

## Subject: Mathematics

---

1. For which of the following curves, the line  $x + \sqrt{3}y = 2\sqrt{3}$  is the tangent at the point  $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$  ?

- ☒ **A.**  $x^2 + 9y^2 = 9$
- ☐ **B.**  $2x^2 - 18y^2 = 9$
- ☐ **C.**  $y^2 = \frac{1}{6\sqrt{3}}x$
- ☐ **D.**  $x^2 + y^2 = 7$

Tangent to  $x^2 + 9y^2 = 9$  at point  $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$  is  $T = 0$

i.e.,  $x \left(\frac{3\sqrt{3}}{2}\right) + 9y \left(\frac{1}{2}\right) - 9 = 0$

$$\Rightarrow 3\sqrt{3}x + 9y = 18$$

$$\Rightarrow x + \sqrt{3}y = 2\sqrt{3}$$

2. Let  $L$  be a tangent line to the parabola  $y^2 = 4x - 20$  at  $(6, 2)$ . If  $L$  is also a tangent to the ellipse  $\frac{x^2}{2} + \frac{y^2}{b} = 1$  then the value of  $b$  is equal to :

- ☐ A. 20
- ☒ B. 14
- ☐ C. 16
- ☐ D. 11

Parabola Equation is  $y^2 = 4x - 20$

Tangent at  $P(6, 2)$  will be

$$2y = 4 \left( \frac{x+6}{2} \right) - 20$$

$$\Rightarrow 2y = 2x + 12 - 20$$

$$\Rightarrow x - y - 4 = 0 \quad \dots (1)$$

This is also tangent to ellipse  $\frac{x^2}{2} + \frac{y^2}{b} = 1$

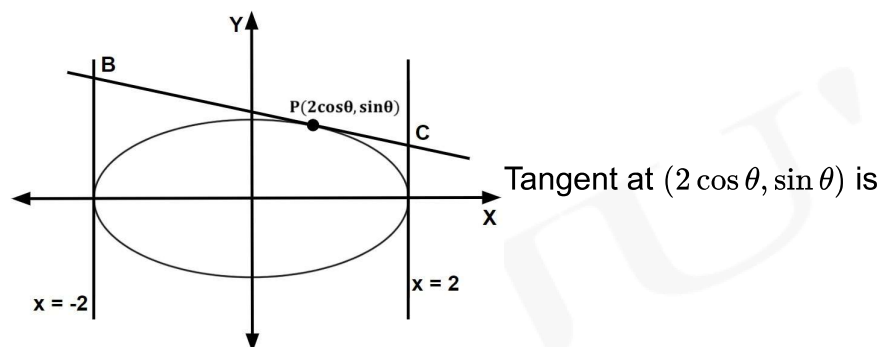
Applying  $c^2 = a^2m^2 + b^2$

$$(-4)^2 = (2)(1) + b$$

$$\Rightarrow b = 14$$

3. If a tangent to the ellipse  $x^2 + 4y^2 = 4$  meets the tangents at the extremities of its major axis at  $B$  and  $C$ , then the circle with  $BC$  as diameter passes through the point

- ☐ A.  $(-1, 1)$   
☒ B.  $(\sqrt{3}, 0)$   
☐ C.  $(1, 1)$   
☐ D.  $(\sqrt{2}, 0)$



$$\frac{x(\cos \theta)}{2} + \frac{y(\sin \theta)}{1} = 1$$

Coordinates of  $B \left( -2, \cot \frac{\theta}{2} \right)$  and  $C \left( 2, \tan \frac{\theta}{2} \right)$

Now, the equation of circle with  $BC$  as diameter is

$$(x - 2)(x + 2) + \left( y - \cot \frac{\theta}{2} \right) \left( y - \tan \frac{\theta}{2} \right)$$

$$\Rightarrow x^2 + y^2 - y \left( \tan \frac{\theta}{2} + \cot \frac{\theta}{2} \right) - 3 = 0$$

At  $y = 0, x = \pm\sqrt{3}$

Hence, circle passes through the point  $(\sqrt{3}, 0)$ .

4. A tangent is drawn to the parabola  $y^2 = 6x$  which is perpendicular to the line  $2x + y = 1$ . Which of the following points does **NOT** lie on it ?

☒ A.  $(0, 3)$

☒ B.  $(-6, 0)$

☒ C.  $(4, 5)$

☒ D.  $(5, 4)$

For  $y^2 = 6x$ ,  $a = \frac{3}{2}$

equation of tangent :  $y = mx + c$

$$\Rightarrow y = mx + \frac{3}{2m}$$

$$\therefore c = \frac{a}{m}$$

and

$$m = \frac{1}{2} \quad (\because \text{perpendicular to line } 2x + y = 1)$$

$$\therefore \text{tangent is : } y = \frac{x}{2} + \frac{3}{1} \Rightarrow x - 2y + 6 = 0$$

5. Let  $C$  be the locus of the mirror image of a point on the parabola  $y^2 = 4x$  with respect to the line  $y = x$ . Then the equation of tangent to  $C$  at  $P(2, 1)$  is :

- ☐ A.  $2x + y = 5$
- ☐ B.  $x + 2y = 4$
- ☐ C.  $x + 3y = 5$
- ☒ D.  $x - y = 1$

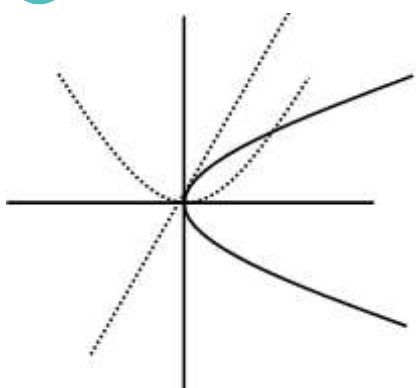


Image of  $y^2 = 4x$  w.r.t.  $y = x$  is  $x^2 = 4y$

Tangent from  $(2, 1)$  is  $T = 0$

$$xx_1 = 2(y + y_1)$$

$$\Rightarrow 2x = 2(y + 1)$$

$$\Rightarrow x = y + 1$$

6. If the common tangent to the parabolas,  $y^2 = 4x$  and  $x^2 = 4y$  also touches the circle,  $x^2 + y^2 = c^2$ , then  $c$  is equal to:

- ☒ A.  $\frac{1}{2}$   
☒ B.  $\frac{1}{4}$   
☒ C.  $\frac{1}{\sqrt{2}}$   
☒ D.  $\frac{1}{2\sqrt{2}}$

$$\begin{aligned}
 y &= mx + \frac{1}{m} \\
 \Rightarrow x^2 &= 4 \left( mx + \frac{1}{m} \right) \\
 \Rightarrow x^2 - 4mx - \frac{4}{m} &= 0 \\
 D &= 0 \\
 \Rightarrow 16m^2 + \frac{16}{m} &= 0 \\
 \Rightarrow 16 \left( \frac{m^3 + 1}{m} \right) &= 0 \\
 \Rightarrow m &= -1 \\
 \Rightarrow y &= -x - 1 \\
 \Rightarrow x + y + 1 &= 0
 \end{aligned}$$

Perpendicular distance of the tangent from the centre of circle is equal to radius.

$$\therefore p = c$$

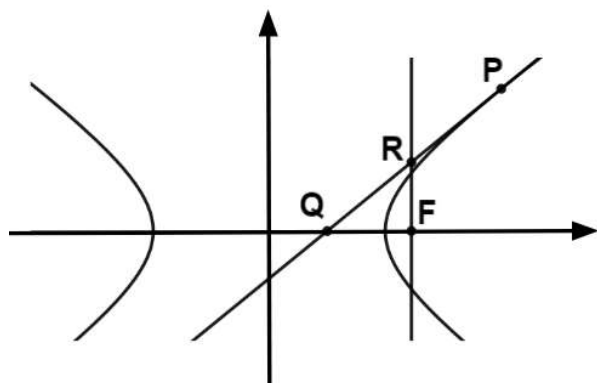
$$\begin{aligned}
 \text{Now, } \left| \frac{0 + 0 - 1}{\sqrt{2}} \right| &= |c| \\
 \Rightarrow c &= \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$

7. Consider a hyperbola  $H : x^2 - 2y^2 = 4$ . Let the tangent at a point  $P(4, \sqrt{6})$  meet the  $x$ -axis at  $Q$  and latus rectum at  $R(x_1, y_1)$ ,  $x_1 > 0$ . If  $F$  is a focus of  $H$  which is nearer to the point  $P$ , then the area of  $\Delta QFR$  is equal to:

- ☒ A.  $\sqrt{6} - 1$
- ☒ B.  $4\sqrt{6} - 1$
- ☒ C.  $4\sqrt{6}$
- ☒ D.  $\frac{7}{\sqrt{6}} - 2$

Given :  $H : x^2 - 2y^2 = 4$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{2} = 1$$



Tangent at  $P(4, \sqrt{6})$

$$T = 0$$

$$\Rightarrow \frac{4x}{4} - \frac{\sqrt{6}y}{2} = 1$$

$$\Rightarrow 2x - \sqrt{6}(y) = 2 \quad \dots (1)$$

Putting  $y = 0$ , we get

$$Q = (1, 0)$$

Eccentricity

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$F = (\sqrt{6}, 0)$$

Equation of latus rectum:

$$x = ae = 2\sqrt{\frac{3}{2}} = \sqrt{6} \quad \dots (2)$$

Solving equations (1) and (2), we get

$$R = \left( \sqrt{6}, \frac{2\sqrt{6} - 2}{\sqrt{6}} \right)$$

Therefore, area of  $\triangle QFR$

$$\Delta = \frac{1}{2} \times QF \times FR$$

$$\Rightarrow \Delta = \frac{1}{2} \times (\sqrt{6} - 1) \times \left( \frac{2\sqrt{6} - 2}{\sqrt{6}} \right)$$

$$\Rightarrow \Delta = \frac{7}{\sqrt{6}} - 2$$



8. Let a line  $L : 2x + y = k, k > 0$  be a tangent to the hyperbola  $x^2 - y^2 = 3$ . If  $L$  is also a tangent to the parabola  $y^2 = \alpha x$ , then  $\alpha$  is equal to:

☒ A. -24

☐ B. 24

☐ C. 12

☐ D. -12

Given line is  $L: 2x + y = k, k > 0$

$\Rightarrow L : y = -2x + k$  is tangent to  $x^2 - y^2 = 3$

$\therefore x^2 - (k - 2x)^2 = 3 \Rightarrow 3x^2 - 4kx + k^2 + 3 = 0$

Here  $\Delta = b^2 - 4ac = 0$

$\therefore k^2 = 9$

$\therefore L = 0$  is also tangent to  $y^2 = \alpha x$

$\therefore k = \frac{\alpha/4}{-2}$

$\therefore \alpha = -8k \Rightarrow \alpha = -24$

9. A line parallel to the straight line  $2x - y = 0$  is tangent to the hyperbola  $\frac{x^2}{4} - \frac{y^2}{2} = 1$  at the point  $(x_1, y_1)$ . Then  $x_1^2 + 5y_1^2$  is equal to :

☒ A. 6

☐ B. 10

☐ C. 8

☐ D. 5

$$T : \frac{xx_1}{4} - \frac{yy_1}{2} = 1 \quad \dots (1)$$

$t : 2x - y = 0$  is parallel to  $T$

$$\Rightarrow T : 2x - y = \lambda \quad \dots (2)$$

Now comparing the slopes

$$x_1 = 4y_1$$

$(x_1, y_1)$  lies on hyperbola

$$\Rightarrow 4y_1^2 - \frac{y_1^2}{2} = 1 \Rightarrow 7y_1^2 = 2$$

Now

$$x_1^2 + 5y_1^2 = 21y_1^2 = 6$$

10. Equation of a common tangent to the parabola  $y^2 = 4x$  and the hyperbola  $xy = 2$  is:

☐ A.  $x - 2y + 4 = 0$

☐ B.  $x + y + 1 = 0$

☐ C.  $4x + 2y + 1 = 0$

☒ D.  $x + 2y + 4 = 0$

Equation of tangent to  $y^2 = 4x$  is  $y = mx + \frac{1}{m}$

This is also tangent to  $xy = 2$

$$x \left( mx + \frac{1}{m} \right) = 2$$

$$\Rightarrow mx^2 + \frac{x}{m} - 2 = 0$$

For common tangent,  $D = 0$

$$\Rightarrow \left( \frac{1}{m} \right)^2 - 4 \times m(-2) = 0$$

$$\Rightarrow \frac{1}{m^2} + 8m = 0$$

$$\Rightarrow 8m^3 + 1 = 0$$

$$\Rightarrow m = -\frac{1}{2}$$

$\therefore$  Equation of tangent is  $x + 2y + 4 = 0$

11. The locus of the mid-point of the line segment joining the focus of the parabola  $y^2 = 4ax$  to a moving point of the parabola, is another parabola whose directrix is:

- ☐ A.  $x = a$
- ☒ B.  $x = 0$
- ☐ C.  $x = -\frac{a}{2}$
- ☐ D.  $x = \frac{a}{2}$

Any point on the parabola  $y^2 = 4ax$  be  $(at^2, 2at)$   
 Let mid point of focus and variable point be  $(h, k)$

$$\therefore h = \frac{at^2 + a}{2}, k = \frac{2at + 0}{2}$$

$$\Rightarrow t^2 = \frac{2h - a}{a} \text{ and } t = \frac{k}{a}$$

$$\Rightarrow \frac{k^2}{a^2} = \frac{2h - a}{a}$$

$$\Rightarrow \text{Locus of } (h, k) \text{ is } y^2 = a(2x - a)$$

$$\Rightarrow y^2 = 2a \left( x - \frac{a}{2} \right)$$

Its directrix is

$$x - \frac{a}{2} = -\frac{a}{2} \Rightarrow x = 0$$

12. A ray of light through  $(2, 1)$  is reflected at a point  $P$  on the  $y$ -axis and then passes through the point  $(5, 3)$ . If this reflected ray is the directrix of an ellipse with eccentricity  $\frac{1}{3}$  and the distance of the nearer focus from this directrix is  $\frac{8}{\sqrt{53}}$ , then the equation of the other directrix can be

- ☒ **A.**  $2x - 7y + 29 = 0$  or  $2x - 7y - 7 = 0$   
☐ **B.**  $11x + 7y + 8 = 0$  or  $11x + 7y - 15 = 0$   
☐ **C.**  $2x - 7y - 39 = 0$  or  $2x - 7y - 7 = 0$   
☐ **D.**  $11x - 7y - 8 = 0$  or  $11x + 7y + 15 = 0$

Image of  $(2, 1)$  w.r.t.  $y$ -axis is  $(-2, 1)$

$\therefore$  equation of reflected ray is

$$y - 1 = \frac{3 - 1}{5 + 2}(x + 2)$$

$$\Rightarrow 2x - 7y + 11 = 0 \quad \dots (1)$$

Let the equation of other directrix be  $2x - 7y + k = 0$

Distance of directrix from focus

$$\frac{a}{e} - ae = \frac{8}{\sqrt{53}}$$

$$\Rightarrow a \left( 3 - \frac{1}{3} \right) = \frac{8}{\sqrt{53}}$$

$$\Rightarrow a = \frac{3}{\sqrt{53}}$$

Now, distance between two directrices

$$= \frac{2a}{e} = \frac{18}{\sqrt{53}}$$

$$\text{and } \left| \frac{k - 11}{\sqrt{53}} \right| = \frac{18}{\sqrt{53}}$$

$$\Rightarrow |k - 11| = 18$$

$$\therefore k = 29 \text{ or } -7$$

$\therefore$  Equation of directrix

$$2x - 7y + 29 = 0 \text{ or } 2x - 7y - 7 = 0$$

13. The locus of the centroid of the triangle formed by any point  $P$  on the hyperbola  $16x^2 - 9y^2 + 32x + 36y - 164 = 0$ , and its foci is

☒ A.  $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

☐ B.  $9x^2 - 16y^2 + 36x + 32y - 144 = 0$

☐ C.  $9x^2 - 16y^2 + 36x + 32y - 36 = 0$

☐ D.  $16x^2 - 9y^2 + 32x + 36y - 144 = 0$

$$H \equiv 16(x^2 + 2x + 1) - 9(y^2 - 4y + 4) = 144$$

$$\equiv \frac{(x+1)^2}{\frac{9}{16}} - \frac{(y-2)^2}{16} = 1$$

$$e = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$

$$\text{Foci } (\pm ae - 1, 2) = (\pm 5 - 1, 2)$$

$$\text{Foci are } (4, 2) \text{ and } (-6, 2)$$

$$\text{Let a general point be } P(-1 + 3 \sec \theta, 2 + 4 \tan \theta)$$

$$\text{Let centroid be } (h, k)$$

$$h = \frac{4 - 6 - 1 + 3 \sec \theta}{3}, k = \frac{2 + 2 + 2 + 4 \tan \theta}{3}$$

$$\Rightarrow h = -1 + \sec \theta \text{ and}$$

$$3k = 6 + 4 \tan \theta$$

$$\Rightarrow \sec \theta = (h + 1) \text{ and } \frac{3}{4}(k - 2) = \tan \theta$$

$$\text{Now } \sec^2 \theta - \tan^2 \theta = 1$$

$$\Rightarrow 16(h + 1)^2 - 9(k - 2)^2 = 16$$

$$\Rightarrow 16x^2 - 9y^2 + 32x + 36y - 36 = 0$$

14. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is

☐ A.  $2\sqrt{3}$

☐ B.  $\sqrt{3}$

☐ C.  $\frac{3}{\sqrt{2}}$

☒ D.  $3\sqrt{2}$

Let the equation of ellipse be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  ( $a > b$ )

Now,  $2ae = 6$  and  $\frac{2a}{e} = 12$

$$\Rightarrow ae = 3 \text{ and } \frac{a}{e} = 6$$

$$\Rightarrow a^2 = 18$$

$$a^2e^2 = c^2 = a^2 - b^2 = 9$$

$$\Rightarrow b^2 = 9$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2 \times 9}{\sqrt{18}} = 3\sqrt{2}$$

15. If  $e_1$  and  $e_2$  are the eccentricities of the ellipse,  $\frac{x^2}{18} + \frac{y^2}{4} = 1$  and the hyperbola,  $\frac{x^2}{9} - \frac{y^2}{4} = 1$  respectively and  $(e_1, e_2)$  is a point on the ellipse,  $15x^2 + 3y^2 = k$ . Then  $k$  is equal to :

☐ A. 14

☐ B. 15

☐ C. 17

☒ D. 16

$$e_1 = \sqrt{1 - \frac{4}{18}} = \frac{\sqrt{7}}{3} \text{ \& } e_2 = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

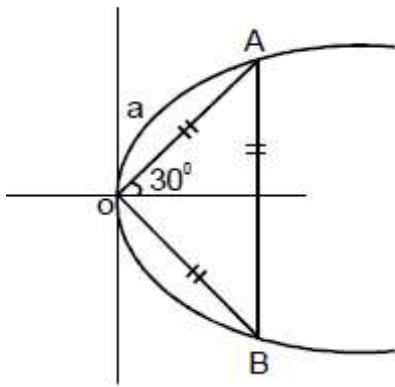
$\therefore (e_1, e_2)$  lies on the ellipse  $15x^2 + 3y^2 = k$

$$\therefore 15e_1^2 + 3e_2^2 = k$$

$$\Rightarrow 15 \times \frac{7}{9} + 3 \times \frac{13}{9} = k \Rightarrow k = 16$$

16. The area (in sq. units) of an equilateral triangle inscribed in the parabola  $y^2 = 8x$ , with one of its vertices on the vertex of this parabola, is:

- ☐ A.  $128\sqrt{3}$
- ☒ B.  $192\sqrt{3}$
- ☐ C.  $64\sqrt{3}$
- ☐ D.  $256\sqrt{3}$



$A : (a \cos 30^\circ, a \sin 30^\circ)$  lies on the parabola  $\frac{a^2}{4} = 8 \cdot \frac{a \cdot \sqrt{3}}{2}$   
 $\Rightarrow a = 16\sqrt{3}$

Area of equilateral  $\triangle = \frac{\sqrt{3}}{4}a^2$

$\triangle = \frac{\sqrt{3}}{4} \cdot 16 \cdot 16 \cdot 3$

$\triangle = 192\sqrt{3}$

17. The shortest distance between the point  $\left(\frac{3}{2}, 0\right)$  and the curve  $y = \sqrt{x}, (x > 0)$ , is :

☒ A.  $\frac{\sqrt{5}}{2}$

☐ B.  $\frac{\sqrt{3}}{2}$

☐ C.  $\frac{3}{2}$

☐ D.  $\frac{5}{4}$

Any point on the curve  $y = \sqrt{x}$  will be of the form  $(t^2, t), t > 0$

Using distance formula

$$d = \sqrt{(t - 0)^2 + \left(t^2 - \frac{3}{2}\right)^2}$$

$$d = \sqrt{t^2 + t^4 - 3t^2 + \frac{9}{4}}$$

$$d = \sqrt{(t^2 - 1)^2 + \frac{5}{4}}$$

$d$  will be minimum when  $t^2 = 1$

$$d_{\min} = \frac{\sqrt{5}}{2}$$



18. Let the length of the latus rectum of an ellipse with its major-axis along  $x$ -axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it?

☐ A.  $(4\sqrt{2}, 2\sqrt{2})$

☐ B.  $(4\sqrt{2}, 2\sqrt{3})$

☐ C.  $(4\sqrt{3}, 2\sqrt{3})$

☒ D.  $(4\sqrt{3}, 2\sqrt{2})$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

$$\text{Length of latus ractum} = \frac{2b^2}{a} = 8 \quad \dots (1)$$

and distance between foci = length of minor axis

$$2ae = 2b$$

$$\Rightarrow ae = b \dots (2)$$

We know that,

$$a^2e^2 = a^2 - b^2$$

From equation (1), we get

$$\Rightarrow b^2 = a^2 - b^2$$

$$\Rightarrow 2b^2 = a^2 \dots (3)$$

From equation (1) and (3), we have

$$\frac{a^2}{a} = 8$$

$$\Rightarrow a = 8$$

From equation (3), we get

$$b^2 = 32$$

Thus, equation of ellipse is

$$\Rightarrow \frac{x^2}{64} + \frac{y^2}{32} = 1$$

Clearly,  $(4\sqrt{3}, 2\sqrt{2})$  lies on the ellipse.

19. A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the  $x$ -axis. Then the eccentricity of the hyperbola is :

☐ A.  $\sqrt{3}$

☒ B.  $\frac{2}{\sqrt{3}}$

☐ C. 2

☐ D.  $\frac{3}{2}$

Given  $2a = 4$   
 $\Rightarrow a = 2$

The hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  passes through (4, 2)

So,  $\frac{4^2}{2^2} - \frac{2^2}{b^2} = 1$

$\Rightarrow b^2 = \frac{4}{3}$

$\therefore c^2 = a^2 + b^2 = 2^2 + \frac{4}{3}$

$\Rightarrow c^2 = \frac{16}{3}$

$\Rightarrow e = \frac{c}{a} = \frac{\sqrt{\frac{16}{3}}}{2} = \frac{2}{\sqrt{3}}$

20. If the three normals drawn to the parabola,  $y^2 = 2x$  pass through the point  $(a, 0)$ ,  $a \neq 0$  then 'a' must be greater than:

- ☒ A. 1
- ☐ B.  $\frac{1}{2}$
- ☐ C.  $-\frac{1}{2}$
- ☐ D. -1

Let the equation of the normal is

$$y = mx - 2am - am^3$$

Here

$$4a = 2 \Rightarrow a = \frac{1}{2}$$

$$y = mx - m - \frac{1}{2}m^3$$

It passes through  $A(a, 0)$  then

$$0 = am - m - \frac{1}{2}m^3$$

$$m = 0, m^2 - 2(a - 1) = 0$$

For real values of  $m$

$$2(a - 1) > 0$$

$$\therefore a > 1$$

21. Let  $x = 4$  be a directrix to an ellipse whose centre is at the origin and its eccentricity is  $\frac{1}{2}$ . If  $P(1, \beta)$ ,  $\beta > 0$  is a point on this ellipse, then the equation of the normal to it at  $P$  is

☐ A.  $8x - 2y = 5$

☒ B.  $4x - 2y = 1$

☐ C.  $7x - 4y = 1$

☐ D.  $4x - 3y = 2$

Given:  $e = \frac{1}{2}$

We know that directrix to an ellipse whose centre is at the origin is

$$x = \frac{a}{e} = 4$$

$$\Rightarrow a = 2$$

$$\text{and } e^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \frac{1}{4} = 1 - \frac{b^2}{4}$$

$$\Rightarrow \frac{b^2}{4} = \frac{3}{4}$$

$$\Rightarrow b^2 = 3$$

$$\text{Ellipse : } \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Since  $P(1, \beta)$  is a point on this ellipse.

$$\therefore \frac{1}{4} + \frac{\beta^2}{3} = 1$$

$$\Rightarrow \beta = \frac{3}{2}$$

$$\therefore P\left(1, \frac{3}{2}\right)$$

Now, equation of normal at point  $P\left(1, \frac{3}{2}\right)$

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\Rightarrow \frac{4x}{1} - \frac{3y}{3/2} = 4 - 3$$

$$\therefore 4x - 2y = 1$$

22. Two sets  $A$  and  $B$  are as under :

$$A = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - 5| < 1 \text{ and } |b - 5| < 1\};$$

$$B = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}. \text{ Then}$$

- ☒ A. neither  $A \subset B$  nor  $B \subset A$
- ☒ B.  $B \subset A$
- ☒ C.  $A \subset B$
- ☒ D.  $A \cap B = \phi$  (an empty set)

Given:

$$A = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - 5| < 1 \text{ and } |b - 5| < 1\};$$

$$\text{Since } |a - 5| < 1$$

$$\therefore -1 < a - 5 < 1$$

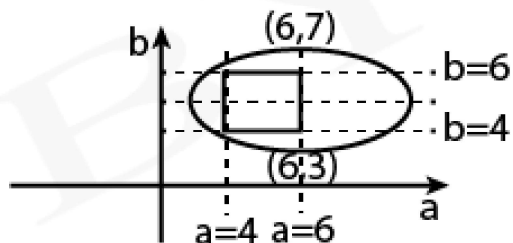
$$\Rightarrow a \in (4, 6)$$

$$\text{Similarly } b \in (4, 6)$$

$$\text{and } B = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$$

$$\text{Since } 4(a - 6)^2 + 9(b - 5)^2 \leq 36$$

$$\therefore \frac{(a - 6)^2}{9} + \frac{(b - 5)^2}{4} \leq 1$$



From diagram, it is clear that  $A \subset B$

23. An ellipse is drawn by taking a diameter of the circle  $(x - 1)^2 + y^2 = 1$  as its semi-minor axis and a diameter of the circle  $x^2 + (y - 2)^2 = 4$  as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

- ☐ A.  $4x^2 + y^2 = 4$
- ☐ B.  $x^2 + 4y^2 = 8$
- ☐ C.  $x^2 + 4y^2 = 8$
- ☒ D.  $x^2 + 4y^2 = 16$

Given:  $(x - 1)^2 + y^2 = 1$

Radius = 1

$\therefore$  Diameter = 2 =  $b$  (semi minor axis)

Also,  $x^2 + (y - 2)^2 = 4$

Radius = 2

$\therefore$  Diameter = 4 =  $a$  (semi major axis)

Now, Equation of ellipse is

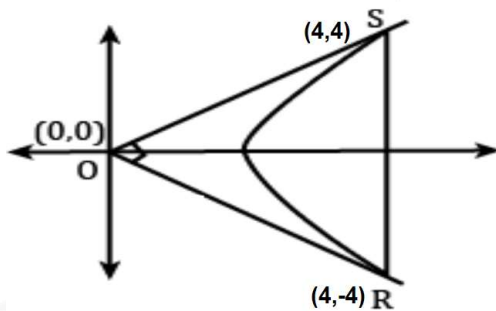
$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

$$\therefore x^2 + 4y^2 = 16$$

24. Let a parabola  $P$  be such that its vertex and focus lie on the positive  $x$ -axis at a distance 2 and 4 units from the origin, respectively. If tangents are drawn from  $O(0, 0)$  to the parabola  $P$  which meet  $P$  at  $S$  and  $R$ , then the area (in sq. units) of  $\triangle SOR$  is equal to

- ☒ A. 16  
☐ B. 32  
☐ C.  $16\sqrt{2}$   
☐ D.  $8\sqrt{2}$

From given condition  $y$ -axis is directrix and equation of parabola is  $y^2 = 8(x - 2)$



Let  $y = mx$  be tangent

$$\therefore m^2x^2 - 8x + 16 = 0 \Rightarrow m = \pm 1$$

So that lines  $y = x$  and  $y = -x$  are tangents.

$\therefore$  Coordinate of  $S$  and  $R$  be  $(4, 4)$  and  $(4, -4)$

$$\therefore \text{Area of } \triangle SOR = \left( \frac{1}{2} \times 4 \times 4 \right) \times 2 = 16$$

25. If the line  $y = mx + c$  is a common tangent to the hyperbola  $\frac{x^2}{100} - \frac{y^2}{64} = 1$  and the circle  $x^2 + y^2 = 36$ , then which one of the following is true?

- ☒ A.  $4c^2 = 369$
- ☐ B.  $c^2 = 369$
- ☐ C.  $8m + 5 = 0$
- ☐ D.  $5m = 4$

$$c = \pm \sqrt{a^2 m^2 - b^2}$$

$$\Rightarrow c = \pm \sqrt{100m^2 - 64} \dots (i)$$

$$y = mx \pm \sqrt{100m^2 - 64}$$

$$d|_{(0,0)} = 6$$

$$\Rightarrow \left| \frac{\sqrt{100m^2 - 64}}{\sqrt{m^2 + 1}} \right| = 6$$

$$\Rightarrow 100m^2 - 64 = 36m^2 + 36$$

$$\Rightarrow 64m^2 = 100$$

$$\Rightarrow m = \pm \frac{5}{8} \dots (ii)$$

From (i) and (ii)

$$c^2 = 100 \times \frac{100}{64} - 64 = \frac{(164)(36)}{64}$$

$$\Rightarrow 4c^2 = 369$$



## Subject: Mathematics

---

1. Let  $L$  be a common tangent line to the curves  $4x^2 + 9y^2 = 36$  and  $(2x)^2 + (2y)^2 = 31$ . Then the square of the slope of the line  $L$  is

Accepted Answers

3      3.0      3.00

Solution:

$$E : \frac{x^2}{9} + \frac{y^2}{4} = 1, \quad C : x^2 + y^2 = \frac{31}{4}$$

equation of tangent to ellipse is

$$y = mx \pm \sqrt{9m^2 + 4} \dots (1)$$

equation of tangent to circle is

$$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}} \dots (2)$$

Comparing equation (1) & (2)

$$9m^2 + 4 = \frac{31}{4}m^2 + \frac{31}{4}$$

$$\Rightarrow 36m^2 + 16 = 31m^2 + 31$$

$$\Rightarrow 5m^2 = 15 \Rightarrow m^2 = 3$$

2. A line is a common tangent to the circle  $(x-3)^2 + y^2 = 9$  and the parabola  $y^2 = 4x$ . If the two points of contact  $(a, b)$  and  $(c, d)$  are distinct and lie in the first quadrant, then  $2(a + c)$  is equal to

Accepted Answers

9      9.0      9.00

Solution:

Circle :  $(x - 3)^2 + y^2 = 9$

Parabola :  $y^2 = 4x$

Let common tangent equation be  $y = mx + \frac{a}{m}$

$\Rightarrow y = mx + \frac{1}{m}$

$\Rightarrow m^2x - my + 1 = 0$

the above line is also tangent to circle

$(x - 3)^2 + y^2 = 9$

So, perpendicular distance from (3, 0) to line = 3

$\Rightarrow \left| \frac{3m^2 - 0 + 1}{\sqrt{m^2 + m^4}} \right| = 3$

$\Rightarrow (3m^2 + 1)^2 = 9(m^2 + m^4)$

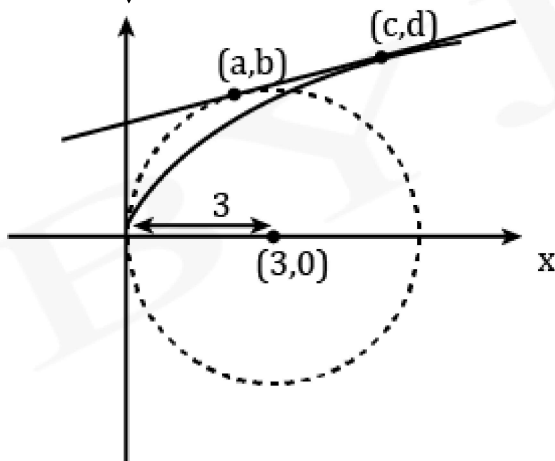
$\Rightarrow m = \pm \frac{1}{\sqrt{3}}$

$\therefore$  tangent equation is

$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$

slope should not be negative for point of contacts in first quadrant

$\therefore m = \frac{1}{\sqrt{3}}$



For Parabola, point of contact is  $\left( \frac{a}{m^2}, \frac{2a}{m} \right) \equiv (3, 2\sqrt{3}) \equiv (c, d)$

Solving Circle  $(x - 3)^2 + y^2 = 9$  & line equation  $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$

$(x - 3)^2 + \left( \frac{1}{\sqrt{3}}x + \sqrt{3} \right)^2 = 9$

$\Rightarrow x^2 + 9 - 6x + \frac{1}{3}x^2 + 3 + 2x = 9$

$\Rightarrow \frac{4}{3}x^2 - 4x + 3 = 0$

$\Rightarrow x = \frac{3}{2} = a$

$\therefore 2(a + c) = 2 \left( \frac{3}{2} + 3 \right) = 9$

3. Let  $A(\sec \theta, 2 \tan \theta)$  and  $B(\sec \phi, 2 \tan \phi)$ , where  $\theta + \phi = \frac{\pi}{2}$ , be two point on the hyperbola  $2x^2 - y^2 = 2$ . If  $(\alpha, \beta)$  is the point of the intersection of the normals to the hyperbola at  $A$  and  $B$ , then  $(2\beta)^2$  is equal to

Accepted Answers

\*

Solution:

Point  $A(\sec \theta, 2 \tan \theta)$  lies on the hyperbola  $2x^2 - y^2 = 2$ .

$$\Rightarrow 2 \sec^2 \theta - 4 \tan^2 \theta = 2$$

$$\Rightarrow 2 + 2 \tan^2 \theta - 4 \tan^2 \theta = 2$$

$$\Rightarrow \tan^2 \theta = 0$$

$$\Rightarrow \theta = \pi n, \quad n \in \mathbb{Z}$$

Similarly, point  $B(\sec \phi, 2 \tan \phi)$  lies on the hyperbola  $2x^2 - y^2 = 2$ .

$$\Rightarrow \phi = \pi n, \quad n \in \mathbb{Z}$$

But in question, it is given that  $\theta + \phi = \frac{\pi}{2}$ , which is not possible.

Hence, it is a Bonus question.

4. Let  $y = mx + c, m > 0$  be the focal chord of  $y^2 = -64x$ , which is tangent to  $(x + 10)^2 + y^2 = 4$ . Then the value of  $4\sqrt{2}(m + c)$  is equal to

Accepted Answers

34    34.0    34.00

Solution:

For  $y^2 = -64x$  focus is  $(-16, 0)$

$\therefore y = mx + c$  passes through  $(-16, 0)$

then  $c = 16m \dots (1)$

Also  $y = mx + c$  touches the given circle

$$\text{So, } \left| \frac{-10m + c}{\sqrt{1 + m^2}} \right| = 2$$

From equation (1)

$$\Rightarrow |3m| = \sqrt{1 + m^2}$$

$$\Rightarrow m = \frac{1}{2\sqrt{2}} \text{ and } c = 4\sqrt{2}$$

$$\text{Now, } 4\sqrt{2}(m + c) = 2 \cdot 17 = 34$$

5. Let  $E$  be an ellipse whose axes are parallel to the co-ordinates axes, having its center at  $(3, -4)$ , one focus at  $(4, -4)$  and one vertex at  $(5, -4)$ . If  $mx - y = 4, m > 0$  is a tangent to the ellipse  $E$ , then the value of  $5m^2$  is equal to

Accepted Answers

3 3.0 3.00 03

Solution:

Given: center  $C(3, -4)$ , focus  $F_1(4, -4)$ , vertex  $V_1(5, -4)$

$\therefore ae = CF_1 = 1$  and  $a = CV_1 = 2$  so,  $b = \sqrt{3}$

$$E : \frac{(x - 3)^2}{4} + \frac{(y + 4)^2}{3} = 1$$

Equation of tangent

$$y + 4 = m(x - 3) \pm \sqrt{4m^2 + 3}$$

$$\Rightarrow y = mx - 3m - 4 \pm \sqrt{4m^2 + 3}$$

Comparing with  $y = mx - 4$

$$\text{We get } -3m \pm \sqrt{4m^2 + 3} = 0$$

$$\Rightarrow 9m^2 = 4m^2 + 3$$

$$\Rightarrow 5m^2 = 3$$