Full Syllabus Test 1

Subject: Mathematics

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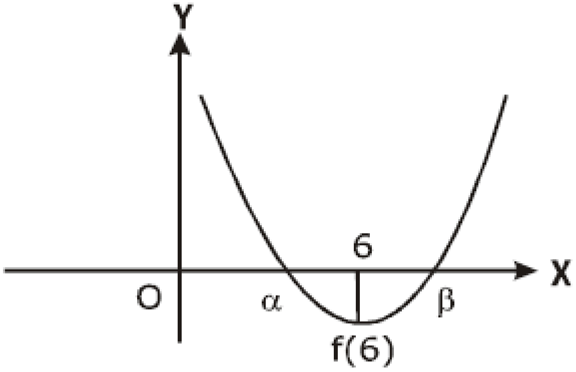
1. The values of a for which the number 6 lies in between the roots of the equation $x^2 + 2(a-3)x + 9 = 0$, belong to

✗ **A.** $\left(\frac{3}{4}, \infty\right)$

✓ **B.** $\left(-\infty, -\frac{3}{4}\right)$

✗ **C.** $(-\infty, 0) \cup (6, \infty)$

✗ **D.** $(-\infty, 0) \cup (3, \infty)$



$f(x) = x^2 + 2(a-3)x + 9$ is a parabola facing upwards as shown in the figure.
Let the roots be α and β with $\alpha < \beta$
If 6 lies between α and β,
$f(6) < 0$
$\Rightarrow (6)^2 + 2(a-3)(6) + 9 < 0$
$\Rightarrow 12a + 9 < 0$
$\Rightarrow a < -\frac{3}{4}$

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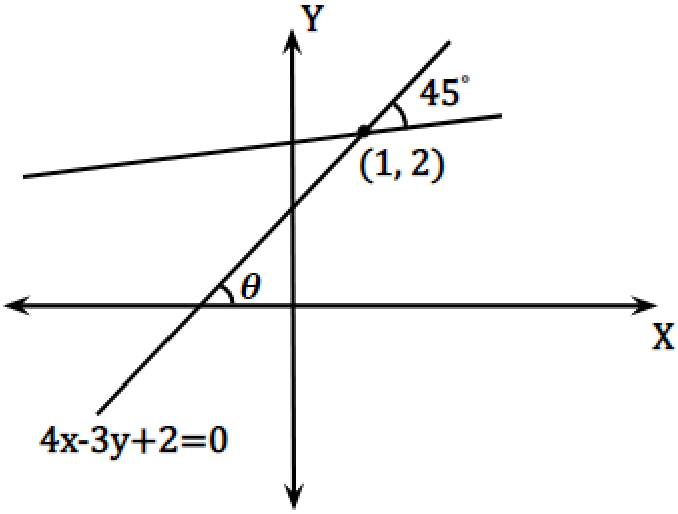
2. The line $4x-3y+2=0$ is rotated through an angle of $\frac{\pi}{4}$ in clockwise direction about the point $(1,2)$. The equation of the line in its new position is

✓ **A.** $x-7y+13=0$

✗ **B.** $y-7x+5=0$

✗ **C.** $x+7y-15=0$

✗ **D.** $y+7x-15=0$



$(1,2)$ lies on the line $4x-3y+2=0$

The slope of the given line is $\tan\theta = \dfrac{4}{3}$

Let the slope of the new line be m.
Angle between these two lines is 45°

$$\Rightarrow \tan 45^\circ = \left|\frac{\frac{4}{3}-m}{1+\frac{4}{3}m}\right|$$

$$\Rightarrow \frac{4-3m}{3+4m} = \pm 1$$

$$\Rightarrow 4-3m = \pm(3+4m)$$

$$\Rightarrow m = \frac{1}{7} \quad (m=-7 \text{ rejected})$$

Required equation of line is

$$y-2=\frac{1}{7}(x-1)$$

$$\Rightarrow x-7y+13=0$$

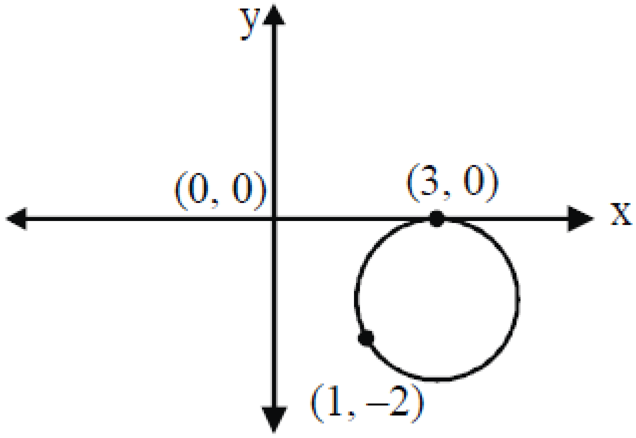
3. The circle passing through $(1,-2)$ and touching the x-axis at $(3,0)$ also passes through the point

✗ **A.** $(-2,-2)$

✗ **B.** $(2,-5)$

✓ **C.** $(5,-2)$

✗ **D.** $(-2,5)$



As the circle is touching the $x-$axis at $(3,0)$,
let the centre of the circle be $C(3,k)$
Radius, $r=|k|$
Equation of the circle is $(x-3)^2+(y-k)^2=k^2$
Above circle passes through $(1,-2)$.
Then, $(1-3)^2+(-2-k)^2=k^2$
$\Rightarrow k=-2$
Hence, equation of the circle is $(x-3)^2+(y+2)^2=4$
Clearly, $(5,-2)$ satisfies it.

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4. Let the coefficients of powers of x in the second, third and fourth terms in the binomial expansion of $(1+x)^n$, where n is a positive integer, be in arithmetic progression. The sum of the coefficients of odd powers of x in the expansion is

☒ **A.** 32

☑ **B.** 64

☒ **C.** 128

☒ **D.** 256

Given that ${}^nC_1,\ {}^nC_2,\ {}^nC_3$ are in A.P.
$\Rightarrow 2\ {}^nC_2 = {}^nC_1 + {}^nC_3$
$\Rightarrow \dfrac{2n(n-1)}{2} = n + \dfrac{n(n-1)(n-2)}{6}$
$\Rightarrow n^2 - 9n + 14 = 0$
$\Rightarrow n = 7 \quad [n = 2 \text{ rejected}]$
Sum of the coefficients of odd powers of x in the expansion is
$2^{n-1} = 2^6 = 64$

5. If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then the locus of z is

☒ **A.** a circle

☑ **B.** a straight line

☒ **C.** an ellipse

☒ **D.** a parabola

Let $z = x + iy$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$$

$$= \frac{(2x+1)+2iy}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)+2iy}{(1-y)^2+x^2} \times \{(1-y)-ix\}$$

$\therefore$ Imaginary part

$$= \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2} = -2$$

$\Rightarrow 2y - 2y^2 - 2x^2 - x = -2x^2 - 2 + 4y - 2y^2$

$\Rightarrow x + 2y - 2 = 0$ which is a straight line.

6. A geometric progression with common ratio r, consists of an even number of terms. If the sum of all terms is 5 times the sum of the terms occupying the odd places, then $\sum_{i=1}^{4}(ir)^2$ is

✗ **A.** 1456

✗ **B.** 120

✗ **C.** 1172

✓ **D.** 480

Let the G.P. be $a, ar, ar^2, \dots, ar^{2n-1}$.
Then, $S_{2n} = 5(a + ar^2 + \cdots + n \text{ terms})$
$$\Rightarrow \frac{a}{1-r}(1-r^{2n}) = 5 \cdot \frac{a}{1-r^2}(1-(r^2)^n)$$
$$\Rightarrow r + 1 = 5$$
$$\Rightarrow r = 4$$

Now, $\sum_{i=1}^{4}(ir)^2$
$$= 16\sum_{i=1}^{4} i^2$$
$$= 16 \times \frac{4 \times 5 \times 9}{6}$$
$$= 480$$

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7. If a, b, c, x are positive integers, then $\begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by

☑ **A.** x^2

☒ **B.** x^3

☒ **C.** x^4

☒ **D.** $a^2+b^2+c^2$

$$\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^3+ax & a^2b & a^2c \\ ab^2 & b^3+bx & b^2c \\ ac^2 & bc^2 & c^3+cx \end{vmatrix}$$

$$= \begin{vmatrix} a^2+x & a^2 & a^2 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$,

$$\Delta = \begin{vmatrix} a^2+b^2+c^2+x & a^2+b^2+c^2+x & a^2+b^2+c^2+x \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

$$= (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & x & 0 \\ c^2 & 0 & x \end{vmatrix}$$

$$= (a^2+b^2+c^2+x)x^2$$

$\therefore$ determinant is divisible by x^2

8. An open cylindrical can has to be made with 100 square units of tin. If its volume is maximum, then the ratio of its base radius and the height is

✗ **A.** $2:1$

✓ **B.** $1:1$

✗ **C.** $1:2$

✗ **D.** $\sqrt{2}:1$

Let r be the base radius and h be the height of the cylinder.
Then, $2\pi rh + \pi r^2 = 100$

$$\Rightarrow h = \frac{50}{\pi r} - \frac{r}{2}$$

Volume of cylinder, $V = \pi r^2 h = \pi r^2 \left(\frac{50}{\pi r} - \frac{r}{2}\right) = 50r - \frac{\pi r^3}{2}$

$$\frac{dV}{dr} = 50 - \frac{3\pi r^2}{2}$$

$$\frac{dV}{dr} = 0$$

$$\Rightarrow r = \frac{10}{\sqrt{3\pi}}$$

$\frac{d^2V}{dr^2} = -3\pi r < 0$ at $r = \frac{10}{\sqrt{3\pi}}$

Hence, V is maximum when $r = \frac{10}{\sqrt{3\pi}}$

$$\therefore h = \frac{50}{\pi \cdot \frac{10}{\sqrt{3\pi}}} - \frac{10}{2\sqrt{3\pi}} = \frac{10}{\sqrt{3\pi}}$$

So, when V is maximum, $r : h = 1 : 1$

9. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then the sum of the solutions in x is

☒ **A.** 1

☑ **B.** -1

☒ **C.** 0

☒ **D.** not finite

$$(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$$

Since $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$,

So, $(\tan^{-1} x)^2 + \left(\frac{\pi}{2} - \tan^{-1} x\right)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow (\tan^{-1} x)^2 + \frac{\pi^2}{4} - \pi \tan^{-1} x + (\tan^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1} x)^2 - \pi \tan^{-1} x + \frac{\pi^2}{4} - \frac{5\pi^2}{8} = 0$$

$$\Rightarrow 16(\tan^{-1} x)^2 - 8\pi \tan^{-1} x - 3\pi^2 = 0$$

Above equation is quadratic in $\tan^{-1} x$

$$\tan^{-1} x = \frac{8\pi \pm \sqrt{64\pi^2 + 64 \cdot 3\pi^2}}{2 \cdot 16}$$

$$\Rightarrow \tan^{-1} x = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow \tan^{-1} x = \frac{3\pi}{4}, \frac{-\pi}{4}$$

$$\Rightarrow \tan^{-1} x = \frac{-\pi}{4}$$

$$\Rightarrow x = -1$$

10. If in a $\triangle ABC$, $\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$, then $\triangle ABC$ is

- ✗ **A.** isosceles
- ✗ **B.** equilateral
- ✓ **C.** right-angled isosceles
- ✗ **D.** right-angled but not isosceles

$$\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$$
$$\Rightarrow \sin C + \sin(2B + C) + \cos C - \cos(2B + C) = 2\sqrt{2}$$
$$\Rightarrow 2\sin(B + C)\cos B + 2\sin B \sin(B + C) = 2\sqrt{2}$$
$$\Rightarrow 2\sin(\pi - A)[\cos B + \sin B] = 2\sqrt{2} \quad [\because A + B + C = \pi]$$

$$\Rightarrow \sin A\left[\sqrt{2}\left(\sin B \cdot \frac{1}{\sqrt{2}} + \cos B \cdot \frac{1}{\sqrt{2}}\right)\right] = \sqrt{2}$$

$$\Rightarrow \sin A \cdot \sin\left(B + \frac{\pi}{4}\right) = 1$$

It is possible only if $\sin A = 1$ and $\sin\left(B + \frac{\pi}{4}\right) = 1$

So, $A = \frac{\pi}{2}$ and $B + \frac{\pi}{4} = \frac{\pi}{2}$
$$\Rightarrow A = \frac{\pi}{2}, \; B = C = \frac{\pi}{4}$$

11. If the chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$, then the locus of the midpoints of the chords is the curve

⊗ **A.** $y^2(x+a) = x^3$

✓ **B.** $y^2(x-a) = x^3$

⊗ **C.** $y^2(x+2a) = 3x^3$

⊗ **D.** $y^2(x-2a) = 2x^3$

If (x_1, y_1) is the midpoint of the chord to the hyperbola $x^2 - y^2 = a^2$,
its equation is $T = S_1$
i.e., $xx_1 - yy_1 - a^2 = x_1^2 - y_1^2 - a^2$
$\Rightarrow xx_1 - yy_1 = x_1^2 - y_1^2$
$$\Rightarrow y = \frac{x_1}{y_1}x + \frac{y_1^2 - x_1^2}{y_1}$$

If this is a tangent to $y^2 = 4ax$,
then $c = \dfrac{a}{m}$
$$\Rightarrow \frac{y_1^2 - x_1^2}{y_1} = \frac{ay_1}{x_1}$$
$\Rightarrow x_1^3 = y_1^2(x_1 - a)$
$\therefore$ Locus of (x_1, y_1) is $x^3 = y^2(x-a)$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^3 + x^2 + 3x + \sin x$. Then f is

☑ **A.** injective and surjective

☒ **B.** injective but not surjective

☒ **C.** surjective but not injective

☒ **D.** neither injective nor surjective

$f(x) = x^3 + x^2 + 3x + \sin x,\ x \in \mathbb{R}$
$f'(x) = 3x^2 + 2x + 3 + \cos x$
$f'(x) = g(x) + \cos x$
$g(x) > 0$ as $D = 4 - 36 = -32 < 0$
Range of g is $\left[\frac{-D}{4a}, \infty\right)$
i.e., range of g is $\left[\frac{+32}{12}, \infty\right) = \left[\frac{8}{3}, \infty\right)$

Also, $-1 \le \cos x \le 1$
$\therefore f'(x) > 0$
Hence, function f is strictly increasing.
$\Rightarrow f$ is injective.

$\lim\limits_{x \to \infty} f(x) = \infty$
and $\lim\limits_{x \to -\infty} f(x) = -\infty$
Also, f is continuous over $\mathbb{R}$.
$\Rightarrow$ Range of f is $\mathbb{R}$
$\therefore f$ is surjective.

13. If a variable tangent of the circle $x^2+y^2=1$ intersects the ellipse $x^2+2y^2=4$ at points P and Q, then the locus of the point of intersection of tangent at P and Q is

☒ **A.** a circle of radius 2 units

☒ **B.** a parabola with focus at $(2,3)$

☑ **C.** an ellipse with latus rectum 2 units

☒ **D.** a hyperbola with eccentricity $\frac{3}{2}$

Let the intersection of the tangent at P and Q to the ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$ be (x_1,y_1)

Then the equation of PQ is $T=0$

$$\frac{xx_1}{4}+\frac{yy_1}{2}=1$$

i.e., $y=-\frac{xx_1}{2y_1}+\frac{2}{y_1}$

This is a tangent to the circle $x^2+y^2=1$

So, $c^2=a^2(1+m^2)$

$$\Rightarrow \frac{4}{y_1^2}=1\left(1+\frac{x_1^2}{4y_1^2}\right)$$

$$\Rightarrow 16=4y_1^2+x_1^2$$

$$\Rightarrow \frac{x_1^2}{16}+\frac{y_1^2}{4}=1$$

which is the equation of an ellipse.

Length of latus rectum $=\frac{2b^2}{a}=\frac{8}{4}=2$

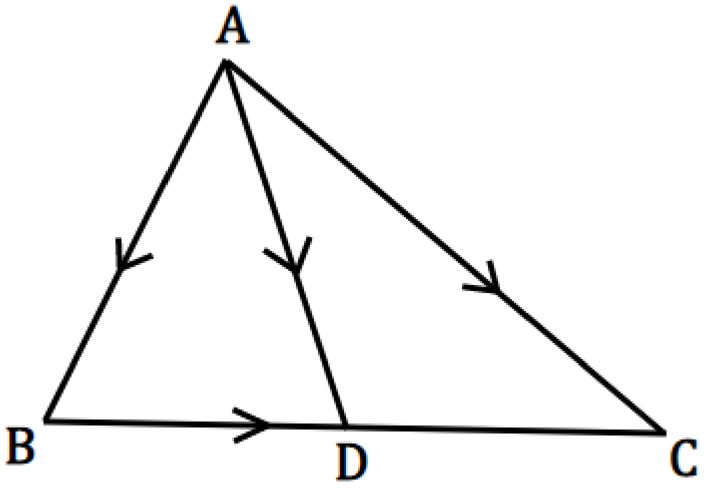
14. The two vectors $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}+5\hat{k}$ represent the two sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of a triangle ABC. The length of the median through A is

☒ **A.** 7

☒ **B.** 14

☑ **C.** $\sqrt{14}$

☒ **D.** $\frac{\sqrt{14}}{2}$



$\overrightarrow{AB}=\hat{i}+\hat{j}+\hat{k}$

$\overrightarrow{AC}=\hat{i}+3\hat{j}+5\hat{k}$

$\Rightarrow \overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}=2\hat{j}+4\hat{k}$

Since D is the mid-point of $\overrightarrow{BC}$,

$\overrightarrow{BD}=\frac{\overrightarrow{BC}}{2}=\hat{j}+2\hat{k}$

Now, in $\triangle ABD$,

$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}=\hat{i}+2\hat{j}+3\hat{k}$

$\therefore |\overrightarrow{AD}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

15. The logical statement $(p \to q) \to ((\sim p \to q) \to q)$ is

- ☑ **A.** a tautology
- ☒ **B.** equivalent to $\sim p \to q$
- ☒ **C.** equivalent to $p \to \sim q$
- ☒ **D.** a fallacy

The truth table of given expression is given below

p	q	$x \equiv p \to q$	$\sim p$	$\sim p \to q$	$y \equiv (\sim p \to q) \to q$	$x \to y$
T	T	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

For all possible truth values of p and q, the statement $(p \to q) \to ((\sim p \to q) \to q)$ is true.
Hence, the given statement is a tautology.

16. $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$ is equal to

✓ **A.** $e^{1/3}$

✗ **B.** 1

✗ **C.** e

✗ **D.** 0

$$L=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}\qquad [1^{\infty}\text{ form}]$$

$$=\lim_{x\to 0}\left(1+\frac{\tan x}{x}-1\right)^{1/x^2}$$

$$=\exp\left(\lim_{x\to 0}\left(\frac{\tan x}{x}-1\right)\frac{1}{x^2}\right)$$

$$=\exp\left(\lim_{x\to 0}\frac{\tan x-x}{x^3}\right)$$

$$=\exp\left(\lim_{x\to 0}\frac{x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots-x}{x^3}\right)$$

$$=\exp\left(\lim_{x\to 0}\frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{x^3}\right)$$

$$=e^{1/3}$$

17. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=\max\left\{x,\ x^3\right\}$. Then the set of all points where f is not differentiable, is

✗ **A.** $\{-1,0\}$

✓ **B.** $\{-1,0,1\}$

✗ **C.** $\{0,1\}$

✗ **D.** $\{-1,1\}$

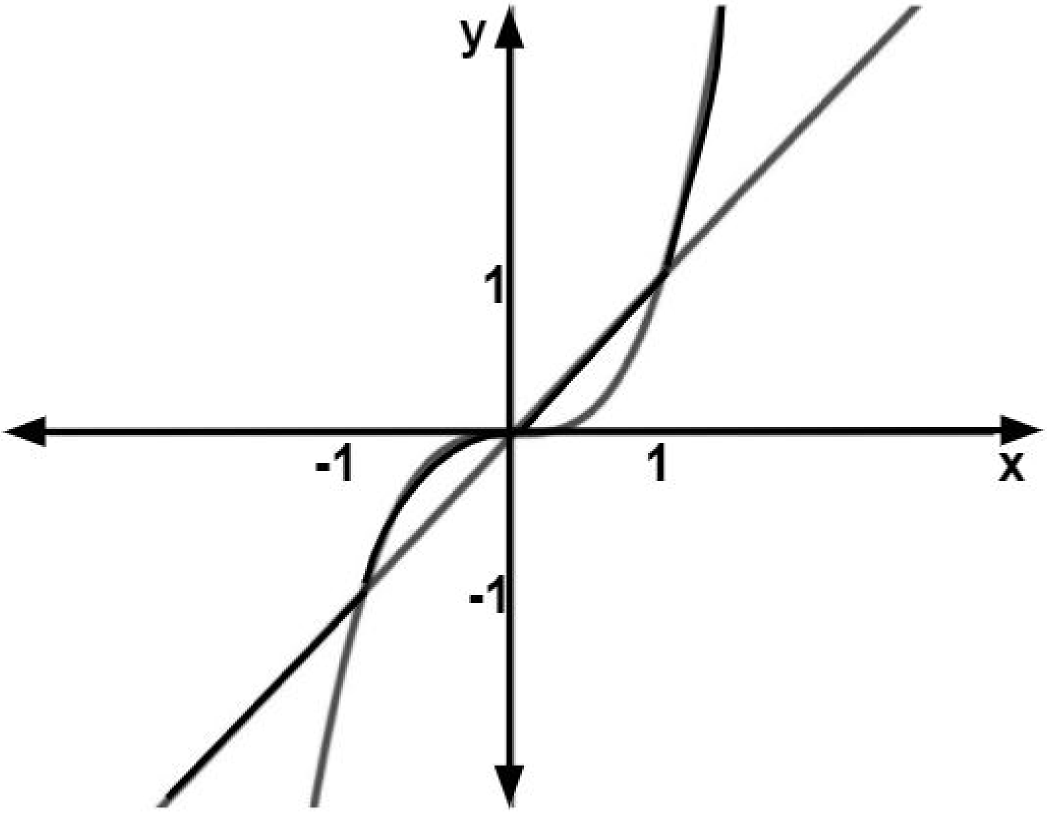
$$f(x)=\begin{cases} x, & -\infty<x\le -1 \text{ and } 0\le x\le 1 \\ x^3, & -1<x<0 \text{ and } x>1 \end{cases}$$

At $x=-1$,
$LHD=1,\ RHD=3x^2=3$

At $x=0$,
$LHD=3x^2=0,\ RHD=1$

At $x=1$,
$LHD=1,\ RHD=3x^2=3$
Hence f is not differentiable at $-1,0,1$

Alternative solution :



Clearly, we can observe from the graph that the function f is not differentiable at $x=-1,0,1$

18. Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then

✗ **A.** f' does not exist at $x = 0$

✗ **B.** f' exists and is continuous at $x = 0$

✓ **C.** f' exists but not continuous at $x = 0$

✗ **D.** f' does not exist at any point

Clearly, $f'(x)$ exists for all $x \neq 0$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

$\therefore f'(0)$ exists and equals 0

When $x \neq 0$,

$$f'(x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$$

$\lim\limits_{h \to 0} f'(0^-)$ and $\lim\limits_{h \to 0} f'(0^+)$ do not exist.

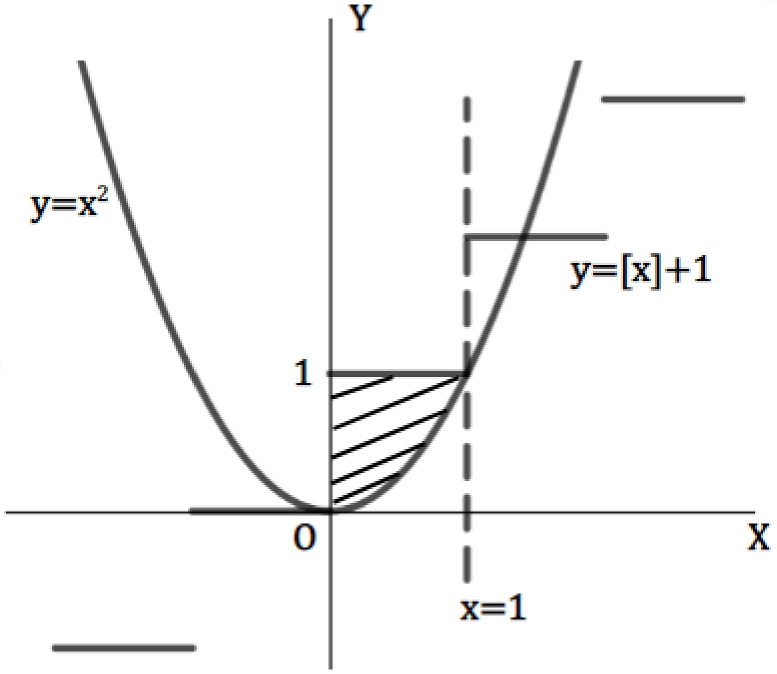
Hence, f' is not continuous at $x = 0$

19. The area bounded by $y = x^2$, $y = [x + 1]$, $x \leq 1$ and the y-axis, where $[.]$ represents the greatest integer function, is

- ✓ **A.** $\frac{2}{3}$
- ✗ **B.** $\frac{1}{3}$
- ✗ **C.** $\frac{7}{3}$
- ✗ **D.** 1

$y = x^2, \ y = [x + 1] = [x] + 1$

For $0 \leq x < 1, \ y = 0 + 1 = 1$



Hence, the required area $=$ the shaded area

$$= \int_0^1 x dy = \int_0^1 \sqrt{y}\, dy$$

$$= \frac{2}{3}$$

20. The solution of the differential equation $ydx - xdy = y^2 \tan\left(\frac{x}{y}\right)dx$ is (C is constant of integration)

☒ **A.** $\frac{x}{y} = Ce^x$

☑ **B.** $\sin\left(\frac{x}{y}\right) = Ce^x$

☒ **C.** $\cos\left(\frac{x}{y}\right) = Ce^x$

☒ **D.** $x = Cy$

$$ydx - xdy = y^2 \tan\left(\frac{x}{y}\right)dx$$

$$\Rightarrow \cot\frac{x}{y}\left(\frac{ydx - xdy}{y^2}\right) = dx$$

$$\Rightarrow \cot\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) = dx$$

Integrating both sides, we get

$$\log\left(\sin\frac{x}{y}\right) = x + \log C$$

$$\Rightarrow \sin\frac{x}{y} = e^{x+\log C} = Ce^x$$

21. The total number of numbers greater than $4,00,000$ that can be formed by using the digits $0, 2, 2, 4, 4, 5$ is

Accepted Answers

90 90.0 90.00

Solution:

Suppose, the six-digit number is denoted by $a_1a_2a_3a_4a_5a_6$
For the number to be greater than $4,00,000$,
a_1 should be either 4 or 5.

If a_1 is 4, then $a_2a_3a_4a_5a_6$ can be arranged in $\frac{5!}{2!} = 60$ ways

If a_1 is 5, then $a_2a_3a_4a_5a_6$ can be arranged in $\frac{5!}{2!\,2!} = 30$ ways

$\therefore$ Total number of required numbers $= 90$

22. Three persons A, B, C are to speak at a function along with 5 other persons. If the persons speak in random order, the probability that A speaks before B and B speaks before C is $\frac{p}{q}$, where p, q are co-prime. Then $p+q$ is

Accepted Answers

7 7.0 7.00

Solution:

Total number of ways in which 8 persons can speak is $8!$

Number of ways in which A, B and C can be arranged in the specified speak order is ${}^8C_3 \times 1$ as the order of A, B, C is already fixed.
Remaining 5 persons can speak in $5!$ ways.
So the favourable number of ways $= {}^8C_3 \times 5!$
Hence, required probability $= \frac{{}^8C_3 \times 5!}{8!} = \frac{1}{6}$

23. If $x = \log_{24} 12$, $y = \log_{36} 24$ and $z = \log_{48} 36$, then $(1 + xyz)$ equals k times yz. The value of k is

Accepted Answers

2 2.0 2.00

Solution:

$$1 + xyz = 1 + \frac{\log 12}{\log 24} \cdot \frac{\log 24}{\log 36} \cdot \frac{\log 36}{\log 48}$$
$$= 1 + \frac{\log 12}{\log 48}$$
$$= \frac{\log(48 \times 12)}{\log 48} = \frac{\log(24)^2}{\log 48} = \frac{2 \log 24}{\log 48}$$
$$= 2 \cdot \frac{\log 24}{\log 36} \cdot \frac{\log 36}{\log 48}$$
$$= 2yz = kyz$$
Hence, $k = 2$

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24. The plane $P_1 : 4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $P_2 : 5x + 3y + 10z = 25$. If the plane in its new position be denoted by P and the distance of plane P from the origin is d units, then the value of $[d/2]$, where $[.]$ represents the greatest integer function, is

Accepted Answers

7 7.0 7.00

Solution:

$P_1 : 4x + 7y + 4z + 81 = 0$
$P_2 : 5x + 3y + 10z = 25$

Equation of plane passing through line of intersection of P_1 and P_2 is
$P : (4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$
$\Rightarrow (4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0$
This plane is perpendicular to P_1.
So $4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$
$\Rightarrow \lambda = -1$
Hence, equation of the plane P is $-x + 4y - 6z + 106 = 0$

Distance of plane P from $(0, 0, 0)$ is
$$d = \frac{106}{\sqrt{1 + 16 + 36}} = \frac{106}{\sqrt{53}}$$
Thus, $[d/2] = 7$

25. If $\int_{\ln 2}^{k} \frac{1}{\sqrt{e^x - 1}} dx = \frac{\pi}{6}$, then $k = \ln p$. The value of $p + 1$ is

Accepted Answers

5 5.0 5.00

Solution:

$$\int_{\ln 2}^{k} \frac{1}{\sqrt{e^x - 1}} dx$$

Let $e^x - 1 = t^2$

$\Rightarrow e^x dx = 2t dt$

$\Rightarrow dx = \frac{2t}{t^2 + 1} dt$

$$\therefore \int \frac{1}{\sqrt{e^x - 1}} dx$$

$$= \int \frac{2}{t^2 + 1} dt = 2 \tan^{-1} t$$

Now, $\int_{\ln 2}^{k} \frac{1}{\sqrt{e^x - 1}} dx = \frac{\pi}{6}$

$\Rightarrow 2 \tan^{-1} \sqrt{e^k - 1} - 2 \tan^{-1} \sqrt{e^{\ln 2} - 1} = \frac{\pi}{6}$

$\Rightarrow 2 \tan^{-1} \sqrt{e^k - 1} - 2 \tan^{-1} 1 = \frac{\pi}{6}$

$\Rightarrow 2 \tan^{-1} \sqrt{e^k - 1} - \frac{\pi}{2} = \frac{\pi}{6}$

$\Rightarrow \tan^{-1} \sqrt{e^k - 1} = \frac{\pi}{3}$

$\Rightarrow \sqrt{e^k - 1} = \sqrt{3}$

$\Rightarrow e^k = 4$

$\Rightarrow k = \ln 4$

Thus $p = 4$

$\Rightarrow p + 1 = 5$

26. If the mean and variance of eight numbers $3, 7, 9, 12, 13, 20, x$ and y be 10 and 25 respectively, then xy is equal to

Accepted Answers

54 54.0 54.00

Solution:

Mean $= \dfrac{64 + x + y}{8} = 10$
$\Rightarrow x + y = 16$

Variance $= \dfrac{\sum x_i^2}{n} - (\bar{x})^2$
$\Rightarrow 25 = \dfrac{3^2 + 7^2 + 9^2 + 12^2 + 13^2 + 20^2 + x^2 + y^2}{8} - 100$
$\Rightarrow 1000 = 852 + x^2 + y^2$
$\Rightarrow x^2 + y^2 = 148$
$\Rightarrow (x + y)^2 - 2xy = 148$
$\Rightarrow 256 - 2xy = 148$
$\therefore xy = 54$

27. The distance from the origin to the normal of the curve $x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$ is

Accepted Answers

2 2.0 2.00

Solution:

$x = 2\cos t + 2t\sin t$

$\frac{dx}{dt} = 2t\cos t$

$y = 2\sin t - 2t\cos t$

$\frac{dy}{dt} = 2t\sin t$

$\Rightarrow \frac{dy}{dx} = \tan t$

So, slope of normal at $t = \frac{\pi}{4}$ is $-\frac{1}{\tan t} = -1$

At $t = \frac{\pi}{4}$, $x = \sqrt{2}\left(1 + \frac{\pi}{4}\right)$ and $y = \sqrt{2}\left(1 - \frac{\pi}{4}\right)$

Hence, equation of normal at $t = \frac{\pi}{4}$ is $x + y - 2\sqrt{2} = 0$

Distance of normal from the origin $= \left|\frac{-2\sqrt{2}}{\sqrt{1+1}}\right| = 2$ units

28. If $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+\dfrac{\sin x}{1+\dfrac{\cos x}{1+\cdots\infty}}}}$, then $\dfrac{dx}{dy}$ at $x=\dfrac{\pi}{2}$ is

Accepted Answers

2 2.0 2.00

Solution:

$$y=\frac{\sin x}{1+\dfrac{\cos x}{1+y}}$$
$$\Rightarrow y=\frac{(1+y)\sin x}{1+y+\cos x}$$
$$\Rightarrow y+y^2+y\cos x=\sin x+y\sin x$$
$$\Rightarrow \frac{dy}{dx}+2y\frac{dy}{dx}+y(-\sin x)+\cos x\frac{dy}{dx}=\cos x+y\cos x+\sin x\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx}(1+2y+\cos x-\sin x)=y\sin x+(1+y)\cos x$$
$$\Rightarrow \frac{dy}{dx}=\frac{y\sin x+(1+y)\cos x}{1+2y+\cos x-\sin x}$$

At $x=\dfrac{\pi}{2}$,
$$\frac{dy}{dx}=\frac{y}{2y}=\frac{1}{2}$$
$$\Rightarrow \frac{dx}{dy}=2$$

29. If $f(x)=\int \frac{2x^5+5x^4}{\left(4+2x+3x^5\right)^2}dx, (x\geq 0)$ and $f(0)=0$, then the value of $72\cdot f(1)$ is

Accepted Answers

2 2.0 2.00

Solution:

Given : $f(x)=\int \frac{2x^5+5x^4}{\left(4+2x+3x^5\right)^2}dx$

Taking x^{10} common from numerator and denominator, we get

$f(x)=\int \frac{2x^{-5}+5x^{-6}}{\left(4x^{-5}+2x^{-4}+3\right)^2}dx$

Taking $4x^{-5}+2x^{-4}+3=z$

$\Rightarrow \left(-20x^{-6}-8x^{-5}\right)dx=dz$

$\Rightarrow f(z)=-\frac{1}{4}\int \frac{dz}{z^2}=\frac{1}{4z}+C$

$\Rightarrow f(x)=\frac{x^5}{4\left(3x^5+2x+4\right)}+C$

As $f(0)=0 \Rightarrow C=0$

So, $72\cdot f(1)=72\times \frac{1}{36}=2$

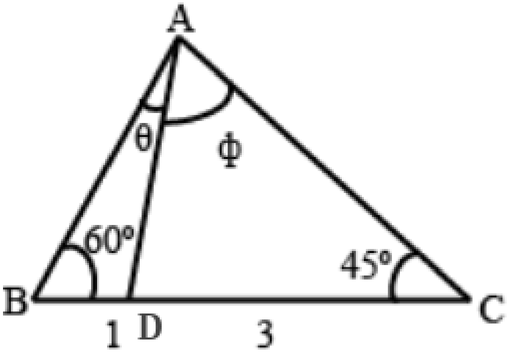
Full Syllabus Test 1

30. In triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio $1:3$ and the angles $\angle BAD = \theta$, $\angle CAD = \phi$. The value of $\frac{\sin^2\phi}{\sin^2\theta} =$

Accepted Answers

6 6.0 6.00 06

Solution:



Using sine rule in ΔABD,

$$\frac{\sin\theta}{BD} = \frac{\sin 60^\circ}{AD}$$

$$\Rightarrow \frac{AD}{BD} = \frac{\sin 60^\circ}{\sin\theta} \quad \cdots (i)$$

Using sine rule in ΔACD,

$$\frac{\sin\phi}{DC} = \frac{\sin 45^\circ}{AD}$$

$$\Rightarrow \frac{DC}{AD} = \frac{\sin\phi}{\sin 45^\circ} \quad \cdots (ii)$$

Multiplying equation (i), (ii), we get

$$\frac{DC}{BD} = \frac{\sin 60^\circ \sin\phi}{\sin 45^\circ \sin\theta}$$

Hence, $\frac{\sin^2\phi}{\sin^2\theta} = 6$