

EXERCISE 13.1

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1. Evaluate the Given limit: $\lim_{x \rightarrow 3} x + 3$

Solution:

Given

$$\lim_{x \rightarrow 3} x + 3$$

Substituting $x = 3$, we get

$$= 3 + 3$$

$$= 6$$

2. Evaluate the Given limit: $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

Solution:

Given limit: $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

Substituting $x = \pi$, we get

$$\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$$

3. Evaluate the Given limit: $\lim_{r \rightarrow 1} \pi r^2$

Solution:

Given limit: $\lim_{r \rightarrow 1} \pi r^2$

Substituting $r = 1$, we get

$$\lim_{r \rightarrow 1} \pi r^2 = \pi(1)^2$$

$$= \pi$$

4. Evaluate the Given limit: $\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2}$

Solution:

Given limit: $\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2}$

Substituting $x = 4$, we get

$$\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2} = \frac{4(4) + 3}{4 - 2}$$

$$= \frac{16 + 3}{2}$$

$$= \frac{19}{2}$$

5. Evaluate the Given limit: $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Solution:

Given limit: $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Substituting $x = -1$, we get

$$\begin{aligned} & \lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1} \\ &= [(-1)^{10} + (-1)^5 + 1] / (-1 - 1) \\ &= (1 - 1 + 1) / -2 \\ &= -1 / 2 \end{aligned}$$

6. Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

Solution:

Given limit: $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

$$\begin{aligned} &= [(0 + 1)^5 - 1] / 0 \\ &= 0 \end{aligned}$$

Since, this limit is undefined

Substitute $x + 1 = y$, then $x = y - 1$

$$\lim_{y \rightarrow 1} \frac{(y)^5 - 1}{y - 1}$$

$$= \lim_{y \rightarrow 1} \frac{(y)^5 - 1^5}{y - 1}$$

We know that,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Hence,

$$\begin{aligned} & \lim_{y \rightarrow 1} \frac{(y)^5 - 1^5}{y - 1} \\ &= 5(1)^{5-1} \\ &= 5(1)^4 \\ &= 5 \end{aligned}$$

7. Evaluate the Given limit: $\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Solution:

By evaluating the limit at $x = 2$, we get

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4} = [3(2)^2 - x - 10] / 4 - 4$$
$$= 0$$

Now, by factorising numerator, we get

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{3x^2 - 6x + 5x - 10}{x^2 - 2^2}$$

We know that,

$$a^2 - b^2 = (a - b)(a + b)$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$= \lim_{x \rightarrow 2} \frac{(3x+5)}{(x+2)}$$

By substituting $x = 2$, we get,

$$= [3(2) + 5] / (2 + 2)$$
$$= 11 / 4$$

8. Evaluate the Given limit: $\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Solution:

First substitute $x = 3$ in the given limit, we get

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{(3)^4 - 81}{2(3)^2 - 5 \times 3 - 3} \\ &= \frac{81 - 81}{18 - 18} \\ &= 0 / 0 \end{aligned}$$

Since the limit is of the form $0 / 0$, we need to factorise the numerator and denominator

$$\lim_{x \rightarrow 3} \frac{(x^2 - 9)(x^2 + 9)}{2x^2 - 6x + x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(2x + 1)(x - 3)}$$

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

Now substituting $x = 3$, we get

$$\begin{aligned} & \frac{(3 + 3)(3^2 + 9)}{(2 \times 3 + 1)} \\ &= 108 / 7 \end{aligned}$$

Hence,

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = 108 / 7$$

9. Evaluate the Given limit:

$$\lim_{x \rightarrow 0} \frac{ax + b}{cx + 1}$$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{ax + b}{cx + 1} \\ &= [a(0) + b] / c(0) + 1 \\ &= b / 1 \\ &= b \end{aligned}$$

10. Evaluate the Given limit:

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

Solution:

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = (1 - 1) / (1 - 1)$$

$$= 0$$

Let the value of $z^{1/6}$ be x

$$(z^{1/6})^2 = x^2$$

$$z^{1/3} = x^2$$

Now, substituting $z^{1/3} = x^2$ we get

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{x^2 - 1^2}{x - 1}$$

We know that,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1^2}{x - 1} = 2(1)^{2-1}$$

$$= 2$$

11. Evaluate the Given limit: $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Solution:

Given limit: $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Substituting $x = 1$

$$\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$$

$$= [a(1)^2 + b(1) + c] / [c(1)^2 + b(1) + a]$$

$$= (a + b + c) / (a + b + c)$$

Given

$$[a + b + c \neq 0]$$

$$= 1$$

12. Evaluate the Given limit: $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Solution:

By substituting $x = -2$, we get

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2} = 0 / 0$$

Now,

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2} &= \frac{\frac{2+x}{2x}}{x+2} \\ &= 1 / 2x \\ &= 1 / 2(-2) \\ &= -1 / 4 \end{aligned}$$

13. Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$

Solution:

Given $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$

Formula used here

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

By applying the limits in the given expression

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{0}{0}$$

By multiplying and dividing by 'a' in the given expression, we get

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx} \times \frac{a}{a}$$

We get,

$$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times \frac{a}{b}$$

We know that,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$= \frac{a}{b} \lim_{ax \rightarrow 0} \frac{\sin ax}{ax} = \frac{a}{b} \times 1$$

$$= a / b$$

Act
Go i

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$$

14. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = 0 / 0$$

By multiplying ax and bx in numerator and denominator, we get

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$$

Now, we get

$$\frac{a \lim_{ax \rightarrow 0} \frac{\sin ax}{ax}}{b \lim_{bx \rightarrow 0} \frac{\sin bx}{bx}}$$

We know that,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\begin{aligned} \text{Hence, } a / b \times 1 \\ = a / b \end{aligned}$$

15. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{\pi - x \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)} \times \frac{1}{\pi}$$

$$= \frac{1}{\pi} \lim_{\pi - x \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)}$$

We know that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\frac{1}{\pi} \lim_{\pi - x \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)} = \frac{1}{\pi} \times 1$$

$$= 1 / \pi$$

16. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0}$$

$$= 1 / \pi$$

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$$

17. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \frac{0}{0}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1}$$

$$(\cos 2x = 1 - 2 \sin^2 x)$$

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\frac{\sin^2 x \times x^2}{x^2}}{\frac{\sin^2 \frac{x}{2} \times \frac{x^2}{4}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4 \frac{\lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}}{\lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4 \frac{\lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2}\right)^2}{\lim_{x \rightarrow 0} \left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2}$$

We know that,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$= 4 \times 1^2 / 1^2$$

$$= 4$$

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

18. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{0}{0}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{1}{b} \lim_{x \rightarrow 0} \frac{x(a + \cos x)}{\sin x}$$

$$= \frac{1}{b} \lim_{x \rightarrow 0} x \times \lim_{x \rightarrow 0} (a + \cos x)$$

$$= \frac{1}{b} \times \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \times \lim_{x \rightarrow 0} (a + \cos x)$$

We know that,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$= \frac{1}{b} \times (a + \cos 0)$$

$$= (a + 1) / b$$

19. Evaluate the given limit: $\lim_{x \rightarrow 0} x \sec x$
Solution:

$$\lim_{x \rightarrow 0} x \sec x = \lim_{x \rightarrow 0} \frac{x}{\cos x}$$

$$= \lim_{x \rightarrow 0} \frac{0}{\cos 0} = \frac{0}{1}$$

$$= 0$$

20. Evaluate the given limit: $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a + b \neq 0$
Solution:

$$\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} = \frac{0}{0}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \rightarrow 0} \frac{(\sin \frac{ax}{ax})ax + bx}{ax + (\sin \frac{bx}{bx})}$$

$$= \frac{(\lim_{ax \rightarrow 0} \sin \frac{ax}{ax}) \times \lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx \times (\lim_{bx \rightarrow 0} \sin \frac{bx}{bx})}$$

We know that,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$= \frac{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx}$$

We get,

$$= \frac{\lim_{x \rightarrow 0} (ax + bx)}{\lim_{x \rightarrow 0} (ax + bx)}$$

$$= 1$$

21. Evaluate the given limit: $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

Solution:

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$$

Applying the formulas for cosec x and cot x, we get

$$\operatorname{cosec} x = \frac{1}{\sin x} \text{ and } \cot x = \frac{\cos x}{\sin x}$$

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x}$$


Now, by applying the formula we get,

$$1 - \cos x = 2 \sin^2 \frac{x}{2} \text{ and } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) = \lim_{x \rightarrow 0} \tan \frac{x}{2}$$

$$= 0$$



$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

22. Evaluate the given limit:

Solution:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \frac{0}{0}$$

Let $x - (\pi / 2) = y$

Then, $x \rightarrow (\pi/2) = y \rightarrow 0$

Now, we get

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{y \rightarrow 0} \frac{\tan 2(y + \frac{\pi}{2})}{y}$$

$$= \lim_{y \rightarrow 0} \frac{\tan(2y + \pi)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{\tan(2y)}{y}$$

We know that,

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \lim_{y \rightarrow 0} \frac{\sin 2y}{y \cos 2y}$$

By multiplying and dividing by 2, we get

$$= \lim_{y \rightarrow 0} \frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}$$

$$= \lim_{2y \rightarrow 0} \frac{\sin 2y}{2y} \times \lim_{y \rightarrow 0} \frac{2}{\cos 2y}$$

$$= 1 \times 2 / \cos 0$$

$$= 1 \times 2 / 1$$

$$= 2$$

Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \begin{cases} 2x + 3 & x \leq 0 \\ 3(x + 1) & x > 0 \end{cases}$

23.

Solution:

$$\text{Given function is } f(x) = \begin{cases} 2x + 3 & x \leq 0 \\ 3(x + 1) & x > 0 \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (2x + 3)$$

$$= 2(0) + 3$$

$$= 0 + 3$$

$$= 3$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} 3(x + 1)$$

$$= 3(0 + 1)$$

$$= 3(1)$$

$$= 3$$

$$\text{Hence, } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 3$$

Now, for $\lim_{x \rightarrow 1} f(x)$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} 3(x + 1)$$

$$= 3(1 + 1)$$

$$= 3(2)$$

$$= 6$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} 3(x + 1)$$

$$= 3(1 + 1)$$

$$= 3(2)$$

$$= 6$$

Hence, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = 6$

$$\lim_{x \rightarrow 0} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} f(x) = 6$$

24. Find $\lim_{x \rightarrow 1} f(x)$, where

$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1} f(x):$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} x^2 - 1$$

$$= 1^2 - 1$$

$$= 1 - 1$$

$$= 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (-x^2 - 1)$$

$$= (-1^2 - 1)$$

$$= -1 - 1$$

$$= -2$$

We find,

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

Hence, $\lim_{x \rightarrow 1} f(x)$ does not exist

25. Evaluate $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ x & x = 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

We know that,

$$\lim_{x \rightarrow a} f(x) \text{ exists only when } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Now, we need to prove that: $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$

We know,

$$|x| = x, \text{ if } x > 0 \quad = -x, \text{ if } x < 0$$

Hence,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1)$$

$$= -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{|x|}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} (1)$$

$$= 1$$

We find here,

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

26. Find $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

Given function is:

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0} f(x):$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x}{|x|}$$

$$= \lim_{x \rightarrow 0^-} \frac{x}{-x} = \lim_{x \rightarrow 0^-} -1$$

$$= -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x}{|x|}$$

$$\lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} (1)$$

$$= 1$$

We find here,

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

27. Find $\lim_{x \rightarrow 5} f(x)$, where $f(x) = |x| - 5$

Solution:

Given function is:

$$f(x) = |x| - 5$$

$$\lim_{x \rightarrow 5} f(x):$$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} |x| - 5$$

$$= \lim_{x \rightarrow 5} (x - 5) = 5 - 5$$

$$= 0$$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} |x| - 5$$

$$= \lim_{x \rightarrow 5} (x - 5)$$

$$= 5 - 5$$

$$= 0$$

Hence, $\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5} f(x) = 0$

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$$

28. Suppose
a and b

and if $\lim_{x \rightarrow 1} f(x) = f(1)$ what are possible values of

Solution:

Given function is:

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases} \text{ and}$$

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} a + bx$$

$$= a + b(1)$$

$$= a + b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} b - ax$$

$$= b - a(1)$$

$$= b - a$$

Here,

$$f(1) = 4$$

Hence, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$

Then, $a + b = 4$ and $b - a = 4$

By solving the above two equations, we get,

$$a = 0 \text{ and } b = 4$$

Therefore, the possible values of a and b is 0 and 4 respectively

29. Let a_1, a_2, \dots, a_n be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$, compute $\lim_{x \rightarrow a} f(x)$

Solution:

Given function is:

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

$$\lim_{x \rightarrow a_1} f(x):$$

$$\lim_{x \rightarrow a_1} f(x) = \lim_{x \rightarrow a_1} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

$$= \left[\lim_{x \rightarrow a_1} (x - a_1) \right] \left[\lim_{x \rightarrow a_1} (x - a_2) \right] \dots \left[\lim_{x \rightarrow a_1} (x - a_n) \right]$$

We get,

$$= (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n) = 0$$

Hence, $\lim_{x \rightarrow a_1} f(x) = 0$

$$\lim_{x \rightarrow a} f(x):$$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

$$= \left[\lim_{x \rightarrow a} (x - a_1) \right] \left[\lim_{x \rightarrow a} (x - a_2) \right] \dots \left[\lim_{x \rightarrow a} (x - a_n) \right]$$

We get,

$$= (a - a_1)(a - a_2) \dots (a - a_n)$$

Hence, $\lim_{x \rightarrow a} f(x) = (a - a_1)(a - a_2) \dots (a - a_n)$

Therefore, $\lim_{x \rightarrow a_1} f(x) = 0$ and $\lim_{x \rightarrow a} f(x) = (a - a_1)(a - a_2) \dots (a - a_n)$

$$f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$$

30. If

Solution:

Given function is:

$$f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$$

There are three cases.

Case 1:

When $a = 0$

$\lim_{x \rightarrow 0} f(x)$:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (|x| + 1)$$

$$= \lim_{x \rightarrow 0} (-x + 1) = -0 + 1$$

$$= 1$$

For what value (s) of a does $\lim_{x \rightarrow a} f(x)$ exists?

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (|x| - 1)$$

$$= \lim_{x \rightarrow 0} (x - 1) = 0 - 1$$

$$= -1$$

Here, we find

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Case 2:

When $a < 0$

$$\lim_{x \rightarrow a} f(x):$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| + 1)$$

$$= \lim_{x \rightarrow a} (-x + 1) = -a + 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| + 1)$$

$$= \lim_{x \rightarrow a} (-x + 1) = -a + 1$$

$$\text{Hence, } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = -a + 1$$

Therefore, $\lim (f(x))$ exists at $x = a$ and $a < 0$

Case 3:

When $a > 0$

$\lim_{x \rightarrow a} f(x)$:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| - 1)$$

$$= \lim_{x \rightarrow a^-} (x - 1) = a - 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| - 1)$$

$$= \lim_{x \rightarrow a^+} (x - 1) = a - 1$$

Hence, $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = a - 1$

Therefore, $\lim_{x \rightarrow a} (f(x))$ exists at $x = a$ when $a > 0$

31. If the function $f(x)$ satisfies $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \rightarrow 1} f(x)$

Solution:

Given function that $f(x)$ satisfies $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$

$$\frac{\lim_{x \rightarrow 1} f(x) - 2}{\lim_{x \rightarrow 1} x^2 - 1} = \pi$$

$$\lim_{x \rightarrow 1} (f(x) - 2) = \pi (\lim_{x \rightarrow 1} (x^2 - 1))$$

Substituting $x = 1$, we get,

$$\lim_{x \rightarrow 1} (f(x) - 2) = \pi(1^2 - 1)$$

$$\lim_{x \rightarrow 1} (f(x) - 2) = \pi(1 - 1)$$

$$\lim_{x \rightarrow 1} (f(x) - 2) = 0$$

$$\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} 2 = 0$$

$$\lim_{x \rightarrow 1} f(x) - 2 = 0$$

$$= 2$$

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$$

32. If

and $\lim_{x \rightarrow 1} f(x)$ exist?

Solution:

For what integers m and n does both $\lim_{x \rightarrow 0} f(x)$

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$$

Given function is

$$\lim_{x \rightarrow 0} f(x):$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (mx^2 + n)$$

$$= m(0) + n$$

$$= 0 + n$$

$$= n$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (nx + m)$$

$$= n(0) + m$$

$$= 0 + m$$

$$= m$$

Hence,

$\lim_{x \rightarrow 0} f(x)$ exists if $n = m$.

Now,

$\lim_{x \rightarrow 1} f(x)$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (nx + m)$$

$$= n(1) + m$$

$$= n + m$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (nx^3 + m)$$

$$= n(1)^3 + m$$

$$= n(1) + m$$

$$= n + m$$

Therefore $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x)$

Hence, for any integral value of m and n $\lim_{x \rightarrow 1} f(x)$ exists.

EXERCISE 13.2

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1. Find the derivative of $x^2 - 2$ at $x = 10$

Solution:

Let $f(x) = x^2 - 2$

From first principle

From first principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Put $x = 10$, we get

$$f'(10) = \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{10^2 + 2 \times 10 \times h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \rightarrow 0} (20 + h)$$

$$= 20 + 0$$

$$= 20$$

2. Find the derivative of x at $x = 1$.

Solution:

Let $f(x) = x$

Then,

From first principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = x$

From first principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(10)}{h}$$

Put $x = 1$, we get

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1+h-1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h}$$

$$= \lim_{h \rightarrow 0} 1$$

$$= 1$$

3. Find the derivative of $99x$ at $x = 100$.

Solution:

Let $f(x) = 99x$,

From first principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Put $x = 100$, we get

$$f'(100) = \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \rightarrow 0} \frac{99 \times h}{h}$$

$$= \lim_{h \rightarrow 0} 99$$

$$= 99$$

4. Find the derivative of the following functions from first principle

(i) $x^3 - 27$

(ii) $(x - 1)(x - 2)$

(iii) $1/x^2$

(iv) $x + 1/x - 1$

Solution:

(i) Let $f(x) = x^3 - 27$

From first principle

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \\
 &= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\
 &= 0 + 3x^2 \\
 &= 3x^2
 \end{aligned}$$

(ii) Let $f(x) = (x-1)(x-2)$
From first principle

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{hx + hx + h^2 - 2h - h}{h} \\
 &= \lim_{h \rightarrow 0} (h + 2x - 3)
 \end{aligned}$$

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$$= 0 + 2x - 3$$

$$= 2x - 3$$

(iii) Let $f(x) = 1/x^2$

From first principle, we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-h - 2x}{x^2(x+h)^2} \right]$$

$$= (0 - 2x) / [x^2(x+0)^2]$$

$$= (-2/x^3)$$

(iv) Let $f(x) = x + 1/x - 1$

From first principle, we get

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx + x - x + h - 1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-2h}{h(x-1)(x+h-1)} \\
 &= \lim_{h \rightarrow 0} \frac{-2}{(x-1)(x+h-1)} \\
 &= -\frac{2}{(x-1)(x-1)} \\
 &= -\frac{2}{(x-1)^2}
 \end{aligned}$$

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5. For the function
Solution:

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

.Prove that $f'(1) = 100 f'(0)$.

Given function is:

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

By differentiating both sides, we get

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1 \right] \\ &= \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx} (x) + \frac{d}{dx} (1) \end{aligned}$$

We know that,

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

$$\therefore \frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

$$f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At $x = 0$, we get

$$f'(0) = 0 + 0 + \dots + 0 + 1$$

$$f'(0) = 1$$

At $x = 1$, we get

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = [1 + 1 + \dots + 1] \text{ 100 times} = 1 \times 100 = 100$$

Hence, $f'(1) = 100 f'(0)$

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$ for some fixed real number a .

Solution:

Given function is:

$$f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$$

By differentiating both sides, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n) \\ &= \frac{d}{dx} (x^n) + a \frac{d}{dx} (x^{n-1}) + a^2 \frac{d}{dx} (x^{n-2}) + \dots + a^{n-1} \frac{d}{dx} (x) + a^n \frac{d}{dx} (1) \end{aligned}$$

We know that,

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n(0)$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1}$$

7. For some constants a and b, find the derivative of

(i) $(x - a)(x - b)$

(ii) $(ax^2 + b)^2$

(iii) $x - a / x - b$

Solution:

(i) $(x - a)(x - b)$

$$\text{Let } f(x) = (x - a)(x - b)$$

$$f(x) = x^2 - (a + b)x + ab$$

Now, by differentiating both sides, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2 - (a + b)x + ab) \\ &= \frac{d}{dx}(x^2) - (a + b)\frac{d}{dx}(x) + \frac{d}{dx}(ab) \end{aligned}$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = 2x - (a + b) + 0$$

$$= 2x - a - b$$

(ii) $(ax^2 + b)^2$

Let $f(x) = (ax^2 + b)^2$

$$f(x) = a^2x^4 + 2abx^2 + b^2$$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$f'(x) = \frac{d}{dx}(x^4) + (2ab)\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = a^2 \times 4x^3 + 2ab \times 2x + 0$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax(ax^2 + b)$$

(iii) $x - a / x - b$

$$\text{Let } f(x) = \frac{(x-a)}{(x-b)}$$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

By further calculation, we get

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

$$\frac{x^n - a^n}{x - a}$$

8. Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a.

Solution:

$$\text{Let } f(x) = \frac{x^n - a^n}{x - a}$$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)$$

$$f'(x) = \frac{(x-a) \frac{d}{dx} (x^n - a^n) - (x^n - a^n) \frac{d}{dx} (x-a)}{(x-a)^2}$$

By further calculation, we get

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9. Find the derivative of

(i) $2x - 3/4$

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$

(vi) $(2/x + 1) - x^2/3x - 1$

Solution:

(i)

$$\text{Let } f(x) = 2x - \frac{3}{4}$$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx} \left(2x - \frac{3}{4} \right)$$

$$= 2 \frac{d}{dx}(x) - \frac{d}{dx} \left(\frac{3}{4} \right)$$

$$= 2 - 0$$

$$= 2$$

(ii)

$$\text{Let } f(x) = (5x^3 + 3x - 1)(x - 1)$$

By differentiating both sides and using the product rule, we get

$$f'(x) = (5x^3 + 3x - 1) \frac{d}{dx}(x - 1) + (x - 1) \frac{d}{dx}(5x^3 + 3x - 1)$$

$$= (5x^3 + 3x - 1) \times 1 + (x - 1) \times (15x^2 + 3)$$

$$= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$$

$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$

$$= 20x^3 - 15x^2 + 6x - 4$$

(iii)

$$\text{Let } f(x) = x^{-3}(5 + 3x)$$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-3} \frac{d}{dx}(5 + 3x) + (5 + 3x) \frac{d}{dx}(x^{-3})$$

$$= x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1})$$

By further calculation, we get

$$= x^{-3}(3) + (5 + 3x)(-3x^{-4})$$

$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$

$$= -6x^{-3} - 15x^{-4}$$

$$= -3x^{-3} \left(2 + \frac{5}{x} \right)$$

$$= \frac{-3x^{-3}}{x} (2x + 5)$$

$$= \frac{-3}{x^4} (5 + 2x)$$

(iv)

Let $f(x) = x^5 (3 - 6x^{-9})$

By differentiating both sides and using Leibnitz product rule, we get

$$\begin{aligned} f'(x) &= x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9}) \frac{d}{dx}(x^5) \\ &= x^5 \{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4) \end{aligned}$$

By further calculation, we get

$$\begin{aligned} &= x^5 (54x^{-10}) + 15x^4 - 30x^{-5} \\ &= 54x^{-5} + 15x^4 - 30x^{-5} \\ &= 24x^{-5} + 15x^4 \\ &= 15x^4 + \frac{24}{x^5} \end{aligned}$$

(v)

Let $f(x) = x^{-4} (3 - 4x^{-5})$

By differentiating both sides and using Leibnitz product rule, we get

$$\begin{aligned} f'(x) &= x^{-4} \frac{d}{dx}(3 - 4x^{-5}) + (3 - 4x^{-5}) \frac{d}{dx}(x^{-4}) \\ &= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1} \end{aligned}$$

By further calculation, we get

$$\begin{aligned} &= x^{-4} (20x^{-6}) + (3 - 4x^{-5})(-4x^{-5}) \\ &= 20x^{-10} - 12x^{-5} + 16x^{-10} \\ &= 36x^{-10} - 12x^{-5} \\ &= -\frac{12}{x^5} + \frac{36}{x^{10}} \end{aligned}$$

(vi)

$$f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$$

Let

By differentiating both sides we get,

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} - \frac{x^2}{3x-1} \right)$$

Using quotient rule we get,

$$f'(x) = \left[\frac{(x+1) \frac{d}{dx}(2) - 2 \frac{d}{dx}(x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2} \right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[\frac{(3x-1)(2x) - (x^2) \times 3}{(3x-1)^2} \right]$$

$$= -\frac{2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right]$$

$$= -\frac{2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Find the derivative of $\cos x$ from first principle

Solution:

Let $f(x) = \cos x$

Accordingly, $f(x + h) = \cos(x + h)$

By first principle, we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

So, we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x + h) - \cos(x)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{x + h + x}{2}\right) \sin\left(\frac{x + h - x}{2}\right) \right] \end{aligned}$$

By further calculation, we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{2x + h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\ &= \lim_{h \rightarrow 0} -\sin\left(\frac{2x + h}{2}\right) \times \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\ &= -\sin\left(\frac{2x + 0}{2}\right) \times 1 \\ &= -\sin(2x / 2) \\ &= -\sin(x) \end{aligned}$$

11. Find the derivative of the following functions:

- (i) $\sin x \cos x$
- (ii) $\sec x$
- (iii) $5 \sec x + 4 \cos x$
- (iv) $\operatorname{cosec} x$
- (v) $3 \cot x + 5 \operatorname{cosec} x$
- (vi) $5 \sin x - 6 \cos x + 7$
- (vii) $2 \tan x - 7 \sec x$

Solution:

- (i) $\sin x \cos x$

Let $f(x) = \sin x \cos x$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [2\sin(x+h)\cos(x+h) - 2\sin x \cos x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2\cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right]
 \end{aligned}$$

By further calculation, we get

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos \frac{4x+2h}{2} \sin \frac{2h}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\cos(2x+h)\sin h] \\
 &= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos(2x+0) \cdot 1 \\
 &= \cos 2x
 \end{aligned}$$

(ii) $\sec x$

$$\text{Let } f(x) = \sec x$$

$$= 1 / \cos x$$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx} \left(\frac{1}{\cos x} \right)$$

Using quotient rule, we get

$$\begin{aligned} f'(x) &= \frac{\cos x \frac{d}{dx}(1) - 1 \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x} \end{aligned}$$

We get

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$= \tan x \sec x$$

(iii) $5 \sec x + 4 \cos x$

Let $f(x) = 5 \sec x + 4 \cos x$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx} (5 \sec x + 4 \cos x)$$

By further calculation, we get

$$= 5 \frac{d}{dx} (\sec x) + 4 \frac{d}{dx} (\cos x)$$

$$= 5 \sec x \tan x + 4 \times (-\sin x)$$

$$= 5 \sec x \tan x - 4 \sin x$$

(iv) cosec x

Let $f(x) = \operatorname{cosec} x$

Accordingly $f(x+h) = \operatorname{cosec}(x+h)$

By first principle, we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right)$$



$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right]
 \end{aligned}$$

By further calculation, we get

$$\begin{aligned}
 &= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-\sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin(x+h)} \right] \\
 &= -\frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \\
 &= -\frac{1}{\sin x} \times 1 \times \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)} \\
 &= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\
 &= -\operatorname{cosec} x \cot x
 \end{aligned}$$

(v) $3 \cot x + 5 \operatorname{cosec} x$

Let $f(x) = 3 \cot x + 5 \operatorname{cosec} x$

$$f'(x) = 3 (\cot x)' + 5 (\operatorname{cosec} x)'$$

Let $f_1(x) = \cot x$,

Accordingly $f_1(x+h) = \cot(x+h)$

By using first principle, we get

$$f_1'(x) = \lim_{x \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right)$$

By further calculation, we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right)$$

$$= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin(x+h)} \right]$$

$$= -\frac{1}{\sin x} \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x+h)} \right)$$

$$= -\frac{1}{\sin x} \times 1 \times \frac{1}{\sin(x+0)}$$

$$= -\frac{1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

Let $f_2(x) = \operatorname{cosec} x$,

Accordingly $f_2(x+h) = \operatorname{cosec}(x+h)$

By using first principle, we get

$$f_2'(x) = \lim_{h \rightarrow 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right]$$

By further calculation, we get

$$= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \lim_{h \rightarrow 0} \left[\frac{-\sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin(x+h)} \right]$$

$$= -\frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= -\frac{1}{\sin x} \times 1 \times \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cot x$$

Now, substitute the value of $(\cot x)'$ and $(\operatorname{cosec} x)'$ in $f'(x)$, we get

$$f'(x) = 3 (\cot x)' + 5 (\operatorname{cosec} x)'$$

$$f'(x) = 3 \times (-\operatorname{cosec}^2 x) + 5 \times (-\operatorname{cosec} x \cot x)$$

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

$$(vi) 5 \sin x - 6 \cos x + 7$$

$$\text{Let } f(x) = 5 \sin x - 6 \cos x + 7$$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [5 \sin(x+h) - 6 \cos(x+h) + 7 - 5 \sin x + 6 \cos x - 7] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [5 \{\sin(x+h) - \sin x\} - 6 \{\cos(x+h) - \cos x\}] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x]
 \end{aligned}$$

By further calculation, we get

$$\begin{aligned}
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x (1 - \cos h) - \sin x \sin h}{h} \right]
 \end{aligned}$$

Now, we get

$$\begin{aligned}
 &= 5 \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
 &= 5 \left[\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \left[(-\cos x) \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \right] \\
 &= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
 &= 5 \cos x + 6 \sin x
 \end{aligned}$$

(vii) $2 \tan x - 7 \sec x$

Let $f(x) = 2 \tan x - 7 \sec x$

Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [2 \{ \tan(x+h) - \tan x \} - 7 \{ \sec(x+h) - \sec x \}] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x]
 \end{aligned}$$

By further calculation, we get

$$\begin{aligned}
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right]
 \end{aligned}$$

Now, we get

$$\begin{aligned}
 &= 2 \lim_{h \rightarrow 0} \left[\left(\frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 2 \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left(\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right)
 \end{aligned}$$

$$\begin{aligned} &= 2.1 \cdot \frac{1}{\cos x \cos x} - 7.1 \left(\frac{\sin x}{\cos x \cos x} \right) \\ &= 2 \sec^2 x - 7 \sec x \tan x \end{aligned}$$



MISCELLANEOUS EXERCISE

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1. Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x + 1)$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Solution:

(i) $-x$

Let $f(x) = -x$

Accordingly, $f(x + h) = -(x + h)$

Using first principle, we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h}$$

Now, we get

$$= \lim_{h \rightarrow 0} \frac{-x - h + x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h}$$

$$= \lim_{h \rightarrow 0} (-1) = -1$$

(ii) $(-x)^{-1}$

Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$

Accordingly, $f(x + h) = \frac{-1}{(x + h)}$

Using first principle, we get

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{x+h} - \left(\frac{-1}{x} \right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{x+h} + \frac{1}{x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-x + (x+h)}{x(x+h)} \right]
 \end{aligned}$$

By further calculation, we get

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-x + x + h}{x(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{x(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{x(x+h)} \\
 &= \frac{1}{x \cdot x} \\
 &= 1/x^2
 \end{aligned}$$

(iii) $\sin(x+1)$

Let $f(x) = \sin(x+1)$

Accordingly, $f(x+h) = \sin(x+h+1)$

By using first principle, we get

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+1) - \sin(x+1)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+1+x+1}{2} \right) \sin \left(\frac{x+h+1-x-1}{2} \right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+h+2}{2} \right) \sin \left(\frac{h}{2} \right) \right]
 \end{aligned}$$

$$= \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right]$$

We get,

$$= \lim_{h \rightarrow 0} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

We know that,

$$h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0$$

$$= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1$$

$$= \cos(x+1)$$

$$(iv) \cos\left(x - \frac{\pi}{8}\right)$$

$$\text{Let } f(x) = \cos\left(x - \frac{\pi}{8}\right)$$

$$\text{Accordingly, } f(x+h) = \cos\left(x+h - \frac{\pi}{8}\right)$$

By using first principle, we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(x+h - \frac{\pi}{8}\right) - \cos\left(x - \frac{\pi}{8}\right) \right] \end{aligned}$$

We get,

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin \left(\frac{x+h-\frac{\pi}{8}+x-\frac{\pi}{8}}{2} \right) \sin \left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2} \right) \right]$$

Further we get,

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin \left(\frac{2x+h-\frac{\pi}{4}}{2} \right) \sin \frac{h}{2} \right]$$

So,

$$= \lim_{h \rightarrow 0} \left[-\sin \left(\frac{2x+h-\frac{\pi}{4}}{2} \right) \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right]$$

$$= \lim_{h \rightarrow 0} \left[-\sin \left(\frac{2x+h-\frac{\pi}{4}}{2} \right) \right] \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)}$$

$$\left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right]$$

$$= -\sin \left(\frac{2x+0-\frac{\pi}{4}}{2} \right) \cdot 1$$

Hence, we get

$$= -\sin \left(x - \frac{\pi}{8} \right)$$

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

2. $(x + a)$

Solution:

$$\text{Let } f(x) = x + a$$

$$\text{Accordingly, } f(x + h) = x + h + a$$

Using first principle, we get

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

So, now we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{x + h + a - x - a}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{h}{h} \right) \\ &= \lim_{h \rightarrow 0} (1) \\ &= 1 \end{aligned}$$

3. (px + q) (r / x + s)

Solution:

$$\text{Let } f(x) = (px + q) \left(\frac{r}{x} + s \right)$$

Using Leibnitz product rule, we get

$$f'(x) = (px + q) \left(\frac{r}{x} + s \right)' + \left(\frac{r}{x} + s \right) (px + q)'$$

We get,

$$= (px + q) (rx^{-1} + s)' + \left(\frac{r}{x} + s \right) (p)$$

By further calculation, we get

$$= (px + q) (-rx^{-2}) + \left(\frac{r}{x} + s \right) p$$

$$= (px + q) \left(\frac{-r}{x^2} \right) + \left(\frac{r}{x} + s \right) p$$

Now, we get

$$\begin{aligned} &= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\ &= ps - \frac{qr}{x^2} \end{aligned}$$

4. $(ax + b)(cx + d)^2$

Solution:

$$\text{Let } f(x) = (ax + b)(cx + d)^2$$

By using Leibnitz product rule, we get

$$f'(x) = (ax + b) \frac{d}{dx} (cx + d)^2 + (cx + d)^2 \frac{d}{dx} (ax + b)$$

We get,

$$= (ax + b) \frac{d}{dx} (c^2x^2 + 2cdx + d^2) + (cx + d)^2 \frac{d}{dx} (ax + b)$$

By differentiating separately, we get

$$= (ax + b) \left[\frac{d}{dx} (c^2x^2) + \frac{d}{dx} (2cdx) + \frac{d}{dx} d^2 \right] + (cx + d)^2 \left[\frac{d}{dx} ax + \frac{d}{dx} b \right]$$

So,

$$\begin{aligned} &= (ax + b)(2c^2x + 2cd) + (cx + d)^2 a \\ &= 2c(ax + b)(cx + d) + a(cx + d)^2 \end{aligned}$$

5. $(ax + b) / (cx + d)$

Solution:

$$\text{Let } f(x) = \frac{ax + b}{cx + d}$$

Using quotient rule, we get

$$f'(x) = \frac{(cx+d) \frac{d}{dx}(ax+b) - (ax+b) \frac{d}{dx}(cx+d)}{(cx+d)^2}$$

Further we get

$$= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

So, now we get

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

Hence,

$$= \frac{ad - bc}{(cx+d)^2}$$

6. $(1 + 1/x) / (1 - 1/x)$

Solution:

$$\text{Let } f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}, \text{ where } x \neq 0$$

Using quotient rule, we get

$$f'(x) = \frac{(x-1) \frac{d}{dx}(x+1) - (x+1) \frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1$$

Further, we get

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0, 1$$

So,

$$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{-2}{(x-1)^2}, x \neq 0, 1$$

7. $1 / (ax^2 + bx + c)$

Solution:

$$\text{Let } f(x) = \frac{1}{ax^2 + bx + c}$$

Using quotient rule, we get

$$f'(x) = \frac{(ax^2 + bx + c) \frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2}$$

By further calculation, we get

$$= \frac{(ax^2 + bx + c)(0) - (2ax + b)}{(ax^2 + bx + c)^2}$$

$$= \frac{-(2ax + b)}{(ax^2 + bx + c)^2}$$

8. $(ax + b) / px^2 + qx + r$

Solution:

$$\text{Let } f(x) = \frac{ax + b}{px^2 + qx + r}$$

Using quotient rule, we get

$$f'(x) = \frac{(px^2 + qx + r) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2}$$

Further we get,

$$= \frac{(px^2 + qx + r)(a) - (ax + b)(2px + q)}{(px^2 + qx + r)^2}$$

Again by further calculation, we get

$$= \frac{apx^2 + aqx + ar - 2apx^2 - aqx - 2bpx - bq}{(px^2 + qx + r)^2}$$

$$= \frac{-apx^2 - 2bpx + ar - bq}{(px^2 + qx + r)^2}$$

9. $(px^2 + qx + r) / ax + b$

Solution:

$$\text{Let } f(x) = \frac{px^2 + qx + r}{ax + b}$$

Using quotient rule, we get

$$f'(x) = \frac{(ax + b) \frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r) \frac{d}{dx}(ax + b)}{(ax + b)^2}$$

By further calculation, we get

$$= \frac{(ax + b)(2px + q) - (px^2 + qx + r)(a)}{(ax + b)^2}$$

So, we get

$$= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ar}{(ax + b)^2}$$

$$= \frac{apx^2 + 2bpx + bq - ar}{(ax + b)^2}$$

10. $(a / x^4) - (b / x^2) + \cos x$

Solution:

$$\text{Let } f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$$

By differentiating we get,

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(\cos x)$$

On further calculation, we get

$$= a \frac{d}{dx}(x^{-4}) - b \frac{d}{dx}(x^{-2}) + \frac{d}{dx}(\cos x)$$

We know that,

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} \text{ and } \frac{d}{dx}(\cos x) = -\sin x \right]$$

So,

$$\begin{aligned} &= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x) \\ &= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x \end{aligned}$$

11. $4\sqrt{x} - 2$

Solution:

$$\text{Let } f(x) = 4\sqrt{x} - 2$$

By differentiating we get,

$$f'(x) = \frac{d}{dx}(4\sqrt{x} - 2) = \frac{d}{dx}(4\sqrt{x}) - \frac{d}{dx}(2)$$

Further, we get

$$= 4 \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0$$

$$= 4 \left(\frac{1}{2} x^{\frac{1}{2}-1} \right)$$

$$= \left(2x^{-\frac{1}{2}} \right)$$

$$= \frac{2}{\sqrt{x}}$$

12. $(ax + b)^n$

Solution:

$$\text{Let } f(x) = (ax + b)^n$$

$$\text{Accordingly, } f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$$

Using first principle, we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax+ah+b)^n - (ax+b)^n}{h} \end{aligned}$$

Further we get,

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(ax+b)^n \left(1 + \frac{ah}{ax+b}\right)^n - (ax+b)^n}{h} \\ &= (ax+b)^n \lim_{h \rightarrow 0} \frac{\left(1 + \frac{ah}{ax+b}\right)^n - 1}{h} \end{aligned}$$

By using binomial theorem, we get

$$= (ax+b)^n \lim_{h \rightarrow 0} \frac{1}{n} \left[\left\{ 1 + n \left(\frac{ah}{ax+b} \right) + \frac{n(n-1)}{2} \left(\frac{ah}{ax+b} \right)^2 + \dots \right\} - 1 \right]$$

Now, we get

$$= (ax+b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[n \left(\frac{ah}{ax+b} \right) + \frac{n(n-1)a^2h^2}{2(ax+b)^2} + \dots (\text{Terms containing higher degrees of } h) \right]$$

So, we get

$$= (ax+b)^n \lim_{h \rightarrow 0} \left[\frac{na}{(ax+b)} + \frac{n(n-1)a^2h}{2(ax+b)^2} + \dots \right]$$

On further calculation, we get

$$= (ax+b)^n \left[\frac{na}{(ax+b)} + 0 \right]$$

$$= na \frac{(ax+b)^n}{(ax+b)}$$

$$= na(ax+b)^{n-1}$$

13. $(ax + b)^n (cx + d)^m$

Solution:

$$\text{Let } f(x) = (ax + b)^n (cx + d)^m$$

By using Leibnitz product rule, we get

$$f'(x) = (ax + b)^n \frac{d}{dx} (cx + d)^m + (cx + d)^m \frac{d}{dx} (ax + b)^n$$

$$\text{let } f_1(x) = (cx + d)^m$$

$$f_1(x+h) = (cx + ch + d)^m$$

Then,

$$f_1'(x) = \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(cx + ch + d)^m - (cx + d)^m}{h}$$

By taking $(cx + d)^m$ as common, we get

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{ch}{cx + d} \right)^m - 1 \right]$$

On further calculation, we get

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{mch}{(cx + d)} + \frac{m(m-1)}{2} \frac{(c^2h^2)}{(cx + d)^2} + \dots \right) - 1 \right]$$

Now, we get

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{mch}{(cx + d)} + \frac{m(m-1)c^2h^2}{2(cx + d)^2} + \dots (\text{Terms containing higher degrees of } h) \right]$$

We know that,

$$\frac{d}{dx} (cx + d)^m = mc (cx + d)^{m-1}$$

Similarly, $\frac{d}{dx} (ax + b)^n = na (ax + b)^{n-1}$

$$= (cx + d)^m \lim_{h \rightarrow 0} \left[\frac{mc}{(cx + d)} + \frac{m(m-1)c^2h}{2(cx + d)^2} + \dots \right]$$

Now, we get

$$\begin{aligned} &= (cx + d)^m \left[\frac{mc}{cx + d} + 0 \right] \\ &= \frac{mc (cx + d)^m}{(cx + d)} \\ &= mc (cx + d)^{m-1} \end{aligned}$$

Hence, we get

$$\begin{aligned} f'(x) &= (ax+b)^n \{mc(cx+d)^{m-1}\} + (cx+d)^m \{na(ax+b)^{n-1}\} \\ &= (ax+b)^{n-1} (cx+d)^{m-1} [mc(ax+b) + na(cx+d)] \end{aligned}$$

14. $\sin(x+a)$

Solution:

$$\text{Let } f(x) = \sin(x+a)$$

$$f(x+h) = \sin(x+h+a)$$

By using first principle, we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h+a) - \sin(x+a)}{h} \end{aligned}$$

On further calculation, we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+a+x+a}{2}\right) \sin\left(\frac{x+h+a-x-a}{2}\right) \right]$$

So, we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\ &= \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+2a+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \right] \end{aligned}$$

By taking limits, we get

$$= \lim_{h \rightarrow 0} \cos\left(\frac{2x+2a+h}{2}\right) \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\}$$

Hence, we get

$$= \cos\left(\frac{2x+2a}{2}\right) \times 1$$

$$= \cos(x+a)$$

15. cosec x cot x

Solution:

$$\text{Let } f(x) = \text{cosec } x \cot x$$

By using Leibnitz product rule, we get

$$f'(x) = \text{cosec } x (\cot x)' + \cot x (\text{cosec } x)' \quad \dots(1)$$

$$\text{Let } f_1(x) = \cot x.$$

$$\text{Accordingly, } f_1(x+h) = \cot(x+h)$$

By using first principle, we get

$$f_1'(x) = \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h}$$

On further calculation, we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos(x+h) - \cos x}{\sin(x+h) - \sin x} \right)$$

Now, we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \end{aligned}$$

We get

$$\begin{aligned} &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin(x+h)} \right] \\ &= \frac{-1}{\sin x} \cdot \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x+h)} \right) \end{aligned}$$

So, we get

$$\begin{aligned}
 &= \frac{-1}{\sin x} \cdot 1 \cdot \left(\frac{1}{\sin(x+0)} \right) \\
 &= \frac{-1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

Hence, we get

$$(\cot x)' = -\operatorname{cosec}^2 x \quad \dots(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By using first principle, we get

$$\begin{aligned}
 f_2'(x) &= \lim_{h \rightarrow 0} \frac{f_2(x+h) - f_2(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x]
 \end{aligned}$$

By calculating further, we get

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right]
 \end{aligned}$$

So,

$$\begin{aligned}
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{x+x+h}{2} \right) \sin \left(\frac{x-x-h}{2} \right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{2x+h}{2} \right) \sin \left(\frac{-h}{2} \right)}{\sin(x+h)} \right]
 \end{aligned}$$

$$= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \left[\frac{-\sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin(x+h)} \right]$$

We get,

$$\begin{aligned} &= \frac{-1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \\ &= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)} \\ &= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -\operatorname{cosec} x \cdot \cot x \end{aligned}$$

Hence,

$$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \quad \dots(3)$$

From equations (1) (2) and (3) we get,

$$\begin{aligned} f'(x) &= \operatorname{cosec} x (-\operatorname{cosec}^2 x) + \cot x (-\operatorname{cosec} x \cot x) \\ &= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x \end{aligned}$$

16. $\frac{\cos x}{1 + \sin x}$

Solution:

$$\text{Let } f(x) = \frac{\cos x}{1 + \sin x}$$

By using quotient rule, we get

$$\begin{aligned} f'(x) &= \frac{(1 + \sin x) \frac{d}{dx}(\cos x) - (\cos x) \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \end{aligned}$$

We get,

$$\begin{aligned} &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \end{aligned}$$

Now, we get

$$\begin{aligned} &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-1}{(1 + \sin x)} \end{aligned}$$

$$17. \frac{\sin x + \cos x}{\sin x - \cos x}$$

Solution:

$$\text{Let } f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(\sin x - \cos x) \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

On further calculation, we get

$$\begin{aligned} &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\ &= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} \end{aligned}$$

By expanding the terms, we get

$$= \frac{-[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x]}{(\sin x - \cos x)^2}$$

We get

$$\begin{aligned} &= \frac{-[1+1]}{(\sin x - \cos x)^2} \\ &= \frac{-2}{(\sin x - \cos x)^2} \end{aligned}$$

$$\frac{\sec x - 1}{\sec x + 1}$$

18. Solution:

$$\text{Let } f(x) = \frac{\sec x - 1}{\sec x + 1}$$

Now, this can be written as

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By differentiating and using quotient rule, we get

$$\begin{aligned} f'(x) &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\ &= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2} \end{aligned}$$

On multiplying we get

$$\begin{aligned} &= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2} \\ &= \frac{2\sin x}{(1 + \cos x)^2} \end{aligned}$$

This can be written as

$$= \frac{2 \sin x}{\left(1 + \frac{1}{\sec x}\right)^2}$$

On taking L.C.M we get

$$= \frac{2 \sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}}$$

On further calculation, we get

$$\begin{aligned} &= \frac{2 \sin x \sec^2 x}{(\sec x + 1)^2} \\ &= \frac{2 \sin x}{\cos x} \sec x \\ &= \frac{2 \sec x \tan x}{(\sec x + 1)^2} \end{aligned}$$

19. $\sin^n x$

Solution:

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

We know that,

$$\frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx} \sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\text{So, } \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin x)$$

By Leibnitz product rule, we get

$$\begin{aligned} &= (\sin x)' \sin x + \sin x (\sin x)' \\ &= \cos x \sin x + \sin x \cos x \\ &= 2 \sin x \cos x \quad \dots(1) \end{aligned}$$

For $n = 3$, $y = \sin^3 x$.

$$\text{So, } \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin^2 x)$$

By Leibnitz product rule, we get

$$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)'$$

From equation (1) we get

$$= \cos x \sin^2 x + \sin x (2 \sin x \cos x)$$

$$= \cos x \sin^2 x + 2 \sin^2 x \cos x$$

$$= 3 \sin^2 x \cos x$$

We state that, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

For $n = k$, let our assertion be true

$$\text{i.e., } \frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad \dots(2)$$

Now, consider

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^k x)$$

By using Leibnitz product rule, we get

$$= (\sin x)' \sin^k x + \sin x (\sin^k x)'$$

From equation (2) we get

$$= \cos x \sin^k x + \sin x (k \sin^{(k-1)} x \cos x)$$

$$= \cos x \sin^k x + k \sin^k x \cos x$$

$$= (k+1) \sin^k x \cos x$$

Hence, our assertion is true for $n = k + 1$

Therefore, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

▼

20. $\frac{a + b \sin x}{c + d \cos x}$

Solution:

$$\text{Let } f(x) = \frac{a + b \sin x}{c + d \cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(c + d \cos x) \frac{d}{dx}(a + b \sin x) - (a + b \sin x) \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2}$$

$$= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2}$$

On multiplying we get

$$= \frac{cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2}$$

Now, taking bd as common we get

$$= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

$$\frac{\sin(x+a)}{\cos x}$$

21. $\cos x$

Solution:

$$\text{Let } f(x) = \frac{\sin(x+a)}{\cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a) \frac{d}{dx} \cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a)(-\sin x)}{\cos^2 x} \quad \dots (i)$$

Let $g(x) = \sin(x+a)$. Accordingly, $g(x+h) = \sin(x+h+a)$

By using first principle, we get

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+a) - \sin(x+a)] \end{aligned}$$

On further calculation, we get

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+a+x+a}{2} \right) \sin \left(\frac{x+h+a-x-a}{2} \right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right] \\
 &= \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+2a+h}{2} \right) \left\{ \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right\} \right]
 \end{aligned}$$

Now, taking limits we get

$$= \lim_{h \rightarrow 0} \cos \left(\frac{2x+2a+h}{2} \right) \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right\} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right]$$

We know that,

$$\begin{aligned}
 &\left[\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 &= \left(\cos \frac{2x+2a}{2} \right) \times 1 \\
 &= \cos(x+a) \quad \dots \text{(ii)}
 \end{aligned}$$

From equation (i) and (ii) we get

$$\begin{aligned}
 f'(x) &= \frac{\cos x \cdot \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x} \\
 &= \frac{\cos(x+a-x)}{\cos^2 x} \\
 &= \frac{\cos a}{\cos^2 x}
 \end{aligned}$$

22. $x^4 (5 \sin x - 3 \cos x)$

Solution:

$$\text{Let } f(x) = x^4 (5 \sin x - 3 \cos x)$$

By differentiating and using product rule, we get

$$f'(x) = x^4 \frac{d}{dx} (5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4)$$

On further calculation, we get

$$= x^4 \left[5 \frac{d}{dx}(\sin x) - 3 \frac{d}{dx}(\cos x) \right] + (5 \sin x - 3 \cos x) \frac{d}{dx}(x^4)$$

So, we get

$$= x^4 [5 \cos x - 3(-\sin x)] + (5 \sin x - 3 \cos x)(4x^3)$$

By taking x^3 as common, we get

$$= x^3 [5x \cos x + 3x \sin x + 20 \sin x - 12 \cos x]$$

23. $(x^2 + 1) \cos x$

Solution:

$$\text{Let } f(x) = (x^2 + 1) \cos x$$

By differentiating and using product rule, we get

$$f'(x) = (x^2 + 1) \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1)$$

On further calculation, we get

$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$

By multiplying we get

$$= -x^2 \sin x - \sin x + 2x \cos x$$

24. $(ax^2 + \sin x)(p + q \cos x)$

Solution:

$$\text{Let } f(x) = (ax^2 + \sin x)(p + q \cos x)$$

By differentiating and using product rule, we get

$$f'(x) = (ax^2 + \sin x) \frac{d}{dx}(p + q \cos x) + (p + q \cos x) \frac{d}{dx}(ax^2 + \sin x)$$

On further calculation, we get

$$= (ax^2 + \sin x)(-q \sin x) + (p + q \cos x)(2ax + \cos x)$$

$$= -q \sin x(ax^2 + \sin x) + (p + q \cos x)(2ax + \cos x)$$

25. $(x + \cos x)(x - \tan x)$

Solution:

Let $f(x) = (x + \cos x)(x - \tan x)$

By differentiating and using product rule, we get

$$f'(x) = (x + \cos x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \cos x)$$

$$= (x + \cos x) \left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x) \right] + (x - \tan x)(1 - \sin x)$$

Now, we get

$$= (x + \cos x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x)(1 - \sin x) \quad \dots (i)$$

Let $g(x) = \tan x$. Accordingly, $g(x+h) = \tan(x+h)$

By using first principle, we get

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\tan(x+h) - \tan x}{h} \right)$$

On further calculation, we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right]$$

Now, we get

$$= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)} \right]$$

So, we get

$$= \frac{1}{\cos x} \cdot \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \right)$$

We get

$$\begin{aligned}
 &= \frac{1}{\cos x} \cdot 1 \cdot \frac{1}{\cos(x+0)} \\
 &= \frac{1}{\cos^2 x} \\
 &= \sec^2 x \quad \dots \text{(ii)}
 \end{aligned}$$

Hence, from equation (i) and (ii) we get

$$\begin{aligned}
 f'(x) &= (x + \cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x) \\
 &= (x + \cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x) \\
 &= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)
 \end{aligned}$$

26. $\frac{4x + 5 \sin x}{3x + 7 \cos x}$

Solution:

$$\text{Let } f(x) = \frac{4x + 5 \sin x}{3x + 7 \cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(3x + 7 \cos x) \frac{d}{dx}(4x + 5 \sin x) - (4x + 5 \sin x) \frac{d}{dx}(3x + 7 \cos x)}{(3x + 7 \cos x)^2}$$

On further calculation, we get

$$\begin{aligned}
 &= \frac{(3x + 7 \cos x) \left[4 \frac{d}{dx}(x) + 5 \frac{d}{dx}(\sin x) \right] - (4x + 5 \sin x) \left[3 \frac{d}{dx}x + 7 \frac{d}{dx} \cos x \right]}{(3x + 7 \cos x)^2} \\
 &= \frac{(3x + 7 \cos x)(4 + 5 \cos x) - (4x + 5 \sin x)(3 - 7 \sin x)}{(3x + 7 \cos x)^2}
 \end{aligned}$$

On multiplying we get

$$= \frac{12x + 15x \cos x + 28 \cos x + 35 \cos^2 x - 12x + 28x \sin x - 15 \sin x + 35 \sin^2 x}{(3x + 7 \cos x)^2}$$

We get

$$\begin{aligned}
 &= \frac{15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x + 35(\cos^2 x + \sin^2 x)}{(3x + 7 \cos x)^2} \\
 &= \frac{35 + 15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x}{(3x + 7 \cos x)^2}
 \end{aligned}$$

$$\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$$

27. $\sin x$

Solution:

$$\text{Let } f(x) = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \cos \frac{\pi}{4} \cdot \left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x} \right]$$

By further calculation, we get

$$= \cos \frac{\pi}{4} \cdot \left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x} \right]$$

By taking x as common, we get

$$= \frac{x \cos \frac{\pi}{4} [2 \sin x - x \cos x]}{\sin^2 x}$$

28. $\frac{x}{1 + \tan x}$

Solution:

$$\text{Let } f(x) = \frac{x}{1 + \tan x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1 + \tan x) \frac{d}{dx}(x) - x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{(1 + \tan x) - x \cdot \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \quad \dots (i)$$

Let $g(x) = 1 + \tan x$. Accordingly, $g(x+h) = 1 + \tan(x+h)$.

Using first principle, we get

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{1 + \tan(x+h) - 1 - \tan x}{h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right]$$

By taking L.C.M we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right]$$

We get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)\cos x} \right]$$

So, we get

$$= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(1 + \tan x) = \sec^2 x \quad \dots \text{(ii)}$$

From equation (i) and (ii) we get

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

29. $(x + \sec x)(x - \tan x)$

Solution:

$$\text{Let } f(x) = (x + \sec x)(x - \tan x)$$

By differentiating and using product rule, we get

$$f'(x) = (x + \sec x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \sec x)$$

So, we get

$$= (x + \sec x) \left[\frac{d}{dx}(x) - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[\frac{d}{dx}(x) + \frac{d}{dx} \sec x \right]$$

$$= (x + \sec x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[1 + \frac{d}{dx} \sec x \right] \quad \dots \text{(i)}$$

Let $f_1(x) = \tan x$, $f_2(x) = \sec x$

Accordingly, $f_1(x+h) = \tan(x+h)$ and $f_2(x+h) = \sec(x+h)$

$$\begin{aligned} f_1'(x) &= \lim_{h \rightarrow 0} \left(\frac{f_1(x+h) - f_1(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\tan(x+h) - \tan x}{h} \right) \end{aligned}$$

By further calculation, we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \left[\frac{\tan(x+h) - \tan x}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \end{aligned}$$

Now, by taking L.C.M we get

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)\cos x} \right] \\ &= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \right) \\ &= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

Hence we get

$$\frac{d}{dx} \tan x = \sec^2 x \quad \dots \text{(ii)}$$

Now, take

$$f_2'(x) = \lim_{h \rightarrow 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sec(x+h) - \sec x}{h} \right)$$

This can be written as

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

By taking L.C.M we get

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right]$$

On further calculation, we get

$$= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right]$$

We get

$$= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right]$$

By taking limits, we get

$$= \sec x \cdot \frac{\left\{ \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\lim_{h \rightarrow 0} \cos(x+h)}$$

We get

$$= \sec x \cdot \frac{\sin x \cdot 1}{\cos x}$$

$$\frac{d}{dx} \sec x = \sec x \tan x \quad \dots \text{(iii)}$$

From equation (i) (ii) and (iii) we get

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

30. $\frac{x}{\sin^n x}$

Solution:

$$\text{Let } f(x) = \frac{x}{\sin^n x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

Easily, it can be shown that,

$$\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$$

Hence,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

By further calculation, we get

$$= \frac{\sin^n x \cdot 1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x}$$

By taking common terms, we get

$$= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x}$$

Hence, we get

$$= \frac{\sin x - nx \cos x}{\sin^{n+1} x}$$

