Full Syllabus Test 2

Subject: Mathematics

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1. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X, Z \subseteq X$ and $Y \cap Z$ is empty, is

☒ **A.** 5^3

☒ **B.** 5^2

☑ **C.** 3^5

☒ **D.** 2^5

$Y \cap Z = \phi \Rightarrow Y$ and Z are disjoint subsets of X.
Let $a \in X$
Case 1 : $a \in Y$ and $a \notin Z$
Case 2 : $a \in Z$ and $a \notin Y$
Case 3 : $a \notin Y$ and $a \notin Z$
So, each element has 3 options.
There are 5 possibilities for a.
$\therefore$ Possible number of ordered pairs $= 3^5$

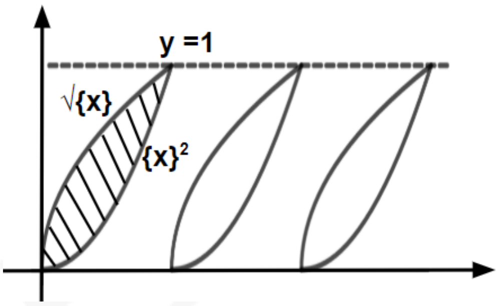
2. Let $f(x) = \begin{cases} \sqrt{\{x\}}\ , & x \notin \mathbb{Z} \\ 1\ , & x \in \mathbb{Z} \end{cases}$ and $g(x) = \{x\}^2$, the area bounded by $f(x)$ and $g(x)$ for $x \in [0, n];\ (n \in \mathbb{Z})$ is denoted by $S(n)$, then which of the following is correct?

✗ **A.** $S(10) = \frac{5}{3}$

✗ **B.** $\frac{S(10)}{S(5)} = 4$

✓ **C.** $\lim\limits_{n\to\infty} \frac{S(n)}{n} = \frac{1}{3}$

✗ **D.** $\sum\limits_{n=1}^{9} S(n) = 30$



The required area is,

$$S(n) = n\int_0^1 \sqrt{\{x\}} - \{x\}^2\, dx$$

$$\Rightarrow S(n) = n\int_0^1 \sqrt{x} - x^2\, dx$$

$$\Rightarrow S(n) = n \times \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{n}{3}$$

$$S(10) = \frac{10}{3}$$

$$\frac{S(10)}{S(5)} = \frac{10}{5} = 2$$

$$\lim_{n\to\infty} \frac{S(n)}{n} = \lim_{n\to\infty} \frac{n/3}{n} = \frac{1}{3}$$

$$\sum_{n=1}^{9} S(n) = \frac{1}{3} \times \frac{9 \times 10}{2} = 15$$

3. Number of 4 digit numbers that can be formed using the digits $0, 1, 2, 3, 4, 5$ which are divisible by 6 when repetition of digits is not allowed are

✗ **A.** 24

✓ **B.** 52

✗ **C.** 72

✗ **D.** 48

For any number to be divisible by 6 the number should be even and divisible by 3. Now, for number to be even the last digit should be $0, 2, 4$.

Now, if the last digit is 0 then first 3 digits can be $(1, 2, 3)$ or $(2, 3, 4)$ or $(3, 4, 5)$ or $(1, 3, 5)$.
So, possible numbers are $= 4 \times 3! = 24$

Now, if the last digit is 2 then first 3 digits can be $(0, 1, 3)$ or $(0, 3, 4)$ or $(1, 4, 5)$.
So, possible numbers are $= 4 + 4 + 3! = 14$

Now, if the last digit is 4 then first 3 digits can be $(0, 2, 3)$ or $(0, 3, 5)$ or $(1, 2, 5)$.
So, possible numbers are $= 4 + 4 + 3! = 14$

Hence, total numbers are $14 + 14 + 24 = 52$

4. The sum of the series of $\cot^{-1}\frac{5}{\sqrt{3}}+\cot^{-1}\frac{9}{\sqrt{3}}+\cot^{-1}\frac{15}{\sqrt{3}}+\cot^{-1}\frac{23}{\sqrt{3}}+\cdots\infty$ is equal to

☒ **A.** $\frac{\pi}{4}$

☑ **B.** $\frac{\pi}{3}$

☒ **C.** $\frac{\pi}{6}$

☒ **D.** $\frac{\pi}{12}$

$$S_n=\cot^{-1}\frac{5}{\sqrt{3}}+\cot^{-1}\frac{9}{\sqrt{3}}+\cot^{-1}\frac{15}{\sqrt{3}}+\cot^{-1}\frac{23}{\sqrt{3}}+\cdots\infty$$

$$\begin{aligned}T_n&=\cot^{-1}\left(\frac{n^2+n+3}{\sqrt{3}}\right)\\&=\tan^{-1}\left(\frac{\sqrt{3}}{n^2+n+3}\right)\\&=\tan^{-1}\left(\frac{\frac{n+1}{\sqrt{3}}-\frac{n}{\sqrt{3}}}{1+\frac{n+1}{\sqrt{3}}\times\frac{n}{\sqrt{3}}}\right)\\&=\tan^{-1}\left(\frac{n+1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{n}{\sqrt{3}}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow S_n=&\tan^{-1}\frac{2}{\sqrt{3}}-\tan^{-1}\frac{1}{\sqrt{3}}\\&+\tan^{-1}\frac{3}{\sqrt{3}}-\tan^{-1}\frac{2}{\sqrt{3}}\\&\quad\vdots\qquad\qquad\vdots\\&+\tan^{-1}\left(\frac{n+1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{n}{\sqrt{3}}\right)\end{aligned}$$

$$\Rightarrow S_n=\tan^{-1}\frac{n+1}{\sqrt{3}}-\tan^{-1}\frac{1}{\sqrt{3}}$$

$$\Rightarrow\lim_{n\to\infty}S_n=\lim_{n\to\infty}\tan^{-1}\left(\frac{n+1}{\sqrt{3}}\right)-\frac{\pi}{6}$$

$$\therefore\lim_{n\to\infty}S_n=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}$$

5. $px+qy=40$ is a chord of minimum length of the circle $(x-10)^2+(y-20)^2=729$. If the chord passes through $(5,15)$, then $p^{2019}+q^{2019}$ is equal to

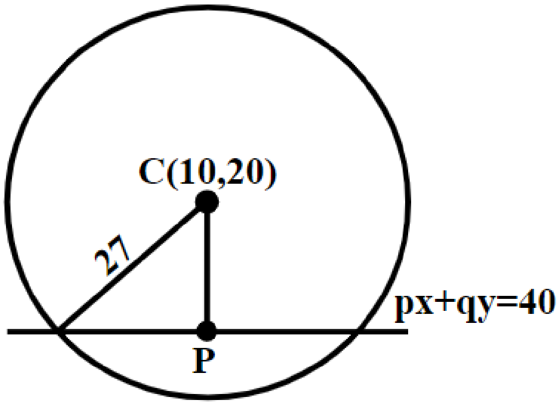
☒ **A.** 0

☒ **B.** 2

☒ **C.** 2^{2019}

☑ **D.** 2^{2020}

Given, equation of circle
$(x-10)^2+(y-20)^2=729$
Centre $(C)\equiv(10,20)$ and radius $=\sqrt{729}=27$



Equation of chord : $px+qy=40$
Slope of chord $=-\dfrac{p}{q}$

Since, chord is of minimum length.
$\therefore P(5,15)$ will be the mid point of given chord.
We know that, any chord and the straight line passing through its midpoint and centre of the circle are perpendicular to each other.
$\Rightarrow m_{CP}\times m_{\text{chord}}=-1$

$$\Rightarrow \frac{20-15}{10-5}\times\frac{-p}{q}=-1$$

$\Rightarrow p=q$

Also, chord passes through $(5,15)$.
$\therefore 5p+15p=40$
$\Rightarrow p=q=2$

$\therefore p^{2019}+q^{2019}=2^{2020}$

6. $\displaystyle\int \frac{\cos x+\sqrt{3}}{1+4\sin\left(x+\frac{\pi}{3}\right)+4\sin^2\left(x+\frac{\pi}{3}\right)}dx$ is

where c is constant of integration

✗ **A.** $\dfrac{\cos x}{1+2\sin\left(x+\frac{\pi}{3}\right)}+c$

✗ **B.** $\dfrac{\sec x}{1+2\sin\left(x+\frac{\pi}{3}\right)}+c$

✓ **C.** $\dfrac{\sin x}{1+2\sin\left(x+\frac{\pi}{3}\right)}+c$

✗ **D.** $\frac{1}{2}\tan^{-1}\left(1+2\sin\left(x+\frac{\pi}{3}\right)\right)+c$

$$I=\int \frac{\cos x+\sqrt{3}}{1+4\sin\left(x+\frac{\pi}{3}\right)+4\sin^2\left(x+\frac{\pi}{3}\right)}dx$$

$$=\int \frac{\cos x+\sqrt{3}}{\left(1+2\sin\left(x+\frac{\pi}{3}\right)\right)^2}dx$$

$$=\int \frac{\cos x+\sqrt{3}}{\left(1+\sin x+\sqrt{3}\cos x\right)^2}dx$$

$$=\int \frac{\cos x+\sqrt{3}}{\sin^2 x\left(\operatorname{cosec} x+1+\sqrt{3}\cot x\right)^2}dx$$

$$=\int \frac{\cot x\operatorname{cosec} x+\sqrt{3}\operatorname{cosec}^2 x}{\left(1+\operatorname{cosec} x+\sqrt{3}\cot x\right)^2}dx$$

Put $1+\operatorname{cosec} x+\sqrt{3}\cot x=t$

$\Rightarrow -(\operatorname{cosec} x\cot x+\sqrt{3}\operatorname{cosec}^2 x)dx=dt$

$$\therefore I=\int -\frac{dt}{t^2}$$

$$=\frac{1}{t}+c$$

$$=\frac{1}{1+\operatorname{cosec} x+\sqrt{3}\cot x}+c$$

$$=\frac{\sin x}{1+2\sin\left(x+\frac{\pi}{3}\right)}+c$$

7. If $z_n = \cos\dfrac{\pi}{(2n+1)(2n+3)} + i\sin\dfrac{\pi}{(2n+1)(2n+3)}, n \in \mathbb{N}$, then the value of $\lim\limits_{k\to\infty}(z_1 \cdot z_2 \cdot z_3 \cdots z_k)$ is

✗ **A.** 1

✓ **B.** $\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$

✗ **C.** $\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

✗ **D.** $\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

$$z_n = \cos\frac{\pi}{(2n+1)(2n+3)} + i\sin\frac{\pi}{(2n+1)(2n+3)}$$

By using Euler's formula, we can write as

$z_n = e^{i\theta_n}$ where $\theta_n = \dfrac{\pi}{(2n+1)(2n+3)}$

Now,

$$(z_1 \cdot z_2 \cdot z_3 \cdots z_k) = e^{i\left(\sum\limits_{n=1}^{n=k}\theta_n\right)}$$
$$= \cos\left(\sum_{n=1}^{n=k}\frac{\pi}{(2n+1)(2n+3)}\right) + i\sin\left(\sum_{n=1}^{n=k}\frac{\pi}{(2n+1)(2n+3)}\right) \quad \cdots(i)$$

Now,

$$S_k = \sum_{n=1}^{n=k}\frac{\pi}{(2n+1)(2n+3)}$$
$$= \frac{\pi}{2}\sum_{n=1}^{n=k}\left[\frac{1}{2n+1} - \frac{1}{2n+3}\right]$$
$$= \frac{\pi}{2}\left[\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)\right]$$
$$= \frac{\pi}{2}\left[\frac{1}{3} - \frac{1}{2k+3}\right]$$
$$S_\infty = \lim_{k\to\infty}\frac{\pi}{2}\left[\frac{1}{3} - \frac{1}{2k+3}\right]$$
$$= \frac{\pi}{6}$$

From equation (i),

$$(z_1 \cdot z_2 \cdot z_3 \cdots z_k) = \cos\left(\frac{\pi}{6} - \frac{\pi}{2(2k+3)}\right) + i\sin\left(\frac{\pi}{6} - \frac{\pi}{2(2k+3)}\right)$$
$$\Rightarrow \lim_{k\to\infty}(z_1 \cdot z_2 \cdot z_3 \cdots z_k) = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

8. If the statement $(p \to q) \to (q \to r)$ is false, then truth values of statement p, q, r respectively, can be

✓ **A.** F T F

✗ **B.** T F T

✗ **C.** T F F

✗ **D.** F T T

As given that $(p \to q) \to (q \to r)$ is false, it means
$p \to q$ must be true and $q \to r$ must be false.
Now, as $(q \to r)$ is false so, q must be true and r must be false and p can be true or false.

Hence, the truth values of statement p, q, r will be F T F or T T F respectively.

Alternate Solution:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \to (q \to r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Clearly, from truth table
$(p \to q) \to (q \to r)$ is false when truth values of statement p, q, r are F T F or T T F respectively.

9. The value of
$(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 2^\circ)(\sqrt{3}+\tan 3^\circ)\cdots\cdots(\sqrt{3}+\tan 28^\circ)(\sqrt{3}+\tan 29^\circ)$ is

✗ **A.** 2^{28}

✗ **B.** $2^{15}+2-\sqrt{3}$

✓ **C.** 2^{29}

✗ **D.** $2^{14}+2-\sqrt{3}$

Given :
$(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 2^\circ)(\sqrt{3}+\tan 3^\circ)\cdots\cdots(\sqrt{3}+\tan 28^\circ)(\sqrt{3}+\tan 29^\circ)$

We know that

$$\sqrt{3}+\tan(30^\circ-x)$$
$$=\sqrt{3}+\frac{\tan 30^\circ-\tan x}{1+\tan 30^\circ\cdot\tan x}$$
$$=\sqrt{3}+\frac{1-\sqrt{3}\tan x}{\sqrt{3}+\tan x}$$
$$=\frac{4}{\sqrt{3}+\tan x}$$

Now,

$$(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 29^\circ)$$
$$=\frac{4}{\sqrt{3}+\tan 29^\circ}\times(\sqrt{3}+\tan 29^\circ)$$
$$=4$$

Similarly,

$$(\sqrt{3}+\tan 2^\circ)(\sqrt{3}+\tan 28^\circ)=4$$

So,

$$(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 2^\circ)(\sqrt{3}+\tan 3^\circ)\cdots(\sqrt{3}+\tan 28^\circ)(\sqrt{3}+\tan 29^\circ)$$
$$=4^{14}\cdot(\sqrt{3}+\tan 15^\circ)$$
$$=4^{14}\cdot\left(\sqrt{3}+\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$$
$$=4^{14}\cdot\left(\frac{2+2\sqrt{3}}{\sqrt{3}+1}\right)$$
$$=2^{29}$$

Full Syllabus Test 2

10. Let x_n, y_n, z_n, w_n denote n^{th} term of four different arithmetic progressions with positive terms. If $x_4 + y_4 + z_4 + w_4 = 8$ and $x_{10} + y_{10} + z_{10} + w_{10} = 20$, then maximum possible value of $x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20}$ is

✓ **A.** 10^4

✗ **B.** 10^8

✗ **C.** 10^{10}

✗ **D.** 10^{20}

Let $x_n = a_x + (n-1)d_x$
$y_n = a_y + (n-1)d_y$
$z_n = a_z + (n-1)d_z$
$w_n = a_w + (n-1)d_w$

Given $x_4 + y_4 + z_4 + w_4 = 8$
$\Rightarrow a_x + a_y + a_z + a_w + 3(d_x + d_y + d_z + d_w) = 8 \cdots (1)$

Also $x_{10} + y_{10} + z_{10} + w_{10} = 20$
$\Rightarrow a_x + a_y + a_z + a_w + 9(d_x + d_y + d_z + d_w) = 20 \cdots (2)$

Solving equation (1) and (2). we get
$d_x + d_y + d_z + d_w = 2$
$a_x + a_y + a_z + a_w = 2$

As the terms of the A.P. is positive, using $A.M \geq G.M$, we get

$$\frac{x_{20} + y_{20} + z_{20} + w_{20}}{4} \geq (x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20})^{1/4}$$

$$\Rightarrow \frac{a_x + a_y + a_z + a_w + 19(d_x + d_y + d_z + d_w)}{4} \geq (x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20})^{1/4}$$

$$\Rightarrow 10 \geq (x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20})^{1/4}$$

$$\Rightarrow x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20} \leq 10^4$$

Hence, the maximum possible value of the given expression is 10^4.

11. The complete solution set of the inequality $\log_5(x^2-2) < \log_5\left(\frac{3}{2}|x|-1\right)$ is

☒ **A.** $\left(-\infty,-\frac{2}{3}\right)\cup\left(\frac{2}{3},\infty\right)$

☒ **B.** $\left(-2,-\frac{2}{3}\right)\cup\left(\frac{2}{3},2\right)$

☑ **C.** $(-2,-\sqrt{2})\cup(\sqrt{2},2)$

☒ **D.** $(-2,-\sqrt{2})\cup\left(\frac{2}{3},2\right)$

$\log_5(x^2-2) < \log_5\left(\frac{3}{2}|x|-1\right)$ is defined if

$x^2-2>0$ and $\frac{3}{2}|x|-1>0$

$\Rightarrow x\in(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty) \quad \cdots(1)$

and $x\in\left(-\infty,-\frac{2}{3}\right)\cup\left(\frac{2}{3},\infty\right) \quad \cdots(2)$

Now, $\log_5(x^2-2) < \log_5\left(\frac{3}{2}|x|-1\right)$

$\Rightarrow x^2-2<\frac{3}{2}|x|-1$

$\Rightarrow x^2-\frac{3}{2}|x|-1<0$

$\Rightarrow 2x^2-3|x|-2<0$

$\Rightarrow 2|x|^2-3|x|-2<0 \quad (\because |x|^2=x^2,\ \forall\, x\in\mathbb{R})$

$\Rightarrow (2|x|+1)(|x|-2)<0$

$\Rightarrow \frac{-1}{2}<|x|<2$

But $|x|\ge 0$

So, $0\le|x|<2$

$\Rightarrow x\in(-2,2) \quad \cdots(3)$

From $(1),(2)$ and (3), we have

$x\in(-2,-\sqrt{2})\cup(\sqrt{2},2)$

Full Syllabus Test 2

12. Let $y = f(x)$ be a real-valued differentiable function on the set of all real numbers $\mathbb{R}$ such that $f(1) = 1$. If $f(x)$ satisfies $xf'(x) = x^2 + f(x) - 2$, then

☒ **A.** $f(x)$ is even function

☒ **B.** $f(x)$ is odd function

☒ **C.** minimum value of $f(x)$ is 0

✓ **D.** $y = f(x)$ represent a parabola with focus $\left(1, \frac{5}{4}\right)$

$xf'(x) = x^2 + f(x) - 2$

Let $y = f(x)$

$\Rightarrow \frac{dy}{dx} = f'(x)$

Now, $\frac{dy}{dx} + \left(\frac{-1}{x}\right) y = x - \frac{2}{x}$

which is a linear differential equation.

$I.F. = \exp\left(\int \frac{-1}{x} dx\right) = e^{-\ln x} = \frac{1}{x}$

The general solution is

$y\left(\frac{1}{x}\right) = \int \left(x - \frac{2}{x}\right) \frac{1}{x} dx$

$\therefore \frac{y}{x} = x + \frac{2}{x} + C$

As $y(1) = 1 \Rightarrow C = -2$

$\therefore \frac{y}{x} = x + \frac{2}{x} - 2$

$\Rightarrow f(x) = x^2 - 2x + 2 = (x-1)^2 + 1$

Clearly, $f(x)$ is neither even nor odd.

Minimum value of $f(x)$ is 1.

$y = (x-1)^2 + 1$

$\Rightarrow (x-1)^2 = 4 \cdot \frac{1}{4}(y-1)$

$\left(a = \frac{1}{4}, h = 1, k = 1\right)$

$\therefore y = f(x)$ represent a parabola with focus $\left(1, \frac{5}{4}\right)$.

13. A vector $(\overrightarrow{d})$ is equally inclined to three vectors $\overrightarrow{a}=\hat{i}-\hat{j}+\hat{k}$, $\overrightarrow{b}=2\hat{i}+\hat{j}$ and $\overrightarrow{c}=3\hat{j}-2\hat{k}$. Let $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ be three vectors in the plane of $\overrightarrow{a}, \overrightarrow{b}; \overrightarrow{b}, \overrightarrow{c}; \overrightarrow{c}, \overrightarrow{a}$, respectively. Then which of the following is INCORRECT?

☒ **A.** $\overrightarrow{x}.\overrightarrow{d}=0$

☒ **B.** $\overrightarrow{y}.\overrightarrow{d}=0$

☒ **C.** $\overrightarrow{z}.\overrightarrow{d}=0$

☑ **D.** none of these

$\overrightarrow{a}=\hat{i}-\hat{j}+\hat{k},$
$\overrightarrow{b}=2\hat{i}+\hat{j}$
$\overrightarrow{c}=3\hat{j}-2\hat{k}$

As,

$$[\overrightarrow{a}\ \overrightarrow{b}\ \overrightarrow{c}]=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & 0\\ 0 & 3 & -2\end{vmatrix}=0$$

Therefore, $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are coplanar vectors.

As $\overrightarrow{d}$ is equally inclined to $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$, so $\overrightarrow{d}$ should be normal to the plane containing vector $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$

$$\therefore \overrightarrow{d}.\overrightarrow{a}=\overrightarrow{d}.\overrightarrow{b}=\overrightarrow{d}.\overrightarrow{c}=0$$
$$\therefore \overrightarrow{d}.\overrightarrow{x}=\overrightarrow{d}.\overrightarrow{y}=\overrightarrow{d}.\overrightarrow{z}=0$$

14. The eccentricity of the conic $4(2y-x-3)^2-9(2x+y-1)^2=80$ is

☒ **A.** $\frac{2}{\sqrt{3}}$

☒ **B.** $\frac{\sqrt{3}}{\sqrt{2}}$

☒ **C.** $\sqrt{2}$

☑ **D.** $\frac{\sqrt{13}}{3}$

Here, $2y-x-3$ and $2x+y-1$ are perpendicular to each other.
Therefore, the equation of the conic can be written as

$$4\times 5\left(\frac{2y-x-3}{\sqrt{2^2+1^2}}\right)^2-9\times 5\left(\frac{2x+y-1}{\sqrt{2^2+1^2}}\right)^2=80$$

$$\Rightarrow 4\left(\frac{2y-x-3}{\sqrt{5}}\right)^2-9\left(\frac{2x+y-1}{\sqrt{5}}\right)^2=16$$

On putting $\frac{2y-x-3}{\sqrt{5}}=X$ and $\frac{2x+y-1}{\sqrt{5}}=Y$, the given equation can be written as

$$4X^2-9Y^2=16$$
$$\Rightarrow \frac{X^2}{4}-\frac{Y^2}{(4/3)^2}=1$$

This is a Hyperbola.
Therefore, eccentricity is given by,

$$e=\sqrt{1+\frac{16/9}{4}}$$
$$=\frac{\sqrt{13}}{3}$$

15.

The integral $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x(\ln(\cos x))(\cos x \sin x - 1)\ \mathrm{cosec}^2 x\ dx$ is equal to

✓ **A.** $\frac{e^{\pi/4}}{2}[2e^{\pi/4} + \ln 2 - 2]$

✗ **B.** $\frac{e^{\pi/4}}{2}[e^{\pi/4} + \ln 2 - 2]$

✗ **C.** $\frac{e^{\pi/4}}{2}[e^{\pi/4} - \ln 2 + 2]$

✗ **D.** $\frac{e^{\pi/4}}{2}[2e^{\pi/4} - \ln 2 + 2]$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x(\ln(\cos x))(\cos x\sin x-1)\ \text{cosec}^2 x\ dx$$

Let

$$I=\int(\ln(\cos x))\cdot e^x(\cot x-\ \text{cosec}^2 x)\ dx$$

We know that,

$$\int e^x(f(x)+f'(x))=e^x f(x)+c$$

$$\Rightarrow \int e^x(\cot x-\ \text{cosec}^2 x)\ dx\ =e^x\cot x+c$$

And using by parts, we get

$$\Rightarrow I=\ln(\cos x)e^x\cot x+\int \tan x\cdot e^x\cdot\cot x\ dx$$

$$\Rightarrow I=\ln(\cos x)e^x\cot x+\int e^x\ dx+c$$

$$\Rightarrow I=e^x[\ln(\cos x)\cdot\cot x+1]+c$$

So,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x(\ln(\cos x))(\cos x\sin x-1)\ \text{cosec}^2 x\ dx$$

$$=\left(e^x[\ln(\cos x)\cdot\cot x+1]\right)_{\pi/4}^{\pi/2}$$

$$=e^{\pi/2}\left[\lim_{x\to\pi/2}\ln(\cos x)\cdot\cot x\right]+e^{\pi/2}-e^{\pi/4}\left[1-\ln\sqrt{2}\right]$$

Now, solving the limit we get

$$\lim_{x\to\pi/2}\ln(\cos x)\cdot\cot x$$

$$=\lim_{x\to\pi/2}\frac{\ln(\cos x)}{\tan x}$$

Using L'Hospital's Rule, we get

$$=\lim_{x\to\pi/2}\frac{-\sin x}{\cos x\cdot\sec^2 x}$$

$$=\lim_{x\to\pi/2}-\sin x\cos x=0$$

Therefore,

$$e^{\pi/2}\left[\lim_{x\to\pi/2}\ln(\cos x)\cdot\cot x\right]+e^{\pi/2}-e^{\pi/4}\left[1-\ln\sqrt{2}\right]$$

$$=e^{\pi/2}-e^{\pi/4}\left[1-\ln\sqrt{2}\right]$$

$$=\frac{e^{\pi/4}}{2}\left[2e^{\pi/4}+\ln 2-2\right]$$

16. 10 is the mean of 25 observations. If $1, 3, 5, \ldots, 49$ are added to corresponding observations, then the new mean is

☒ **A.** 32

☑ **B.** 35

☒ **C.** 39

☒ **D.** 27

We know that, $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$

$$\bar{x}_{\text{new}} = \frac{(x_1+1)+(x_2+3)+(x_3+5)+\ldots+(x_{25}+49)}{25}$$

$$= \frac{x_1+x_2+x_3+\ldots+x_{25}}{25} + \frac{1+3+5+\ldots+49}{25}$$

$$= 10 + \frac{25^2}{25}$$

$$= 35$$

Note:
$1+3+5+\ldots+(2n-1) = n^2$

17. The value of $\lim\limits_{x \to \infty} x^2 \ln(x \cot^{-1} x)$ is

☑ **A.** $-\frac{1}{3}$

☒ **B.** $\frac{1}{3}$

☒ **C.** $\frac{2}{3}$

☒ **D.** $-\frac{2}{3}$

$\lim\limits_{x \to \infty} x^2 \ln(x \cot^{-1} x)$

Assuming $y = \frac{1}{x}$, we get

$\lim\limits_{x \to \infty} x^2 \ln(x \cot^{-1} x)$

$$= \lim_{y \to 0} \frac{1}{y^2} \ln\left(\frac{1}{y} \cot^{-1} \frac{1}{y}\right)$$

$$= \lim_{y \to 0} \frac{\ln\left(\frac{1}{y} \tan^{-1} y\right)}{y^2}$$

$$= \lim_{y \to 0} \frac{\ln\left(\dfrac{\tan^{-1} y}{y}\right)}{y^2}$$

$$= \lim_{y \to 0} \frac{\ln\left(\tan^{-1} y\right) - \ln y}{y^2}$$

Using L'Hospital's Rule, we get

$$= \lim_{y \to 0} \frac{\dfrac{1}{(1+y^2)(\tan^{-1} y)} - \dfrac{1}{y}}{2y}$$

$$= \lim_{y \to 0} \frac{y - (1+y^2)(\tan^{-1} y)}{2y^3(1+y^2)\dfrac{(\tan^{-1} y)}{y}}$$

$$= \lim_{y \to 0} \frac{y - (1+y^2)(\tan^{-1} y)}{2y^3}$$

Using L'Hospital's Rule, we get

$$= \lim_{y \to 0} \frac{1 - 1 - 2y \tan^{-1} y}{6y^2}$$

$$= \lim_{y \to 0} \frac{-\tan^{-1} y}{3y} = -\frac{1}{3}$$

18. A random variable X has probability distribution

X	1	2	3	4	5	6	7	8
$P(X)$	0.13	0.22	0.12	0.21	0.13	0.08	0.06	0.05

If events are $E = \{x \text{ is an odd number}\}$,$F = \{x \text{ is divisible by } 3\}$ and $G = \{x \text{ is less than } 7\}$, then the value of $P(E \cup (F \cap G))$ is

✗ **A.** 0.87

✗ **B.** 0.77

✓ **C.** 0.52

✗ **D.** 0.82

The events are
$E = \{1, 3, 5, 7\}$
$F = \{3, 6\}$
$G = \{1, 2, 3, 4, 5, 6\}$

Now,
$F \cap G = \{3, 6\}$
$E \cup F \cap G = \{1, 3, 5, 6, 7\}$
So,
$P\{E \cup \{F \cap G\}\}$
$= 1 - (0.22 + 0.21 + 0.05)$
$= 0.52$

19. If $f(x) = \sin^4 x + \cos^4 x - \frac{1}{2}\sin 2x$, then the range of $f(x)$ is

☒ **A.** $\left[0, \frac{3}{2}\right]$

☒ **B.** $\left[-\frac{1}{2}, \frac{7}{2}\right]$

☑ **C.** $\left[0, \frac{9}{8}\right]$

☒ **D.** $\left[\frac{3}{4}, \frac{7}{8}\right]$

Given : $f(x) = \sin^4 x + \cos^4 x - \frac{1}{2}\sin 2x$

$$\Rightarrow f(x) = 1 - 2\sin^2 x \cos^2 x - \frac{1}{2}\sin 2x$$

$$\Rightarrow f(x) = 1 - \frac{\sin^2 2x}{2} - \frac{\sin 2x}{2}$$

$$\Rightarrow f(x) = -\frac{\sin^2 2x + \sin 2x - 2}{2}$$

$$\Rightarrow f(x) = -\frac{\left(\sin 2x + \frac{1}{2}\right)^2 - \frac{9}{4}}{2}$$

We know that,
$-1 \le \sin 2x \le 1$

$$\Rightarrow -\frac{1}{2} \le \sin 2x + \frac{1}{2} \le \frac{3}{2}$$

$$\Rightarrow 0 \le \left(\sin 2x + \frac{1}{2}\right)^2 \le \frac{9}{4}$$

$$\Rightarrow -\frac{9}{8} \le \frac{\left(\sin 2x + \frac{1}{2}\right)^2 - \frac{9}{4}}{2} \le 0$$

$$\Rightarrow 0 \le \frac{-\left(\sin 2x + \frac{1}{2}\right)^2 + \frac{9}{4}}{2} \le \frac{9}{8}$$

$$\therefore f(x) \in \left[0, \frac{9}{8}\right]$$

Full Syllabus Test 2

20. If $x+y-2=0$, $2x-y+1=0$ and $px+qy-r=0$ are concurrent lines, then the slope of the member in the family of lines $2px+3qy+4r=0$ which is farthest from the origin is

☒ **A.** $-\frac{1}{2}$

☒ **B.** -2

☒ **C.** $-\frac{2}{3}$

☑ **D.** $-\frac{3}{10}$

Given $x+y-2=0,\ 2x-y+1=0$ and $px+qy-r=0$ are concurrent lines, so

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ p & q & -r \end{vmatrix} = 0$$

$$\Rightarrow p+5q-3r=0 \qquad \cdots(1)$$

Now, $2px+3qy+4r=0$

$$\Rightarrow 2px+3qy+\frac{4}{3}(p+5q)=0$$

$$\Rightarrow p\left(2x+\frac{4}{3}\right)+q\left(3y+\frac{20}{3}\right)=0$$

$$\Rightarrow \left(2x+\frac{4}{3}\right)+\frac{q}{p}\left(3y+\frac{20}{3}\right)=0$$

Point of intersection of the family of lines is

$$x=-\frac{2}{3}, y=-\frac{20}{9}$$

Line farthest from the origin is the line which is perpendicular to the line joining origin and $\left(-\frac{2}{3},-\frac{20}{9}\right)$

Therefore, the slope of required line

$$=-\frac{-\frac{2}{3}-0}{-\frac{20}{9}-0}$$

$$=-\frac{3}{10}$$

21. The number of solutions of the equation $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$, where $\theta \in [0, 2\pi]$ is

Accepted Answers

14 14.0 14.00

Solution:

We have $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$
$\Rightarrow 2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta = 0$
$\Rightarrow \cos 4\theta(\cos 3\theta + \cos\theta) = 0$
$\Rightarrow \cos 4\theta(2\cos 2\theta\cos\theta) = 0$
Now, either $\cos\theta = 0$
$\Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{I}$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$
or $\cos 2\theta = 0$
$\Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in \mathbb{I}$

$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
or $\cos 4\theta = 0$
$\Rightarrow \theta = (2n+1)\frac{\pi}{8}, n \in \mathbb{I}$
$\therefore \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$
So, total number of solutions is 14.

22. If the equation of the plane containing the lines $x - y - z - 4 = 0,\ x + y + 2z - 4 = 0$ and parallel to the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$ is $x + Ay + Bz + C = 0$, then the value of $|A + B + C|$ is

Accepted Answers

11 11.0 11.00

Solution:

Equation of plane containing lines $x - y - z - 4 = 0$ and $x + y + 2z - 4 = 0$ is
$P_1 + \lambda P_2 = 0$
$\Rightarrow (x - y - z - 4) + \lambda(x + y + 2z - 4) = 0$
Normal vector to the plane is
$\overrightarrow{n} = (1 + \lambda)\hat{i} + (-1 + \lambda)\hat{j} + (-1 + 2\lambda)\hat{k}$

Finding the direction ratio of the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$
Putting $x = 0$, we get
$3y + z = 1,\ \ 3y + 2z = 2$
$\Rightarrow z = 1, y = 0$
So, the point of intersection is $(0, 0, 1)$
Now, putting $z = 0$, we get
$2x + 3y = 1,\ \ x + 3y = 2$
$\Rightarrow x = -1, y = 1$
So, the point of intersection is $(-1, 1, 0)$
Therefore, the DR's of the line of intersection $= (1, -1, 1)$
It is perpendicular to the normal vector, so
$(1 + \lambda)\hat{i} + (-1 + \lambda)\hat{j} + (-1 + 2\lambda)\hat{k} \cdot (\hat{i} - \hat{j} + \hat{k}) = 0$
$\Rightarrow 1 + \lambda + 1 - \lambda - 1 + 2\lambda = 0$
$$\Rightarrow \lambda = -\frac{1}{2}$$

Therefore, the equation of plane is
$x - 3y - 4z - 4 = 0$
Comparing with $x + Ay + Bz + C = 0$, we get
$$\frac{1}{1} = \frac{-3}{A} = \frac{-4}{B} = \frac{-4}{C}$$
$\Rightarrow A = -3, B = -4, C = -4$
$\therefore |A + B + C| = 11$

23. If the term independent of x in $\left(1+x+x^{-2}+x^{-3}\right)^{10}$ is n, then the last digit of $(n+2)^3$ is

Accepted Answers

7 7.0 7.00 07

Solution:

$$\left(1+x+\frac{1}{x^2}+\frac{1}{x^3}\right)^{10}$$
$$=\frac{\left(1+x+x^3+x^4\right)^{10}}{x^{30}}$$
$$=\frac{\left((1+x)+x^3(1+x)\right)^{10}}{x^{30}}$$
$$=\frac{\left(1+x^3\right)^{10}(1+x)^{10}}{x^{30}}$$

Coefficient of x^{30} in $\left(1+x^3\right)^{10}(1+x)^{10}$
$={}^{10}C_{10}{}^{10}C_0+{}^{10}C_9{}^{10}C_3+{}^{10}C_8{}^{10}C_6+{}^{10}C_7{}^{10}C_9$
$={}^{10}C_{10}{}^{10}C_0+2({}^{10}C_9{}^{10}C_3)+{}^{10}C_8{}^{10}C_6$
$=1+2400+9450$
$\therefore n=11851$

So, $(n+2)^3=(11853)^3$
Hence, the last digit is 7.

Full Syllabus Test 2

24. The number of integral values of a such that $x^2 + ax + a + 1 = 0$ has integral roots is

Accepted Answers

2 2.0 2.00

Solution:

Given : $x^2 + ax + a + 1 = 0$
As the roots are integer, so discriminant should be perfect square,
$D = a^2 - 4(a+1)$
$\Rightarrow D = a^2 - 4a + 4 - 8$
$\Rightarrow D = (a-2)^2 - 8$
For integral a,
$\Rightarrow D = 1$
$\Rightarrow (a-2)^2 = 9$
$\Rightarrow a - 2 = \pm 3$
$\therefore a = 5, -1$

Hence, there are 2 integral values of a.

25. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then the sum of all the elements of matrix A is

Accepted Answers

3 3.0 3..00

Solution:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$$

$$\Rightarrow A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$\therefore \text{Sum } (A) = 3$

26. Let $f(x) = \begin{cases} \dfrac{b^3+b-2b^2-2}{b^2+5b+6} - x^2\,; & 0 \le x < 1 \\ 3x-4\,; & 1 \le x \le 3 \end{cases}$

where $b \in \mathbb{R}$. If $f(x)$ has minimum value at $x = 1$, then the least integral value of b is

Accepted Answers

2 2.0 2.00

Solution:

$$f'(x) = \begin{cases} -2x; & 0 \le x < 1 \\ 3; & 1 \le x \le 3 \end{cases}$$

f is not differentiable at $x = 1$.
As $f'(x) < 0$ for $x \in (0,1)$ and $f'(x) > 0$ for $x \in (1,3)$,
$\Rightarrow f(x)$ is strictly decreasing on $(0,1)$ and strictly increasing on $(1,3)$.
$\Rightarrow f(x) \ge f(1)$ for $x \in [1,3]$

For $f(x)$ to have the smallest value at $x = 1$ for $x \in [0,3]$, we must have
$\lim\limits_{x\to1^-} f(x) \ge f(1)$

$$\Rightarrow \lim_{x\to1^-} -x^2 + \frac{b^3+b-2b^2-2}{b^2+5b+6} \ge -1$$

$$\Rightarrow -1 + \frac{b^3+b-2b^2-2}{b^2+5b+6} \ge -1$$

$$\Rightarrow \frac{(b-2)(b^2+1)}{(b+2)(b+3)} \ge 0$$

$$\Rightarrow \frac{(b-2)}{(b+2)(b+3)} \ge 0 \text{ as } b^2+1 > 0$$

$\Rightarrow b \in (-3,-2) \cup [2,\infty)$
Therefore, least integral value of b is 2.

27. Let $f(x) = [3 + 2\cos x], x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where $[.]$ denotes the greatest integer function. The number of point(s) of discontinuity of $f(x)$ is

Accepted Answers

3 3.0 3.00

Solution:

$3 < 3 + 2\cos x \leq 5$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$f(x) = [3 + 2\cos x]$ is discontinuous at those points where $3 + 2\cos x$ is an integer. Hence,
$3 + 2\cos x = 4$, if $\cos x = \frac{1}{2}$.
So, x have two values $\frac{\pi}{3}$ and $-\frac{\pi}{3}$
$3 + 2\cos x = 5$, if $\cos x = 1$.
So, $x = 0$
$\therefore$ The number of values of $x = 2 + 1 = 3$

28. If $e^{xy^2} + y\cos x^2 = 5$, then absolute value of $y'(0) =$

Accepted Answers

16 16.0 16.00

Solution:

We have,
$e^{xy^2} + y\cos x^2 = 5 \cdots (1)$
Differentiating both sides w.r.t. x, we get

$$e^{xy^2}\left(y^2 + 2xy\frac{dy}{dx}\right) + \frac{dy}{dx}\cos x^2 - y\sin x^2 \times 2x = 0$$

$$\Rightarrow \frac{dy}{dx}(2xy \cdot e^{xy^2} + \cos x^2) = 2xy\sin x^2 - y^2 \cdot e^{xy^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy\sin x^2 - y^2 \cdot e^{xy^2}}{2xy \cdot e^{xy^2} + \cos x^2}$$

Putting $x = 0$ in equation (1), we get
$e^0 + y\cos(0) = 5$
$\Rightarrow y = 4$

$$\Rightarrow y'(0) = \frac{0 - 16}{1} = -16$$

$\therefore |y'(0)| = 16$

29. If $\triangle = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ and $\triangle_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then the value of $\triangle_1 + \triangle =$

Accepted Answers

0 0.0 0.00

Solution:

Given : $\triangle_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Taking transpose of above determinant, we get

$$\triangle_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$$

Now multiplying x in R_1, y in R_2 and z in R_3, we get

$$\triangle_1 = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$\Rightarrow \triangle_1 = \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$$

Now interchanging C_1 and C_2, we get

$$\triangle_1 = (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\triangle$$

$\therefore \triangle_1 + \triangle = 0$

Full Syllabus Test 2

30. If the area of the triangle ABC is Δ, such that $b^2 \sin 2C + c^2 \sin 2B = k\Delta$, then the value of k is

Accepted Answers

4 4.0 4.00

Solution:

$$
\begin{aligned}
& b^2 \sin 2C + c^2 \sin 2B \\
& = b^2 \cdot 2 \sin C \cos C + c^2 \cdot 2 \sin B \cos B \\
& = 2b \cos C \cdot b \sin C + 2c \cos B \cdot c \sin B \\
& = 2b \cos C \cdot c \sin B + 2c \cos B \cdot c \sin B \\
& \left[\because \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow b \sin C = c \sin B\right] \\
& = 2c \sin B(b \cos C + c \cos B) \\
& = 2c \sin B \cdot a \\
& = 4 \cdot \frac{1}{2} ac \sin B \\
& = 4\Delta \\
& \therefore k = 4
\end{aligned}
$$