CONIC SECTIONS

PARABOLA

1. Conic sections:

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point and perpendicular distance from a fixed straight line is a constant ratio all in same plane.

(a) The fixed point is called the **focus**.

(b) The fixed straight line is called the **DIRECTRIX**.

(c) The constant ratio is called the **ECCENTRICITY** denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the **AXIS.**

(e) A point of intersection of a conic with its axis is called a **VERTEX**.

2. General equation of a conic : Directrix property:

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2$$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. Distinguishing between the conic:

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$ $h^2 = ab$ the lines will coincident.

$e < 1$ $h^2 < ab$ the lines will be imaginary.

Case (ii) When the focus does not lie on the directrix :

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; D \neq 0$ $h^2 = ab$	$0 < e < 1; D \neq 0$ $h^2 < ab$	$D \neq 0; e > 1$ $h^2 > ab$	$e > 1; D \neq 0$ $h^2 > ab$; $a+b=0$

4. Parabola:

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$.

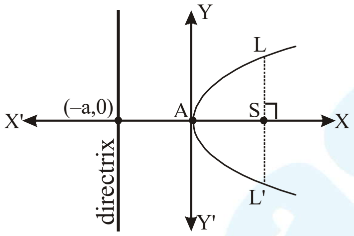
For this parabola :

(i) Vertex (A) is (0, 0)

(ii) Focus (S) is (a, 0)

(iii) Axis is **y = 0**

(iv) Directrix is $x + a = 0$



(a) Focal distance :

The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE With respect to axis.

(d) Latus rectum :

A focal chord perpendicular to the axis of parabola is called the **LATUS RECTUM**.

For $y^2 = 4ax$.

(i) Length of the latus rectum = 4a. (2 × perpendicular distance of focus from the directrix)

(ii) Length of the semi latus rectum = 2a.

(iii) Ends of the latus rectum are L(a,2a) & L'(a, – 2a)

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

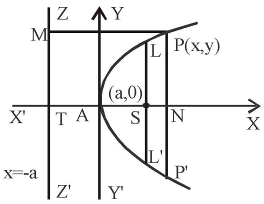
(iii) Two parabolas are said to be equal if they have the same latus rectum of same length.

5. Parametric representation:

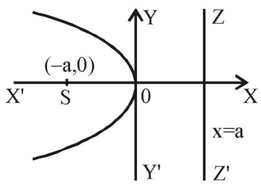
The simplest & the best form of representing the co-ordinates of a point on the parabola is $y^2 = 4ax$ is $(at^2, 2at)$.The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. Type of parabola:

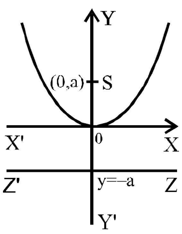
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$;
$x^2 = 4ay$; $x^2 = -4ay$



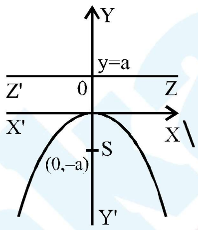
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2, 2at)$	x + a
$y^2 = -4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2, 2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at, at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a– h =0	4a	(h+a, k±2a)	$(h+at^2, k+2at$	x–h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at, q+at^2)$	y–q+b

7. Positon of a point relative to a parabola:

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

8. Chord joining two points:

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

9. Line & Parabola:

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm$ [

$\Rightarrow$ condition of tangency is, $\mathbf{c = \frac{a}{m}}$.

Note : Line $y = mx + c$ will be tangent to parabola

$x^2 = 4ay$ if $\mathbf{c = -am^2}$.

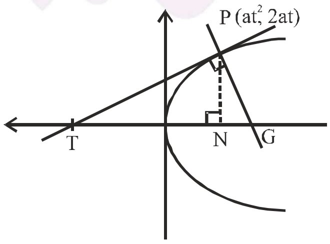
(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line

$y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : Length of the focal chord making an angle α with the x - axis is $4a\,\text{cosec}^2\alpha$.

10. Length of subtangent & subnormal:

PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then



TN = length of subtangent = twice the abscissa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. Tangent to the parabola $y^2 = 4ax$:

(a) Point form :

Equation of the tangent to the given parabola at the point (x_1, y_1) is $yy_1 = 2a(x + x_1)$ or $T = 0$

(b) Slope form :

The equation of tangent of slope m to the given parabola is

$y = mx + \frac{a}{m}, (m \neq 0)$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form : Equation of the tangent to the given parabola at the point P(t) is $ty = x + at^2$,

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1t_2, a(t_1+t_2)]$ i.e. G.M. and A.M. of abscissa and ordinates of the points)

12. Normal to the parabola $y^2 = 4ax$:

(a) Point form :

Equation of the normal to the given parabola at its point (x_1, y_1)

is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

(b) Slope form :

Equation of normal of slope m to the given parabola is $y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$

(c) Parametric form :

Equation of the normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$.

Note :

(i) Point of intersecton of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

(iii) If the normals to the parabola $y^2 = 4ax$ at the ponits t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a,0)$

13. Pair of Tangents :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 \equiv y_1{}^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

14. Chord of contact:

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$ i.e. $T = 0$

Remember that the area of the triangle formed by the tangents from the point (x_1,y_1) & the chord of contact is $\frac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$ Also note that the chord of contact exists only if the point P is not inside.

15. Chord with a given middle ponit :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

i.e. $T = S_1$,

where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1{}^2 - 4ax_1$.

16. Diameter :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$, where m = slope of parallel chords.

17. Conormal points :

Foot of the normals of three concurrent normals are called conormal points.

(i) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero

(ii) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

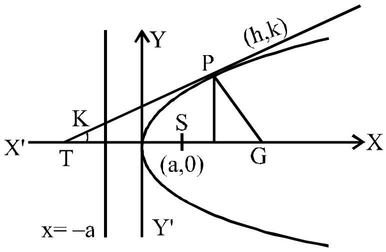
(iii) Centroid of the trangle formed by three co-normal points lies on the axis of parabola.

(iv) If $h > 2a$ & $27ak^2 < 4(h - 2a)^3$ is satisfied then three real and distinct normals are drawn from point (h,k,) on parabola $y^2 = 4ax$.

(v) If three normals are drawn from point (h,0) on parabola $y^2 = 4ax$, then $h > 2a$ and one of the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

18. Important highlights:

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix.



From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

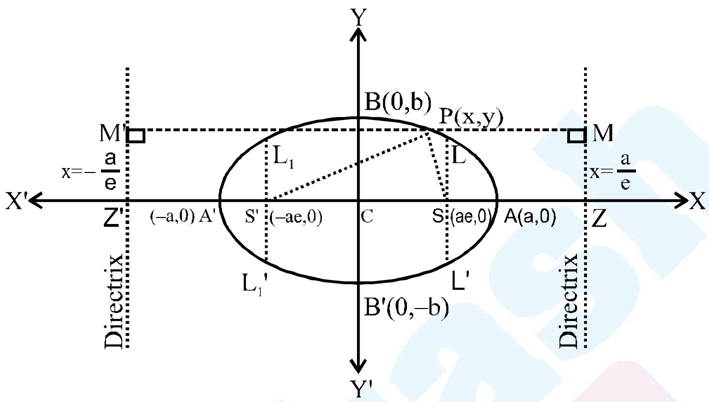
(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord. If PSQ is a focal chord of a parabola, then semi-latus rectum = $\frac{2 \times SP.SQ}{SP + SQ}$.

(f) Image of the focus lies on directrix with respect to any tangent of parabola $y^2 = 4ax$.

ELLIPSE

1. Standard equation & definition:

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where **a > b & b^2 = a^2 (1 – e^2)** $\Rightarrow$ $a^2 - b^2 = a^2 e^2$.
where e = eccentricity **(0 < e < 1)**.



FOCI : S $\equiv$ (ae, 0) & S' $\equiv$ (– ae, 0).

(a) Equation of directrices :

x = $\frac{a}{e}$ **&** $x=-\frac{a}{e}$.

(b) Vertices :

A' $\equiv$ (–a, 0) & A $\equiv$ (a, 0).

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z)** $\left(\pm\frac{a}{e},0\right)$.

(d)Minor Axis : The y-axis intersects the ellipse in the points B\` $\equiv$ (0,- b) & B $\equiv$ (0, b). The line segment B\`B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axes : The major & minor axis together are called **Principal Axes** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn

through it is called the **centre** of the conic. C $\equiv$ (0,0) the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate** with respect to major axis as diameter.

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are

$$L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right) \text{ and } L_1'\left(-ae, -\frac{b^2}{a}\right).$$

(k) Focal radii : $SP = a - ex$ & $S'P = a + ex$

$\Rightarrow$ $SP + S'P = 2a$ = Major axis.

(l) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

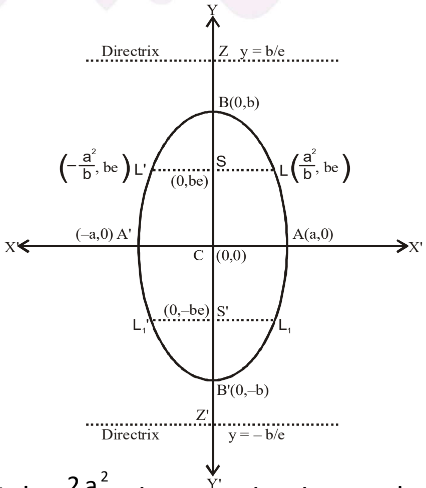
2. Ellipse of the form

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **, a < b:**

(a) AA' = Minor axis = 2a

(b) BB' = Major axis = 2b

(c) $a^2 = b^2(1 - e^2)$



(d) Latus rectum $LL' = L_1L_1' = \frac{2a^2}{b}$, its equation is $y = \pm be$

(e) Ends of the latus rectum are :

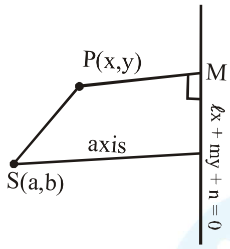
$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) Equation of directrix y = ± b/e

(g) Eccentricity : $e = \sqrt{1 - \frac{a^2}{b^2}}$

3. General Equation of an Ellipse:

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix. Let P(x, y) be any point on the ellipse. Then by definition.



$\Rightarrow$ SP = e PM (e is the eccentricity)

$\Rightarrow SP^2 = e^2 \times PM^2$

$\Rightarrow (x - a)^2 + (y - b)^2 = e^2 \frac{(lx + my + n)^2}{(l^2 + m^2)}$

$\Rightarrow (l^2 + m^2)\{(x - a)^2 + (y - b)^2\} = e^2\{lx + my + n\}^2$

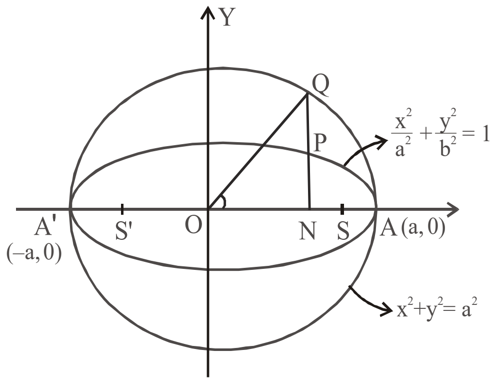
4. Positon a point W.R.R. an ellpse:

The point **P(x_1, y_1)** lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ >< or = 0. i.e. S_1 > 0, < 0 or = 0 respectively.

5. Auxilliary circe/ecdcentric angle:

A circle described on major axis as diameter is called the **auxiliary circle.** Let P be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PN perpendicular from P on the major axis of the ellipse and produce PN to meet auxiliary circle in Q. Join OQ, the angle $\angle NOQ = \theta$ is called the eccentric angle of the point P on the ellipse.

Note: $\angle PON$ is NOT eccentric angle.



Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle"

6. Paramaetric representation:

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

where θ is a parameter (eccentric angle).

Note that if P(θ) $\equiv$ (a cos θ, b sin θ) is on the ellipse then ;

Q(θ**)** $\equiv$ (a cosθ, a sinθ) is on the auxiliary circle.

7. Line and an Ellipse

The line y = mx + c meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two real points ,

coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$ respectively.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

8. Tangent to the ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Point form :

Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ i.e. T = 0

(b) Slope form :

Equation of tangent to the given ellipse whose slope is 'm', is $y = mx \pm \sqrt{a^2m^2+b^2}$

Points of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(c) Parametric form : Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

9. Normal to the ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Point form :
Equation of the normal to the given ellipse at

(x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

(b) Slope form : Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

(c) Parametric form : Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\,\theta = (a^2 - b^2)$.

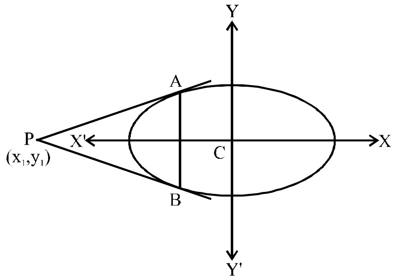
10. Chord of Contact :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ **or** T = 0 (at x_1,y_1).

11. Pair of Tangents :

If $P(x_1,y_1)$ be any point which lies outside the ellipse , $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

i.e. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$

12. Director circle :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

13. Equation of Chord with Mid point (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

whose mid-point be (x_1,y_1) is $T = S_1$

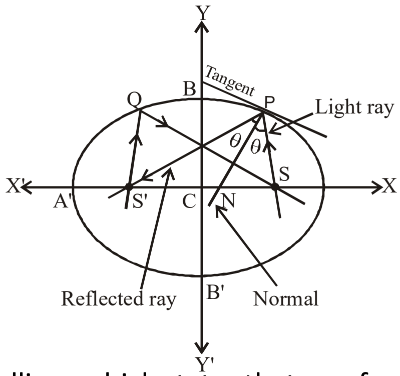
where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$, $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$,

i.e. $\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$

14. Important points for

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(I) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known



reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice versa .

(II) Point of intersection of the tangents at the point α & β is $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(III) If $A(\alpha), B(\beta), C(\gamma)$ & $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$.

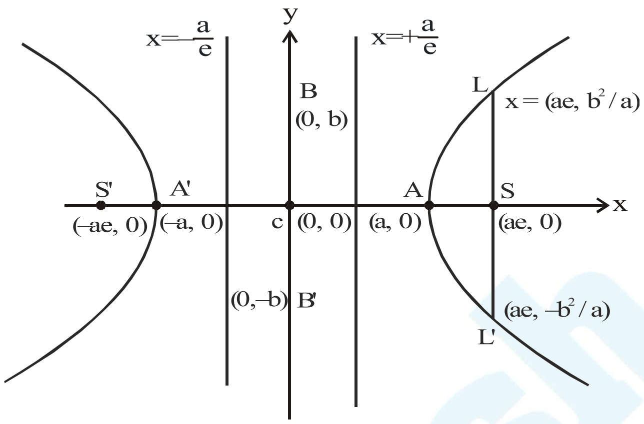
(IV) If $A(\alpha), B(\beta), C(\gamma)$ & $D(\delta)$ are four concyclic points then sum of their eccentric angles is even multiple of π. i.e. $\alpha + \beta + \gamma + \delta = 2n\pi$.

(V) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity (e > 1).

1. Standard Equation & Definition(s) :



Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$ or $a^2e^2 = a^2 + b^2$

i.e. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ (distance from focus to directrix)

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right)$; $\left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A'A of length 2a in which the foci S' & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :
The line segment B′B between the two points B' ≡ (0, – b) & B ≡ (0, b) is called as the Conjugate Axis of the Hyperbola.
The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axes of the hyperbola.

(f) Focal Property :
The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left|\,|PS| - |PS'|\,\right| = 2a$.
The distance SS' = focal length.

(g) Focal distance :
Distance of any point P(x, y) on Hyperbola from foci PS = ex – a &
PS' = ex + a.

2. Conjugate Hyperbola :

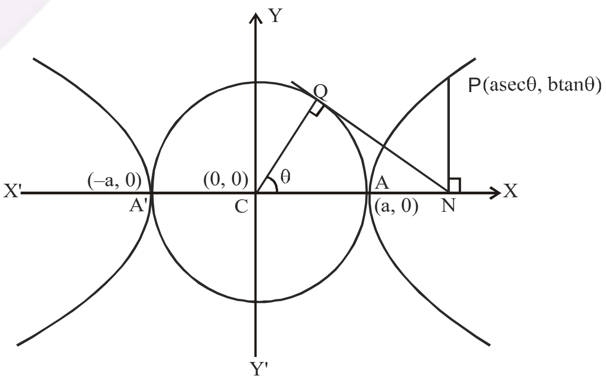
Two hyperbolas such that transverse & conjugate axis of one hyperbola are respectively the conjugate & the transverse axis of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are conjugate hyperbolas of each other .

Note that :
(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.
(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

3. Rectangular Or Equilateral Hyperbola :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of it's latus rectum is equal to it's transverse or conjugate axis.

4. Auxiliary Circle :



A circle drawn with centre C & transverse axis as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Let P(x,y) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular from P on CX, and then a tangent NQ from N to the circle described on AA\` as diameter. Join CQ, then $\theta = \angle QCN$ is called eccentric angle of the point P, $0 \le \theta < 2\pi$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

5. Position Of A Point 'P' w.r.t. A Hyperbola :

The quantity $\frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2} = 1$ is positive , zero or negative according as the point (x_1, y_1) lies within, upon or outside the curve respectively.

6. Line And A Hyperbola :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$ respectively.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

7. Tangent to the hyperbola :

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. i.e. T = 0.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^{\,2} - a^2)m^2 - 2x_1y_1m + y_1^{\,2} + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1 y_1)$ to the hyperbola.

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = mx \pm \sqrt{a^2m^2 - b^2}$. Points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

(c) Parametric form : Equation of the tangent to the given hyperbola at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is

$$x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}, \quad y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$$

8. Normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : The equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

(b) Slope form : The equation of normal of slope m to the given hyperbola is $y = mx\ m\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ feet of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, m\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

(c) Parametric form : The equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

9. Director Circle :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, this circle is real ; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is **no** such circle & so no tangents at right angle can be drawn to the curve.

10. Chord of contact:

If PA and PB be the tangents from point $P(x_1,y_1)$ to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then the equation of the chord of contact AB is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$ at $(x_1 y_1)$

11. Pair or tangents:

If $P(x_1,y_1)$ be any point lies outside the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and pair of tangents PA, PB can be drawn to it from P, Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$.

where $S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T = \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$

12. Equation of chord with mid ponit (x_1,y_1):

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose mid-point be (x_1,y_1) is $T = S_1$.

where $T = \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$, $S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$

13. Asymptotes :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

Note: An asymptote to a curve is a line, at a finite distance from the origin to which the tangent to a curve tends as the point of contact goes to infinity. i.e. Asymptote to a curve touches the curve at infinity.

Combined equation of asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

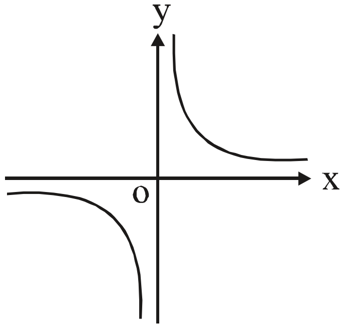
14. rectangular hyperbola :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $\mathbf{xy = c^2}$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1+t_2)$ with slope, $\mathbf{m = \frac{-1}{t_1t_2}}$

8. CONIC SECTIONS



(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

15. Important highlights:

(i) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii,

(ii) Reflection property of the hyperbola : An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.