CIRCULAR MOTION

1. Kinematics of Circular Motion :

- Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} = \vec{a}_t + \vec{a}_c$
- Tangential acceleration $a_t = \frac{d|\vec{v}|}{dt} = \alpha r$

  [$\vec{a}_t$ = component of $\vec{a}$ along $\vec{v} = (\vec{a}\cdot\hat{v})\hat{v}$]
- Centripetal acceleration $a_c = \omega v = \frac{v^2}{r} = \omega^2 r$ or $\vec{a}_c = \omega^2 r(-\hat{r})$
- Magnitude of net acceleration $a = \sqrt{a_c^2 + a_1^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$
- The concept of radius of curvature : The normal on tangent at a point on the curve gives the direction of radius.

  i.e., $R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

2. Centripetal and Centrifugal Forces :

2.1 Centripetal Force :

Centripetal force can be expressed as

$$\vec{F} = -m\omega^2\vec{r} = -m\omega^2\hat{r} = -\left(\frac{mv^2}{r}\right)\hat{r}$$

(a) If the body comes to rest on a circular path i.e., $\vec{v} \to 0$, the body will move along the radius towards the centre and if a_r vanishes, the body will fly off tangentially, so a tangential velocity and radial acceleration are necessary for uniform circular motion.

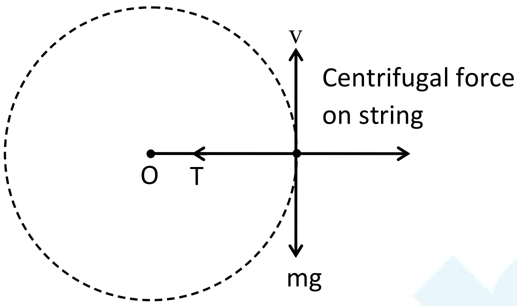
(b) As $F \neq 0$, so the body is not in equilibrium and linear momentum of the particle does not reamin conserved but angular momentum is conserved as the force is central i.e. $\tau = 0$

(c) In the case of circular motion, centripetal force changes only the direction of velocity of the particle.

2.2 Centrifugal Force :

(a) In a non-inertial frame Centrifugal force exist which is equal and opposite to centripetal force which exist in an inertial frame.

(b) Under centrifugal force body moves only along a sraight line. It appears when centripetal force ceases to exist. Since the body is viewed from a non-inertial frame.



(c) In an inertial frame, the centrifugal force does not act on the object.

(d) In non-inertial frames, centrifugal force arises as pseudo forces and need to be considered.

(e) Under centrifugal forces arises as pseudo force and need to be considered.

3. Maximum Speed of Vehicle on Circular Turning Roads

- **On Unbanked Roads (Friction Only) :**

$$v_{max} = \sqrt{\mu R g}$$

- **On Frictionless Banked Road :**

$$v_{max} = \sqrt{Rg\tan\theta}$$

where $\theta \rightarrow$ Banking angle

- **On Frictional Banked Road :**

$$V_{min} = \sqrt{Rg\left(\frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right)} = \sqrt{Rg\tan(\theta-\phi)}$$

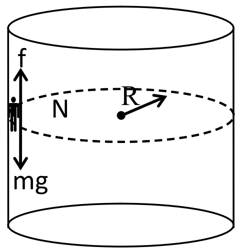
$$v_{max} = \sqrt{Rg\left(\frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}\right)} = \sqrt{Rg\tan(\theta+\phi)}$$

Where, ϕ = angle of friction = $\tan^{-1}(\mu)$
θ = angle of banking

- **Death Well :**

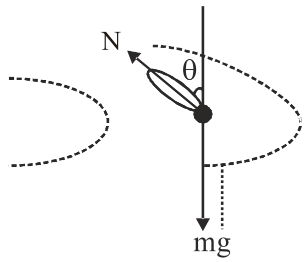
$$f = mg$$

$$N = \frac{mv^2}{R} = mR\omega^2$$



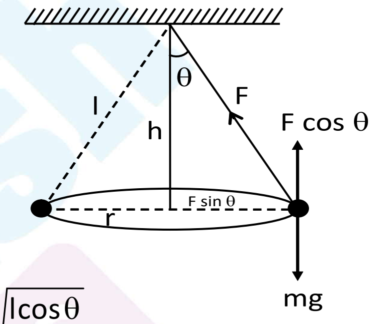
- **Bending of Cyclist** :

$$\tan\theta = \frac{v^2}{rg}$$



4. Conical Pendulum

If the bob of a simple pendulum is pulled to a side and whirled to move along a circle in horizontal plane, the string sweeps a cone and this arrangment is called conical pendulum. If *l* is length of the pendulum, F is tension in the string, r is radius of the horizontal circle and θ is the semivertical angle of the cone, then



$$F\sin\theta = mr\omega^2 \text{ and } F\cos\theta = mg$$

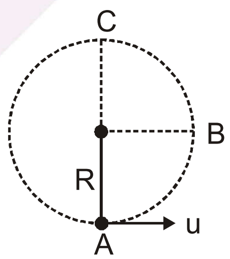
$$\tan\theta = \frac{r\omega^2}{g} \Rightarrow \omega = \sqrt{\frac{g\tan\theta}{r}}$$

$$\text{Time period } T = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

(from $r = l\sin\theta$)

5. Circular Motion in Vertical Plane

A. Condition to Complete Vertical Circle :



$$u \geq \sqrt{5gR}$$

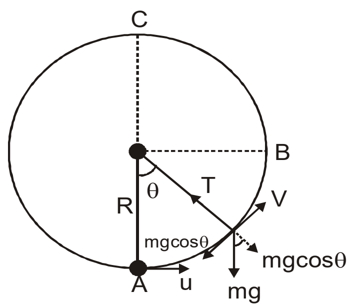
If $u \geq \sqrt{5gR}$ then Tension at C is equal to 0 and tension at A is equal to 6 mg

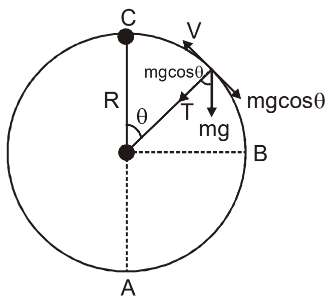
Velocity at B : $v_B = \sqrt{3gR}$

Velocity at C : $v_C = \sqrt{gR}$

From A to B : $T = mg\cos\theta + \dfrac{mv^2}{R}$

From B to C : $T = \dfrac{mv^2}{R} - mg\cos\theta$





B. Condition to Pendulum Motion (Oscillation Condition)

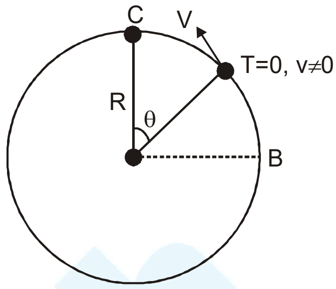
$u \leq \sqrt{2gR}$ (in between A to B)

Velocity can be zero but T never be zero between A & B.

Because T is given by $T = mg\cos\theta + \frac{mv^2}{R}$

C. Condition for Leaving Path :

$$\sqrt{2gR} < u < \sqrt{5gR}$$



Particle crosses the point B but not complete the vertical circle.
Tension will be zero in between B to C & the angle where T = 0

$$\cos\theta = \frac{u^2 - 2gR}{3gR}$$

θ is from vertical line.

Note : After leaving the circle the particle will follow a parabolic path.

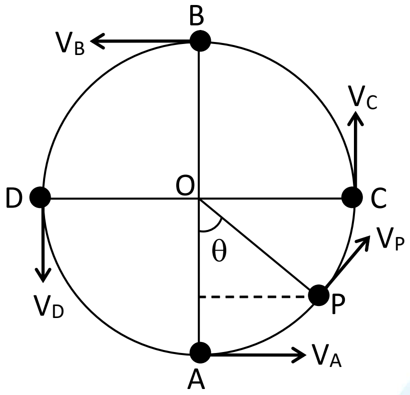
- When a body is to move along a vertical circle it should have certain critical velocities at various points. Tension in string also changes from point to point. Tension will be maximum when the body is at the lower position.
- If the body has velocity less than the critical velocity, body cannot complete vertical circle. In such a case body may execute simple harmonic motion or it may leave the circle.
- Consider a body of m tied to one end of a string be whirled in a vertical circle of radius r in the vertical plane.

(a) For just completing vertical circle,

$$\text{Velocity of body}\begin{Bmatrix} \text{at the lowest point A is } V_A = \sqrt{5gr}\ \left(\text{i.e. } V_A \ge \sqrt{5gr}\right), \\ \text{at highest point B is } V_B = \sqrt{gr}\ \left(\text{i.e. } V_B \ge \sqrt{gr}\right), \\ \text{at horizontal point C or D is } \left(V_C \text{ or } V_D\right) = \sqrt{3gr}\ \left(\text{i.e. } V_C \text{ or } V_D \ge \sqrt{3gr}\right) \end{Bmatrix}$$

(b) For the body to complete the veritcal circle,

$$\text{Tension in the string} \begin{cases} \text{at the lowest point A is } T_A \geq 6 \text{ mg,} \\ \text{at height point B is } T_B \geq 0, \\ \text{at horizontal point C or D is } (T_C \text{ or } T_D) \geq 3\text{mg} \end{cases}$$



(c) If the body at point P has critical velocity for just completing vertical circle

$V_P = \sqrt{gr(3+2\cos\theta)}$ and tension in the string is $T_P = 3$ mg $(1 + \cos\theta)$

$$\Rightarrow \quad T_P = \frac{M\{V_P^2 + g(r-h)\}}{r} \qquad \text{(where h is height of P above A)}$$

Here at A, $\theta = 0°$ and $V_P = V_A = \sqrt{5gr}$

at B, $\theta = 180°$ and $V_P = V_B = \sqrt{gr}$

at C, $\theta = 90°$ and $V_P = V_C = \sqrt{3gr}$

We can obtain tension also.

(d) $\sqrt{gr}$ is the least velocity at the top for the body to describe vertical circle if $V < \sqrt{gr}$ string slackens

(e) If the velocity of the body at the lowest point is less than $\sqrt{5gr}$, it cannot complete vertical cirlce.

(i) If $V_A < \sqrt{2gr}$, velocity of the body becomes zero at a certain point before tension of the string vanishes. So, body will oscillate about A.

(ii) If $V_A = \sqrt{2gr}$ here also body will oscilatles about A but along semcircle DAC.

(iii) If $\sqrt{2gr} < V_A < \sqrt{5gr}$, tension in the string vanishes before velocity of the body becomes zero. Then the body will leave the circle along the tangent at that point (This occurs at a point between C and B).

- When a body moves along a vertical circle by first maintaining its critical velocities at various points, then difference in the tension of the string when it is at the bottom most and top most points is equal to six times the weight of the body. Here tension in the string when the body is at the top most point is zero.
  If the body just complete vertical circle.
  $$T_A - T_B = 6\text{ mg}$$
  $$T_A - T_C = 3\text{ mg}$$

- A body can move along a vertical circle with uniform speed but the tension in the string should be adjusted from point to point.

- When a body moves along a vertical circle with uniform speed, difference in the tension in the string when the body is at the lowest and top most positioin is equal to twice the weight of the body.

- When a body just moves in a vertical circle, its total energy only is constant. Its speed, linear velocity, linear momentum, angular momentum, angular velocity, P.E., K.E., centripetal force, tension in the string all the variable.

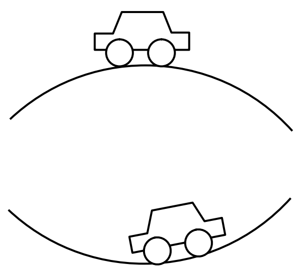
- When a body slides along an inclined plane of height 'h' and describes vertical cirlce of radius 'r' on reaching the bottom, then h = 5r/2.

  Here $mgh = \frac{1}{2}mv^2$ where $V = \sqrt{5gr}$ $\left(\because h = \frac{5r}{2}\right)$

- When a vehicle moving with certain speed is at the top of a convex shaped bridge or speed braker, the normal force on it is less than its weight. If that vehicle is at the lowest portion of a dip or concave shaped bridge, the normal force on it is greater than its weight.

- A car moving with speed V enters on a concave bridge of radius of curvature r. At the top most point of that bridge, normal reaction on the car is $N = mg - \frac{mV^2}{r}$.

  If the car moves on concave bridge and the bottom most point, $N = mg + \frac{mV^2}{r}$

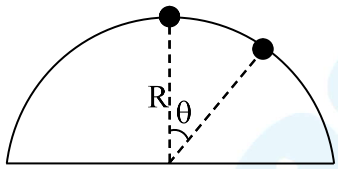


- If a bucket filled with water is whirled in a vertical circle at the end of a rope, water will not fall down when it is at the highest point if its velocity at the point is $V \geq \sqrt{gr}$.

Here its period of revolution is $T \leq 2\pi\sqrt{\frac{r}{g}}$, where r is the lenght of rope.

- A block of mass M hangs at the end of a string of length 'l'. A bullet of mass m flying horizontally hits the block and sticks to it. If the block now completes vertical circle, minimum velocity of the bullet is $V = \frac{M+m}{m}\ \sqrt{5gl}$

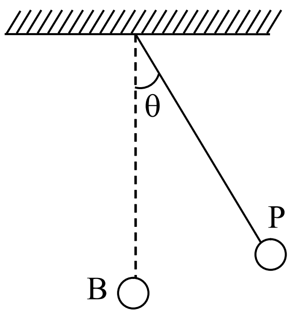
- A particle begins to slide without any friction from the top of a hemisphere of radius R as shown. If leaves the surface of hemisphere at height 'h' above the centre, such that h = 2r/3 and cos θ = 2/3.



If its velocity at the highest point is $\sqrt{gR}$, it leaves the hemisphere along the tangent at that point without sliding down.

- A bob of mass m is suspended from point 'O' using an ideal string of length '*l*'. If the bob is pulled to a position P such that string makes an angle θ to the vertical and released, then velocity of the bob on reaching bottom most point B is $V = \sqrt{2gl(1-\cos\theta)}$ and in this position in the string is

$$T = \frac{mV^2}{l} + mg = mg\,(3-2\cos\theta)$$



- A shell of mass M hangs at the end of a string of length l. It explodes into two pieces, a piece of mass m flies of horizontally and the remaining fragment attached to the string just completes vertical circle.
mu = (M – m) V (numerically)

Where V = $\sqrt{5gl}$

$$\Rightarrow u = \frac{(M-m)\sqrt{5gl}}{m} = \left(\frac{M}{m}-1\right)\sqrt{5gl}$$

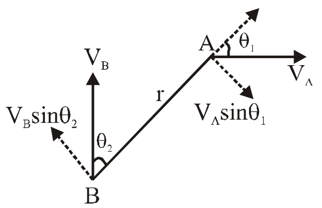
KEY POINTS

- Average angular velocity is a scalar physical quantity whereas instantaneous angular velocity is a vector physical quanity.
- Small Angular displacement $d\vec{\theta}$ is a vector quantity, but large angular displacement is scalar quantity.

  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ But $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$

Relative Angular Velocity

Relative angular velocity of a particle 'A' w.r.t. other moving particle 'B' is the angular velocity of the position vector of 'A' w.r.t. 'B'.



That means it is the rate at which position vector of 'A' w.r.t. 'B' rotates at that instant

$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{\text{Relative velocity of A w.r.t. B perpendicular to line AB}}{\text{seperation between A and B}}$$

here $(v_{AB})_\perp = v_A \sin\theta_1 + v_B \sin\theta_2$

$$\therefore \quad \omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$$

WORK, ENERGY AND POWER

1. Work done :

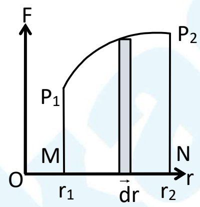
$$W = \int dW = \int \vec{F}.d\vec{r} = \int Fdr\cos\theta$$

For constant force W = $\vec{F}.\vec{d}$ = Fdcosθ

For Unidirectional force

$$W = \int dW = \int Fdx$$ = Area between F – x curve and x-axis.

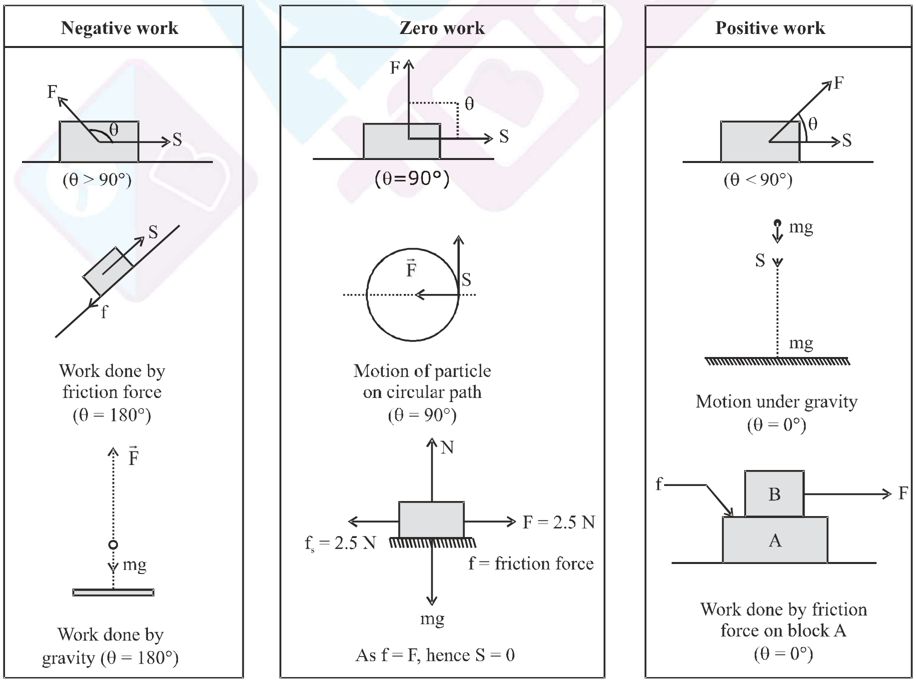
- **Calculation of work done from force-displacement graph :**



total work done, $W = \sum_{r_1}^{r_2} dW = \sum_{r_1}^{r_2} F.dr$ = area of P_1P_2 NM = $\int_{r_1}^{r_2} Fdr$

2. Nature of work done :

Although work done is a scalar quantity, yet its value may be positive,negative or even zero



3. Work done by variable force :

$$\boxed{W_{AB} = \int_A^B \vec{F}.d\vec{s} = \int_A^B F ds \cos\theta}$$

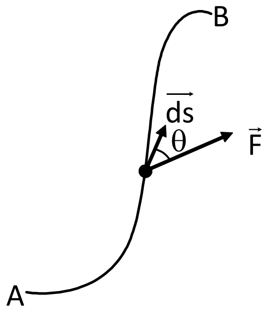
In term of ractangular components

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

then work done is

$$W_{AB} = \int_A^B (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

or $\boxed{W_{AB} = \int_{x_A}^{x_B} F_x d_x + \int_{y_B}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz}$



4. Conservative Forces :

Work done does not depend upon path.

- Work done in a round trip is zero.
- Central forces, spring forces etc. are conservative forces
- When only a conservative force acts within a system, the kinetic energy and potential energy can change into each other. However, their sum, the mechanical energy of the system, doesn't change.
- Work done is completely recoverable.
- If $\vec{F}$ is a conservative force then $\vec{\nabla} \times \vec{F} = 0$ (i.e. curl of $\vec{F}$ is zero)

5. Non-conservative Forces :

- Work done depends upon path.
- Work done in a round trip is not zero.
- Force are velocity- dependent & retarding in nature e.g. friction, viscous force etc.
- Work done against a non-conservative force may be dissipated as heat energy.
- Work done is not recoverable.

6. Kinetic energy :

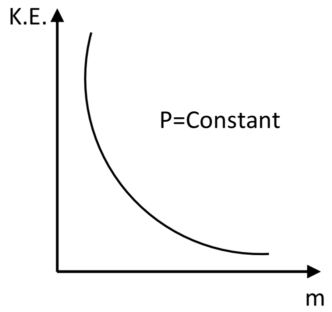
- The energy possessed by a body, by the virtue of its motion is called kinetic energy.

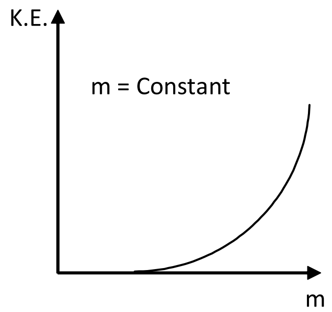
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}.\vec{v})$$

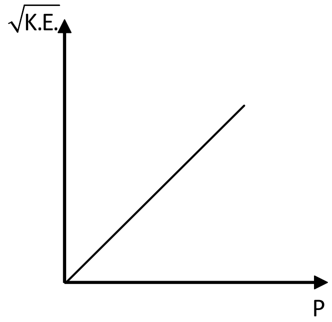
- Kinetic energy can never be negative, it is always positive.
- Relation between kinetic energy (K.E) and linear momentum (P) :

$$\therefore \boxed{K.E. = \frac{P^2}{2m}} \qquad \boxed{P = \sqrt{2m(K.E.)}}$$

Graphs -







7. Work-Energy Theorem :

According to this theorem work done by net force on a body is equal to change in its kinetic energy.

$\boxed{W = \Delta K.E.}$ or $\boxed{W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}$

Note -

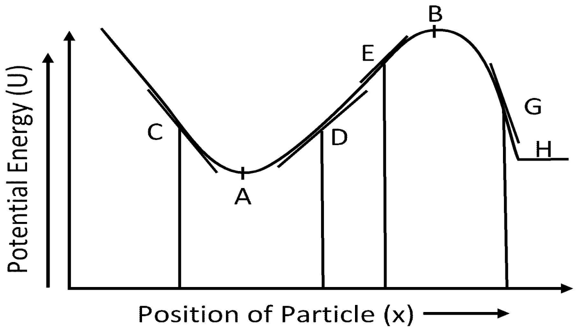
(i) If K.E. of the body decreases then work done is negative i.e. the force opposes the motion.

(ii) If K.E. of the body increases then work done is positive. i.e. the force supports the motion.

8. Potential energy :

- The energy which a body has by virtue of its position or configuration in a conservative force field.
- Potential energy is a relative quantity.
- Potential energy is defined only for conservative force field.
- Relationship between conservative force field and potential energy : $\vec{F} = -\text{grad}(U) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$
- If force varies only with one dimension (along x-axis) then $F = -\frac{dU}{dx} \Rightarrow U = -\int_{x_1}^{x_2} F dx$

9. Potential energy curve and equilibrium :



It is a curve which shows change in potential energy with position of a particle.

- **Nature of Forces :**

  At point **C :** slope $\frac{dU}{dx}$ is negative so F is positive.

  At point **D :** slope $\frac{dU}{dx}$ is positive so F is negative.

  At point **E :** slope $\frac{dU}{dx}$ is positive so F is negative.

  At point **G :** slope $\frac{dU}{dx}$ is negative so F is positive.

- **Stable Equilibrium :**
  When a particle is slightly displaced from equilibrium position and it tends to come back towards equilibrium then it is said to be in stable equilibrium.

  At point **A :** it is the point of stable equilibrium.

  At point **A :** $U = U_{min}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = positive

- **Unstable equilibrium :**
  When a particle is slightly displaced from equilibrium and it tends to move away from equilibrium position then it is said to be in unstable equilibrium.

  At point **B :** it is the point of unstable equilibrium.

  At point **B :** $U = U_{max}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = negative

- **Neutral equilibrium :**
  When a particle is slightly displaced from equilibrium position and no force acts on it then equilibrium is said to be neutral equilibrium. Point H is at neutral equilibrium $\Rightarrow$

  U = constant ; $\frac{dU}{dx} = 0, \frac{d^2U}{dx^2} = 0$

10. Law of conservation of Mechanical energy :

Total mechanical (kinetic + potential) energy of a system remains constant if only conservative forces are acting on the system of particles or the work done by all other forces is zero. From work energy theorem W = ΔKE

Proof :

For internal conservative forces $W_{int} = -\Delta U$

So $W = W_{ext} + W_{int} = 0 + W_{int} = -\Delta U$

$\Rightarrow -\Delta U = \Delta KE \Rightarrow (KE + U) = 0 \Rightarrow KE + U = (\text{constant})$

- Spring force F = –kx, Elastic potential energy stored in spring $U(x) = \frac{1}{2}kx^2$

- Mass and energy are equivalent and are related by $E = mc^2$

11. Power

- Power is a scalar quantity with dimension $M^1L^2T^{-3}$
- SI unit of power is J/s or watt
- 1 horsepower = 746 watt = 550 ft-lb/sec.

  Average power P_{av} = W/t

- **Instantaneous power** $P = \frac{dW}{dt} = \vec{F}.\left(\frac{d\vec{r}}{dt}\right) = \vec{F}.\vec{v}$

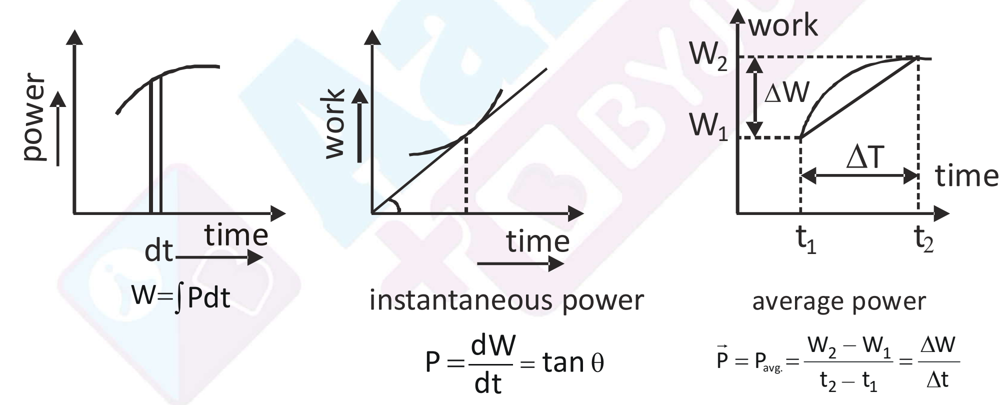


fig. (a) fig. (b) fig. (c)

For a system of varying mass $\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$

If v = constant then $\vec{F} = \vec{v}\frac{dm}{dt}$ then $P = \vec{F}.\vec{v} = v^2\frac{dm}{dt}$

In rotatory motion : $P = \tau.\frac{d\theta}{dt} = \tau\omega$

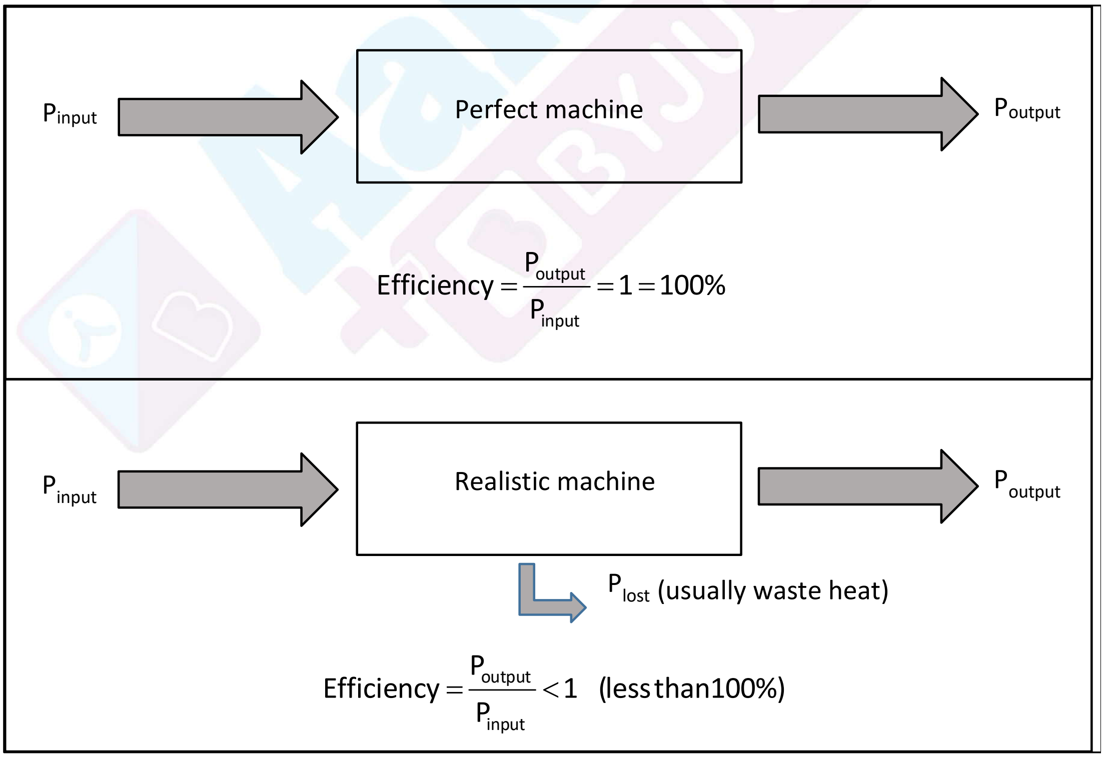
12. Key Points :

- A body may gain kinetic energy and potential energy simultaneously because principle of conservation of mechanical energy may not be valid every time.
- Comets move around the sun in elliptical orbits. The gravitational force on the comet due to sun is not normal to the comet's velocity but the work done by the gravitational force is zero in complete round trip because gravitational force is a conservative force.
- Work done by static friction may be positive because static friction may acts along the direction of motion of an object.

13. Efficiency :

Efficiency of a machine in % $\eta = \dfrac{\text{Work done}}{\text{energy input}} \times 100\%$

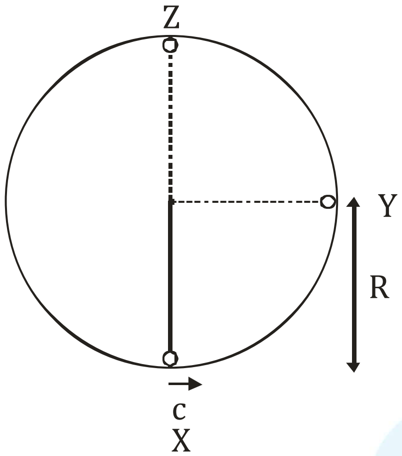
or Efficiency = $\dfrac{P_{output}}{P_{input}} \times 100\%$



14. Circular Motion in a Vertical Plane:

- **Vertical circular motion using a string:**

Suppose a body is tied to a string and rotated in a vertical circle as shown



Between X and Y, tension will balance out weight and hence the string will always be taut. So the velocity required to reach Y can be found out by conserving mechanical energy.

E_x(Energy at X) = $\frac{1}{2}mu^2$

Since the particle just reaches point Y hence Velocity at Y is zero.

E_y=mgR

Equating both we get, u = $\sqrt{2gR}$

Now if want to find the minimum velocity to reach point Z, can I assume velocity to be zero at Z? The answer is no because if the velocity is zero at Z then weight will not be balanced and the string will become slack So at Z, velocity should be such that the weight is equal to the centripetal force making tension just to be zero.

$$\frac{mv^2}{R} = mg \qquad ...(1)$$

$$E_z = mg(2R) + \frac{mv^2}{2}$$

Substituting the value of v we get

E_z= 2.5mgR

Equating E_x and E_z we get,

u = $\sqrt{5gR}$

So now we have our critical values we can frame our cases, Case I : $u < \sqrt{2gR}$;

The ball will oscillate and never reach point Y.

Case II : $\sqrt{2gR} < u < \sqrt{5gR}$

The ball will lose contact somewhere between Y and Z and start projectile motion.

Case III: $u > \sqrt{5gR}$

The string will never become slack and complete the circle.

- **Vertical circular motion using a rod:**
  The situation is similar when the ball is tied to a rod and moved in vertical circle. The only difference is now the velocity at the top can be zero. As now the normal reaction of the rod can balance the weight at that point. Solving similarly as above we get the following cases for a rod:

  Case I : $u < \sqrt{2gR}$

  The body will oscillate and not reach point Y.

  Case II : $\sqrt{2gR} < u < \sqrt{4gR}$

  The ball will oscillate and cross point Y but not reach point Z.

  Case III : $u > \sqrt{4gR}$

  The body will complete the circle.