SIMPLE HARMONIC MOTION

GENERAL EQUATION OF SHM

Displacement $x = A\sin(\omega t + \phi)$

Here $(\omega t + \phi)$ is the pahse of the motion and ϕ is the initial phase of the motion

$\rightarrow$ Two vibrating particles are said to be in same phase if the phas difference between them is an even multiple of π, i.e., $\Delta\phi = 2n\pi$ where n = 0, 1, 2, 3 ......

$\rightarrow$ Two vibrating particle are said to be in opposite phase if the pahse difference between them is an odd multiple of π i.e., $\Delta\phi = (2n+1)\pi$, where n = 0, 1, 2, 3 ....

Angular Frequency

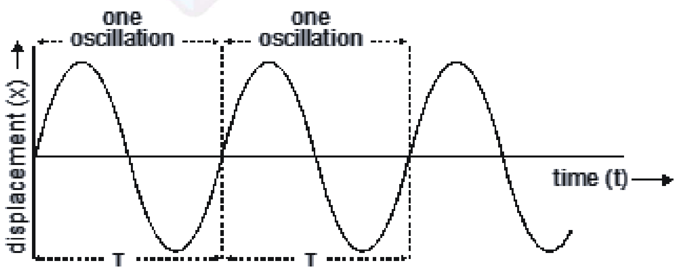
$$\omega = \frac{2\pi}{T} = 2\pi f,$$

where T is the time period

f is the frequency

Time period (T)

$$T = \frac{2\pi}{\omega}$$



- **For linear SHM** $(F \propto -x)$; $F = m\frac{d^2x}{dt^2} = -kx = -m\omega^2 x$, where $\omega = \sqrt{\frac{k}{m}}$

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- **For angular SHM** $(\tau \propto -\theta) : \tau = 1\frac{d^2\theta}{dt^2} = 1\alpha = -k\theta = -m\omega^2\theta$, where $\omega = \sqrt{\frac{k}{m}}$
- **Displacement** $x = A\sin(\omega t + \phi)$,
- **Angular displacement** $\theta = \theta_0 \sin(\omega t + \phi)$
- **Velocity** $v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$
- **Angular velocity** $\frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi)$
- **Acceleration** $a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$
- **Angular acceleration** $\frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi) = -\omega^2\theta$
- **Kinetic energy** $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$
- **Potential energy** $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$
- **Total energy** $E = K + U = \frac{1}{2}m\omega^2A^2 = \text{constant}$
- **Note :**
  (i) Total energy of a particle in S.H.M., is same at all instant and at all displacement.
  (ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

Average energy in S.H.M.

(i) The time average of P.E. and K.E. over one cycle is

(a) $\langle K \rangle_t = \frac{1}{4}kA^2$ (b) $\langle PE \rangle_t = \frac{1}{4}kA^2$ (c) $\langle TE \rangle_t = \frac{1}{2}kA^2$

(ii) The position average of P.E. and K.E. between x = –A to x = A

(a) $\langle K \rangle_x = \frac{1}{3}kA^2$ (b) $\langle PE \rangle_x = \frac{1}{6}kA^2$ (c) $\langle TE \rangle_x = \frac{1}{2}kA^2$

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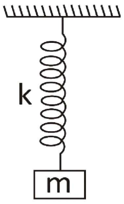
Differential equation of SHM

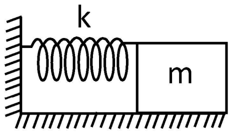
→ Linear SHM $\frac{d^2x}{dt^2}+\omega^2x=0$

→ Angular SHM $\frac{d^2x}{dt^2}+\omega^2\theta=0$

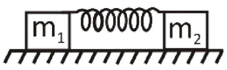
Spring block system



$$T=2\pi\sqrt{\frac{m}{k}}$$

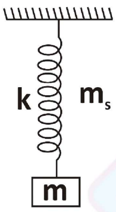


$$T=2\pi\sqrt{\frac{m}{k}}$$



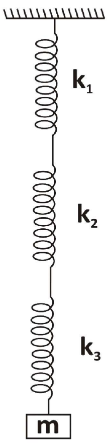
$T=2\pi\sqrt{\frac{\mu}{k}}$, where μ = reduced mass $=\frac{m_1m_2}{m_1+m_2}$

When spring mass is not negligible



$$T=2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$$

Series combination of springs



$T=2\pi\sqrt{\frac{m}{k_{eff}}}$, where $\frac{1}{k_{eff}}=\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}$

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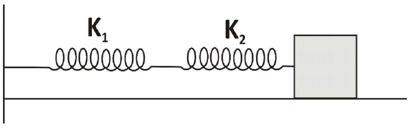
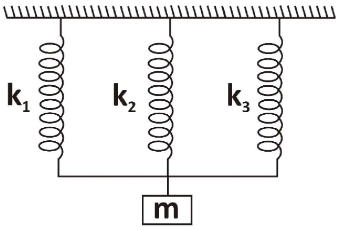
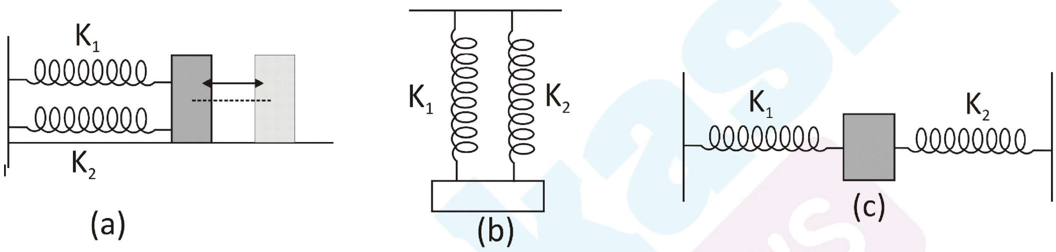


$K_{eq} = k_1 + k_2$

Parallel combination of springs

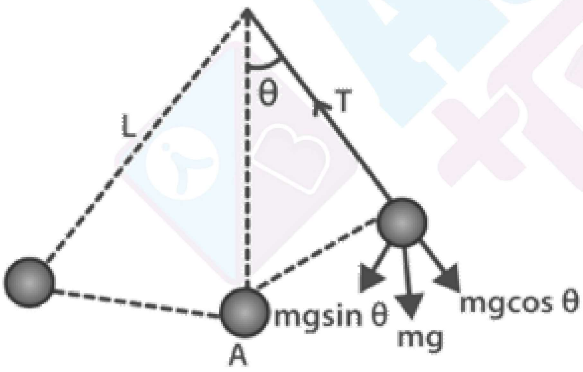


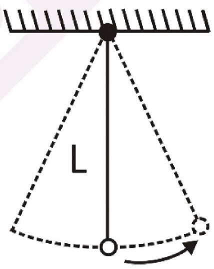
$T = 2\pi\sqrt{\frac{m}{k_{eff}}}$, where $k_{eff} = k_1 + k_2 + k_3$



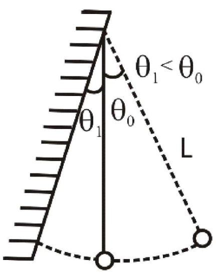
$K_{eff} = K_1 + K_2$

Simple Pendulum





Time period $T = 2\pi\sqrt{\frac{L}{g}}$



Time period $T = \left\{\pi + 2\sin^{-1}\left(\frac{\theta_1}{\theta_0}\right)\right\}\sqrt{\frac{L}{g}}$

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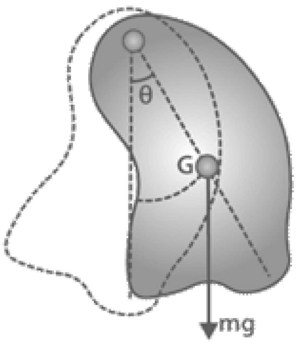
- If the length of simple pendulum is comparable to the radius of the earth R, then

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{\ell}+\frac{1}{R}\right)}}$$

If ℓ << R then T = $2\pi\sqrt{\frac{\ell}{g}}$

If ℓ >> R then T = $2\pi\sqrt{\frac{R}{g}}$ $\approx$ 84 minutes

Physical Pendulum

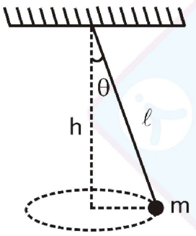


$T = 2\pi\sqrt{I/mg\ell}$

Here $I = I_{cm} + m\ell^2$

ℓ is the distance between the point of suspension and centre of mass

Time period of Conical pendulum



$$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$$

Time period of Torsional pendulum

$$T = 2\pi\sqrt{\frac{I}{k}}$$

where k = torsional constant of the wire,

I = moment of inertia of the body about the vertical axis

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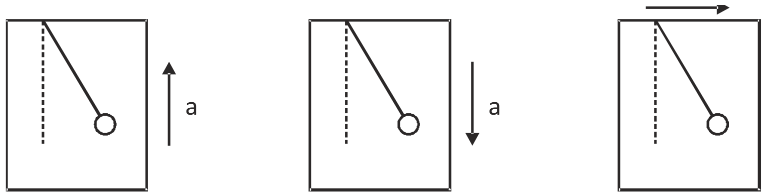
Time Period in Accelerating Cage



$g_{eff} = g + a$

$T = 2\pi\sqrt{\frac{\ell}{g+a}}$

$g_{eff} = g - a$

$T = 2\pi\sqrt{\frac{\ell}{g-a}}$

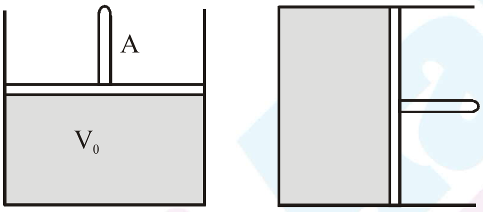
$g_{eff} = \sqrt{g^2 + a^2}$

$T = 2\pi\sqrt{\frac{\ell}{\left(g^2 + a^2\right)^{1/2}}}$

SHM of gas - piston system

Here elastic force is developed due to bulk elasticity of the gas

$$B = \frac{\Delta P}{-\Delta V / V} \Rightarrow F = -\frac{BA^2x}{V_0} \Rightarrow T = 2\pi\sqrt{\frac{m}{BA^2 / V_0}}$$



KEY POINTS

- SHM is the projection of uniform circular motion along one of the diameters of the circle.
- The periodic time of a hard spring is less as compared to that of a soft spring because the spring constant is large for hard spring.
- For a system executing SHM, the mechanical energy remains constant.
- Maximum kinetic energy of a particle in SHM may be greater than mechanical energy as potentia! energy of a system may be negative.
- The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.

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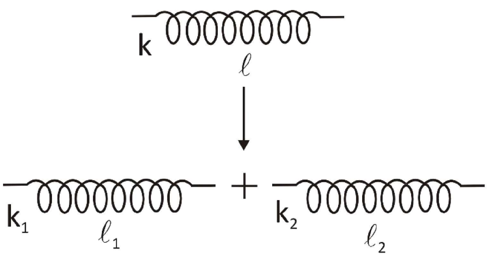
Spring cut into two parts :



Here $\frac{\ell_1}{\ell_2} = \frac{m}{n}$

$\ell_1 = \left(\frac{m}{m+n}\right)\ell$, $\ell_2 = \left(\frac{n}{m+n}\right)\ell$ But $k\ell = k_1\ell_1 = k_2\ell_2$

$\Rightarrow \quad k_1 = \frac{(m+n)}{m}k \; ; \; k_2 = \frac{(m+n)}{n}k$

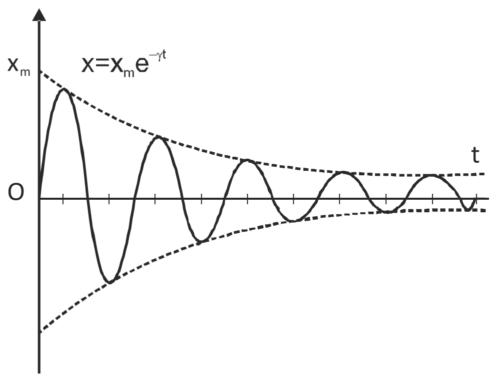
FREE, DAMPED, FORCED OSCILLATION AND RESONANCE

Free oscillation

- The oscillation of a prticle with fundamental frequency under the influence of restoring force are defined as free oscillatios.

Damped oscillations

- The oscillations of a body whose amplitude goes on decreasing with time are defined as damped oscillations.
- In these oscillations the amplitude of oscillations decreases exponentially due to damping forces like frictional force, viscous force etc.
- If initial amplitude is X_m then amplitude after time t will be $x = x_m\, e^{-\gamma t}$ where γ = Damping coefficient



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FORCED OSCILLATION

- The oscillations in which a body oscillates under the influence of an external periodic force (driver) are known as forced oscillations.
- The driven body does not oscillate with its natural frequency rather it oscillates with the frequency of the driver.
- The amplitude of oscillator decreases due to damping forces but on account of the energy gained from the external source (driver) it remains constant.

RESONANCE

- When the frequency of external force (driver) is equal to the natural frequency of the oscillator (driven), then this state of the driver and the driven is known as the state of resonance.
- In the state of resonance, there occurs maximum transfer of energy from the driver to the driven, Hence the amplitude of motion becomes maximum.

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WAVE ON A STRING

CLASSIFICASTION OF WAVES

- **Medium Necessity**
  - Mechanical (Elastic) waves
  - Non-mechanical waves (EM Waves)
- **Propagation of energy**
  - Progressive waves
  - Stationary (standing) waves
- **Vibration of medium particle**
  - Transverse waves
  - Longitudinal waves
- **Dimension**
  - 1-D (Waves on strings)
  - 2-D (Surface waves or ripples on water)
  - 3-D (Sound or light waves)

- In strings, mechanical waves are always transverse.
- In gases and liquids, mechanical waves are always longitudinal because fluids cannot sustain shear.
- In solids mechanical waves (may be sound) can be either transverse or longitudinal depending on the mode of excitation.

Plane progressive waves

- Wave equation : $y = A \sin(\omega t - kx)$ where $k = \frac{2\pi}{\lambda}$ = wave propagration constant.

- Differential equaton : $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

  Wave velocity (phase velocity) $v = \frac{dx}{dt} = \frac{\omega}{k}$

  $\because\ \omega t - kx = \text{constant} \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$

- Particle velocity $v_p = \frac{dy}{dt} = A\omega\cos(\omega t - kx)$

  $v_p = -v \times \text{slope} = -v\left(\frac{dy}{dx}\right)$

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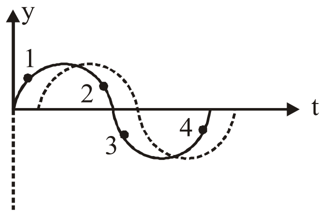
- Particle acceleration : $a_p = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx) = -\omega^2 y$

  For particle 1 : $v_p \downarrow$ and $a_p \downarrow$

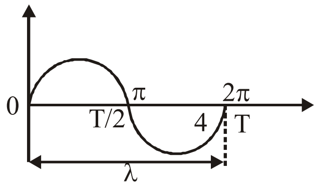
  For particle 2 : $v_p \uparrow$ and $a_p \downarrow$

  For particle 3 : $v_p \uparrow$ and $a_p \uparrow$

  For particle 4 : $v_p \downarrow$ and $a_p \uparrow$



- Relation between phase difference, path difference & time difference.



$$\frac{\Delta\phi}{2\pi} = \frac{\Delta\lambda}{\lambda} = \frac{\Delta T}{T}$$

Energy in wave Motion

- $\frac{KE}{volume} = \frac{1}{2}\left(\frac{\Delta m}{volume}\right)v_p^2 = \frac{1}{2}\rho v_p^2 = \frac{1}{2}\rho\omega^2 A^2\cos^2(\omega t - kx)$

- $\frac{PE}{volume} = \frac{1}{2}\rho v^2\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}\rho\omega^2 A^2\cos^2(\omega t - kx)$

- $\frac{TE}{volume} = \rho\omega^2 A^2\cos^2(\omega t - kx)$

- Pressure energy density [i.e. Average total energy / volume] $u = \frac{1}{2}\rho\omega^2 A^2$

- **Power :** P = (energy density) (volume/time) P = $\left(\frac{1}{2}\rho\omega^2 A^2\right)(Sv)$

  [where S = Area of cross-section]

- **Intensity :** I = $\frac{\text{Power}}{\text{area of cross section}} = \frac{1}{2}\rho\omega^2 A^2 v$

Speed of transverse wave on string

$v = \sqrt{\frac{T}{\mu}}$ where μ = mass/length and T = tension in the string

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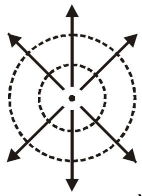
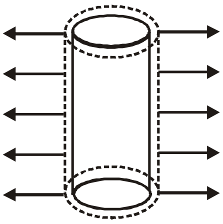
WAVE FRONT

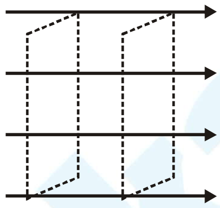
- Spherical wave front (source $\rightarrow$ point source)



- Cylindrical wave front (source $\rightarrow$linear source)



- Plane wave front (source $\rightarrow$ point/linear) source at very large distance



INTENSITY OF WAVE

- Due to point source $I \propto \frac{1}{r^2}$ $\quad$ $y(r,t) = \frac{A}{r}\sin(\omega t - \vec{k}.\vec{r})$
- Due to cylindrical source $I \propto \frac{I}{r}$ $\quad$ $y(r, t) = \frac{A}{\sqrt{r}}\sin(\omega t - \vec{k}.\vec{r})$
- Due to plane source I = constant $\quad$ $y(r, t) = A \sin(\omega t - \vec{k}.\vec{r})$

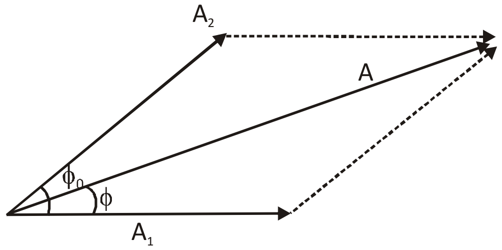
INTERFERENCE OF WAVES

$y_1 = A_1 \sin(\omega t - kx)$ $\quad$ $y_2 = A_2 \sin(\omega t - kx + \phi_0)$

$y = y_1 + y_2 = A \sin(\omega t - kx + \phi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi_0}$

and $\tan\phi = \dfrac{A_2 \sin\phi_0}{A_1 + A_2\cos\phi_0}$



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As $\quad I \propto A^2$

So $\quad I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi_0$

For constructive interference [Maximum intensity]

f_0 = 2np or path difference = nl where n = 0, 1, 2, 3,...

$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$

For destructive interference [Minimum Intensity]

ϕ_0 = (2n + 1)π or path difference = $(2n+1)\frac{\lambda}{2}$

where n = 0, 1, 2, 3, ... $\quad \boxed{I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2}$

- **Degree of hearing** = $\frac{I_{max} - I_{min}}{I_{max} + I_{min}} \times 100$

Reflection and Refraction (transmission) of waves

- The frequency of the wave remain unchanged.
- Amplitude of reflected wave $\rightarrow$ = $A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i$
- Amplitude of transmitted wave $\rightarrow A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_j$
- If $v_2 > v_1$ i.e., medium -2 is rarer

  $A_r > 0 \Rightarrow$ no phase change is reflected wave
- If $v_2 < v_1$ i.e., medium -1 is raer

  $A_r < 0 \Rightarrow$ There is a phase change of p in reflected wave.

Standing waves or Stationary waves

Formation of standing wave is possible only in bounded medium.

- Let two waves are y_1 = A sin(ωt –kx) ; y_2 = A sin(ωt + kx) by principle of superposition

  $y = y_1 + y_2$ = 2A cos kx sin ωt $\leftarrow$ Equation of stationary waves
- **Nodes** $\rightarrow$ amplitude is minimum : cos kx = 0 $\Rightarrow x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$.....
- **Antinodes** $\rightarrow$ amplitude is maixmum : cos kx = 1 $\Rightarrow$ x = 0, $\frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$,...

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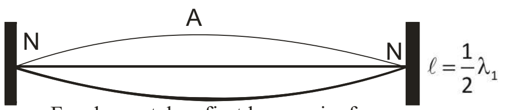
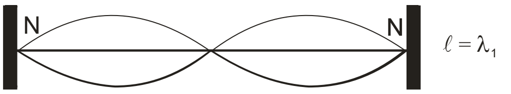
Transverse Stationary Waves In Stretched String

Fixed at both ends

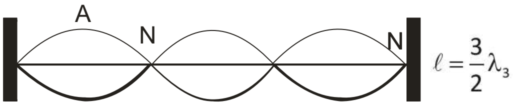
[fixed end → Node & free end → Antinode]



Fundamental or first harmonic, f_1



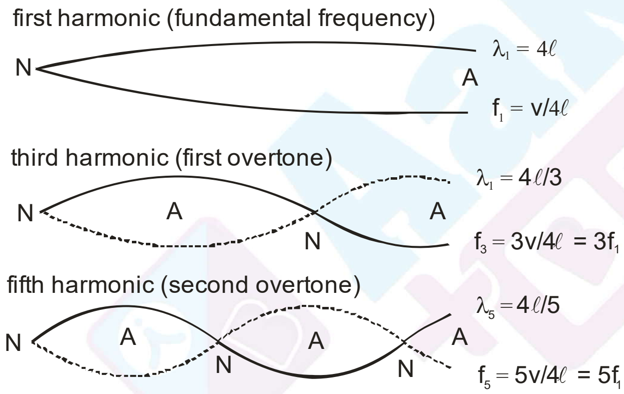
First overtone or second harmonic, $f_2 = 2f_1$



Second overtone or third harmonic, $f_3 = 3f_1$

$$f_n = \frac{v}{\lambda} = n\frac{v}{2L} = n.f_1, \quad (n = 1,2,3,......)$$

Fixed at one end



ratio of frequencies $f_1 : f_2 : f_3 = 1 : 3 : 5$

SOUND WAVE

SOUND WAVES

Velocity of sound in a medium of elasticity E and density ρ is

$$v = \sqrt{\frac{E}{\rho}}$$

Solids (Young's Modulus)

$$v = \sqrt{\frac{Y}{\rho}}$$

Fluid (Bulk Modulus)

$$v = \sqrt{\frac{B}{\rho}}$$

- **Newton's formula :** Sound propagation is isothermal $B = P \Rightarrow v = \sqrt{\frac{P}{\rho}}$
- **Laplace correction :** Sound propagation is adiabatic $B = \gamma P \Rightarrow v = \sqrt{\frac{\gamma P}{\rho}}$

KEY POINTS

- With rise in temperature, velocity of sound in a gas increases as $v = \sqrt{\frac{\gamma RT}{M_w}}$
- With rise in humidity velocity of sound increases due to presence of water in air.
- Pressure has no effect on velocity of sound in a gas as long as temperature remains constant.

Displacement and pressure wave

Displaceme wave	$y = A\sin(\omega t - kx)$
Pressure wave	$p = p_0\cos(\omega t - kx)$
where	$p_0 = ABk = \rho A v \omega$

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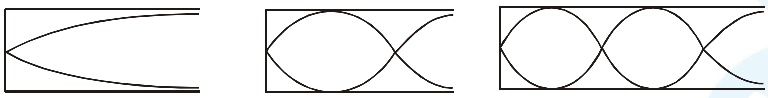
KEY POINTS

- The pressure wave is 90° out of phase ω. r. t. displacement wave, i.e, displacement will be maximum when pressure is minimum and vice-versa,
- Intensity in terms of pressure amplitude $I = \frac{p_0^2}{2\rho v}$

Vibrations of organ pipes

Stationary longitudinal waves closed end $\rightarrow$ displacement node, open end $\rightarrow$ displacement antinode

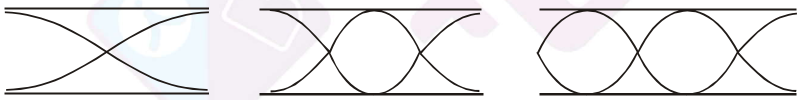
Closed end organ pipe



$\ell = \frac{\lambda}{4} \Rightarrow f = \frac{v}{4\ell}$ $\quad$ $\ell = \frac{3\lambda}{4} \Rightarrow f = \frac{3v}{4\ell}$ $\quad$ $\ell = \frac{5\lambda}{4} \Rightarrow f = \frac{5v}{4\ell}$

- Only odd harmonics are present
- Maximum possible wavelength = 4ℓ
- Frequency of mth overtone = $(2m+1)\frac{v}{4\ell}$

Open end organ pipe



$\ell = \frac{\lambda}{2} \Rightarrow f = \frac{v}{2\ell}$ $\quad$ $\ell = \lambda \Rightarrow f = \frac{2v}{2\ell}$ $\quad$ $\ell = \frac{3\lambda}{2} \Rightarrow f = \frac{3v}{2\ell}$

- All harmonics are present
- Maximum possible wavelength is 2ℓ.
- Frequency of mth overtone = $(m+1)\frac{v}{2\ell}$

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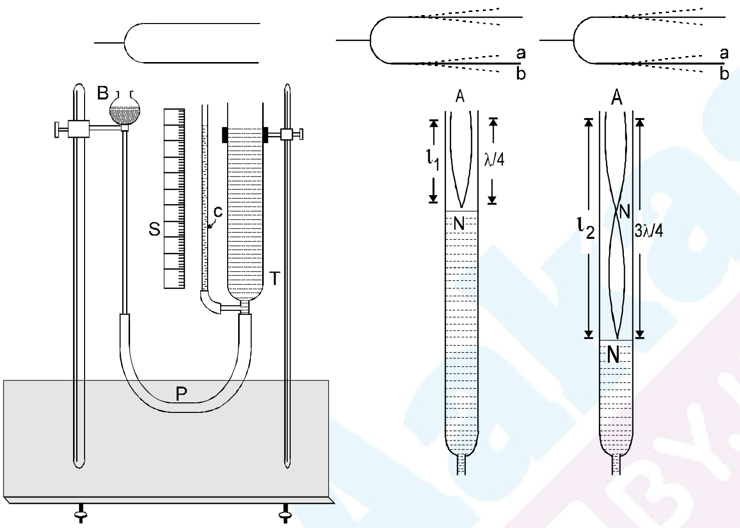
End correction

Due to finite momentum of air molecules in organ pipes reflection takes place not exactly at open end but some what above it, so anti node is not formed exactly at free end but slightly above it.

- In closed organ pipe $f_1 = \frac{v}{4(\ell + e)}$ where, e = 0.6 R (R = radius of pipe)
- In open organ pipe $f_1 = \frac{v}{2(\ell + 2e)}$

Resonance Tube



Wavelength $\lambda = 2(\ell_2 - \ell_1)$ End correction $e = \frac{\ell_2 - 3\ell_1}{2}$

Intensity of sound in decibels

Sound level , SL = $10\log_{10}\left(\frac{I}{I_0}\right)$

Where I_0 = threshold of human ear = 10^{-12} W/m^2

Characteristics of sound

- Loudness $\rightarrow$ Sensation received by the ear due to intensity of sound.
- Pitch $\rightarrow$ Sensation received by the ear due to frequency of sound.
- Quality (or Timbre) $\rightarrow$ Sensation received by the ear due to waveform of sound.

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Doppler's effect

$$f' = \frac{(V \pm V_0)}{(V \pm V_s)} f$$

$f' =$ observed frequency

$f =$ actual frequency

$V =$ velocity of sound waves

$V_0 =$ velocity of observer

$V_s =$ velocity of the source

Source Moving Towards the Observer at Rest	$f' = \frac{V}{(V - V_s)} f$
Source Moving Away from the Observer at Rest	$f' = \frac{V}{(V - (-V_s))} f$
Observer Moving Towards a Stationary Source	$f' = \frac{(V + V_0)}{V} f$
Observer Moving Away from a Stationary Source	$f' = \frac{(V - V_0)}{V} f$

Doppler's effect in light :

- **Case I:**

$$\left.\begin{aligned} &\text{Frequency} \quad v' = \left(\sqrt{\frac{1 + v/c}{1 - v/c}}\right) v \approx \left(1 + \frac{v}{c}\right) v \\ &\text{Wavelength} \quad \lambda' = \left(\sqrt{\frac{1 - v/c}{1 + v/c}}\right) \lambda \approx \left(1 - \frac{v}{c}\right) \lambda \end{aligned}\right\} \text{Violet Shift}$$

- **Case II :**

$$\left.\begin{aligned} &\text{Frequency} \quad v' = \left(\sqrt{\frac{1 - v/c}{1 + v/c}}\right) v \approx \left(1 - \frac{v}{c}\right) v \\ &\text{Wavelength} \quad \lambda' = \left(\sqrt{\frac{1 + v/c}{1 - v/c}}\right) \lambda \approx \left(1 + \frac{v}{c}\right) \lambda \end{aligned}\right\} \text{Red Shift}$$

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