e. $m_1 r_1 = m_2 r_2 \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1}$.

• Two circular discs/sphere of the same material are kept in contact as shown, then distance of centre of mass from the centre of the first disc is $\frac{r_2^2}{(r_1^2 + r_2^2)}(r_1 + r_2)$. Similarly distance of centre of mass from the centre of the second disc is $\frac{r_1^2}{(r_1^2 + r_2^2)}(r_1 + r_2)$.



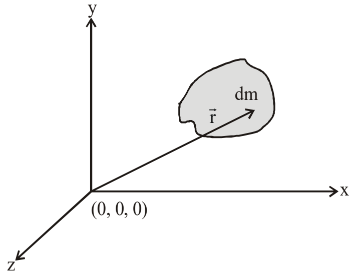
## • Centre of mass of continuous distribution of particles

$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

$$X_{com} = \frac{\int x\,dm}{\int dm} = \frac{1}{M}\int x\,dm$$

$$Y_{com} = \frac{\int y\,dm}{\int dm} = \frac{1}{M}\int y\,dm$$

$$Z_{com} = \frac{\int z\,dm}{\int dm} = \frac{1}{M}\int z\,dm$$




x, y, z are the co-ordinate of the COM of the dm mass.

## • The centre of mass after removal of a part of a body

Original mass (M) – mass of the removed part (m)
= {original mass (M)} + {– mass of the removed part (m)}

When a part is removed from a rigid body. then the position of COM of the remaining portion will be :

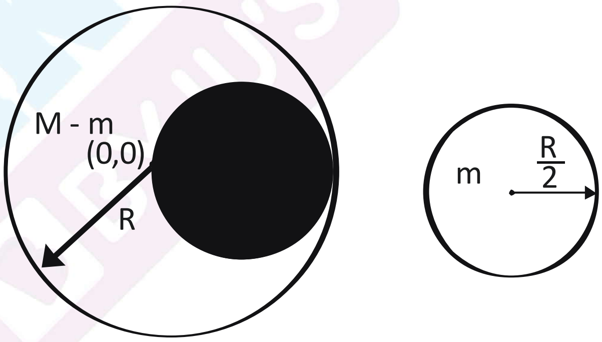
$$\vec{r}_{COM} = \frac{M\vec{r}_1 - m\vec{r}_2}{M - m}$$

The co-oradinates of COM is given by

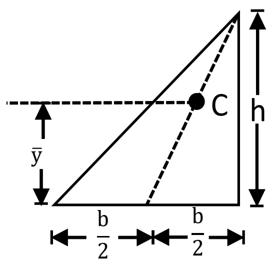
$$x_{COM} = \frac{Mx_1 - mx_2}{M - m}$$

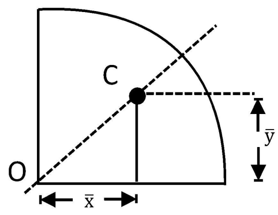
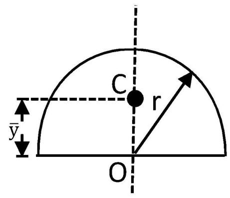
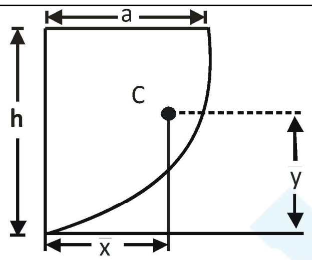
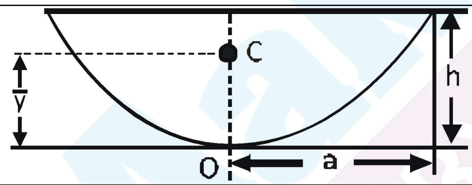
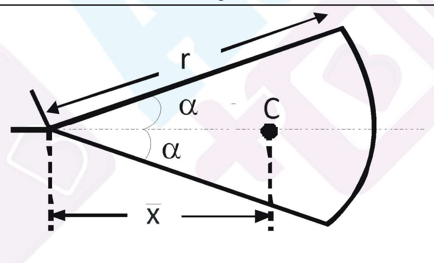
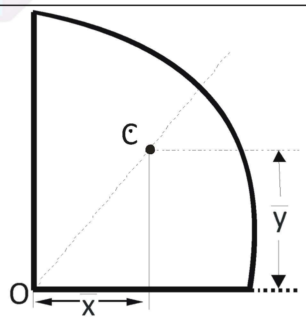
$$y_{COM} = \frac{My_1 - my_2}{M - m}$$

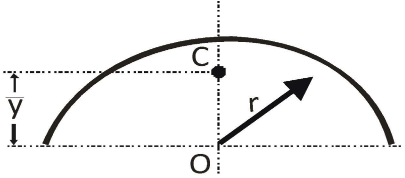
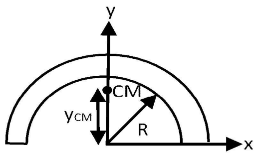
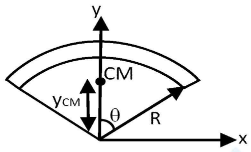
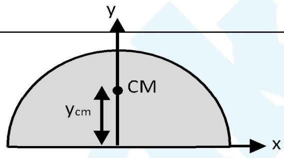
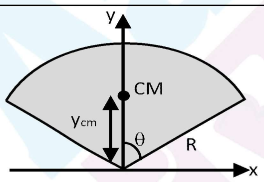
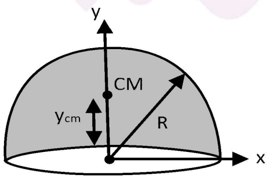
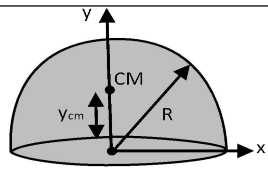
$$z_{COM} = \frac{Mz_1 - mz_2}{M - m}$$

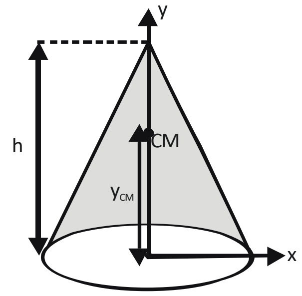
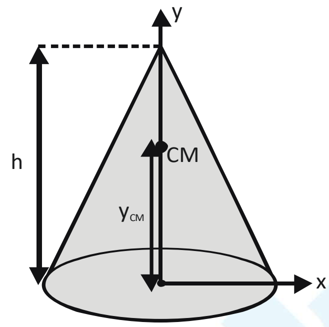



## • Centre of mass of some common objects

| Shape | Figure | $\bar{x}$ | $\bar{y}$ |
|---|---|---|---|
| Triangular area |   | | $\frac{h}{3}$ |

| Shape | Figure | $\bar{x}$ | $\bar{y}$ |
|---|---|---|---|
| Quarter circular area |  | $\frac{4r}{3\pi}$ | $\frac{4r}{3\pi}$ |
| Semi-circular area |  | 0 | $\frac{4r}{3\pi}$ |
| Semi parabolic area |  | $\frac{3a}{8}$ | $\frac{3h}{5}$ |
| Parabolic area |  | 0 | $\frac{3h}{5}$ |
| Circular Sector |  | $\frac{2r\sin\alpha}{3\alpha}$ | 0 |
| Quarter Circular arc |  | $\frac{2r}{\pi}$ | $\frac{2r}{\pi}$ |

| Shape | Figure | $\bar{x}$ | $\bar{y}$ |
|---|---|---|---|
| Semi Circular arc |  | 0 | $\frac{2r}{\pi}$ |
| Half Ring |  | 0 | $y_{cm} = \frac{2R}{\pi}$ |
| Segment of a ring |  | 0 | $y_{cm} = \frac{R\sin\theta}{\theta}$ |
| Half disc (plate) |  | 0 | $y_{cm} = \frac{R\sin\theta}{\theta}$ |
| Segment of a ring |  | 0 | $y_{cm} = \frac{2R\sin\theta}{3\theta}$ |
| Hollow hemisphere |  | 0 | $y_{cm} = \frac{R}{2}$ |
| Solid hemisphere |  | 0 | $y_{cm} = \frac{3R}{8}$ |

| Shape | Figure | $\bar{x}$ | $\bar{y}$ |
|---|---|---|---|
| Hollow Cone |  | 0 | $y_{CM} = \frac{h}{3}$ |
| Solid Cone |  | 0 | $y_{CM} = \frac{h}{4}$ |

## Motion of centre of mass

For a system of particles,

velocity of centre of mass $\vec{v}_{CM} = \frac{d\vec{R}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n}{m_1 + m_2 + \ldots + m_n}$

Similarly acceleration $\vec{a}_{CM} = \frac{d}{dt}\left(\vec{v}_{CM}\right) = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n}{m_1 + m_2 + \ldots + m_n}$

## Analysis of Dynamics of COM

$$\vec{F}_{net} = \left(\sum \vec{F}_{ext}\right)_{sys} = \frac{d\vec{P}_{sys}}{dt} = M\vec{a}_{COM}$$

- COM is rest $\boxed{\begin{array}{l}\vec{F}_{net} = 0 \\ \vec{V}_{COM} = 0\end{array}}$
- COM moves with constant velocity $\boxed{\begin{array}{l}\vec{F}_{net} \neq 0 \\ \vec{V}_{COM} \neq 0\end{array}}$

- COM moves with acceleration $\boxed{\vec{F}_{net} \neq 0}$
- If a system is at rest intially and there is no net external force acting on it then there will be no shift in position of the COM of the system.

$$\boxed{\vec{F}_{net} = 0} + \boxed{\vec{V}_{COM} = 0} \Rightarrow \boxed{\Delta\vec{r}_{COM} = 0}$$

## • Law of conservation of linear momentum

Linear momentum of a system of particles is equal to the product of mass of the system with velocity of its centre of mass.

From Newton's second law $\vec{F}_{ext.} = \dfrac{d(M\vec{v}_{CM})}{dt}$

If $\vec{F}_{ext.} = \vec{0}$, then $M\vec{v}_{cm}$ = constant

If no external force acts on a system the velocity of its centre of mass remains constant, i.e., velocity of centre of mass is unaffected by internal forces.

- **Impulse-Momentum theorem**

Impulse of a force is equal to the change of momentum.

Force-time graph area gives change in momentum. $\boxed{\int_{t_1}^{t_2} \vec{F}dt = \Delta\vec{P}}$

## • Reduced Mass For Two Body System

1. A two body system can be made equivalent to a single body system by introducing the concept of reduced mass.

2. Let $m_1$ and $m_2$ be the masses of two particles with position vectors $\vec{r}_1$ and $\vec{r}_2$ and $\vec{F}_{12}$ be the forces exerted by second body on first body and $\vec{F}_{21}$ by first body on second body respectively.

$\vec{F}_{12} = m_1 \dfrac{d^2\vec{r}_1}{dt^2}$ and $\vec{F}_{21} = m_2 \dfrac{d^2\vec{r}_2}{dt^2}$

As no external force acts on the system, $\vec{F}_{12} = -\vec{F}_{21} = \vec{F}$

$$\Rightarrow \frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \vec{F}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \vec{F}\left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

Let $\vec{r}_1 - \vec{r}_2 = \vec{r}$

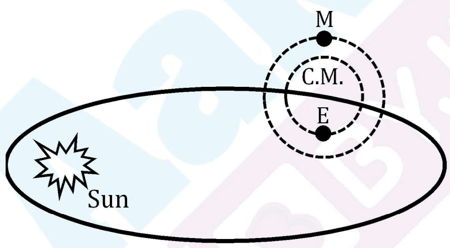
$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \vec{F}\left(\frac{m_1 + m_2}{m_1 m_2}\right) \text{or} \left(\frac{m_1 + m_2}{m_1 m_2}\right)\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

$$\text{or } \mu\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

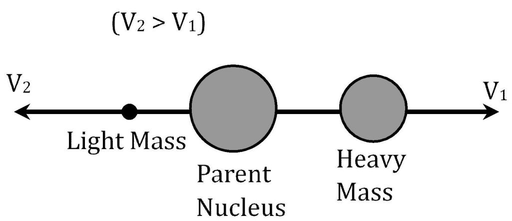
Here $\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right)$ is called reduced mass.

## • Classical Example of Application of COM :-

(1) The earth revolves around the sun in an elliptical orbit whereas the moon revolves round the earth in circular orbit. Both the earth and the moon move in circles about a common centre of mass. The internal force which act on the earth moon system are the gravitational force of attraction on each other. The earth and the moon are always on opposite sides of the centre of mass. Since the earth is heavier than moon, So the centre of mass of the system is very close to the earth. It is this centre of mass which revolves around the sun in an elliptical orbit.

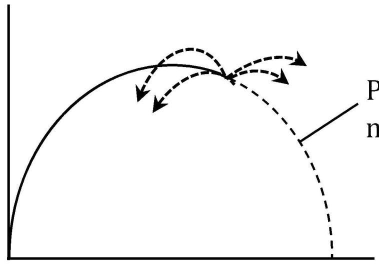



(2) In radioactive decay, the process is caused by the internal forces of the system. Therefore, initial and final momenta are zero. Hence, the decay products fly off in the opposite directions. The centre of mass of the system remains at rest. The heavy mass move with less speed than that of the light mass.




(3) Explosion of a projectile (e.g. fire cracker) in mid air. Let us consider a projectile which explodes in air. Before explosion, the projectile move along a parabolic path. After explosion,

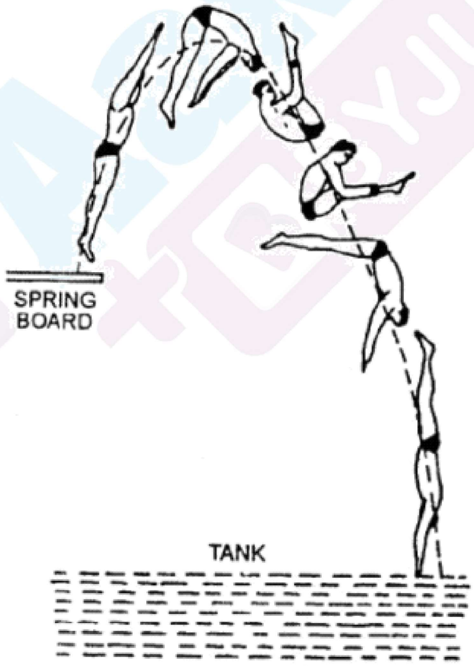
each fragment move along its own parabolic path but the centre of mass of the projectile continues to move in the same parabolic path.




**Explanation.** The projectile follows a parabolic path under the action of gravity (i.e. earth's gravitational force) Explosion of the projectile occurs due to the internal forces i.e, without any external force. These internal forces cannot change the total momentum of the system al though they may change the momenta of the individual fragments. Thus the centre of mass will remain unaffected after the explosion and hence follow the same parabolic path.

(4) When a diver jumps into water form a height, then body can moves in any path but centre of mass of his body traverses in parabolic path.

So centre of mass follow laws of motion.



# COLLISIONS

## 1. Collision of bodies :

The event or the process, in which two bodies either coming in contact with each other or due to mutual interaction at distance apart, affect each others motion (velocity, momentum, energy or direction of motion) is defined as a collision.

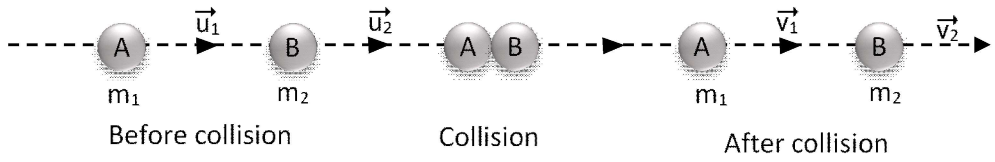
## 2. In collision :

- The particles come closer before collision and after collision they either stick together or move away from each other.
- The particles need not come in contact with each other for a collision.
- The law of conservation of linear momentum is necessarily applicable in a collision, whereas the law of conservation of mechanical energy is not.

**Types of Collisions**

- On the basis of direction
  - One dimensional collision or Head on collision → The collision in which the particles move along the same straight line before and after the collision, is defined as one dimensional collision
  - Two dimensional collision or Oblique collision → The collision, in which the particles move in the same plane at different angles before and after collision, is defined as oblique collision
- On the basis of kinetic energy
  - Elastic collision → A collision is said to be elastic, if the total kinetic energy before and after collision remains the same
  - Inelastic collision → A collision is said to be inelastic, if the total kinetic energy does not remains constant
  - Perfectly inelastic collision → The collision, in which particles get sticked together after the collision. It is called perfectly inelastic collision. In this type of inelastic collision, loss of energy is maximum.

## 2.1 Head on collision :




## 2.2 Elastic Collision :

1. **For one dimensional collision between two bodies**
   Total momentum before collision = Total momentum after collision.

2. $m_1$ and $m_2$ are masses of two bodies moving with velocities $u_1$ and $u_2$ in the same direction ($u_2 < u_1$). After collision their velocities are $v_1$ and $v_2$. Then

   - **For one dimensional collision between two bodies,**
     $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ (conservation of momentum)
   - **If the second body is at rest before collision,** $u_2 = 0$
   - **If they approach each other before collision,** $u_2 = -u_1$
   - **If they move together with velocity v after collision,**
     $m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$

     $\Rightarrow \; v = \dfrac{m_1u_1 + m_2u_2}{(m_1 + m_2)}$ (perfect inelastic collision)

3. Coefficient of restitution e = 1
   $\Rightarrow v_2 - v_1 = u_1 - u_2$

   Or relative velocity of separation after collision is equal to relative velocity of approach before collision.

4. **For perfect elastic collision between two bodies which is head on,**

   $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ and $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

   $(v_2 - v_1) = (u_1 - u_2)$

   Here
   (i) If $m_1 = m_2 = m$, then $v_2 = u_1$ and $v_1 = 0$
   (ii) If $m_2 >> m_1$, then $v_1 = -u_1$ and $v_2 = 0$
   (iii) If $m_1 >> m_2$ then $v_1 = u_1$ and $v_2 = 2u_1$

5. When a lighter body collides with a stationary heavy body elastically, the second body starts moving with the velocity of the first body while the first body stops.

6. When a heavy body collides elastically with a stationary lighter body, then heavy body continues to move with the same velocity but the lighter body starts moving with double the velocity of heavy body.

7. When a lighter body collides with a heavy body at rest, then it returns with the same velocity but heavy body remains at rest.

8. When perfect elastic collision takes place between two bodies of same mass moving along a direction, the two bodies interchange their velocities after collision.

9. For perfect elastic collision between a moving body $m_1$ and stationary body $m_2$
Fraction of K.E transferred to the second body = Fraction of K.E retained by the first body.

10. A ball is dropped from certain height. If the collision is perfectly elastic, it rebounds to the same height.

11. When two bodies of equal mass moving towards each other collide elastically with same velocity in magnitude, after collision, they move away with the same velocity in magnitude.

12. A body makes an oblique elastic collision with another body of same mass at rest. After collision, they will move in mutually perpendicular directions.

13. Collisions between atomic, nuclear and fundamental particles are examples of elastic collisions.

## 2.3 Perfect Inelastic Collision :

1. **For one dimensional collision between two bodies**
Total momentum before collision = Total momentum after collision

2. After perfect inelastic collision the two bodies stick together and move with same velocity

$$\Rightarrow v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

3. The collision between a bullet and a target is perfectly inelastic if the bullet remains embedded in the target.

4. Coefficient of restitution e = 0

$\Rightarrow v_1 = v_2 = v$

5. Only momentum is conserved and kinetic energy is not conserved.

6. If a body of mass $m_1$ collides with a body of mass $m_2$ at rest and the collision is perfectly inelastic, $\frac{\text{Final K.E}}{\text{Initial K.E}} = \frac{m_1}{(m_1+m_2)}$ (for the system)

If K is initial K.E of $m_1$, then loss in K.E $= K\left(\frac{m_2}{m_1 + m_2}\right)$

Fractional loss in K.E is $\left(\frac{m_2}{m_1 + m_2}\right)$

7. Loss in kinetic energy during perfect inelastic collision $= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2$

8. If the two bodies approach each other before collision, common velocity after collision is

$v = \frac{m_1u_1 - m_2u_2}{m_1 + m_2}$ Loss in kinetic energy in the case is $\frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(u_1 + u_2)^2$

**Coefficent of Restitution :**

1. The ratio between relative velocity of separation after collision and relative velocity of approach before collision is know as coefficient of restitution $\mathbf{e} = \frac{v_2 - v_1}{u_1 - u_2}$

2. The value of '**e**' is given by $0 \le \mathbf{e} \le 1$
   If **e** = 0, the collision is perfectly inelastic
   If **e** = 1 the collision is perfectly elastic
   If 0 < **e** < 1 the collision is semi elastic

3. **e** is dimensionless and has no units.

4. The value of **e** is independent of masses and the velocities of the colliding bodies

5. **e** depends on the nature of material of the colliding bodies

6. If a body is dropped from a height 'h' and after first rebound it rises to a height $h_1$ the coefficient of restitution $\mathbf{e} = \sqrt{\frac{h_1}{h}} \Rightarrow h_1 = \mathbf{e}^2 h$ After $n^{th}$ rebound $h_n = \mathbf{e}^{2n} h$

   If the body strikes the ground with velocity v and rebounds with velocity $v_1$ then

$\mathbf{e} = \frac{v_1}{v} \Rightarrow v_1 = \mathbf{e}v$ After nth rebound $v_n = \mathbf{e}^n v$

7. A ball dropped from a height h. It strikes the ground and rebounds. Here '**e**' is coefficient of restitution and this collisions took place repeatedly. The total distance travelled by the ball before coming to rest is $d = h\left(\frac{1+\mathbf{e}^2}{1-\mathbf{e}^2}\right)$ Here total time taken by the ball to come to rest is $t = \frac{\sqrt{2h}}{8}\left(\frac{1+\mathbf{e}}{1-\mathbf{e}}\right)$

8. A ball of mass m is dropped from height h and after hitting the ground it rises to height less than 'h'. If '**e**' is coefficient of restitution, the change in momentum of the ball in magnitude is $m\sqrt{2gh}(1+\mathbf{e})$

9. For one dimensional collision between two bodies of masses $m_1$ and $m_2$ moving with initial velocities $u_1$ and $u_2$ respectively, final velocities after collision are given by

$$v_1 = \left(\frac{m_1 - \mathbf{e}m_2}{m_1 + m_2}\right)u_1 + \frac{m_2(1+\mathbf{e})}{m_1 + m_2}u_2$$

$$v_2 = \left(\frac{m_1 - \mathbf{e}m_1}{m_1 + m_2}\right)u_2 + \frac{m_2(1+\mathbf{e})}{m_1 + m_2}u_1$$

Here loss in kinetic energy during collision is $\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2(1-\mathbf{e})^2$

### 2.4 Head on inelastic collision of two particles :

Let the coefficient of restitution for collision is e

(i) Momentum is conserved $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ .....(i)

(ii) Kinetic energy is not conserved.

(iii) According to Newton's law $e = \frac{v_2 - v_1}{u_1 - u_2}$ ...(ii)

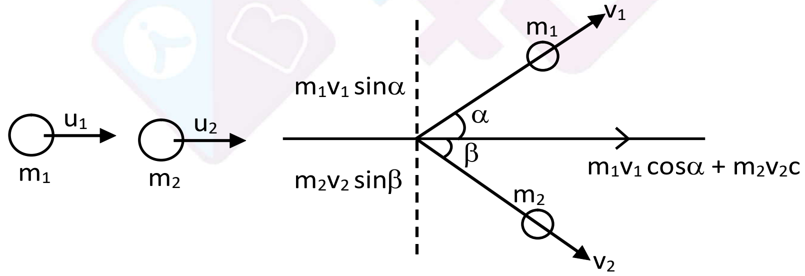
By solving eq. (i) and (ii)

$$v_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)u_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)u_2 = \frac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$$

$$v_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)u_2 + \left(\frac{(1+e)m_1}{m_1 + m_2}\right)u_1 = \frac{m_1u_1 + m_2u_2 - m_1e(u_2 - u_1)}{m_1 + m_2}$$

### 2.5 Two Dimensional Collision :

1. Consider two bodies of masses $m_1$ and $m_2$ moving with velocities $u_1$ and $u_2$ along the same straight line. They collide and after collision they move in directions making angles $\alpha$ and $\beta$ with the initial direction of motion. Let $v_1$ and $v_2$ be their final velocities. Then




From conservation of momentum

$m_1 u_1 + m_2 u_2 = m_1 v_1 \cos\alpha + m_2 v_2 \cos\beta$

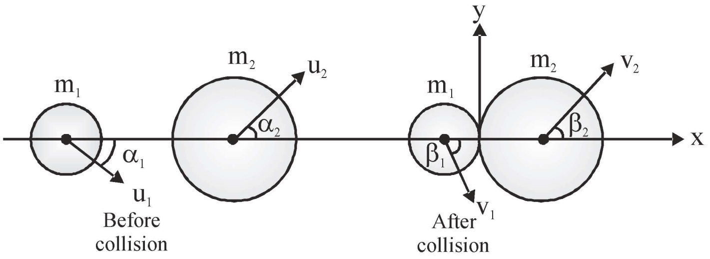
$0 = m_1 v_1 \sin\alpha - m_2 v_2 \sin\beta$

**2.6 Oblique Collision :**

Conserving the momentum of system in directions along normal (x-axis in our case) and tangential (y-axis in our case)

$m_1u_1\cos\alpha_1 + m_2u_2\cos\alpha_2 = m_1v_1\cos\beta_1 + m_2v_2\cos\beta_2$ and

$m_2u_2\sin\alpha_2 - m_1u_1\sin\alpha_1 = m_2v_2\sin\beta_2 - m_1v_1\sin\beta_1$




Since no force is acting on $m_1$ and $m_2$, along the tangent (i.e. y-axis) the individual momentum of $m_1$ and $m_2$ remains conserved.

$m_1u_1\sin\alpha_1 = m_1v_1\sin\beta_1$ & $m_2u_2\sin\alpha_2 = m_2v_2\sin\beta_2$

By using Newton's experimental law along the line of impact

$$e = \frac{v_2\cos\beta_2 - v_1\cos\beta_1}{u_1\cos\alpha_1 - u_2\cos\alpha_2}$$

## 3. Rocket propulsion :

Thrust force on the rocket, $F_{gas} = -v\frac{dm}{dt}$

From Newton's Second law,

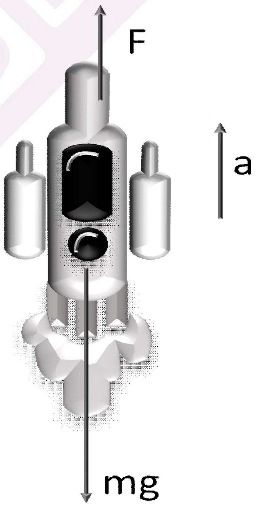
$-F_{gas} = F_{rocket}$

Therefore,

$$F_{rocket} = v\frac{dm}{dt}$$

From free body diagram,

$$v\frac{dm}{dt} - mg = ma$$




Where,

v is the relative velocity of gases w.r.t. rocket

m is the mass of the rocket

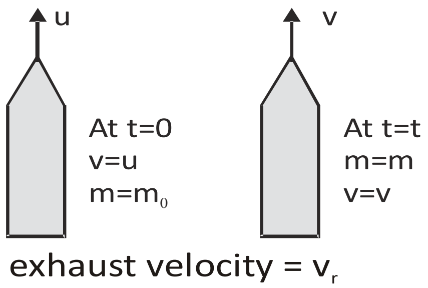
g is the acceleration due to gravity

a is the initial acceleration of the rocket

$\frac{dm}{dt}$ is the rate of consumption of fuel

Velocity of rocket at any instant

$v = u - gt + v_r \ell n\left(\frac{m_0}{m}\right)$




## 4. Ballistic Pendulum :

1. It is used to find velocity of bullet. This arrangement consists of a wooden block suspended using a rope or wire. A bullet fired horizontally into the block, it gets embedded and both move together.

2. Let m be mass of the bullet which strikes the wooden block of mass M with velocity u and gets embedded into it. After this the combined system moves with a velocity v and the system rises to a height *h* above the previous level. Then

$$mu = (M+m)v$$

$$\Rightarrow v = \frac{mu}{(M+m)}$$

As $v = \sqrt{2gh}, \sqrt{2gh} = \frac{mu}{(M+m)}$

$$\Rightarrow u = \left(\frac{M+m}{m}\right)\sqrt{2gh}$$

$$\Rightarrow h = \left(\frac{mu}{M+m}\right)^2 / 2g$$

3. In the previous case if the bullet emerges out from the block with velocity $u_1$ and the block rises to a height h,

$$m(u - u_1) = Mv$$

$$\Rightarrow v = \frac{m(u-u_1)}{M} \text{ or } \sqrt{2gh} = \frac{m(u-u_1)}{M}$$

4. Ballistic pendulum is an example for perfect inelastic collision (if the bullet stops in the block).