1. Let S_1, S_2 and S_3 be three sets defined as
$S_1 = \{z \in \mathbb{C} : |z - 1| \leq \sqrt{2}\}$
$S_2 = \{z \in \mathbb{C} : \text{Re}\,((1 - i)\,z) \geq 1\}$
$S_3 = \{z \in \mathbb{C} : \text{Im}\,(z) \leq 1\}$
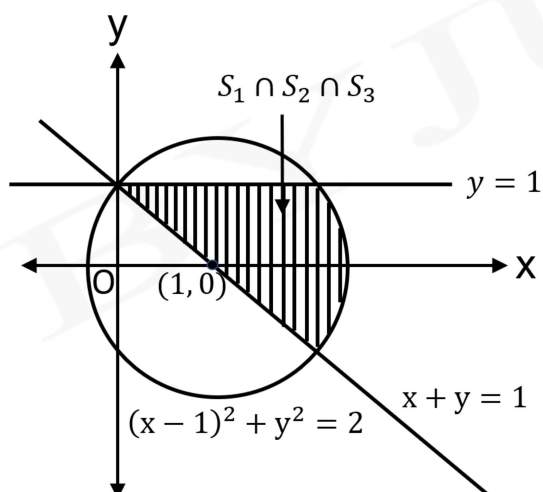Then the set $S_1 \cap S_2 \cap S_3$

✔ **A.** Has infinitely many elements

✘ **B.** Has exactly two elements

✘ **C.** Has exactly three elements

✘ **D.** Is a singleton

Let, $z = x + iy$
$S_1 \equiv (x - 1)^2 + y^2 \leq 2 \quad \cdots (1)$
$S_2 \equiv x + y \geq 1 \quad \cdots (2)$
$S_3 \equiv y \leq 1 \quad \cdots (3)$

$\therefore S_1 \cap S_2 \cap S_3$ Has infinitely many elements.

2. Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to:
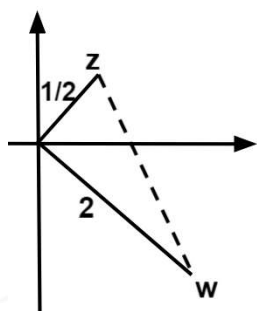
   ✔ A. $\frac{1}{2}$

   ✘ B. 4

   ✘ C. 2

   ✘ D. $\frac{1}{4}$

$w = 1 - \sqrt{3}i \Rightarrow |w| = 2$

$\Rightarrow |zw| = 1$

$\Rightarrow |z| = \frac{1}{|w|} = \frac{1}{2}$

$\arg(z) - \arg(w) = \frac{\pi}{2}$
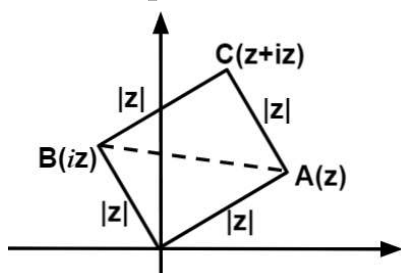

Therefore, the area is

$\Delta = \frac{1}{2} \times \frac{1}{2} \cdot 2$

$\therefore \Delta = \frac{1}{2}$

3. The area of the triangle with vertices $A(z), B(iz)$ and $C(z + iz)$ is:

   ✘ A. $\frac{1}{2}|z + iz|^2$

   ✘ B. 1

   ✘ C. $\frac{1}{2}$

   ✔ D. $\frac{1}{2}|z|^2$

Area of $\triangle ABC$

$= \frac{1}{2}(\text{area of square})$

$= \frac{1}{2}|z|^2$

4. If the equation $a|z|^2 + \overline{\overline{\alpha}z + \alpha\overline{z}} + d = 0$ represents a circle where a, d are real constants, then which of the following condition is correct?

   ✗ **A.** $|\alpha|^2 - ad \neq 0$

   ✓ **B.** $|\alpha|^2 - ad > 0$ and $a \in \mathbb{R} - \{0\}$

   ✗ **C.** $\alpha = 0, a, d \in \mathbb{R}^+$

   ✗ **D.** $|\alpha|^2 - ad \geq 0$ and $a \in \mathbb{R}$

   $a|z|^2 + \overline{\overline{\alpha}z + \alpha\overline{z}} + d = 0$

   $\Rightarrow a|z|^2 + \alpha\overline{z} + \overline{\alpha}z + d = 0$

   $\Rightarrow z\overline{z} + \left(\dfrac{\alpha}{a}\right)\overline{z} + \left(\dfrac{\overline{\alpha}}{a}\right)z + \dfrac{d}{a} = 0$

   Centre $= -\dfrac{\alpha}{a}$

   $r = \sqrt{\left|\dfrac{\alpha}{a}\right|^2 - \dfrac{d}{a}}$

   $\Rightarrow \left|\dfrac{\alpha}{a}\right|^2 \geq \dfrac{d}{a}$

   $\Rightarrow |\alpha|^2 \geq ad$

5. If the equation, $x^2 + bx + 45 = 0 \ (b \in R)$ has conjugate complex roots and they satisfy $|z + 1| = 2\sqrt{10}$, then:

   ✗ **A.** $b^2 + b = 12$

   ✗ **B.** $b^2 - b = 42$

   ✓ **C.** $b^2 - b = 30$

   ✗ **D.** $b^2 + b = 72$

   Given $x^2 + bx + 45 = 0, b \in R$
   Let roots of the equation be $p \pm iq$
   Then, sum of roots $= 2p = -b$
   Product of roots $= p^2 + q^2 = 45$

   As $p \pm iq$ lie on $|z + 1| = 2\sqrt{10}$, we get
   $(p + 1)^2 + q^2 = 40$
   $\Rightarrow p^2 + q^2 + 2p + 1 = 40$
   $\Rightarrow 45 - b + 1 = 40$
   $\Rightarrow b = 6$
   $\Rightarrow b^2 - b = 30.$

6. If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \dfrac{3\pi}{2}$, then $\arg\left(\dfrac{1 - 2\bar{z}\omega}{1 + 3\bar{z}\omega}\right)$ is

(Here $\arg(z)$ denotes the principal argument of complex number z)

✗ **A.** $\dfrac{3\pi}{4}$

✔ **B.** $-\dfrac{3\pi}{4}$

✗ **C.** $\dfrac{\pi}{4}$

✗ **D.** $-\dfrac{\pi}{4}$

$z = re^{i\theta} \qquad \therefore \omega = \dfrac{1}{r}e^{i(\theta - 3\pi/2)}$

$\dfrac{1 - 2\bar{z}\omega}{1 + 3\bar{z}\omega} = \dfrac{1 - 2e^{-i\theta} \cdot e^{i(-3\pi/2 + \theta)}}{1 + 3e^{-i\theta} \cdot e^{i(-3\pi/2 + \theta)}}$

$= \dfrac{1 - 2i}{1 + 3i}$

$\therefore \arg\left(\dfrac{1 - 2i}{1 + 3i}\right) = \arg\left(-\dfrac{5}{10} - \dfrac{5}{10}i\right) = -\dfrac{3\pi}{4}$

7. The region represented by $z = x + iy \in \mathbb{C} : |z| - Re(z) \le 1$ is also given by the inequality:

✔ **A.** $y^2 \le 2\left(x + \dfrac{1}{2}\right)$

✗ **B.** $y^2 \le x + \dfrac{1}{2}$

✗ **C.** $y^2 \le 2(x + 1)$

✗ **D.** $y^2 \le x + 1$

$z = x + iy \in \mathbb{C} : |z| - Re(z) \le 1$

$|z| = \sqrt{x^2 + y^2}$

$Re(z) = x$

$|z| - Re(z) \le 1$

$\Rightarrow \sqrt{x^2 + y^2} - x \le 1$

$\Rightarrow \sqrt{x^2 + y^2} \le 1 + x$

$\Rightarrow x^2 + y^2 \le 1 + x^2 + 2x$

$\Rightarrow y^2 \le 2\left(x + \dfrac{1}{2}\right)$

8. Which of the following boolean expresion is a tautology?

  ❌ **A.** $(p \wedge q) \wedge (p \to q)$

  ❌ **B.** $(p \wedge q) \vee (p \vee q)$

  ❌ **C.** $(p \wedge q) \vee (p \to q)$

  ✔ **D.** $(p \wedge q) \to (p \to q)$

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	T	T	T	T
F	T	F	T	T	T
T	F	F	T	F	T
F	F	F	F	T	T

9. The statement among the following that is a tautology is:

  ❌ **A.** $A \wedge (A \vee B)$

  ❌ **B.** $B \to [A \wedge (A \to B)]$

  ❌ **C.** $A \vee (A \wedge B)$

  ✔ **D.** $[A \vee (A \to B)] \to B$

$A \wedge (\sim A \vee B) \to B$
$= [(A \wedge \sim A) \vee (A \wedge B)] \to B$
$= (A \wedge B) \to B$
$= \sim (A \wedge B) \vee B$
$= t$

10. Let $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$
and $F_2(A, B) = (A \vee B) \vee (B \to \sim A)$ be two logical expression's. Then:

  ✔ **A.** F_1 is not a tautology but F_2 is a tautology

  ❌ **B.** F_1 is a tautology but F_2 is not a tautology

  ❌ **C.** F_1 and F_2 both are tautologies

  ❌ **D.** Both F_1 and F_2 are not tautologies

$F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$
Using the set theory
$(A \cap B') \cup (C' \cap (A \cup B)) \cup A'$
$= (A - A \cap B) \cup (S - A) \cup [(S - C) \cap (A \cup B)]$
$= (S - A \cap B) \cup [A \cup B - C \cap (A \cup)]$
$= S - A \cap B \cap C.$
Hence not a tautology.

Now,
$F_2(A, B) = (A \vee B) \vee (B \to \sim A) = (A \vee B) \vee (\sim B \vee A)$
Using set theory $(A \cup B) \cup (B' \cup A) = S$
Hence it is a tautology.

11. The statement $A \to (B \to A)$ is equivalent to:

   ✖ **A.** $A \to (A \wedge B)$

   ✔ **B.** $A \to (A \vee B)$

   ✖ **C.** $A \to (A \to B)$

   ✖ **D.** $A \to (A \leftrightarrow B)$

A	B	$A \wedge B$	$A \vee B$	$A \to B$	$B \to A$	$A \to (B \to A)$	$A \to (A \wedge B)$	$A \to (A \vee B)$	$A \to (A \to B)$
T	T	T	T	T	T	T	T	T	T
T	F	F	T	F	T	T	F	T	F
F	T	F	T	T	F	T	T	T	T
F	F	F	F	T	T	T	T	T	T

From the above table clearly $A \to (B \to A) \equiv A \to (A \vee B)$

Alternate Solution :
$A \to (B \to A)$
$\Rightarrow A \to (\sim B \vee A)$
$\Rightarrow \sim A \vee (\sim B \vee A)$
$\Rightarrow \sim B \vee (\sim A \vee A)$
$\Rightarrow \sim B \vee (\sim A \vee A)$
$\Rightarrow \sim B \vee t$
$= t$ (tautology)
From options : $A \to (A \vee B)$
$\Rightarrow \sim A \vee (A \vee B)$
$\Rightarrow (\sim A \vee A) \vee B$
$\Rightarrow t \vee B$
$\Rightarrow t$

12. The negation of the statement $\sim p \wedge (p \vee q)$ is

   ✖ **A.** $\sim p \wedge q$

   ✖ **B.** $p \wedge \sim q$

   ✖ **C.** $\sim p \vee q$

   ✔ **D.** $p \vee \sim q$

Negation of $\sim p \wedge (p \vee q)$ is
$\sim [\sim p \wedge (p \vee q)]$
$\equiv p \vee \sim (p \vee q)$
$\equiv p \vee (\sim p \wedge \sim q)$
$\equiv (p \vee \sim p) \wedge (p \vee \sim q)$
$\equiv T \wedge (p \vee \sim q)$, where T is tautology.
$\equiv p \vee \sim q$

13. The Boolean expression $(p \wedge \sim q) \Rightarrow (q \vee \sim p)$ is equivalent to

  ✖ **A.** $q \Rightarrow p$

  ✔ **B.** $p \Rightarrow q$

  ✖ **C.** $p \Rightarrow \sim q$

  ✖ **D.** $\sim q \Rightarrow p$

$\because p \Rightarrow q$ is $\sim p \vee q$
$\therefore (p \wedge \sim q) \Rightarrow (q \vee \sim p)$
$= \sim (p \wedge \sim q) \vee (q \vee \sim p)$
$= (\sim p \vee q) \vee (\sim p \vee q)$
$= \sim p \vee q$
$= p \Rightarrow q$

14. If the truth value of the Boolean expression $((p \vee q) \wedge (q \to r) \wedge (\sim r)) \to (p \wedge q)$ is false, then the truth values of the statements p, q, r respectively can be

  ✖ **A.** $F\ F\ T$

  ✖ **B.** $F\ T\ F$

  ✔ **C.** $T\ F\ F$

  ✖ **D.** $T\ F\ T$

$\therefore ((p \vee q) \wedge (q \to r) \wedge (\sim r)) \to (p \wedge q)$ is false.

$\Rightarrow (p \vee q) \wedge (q \to r) \wedge (\sim r)$ is true and $(p \wedge q)$ is false.

$\Rightarrow (p \vee q)$ is true, $(q \to r)$ is true, $(\sim r)$ is true and $(p \wedge q)$ is false.

$\Rightarrow r$ is false and exactly one out of p and q is false.

$\because q \to r$ is true and r is false, so q is also false and hence p must be true.

15. Consider the statement "The match will be played only if the weather is good and ground is not wet". Select the correct negation from the following

  ✖ **A.** The match will not be played or weather is good and ground is not wet

  ✔ **B.** The match will be played and weather is not good or ground is wet

  ✖ **C.** The match will not be played and weather is not good and ground is wet

  ✖ **D.** If the match will not be played, then either weather is not good or ground is wet

Consider the statements,
p : match will be played
q : weather is good
r : ground is not wet
$\therefore p \to (q \wedge r)$
Now, negation of the given statement is
$\sim [p \to (q \wedge r)]$
$= p \wedge \sim (q \wedge r)$
$= p \wedge (\sim q \vee \sim r)$
The match will be played and weather is not good or ground is wet.

16. Consider the following three statements
    (A) If $3 + 3 = 7$, then $4 + 3 = 8$.
    (B) If $5 + 3 = 8$, then earth is flat.
    (C) If both (A) and (B) are true, then $5 + 6 = 17$.
    Then, which of the following statements is correct:

    ✗ A. (A) and (B) are false while (C) is true

    ✗ B. (A) is false, but (B) and (C) are true

    ✔ C. (A) and (C) are true while (B) is false

    ✗ D. (A) is true while (B) and (C) are false

    $\because 3 + 3 = 7$ is false and $4 + 3 = 8$ is false, then statement (A) is true.
    For (B) $5 + 3 = 8$ is true and earth is flat is false.
    Then statement (B) is false.
    For (C) if A and B are true then $5 + 6 = 17$ is false, then (C) is true.
    $\therefore$ (A) and (C) are true and (B) is false.

17. The mean and standard deviation of 20 observations were calculated as 10 and 2.5 respectively. It was found that by mistake one data value was taken as 25 instead of 35. If α and $\sqrt{\beta}$ are the mean and standard deviation respectively for correct data, then (α, β) is

    ✗ A. $(11, 25)$

    ✗ B. $(11, 26)$

    ✗ C. $(10.5, 25)$

    ✔ D. $(10.5, 26)$

    $\overline{x} = 10 \sqrt{\dfrac{\sum x_i^2}{20} - (\overline{x})^2} = 2.5$

    $\dfrac{\sum x_i^2}{20} - (10)^2 = 6.25$

    $\Rightarrow \sum x_i^2 = 20 \times 106.25 = 2125$

    $\sum x_{i\,(\text{actual})}^2 = 2125 - 25^2 + 35^2$
    $= 2125 + 600 = 2725$

    For $\overline{x}_{(\text{actual})} \Rightarrow \dfrac{\sum x}{n} = 10 \Rightarrow \sum x = 200$

    $\sum x_{(\text{actual})} = 200 - 25 + 35 = 210$

    $\overline{x}_{(\text{actual})} = \dfrac{210}{20} = 10.5$

    $S.D = \sqrt{\dfrac{2725}{20} - (10.5)^2} = \sqrt{136.25 - 110.25} = \sqrt{26}$

18. Let the mean and variance of the frequency distribution

$x : \quad x_1 = 2 \quad x_2 = 6 \quad x_3 = 8 \quad x_4 = 9$

$f : \qquad 4 \qquad\; 4 \qquad\; \alpha \qquad\; \beta$

be 6 and 6.8 respectively. If x_3 is changed from 8 to 7, then the mean for the new data will be

- ✖ **A.** 5
- ✖ **B.** 4
- ✔ **C.** $\dfrac{17}{3}$
- ✖ **D.** $\dfrac{16}{3}$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{8 + 24 + 8\alpha + 9\beta}{8 + \alpha + \beta} = 6$$

$$\Rightarrow 2\alpha + 3\beta = 16 \quad \ldots (1)$$

$$\sigma^2 = \frac{\sum x_i^2 f_i}{\sum f_i} - (\bar{x})^2$$

$$\Rightarrow \frac{16 + 144 + 64\alpha + 81\beta}{8 + \alpha + \beta} - 36 = 6.8$$

$$\Rightarrow 106\alpha + 191\beta = 912 \quad \ldots (2)$$

From (1) and (2), $\alpha = 5$ and $\beta = 2$

Now, new mean when $x_3 = 7$ is

$$\frac{8 + 24 + 35 + 18}{15} = \frac{17}{3}$$

19. The first of the two samples in a group has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{13.44}$, then the standard deviation of the second sample is

    ✖   **A.** 5

    ✖   **B.** 6

    ✔   **C.** 4

    ✖   **D.** 8

$n_1 = 100, \ n_2 = 150$
We know that
$(n_1 + n_2)\overline{X} = n_1\overline{X}_1 + n_2\overline{X}_2$
where
$\overline{X} = $ mean of whole group
$\overline{X}_1 = $ mean of first sample of a group
$\overline{X}_2 = $ mean of second sample of a group
$\Rightarrow 250 \times 15.6 = 100 \times 15 + 150 \times \overline{X}_2$
$\Rightarrow \overline{X}_2 = \dfrac{250 \times 15.6 - 100 \times 15}{150}$
$\therefore \overline{X}_2 = 16$

Standard deviation of whole group, $\sigma = \sqrt{13.44}$
$\Rightarrow$ Standard variance of whole group $\sigma^2 = 13.44$
$\Rightarrow 13.44 = \dfrac{\sum x^2}{n} - (\overline{X})^2$
$\Rightarrow 13.44 = \dfrac{\sum x^2}{250} - (15.6)^2$
$\Rightarrow \sum x^2 = 64200$

For first sample of a group :
Standard deviation of whole group, $\sigma = 3$
$\Rightarrow$ Standard variance of whole group $\sigma^2 = 9$
$\Rightarrow 9 = \dfrac{\sum x_1^2}{n_1} - (\overline{X}_1)^2$
$\Rightarrow 9 = \dfrac{\sum x_1^2}{100} - (15)^2$
$\Rightarrow \sum x_1^2 = 23400$
$\therefore \sum x_2^2 = \sum x^2 - \sum x_1^2$
$\Rightarrow \sum x_2^2 = 64200 - 23400$
$\Rightarrow \sum x_2^2 = 40800$

For second sample of a group :
$\sigma^2 = \dfrac{\sum x_2^2}{n_2} - (\overline{X}_2)^2$
$\Rightarrow \sigma^2 = \dfrac{40800}{150} - (16)^2$
$\Rightarrow \sigma = 4$

20. If the mean and variance of the following data :

$6, 10, 7, 13, a, 12, b, 12$ are 9 and $\dfrac{37}{4}$ respectively, then $(a - b)^2$ is equal to

- ✖ **A.** 32
- ✖ **B.** 12
- ✖ **C.** 24
- ✔ **D.** 16

Given: Mean $= 9$

$\Rightarrow \dfrac{6 + 10 + 7 + 13 + 12 + 12 + (a + b)}{8} = 9$

$\Rightarrow 60 + (a + b) = 72$

$\Rightarrow a + b = 12 \quad \cdots (1)$

and variance $= \dfrac{37}{4}$

$\Rightarrow \dfrac{a^2 + b^2 + 36 + 100 + 49 + 169 + 144 + 144}{8} - 81 = \dfrac{37}{4}$

$\Rightarrow a^2 + b^2 + 642 - 648 = 74$

$\Rightarrow a^2 + b^2 = 80 \quad \cdots (2)$

We know that

$(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow 144 = 80 + 2ab$

$\Rightarrow 2ab = 64$

Now, $(a - b)^2 = a^2 + b^2 - 2ab$

$\Rightarrow (a - b)^2 = 80 - 64$

$\therefore (a - b)^2 = 16$

21. The mean and standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and standard deviation become half of their original values, then q is equal to:

- ✔ **A.** -20
- ✖ **B.** -5
- ✖ **C.** 10
- ✖ **D.** -10

If each observation is multiplied by p and then q is subtracted,

then new mean $\overline{x}' = p\overline{x} - q$

$\therefore \overline{x}' = \dfrac{1}{2}\overline{x}$ and $\overline{x} = 10$

$\Rightarrow 10 = 20p - q \quad \ldots (1)$

If each observation is multiplied by p and then q is subtracted,
then new standard deviation (σ') is $|p|$ times of the initial standard deviation (σ).

$\sigma' = |p|\sigma$

$\Rightarrow \dfrac{1}{2}\sigma = |p|\sigma \Rightarrow |p| = \dfrac{1}{2}$

If $p = \dfrac{1}{2}, q = 0$

If $p = -\dfrac{1}{2}, q = -20.$

22. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is :

(×) **A.** 4.01

(✔) **B.** 3.99

(×) **C.** 3.98

(×) **D.** 4.02

Mean, $\overline{x} = 10$

$\Rightarrow \dfrac{\sum x_i}{20} = 10$

$\Rightarrow \sum x_i = 200$

Variance, $\sigma^2 = 4$

$\Rightarrow \dfrac{\sum x_i^2}{20} - (10)^2 = 4$

$\Rightarrow \sum x_i^2 = 2080$

$\text{Correct mean} = \dfrac{200 - 9 + 11}{20}$

$= \dfrac{202}{20} = 10.1$

$\therefore \text{Correct variance} = \dfrac{2080 - 9^2 + 11^2}{20} - (10.1)^2$

$= 106 - 102.01 = 3.99$

1. If $\left(\dfrac{1+i}{1-i}\right)^{m/2} = \left(\dfrac{1+i}{i-1}\right)^{n/3} = 1, \ (m, n \in \mathbb{N})$, then the greatest common divisor of the least values of m and n is

Accepted Answers

4   4.0   4.00   04

Solution:

$$\left[\dfrac{(1+i)(1+i)}{(1+i)(1-i)}\right]^{m/2} = \left[\left(\dfrac{1+i}{-1+i}\right)\left(\dfrac{-1-i}{-1-i}\right)\right]^{n/3} = 1$$

$$\Rightarrow \left(\dfrac{(1+i)^2}{2}\right)^{m/2} = \left(\dfrac{(1+i)^2}{-2}\right)^{\frac{n}{3}} = 1$$

$\Rightarrow (i)^{m/2} = (-i)^{n/3} = 1$

$\Rightarrow \dfrac{m}{2} = 4k_1$ and $\dfrac{n}{3} = 4k_2$

$\Rightarrow m = 8k_1$ and $n = 12k_2$

Least value of $m = 8$ and $n = 12$

Greatest common divisor of m and n is 4.

2. If the least and the largest real values of α, for which the equation $z + \alpha|z - 1| + 2i = 0 (z \in \mathbb{C}$ and $i = \sqrt{-1})$ has a solution, are p and q respectively; then $4(p^2 + q^2)$ is equal to

Accepted Answers

10   10.0   10.00

Solution:

$x + iy + \alpha\sqrt{(x-1)^2 + y^2} + 2i = 0$
$\Rightarrow y + 2 = 0$ and $x + \alpha\sqrt{(x-1)^2 + y^2} = 0$
$y = -2$ & $x^2 = \alpha^2(x^2 - 2x + 1 + 4)$
$\Rightarrow x^2(\alpha^2 - 1) - 2x\alpha^2 + 5\alpha^2 = 0$
$\because x \in \mathbb{R} \Rightarrow D \geq 0$
$\Rightarrow 4\alpha^4 - 4(\alpha^2 - 1)5\alpha^2 \geq 0$
$\Rightarrow \alpha^2[4\alpha^2 - 20\alpha^2 + 20] \geq 0$
$\Rightarrow \alpha^2[-16\alpha^2 + 20] \geq 0$
$\Rightarrow \alpha^2 \left[\alpha^2 - \dfrac{5}{4}\right] \geq 0$
$\Rightarrow \alpha^2 \in \left[0, \dfrac{5}{4}\right]$
$\Rightarrow \alpha \in \left[-\dfrac{\sqrt{5}}{2}, \dfrac{\sqrt{5}}{2}\right]$

then $4[p^2 + q^2] = 4\left[\dfrac{5}{4} + \dfrac{5}{4}\right] = 10$

3. Let z and w be two compex numbers such that $w = z\bar{z} - 2z + 2$, $\left|\dfrac{z+i}{z-3i}\right| = 1$ and $Re(w)$ has minimum value. Then the minumum value of $n \in \mathbb{N}$ for which w^n is real, is equal to

Accepted Answers

4 4.0 4.00

Solution:

Let $z = x + iy$
$|z+i| = |z-3i|$
$\Rightarrow y = 1$
Now
$w = x^2 + y^2 - 2x - 2iy + 2$
$\Rightarrow w = x^2 + 1 - 2x - 2i + 2$
$Re(w) = x^2 - 2x + 3$
$= (x-1)^2 + 2$
$\therefore Re(w)_{\min}$ at $x = 1 \Rightarrow z = 1 + i$
Now $w = 1 + 1 - 2 - 2i + 2$
$w = 2(1-i) = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$
$w^n = 2\sqrt{2}e^{i\left(\frac{-n\pi}{4}\right)}$
If w^n is real
$\Rightarrow n = 4$

4. The sum of 162th power of the roots of the equation $x^3 - 2x^2 + 2x - 1 = 0$ is

Accepted Answers

3 3.0 3.00

Solution:

Let roots of $x^3 - 2x^2 + 2x - 1 = 0$ be α, β, γ.
$(x^3 - 1) - (2x^2 - 2x) = 0$
$\Rightarrow (x-1)(x^2 - x + 1) = 0$
$\Rightarrow x = 1, -\omega, -\omega^2$

Now, $\alpha^{162} + \beta^{162} + \gamma^{162}$
$= 1 + (-\omega)^{162} + (-\omega^2)^{162}$
$= 1 + (\omega^3)^{54} + (\omega^3)^{108} = 3$

5. If the variance of 10 natural numbers $1, 1, 1, \ldots, 1, k$ is less than 10, then the maximum possible value of k is

Accepted Answers

11   11.0   11.00

Solution:

$$\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n} \right)^2$$

$$\Rightarrow \sigma^2 = \frac{9 + k^2}{10} - \left(\frac{9 + k}{10} \right)^2 < 10$$

$$\Rightarrow 10(9 + k^2) - (81 + k^2 + 18k) < 1000$$

$$\Rightarrow 90 + 10k^2 - k^2 - 18k - 81 < 1000$$

$$\Rightarrow 9k^2 - 18k + 9 < 1000$$

$$\Rightarrow (k - 1)^2 < \frac{1000}{9}$$

$$\Rightarrow k - 1 < \frac{10\sqrt{10}}{3}$$

$$\Rightarrow k < \frac{10\sqrt{10}}{3} + 1$$

$\therefore$ Maximum possible integral value of k is 11.

6. Consider the statistics of two sets of observations as follows :

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\dfrac{17}{9}$, then the value of n is equal to

Accepted Answers

5   5.0   5.00

Solution:

For Observation I :
$$\frac{\sum x_i}{10} = 2 \Rightarrow \sum x_i = 20$$
$$\frac{\sum x_i^2}{10} - (2)^2 = 2 \Rightarrow \sum x_i^2 = 60$$

For Observation II :
$$\frac{\sum y_i^2}{n} - 3^2 = 1 \Rightarrow \sum y_i^2 = 10n$$

Now, combined variance
$$\sigma^2 = \frac{\sum \left(x_i^2 + y_i^2\right)}{10+n} - \left(\frac{\sum \left(x_i + y_i\right)}{10+n}\right)^2$$
$$\Rightarrow \frac{17}{9} = \frac{60+10n}{10+n} - \frac{(20+3n)^2}{(10+n)^2}$$
$$\Rightarrow 17\left(n^2 + 20n + 100\right) = 9\left(n^2 + 40n + 200\right)$$
$$\Rightarrow 8n^2 - 20n - 100 = 0$$
$$\Rightarrow 2n^2 - 5n - 25 = 0$$
$$\Rightarrow n = 5$$

7. Let the mean and variance of four numbers $3, 7, x$ and $y(x > y)$ be 5 and 10 respectively. Then the mean of four numbers $3 + 2x, 7 + 2y, x + y$ and $x - y$ is

Accepted Answers

12   12.0   12.00

Solution:

Numbers $3, 7, x, y$
$\overline{x} = 5, \ \sigma^2 = 10$
$5 = \dfrac{3 + 7 + x + y}{4}$
$\Rightarrow x + y = 10 \cdots (i)$
$10 = \dfrac{1}{4}((3)^2 + (7)^2 + (x)^2 + (y)^2) - (5)^2$
$\Rightarrow 140 = 58 + x^2 + y^2$
$\Rightarrow x^2 + y^2 = 82 \cdots (ii)$

$(x + y)^2 = x^2 + y^2 + 2xy$
$\Rightarrow 100 = 82 + 2xy$
$\Rightarrow xy = 9$
$y = \dfrac{9}{x} \Rightarrow x + \dfrac{9}{x} = 10$
$\Rightarrow (x, y) = (1, 9)$ or $(9, 1)$

Given $x > y \Rightarrow x = 9, \ y = 1$
Now, $3 + 2x, \ 7 + 2y, \ x + y, \ x - y = 21, 9, 10, 8$

$\overline{x} = \dfrac{21 + 9 + 10 + 8}{4} = \dfrac{48}{4} = 12$

8. let n be an odd natural number such that the variance of $1, 2, 3, 4, \ldots, n$ is 14. Then n is equal to

Accepted Answers

13   13.0   13.00

Solution:

Given: variance $= 14$

$$\Rightarrow \frac{\displaystyle\sum_{i=1}^{n} x_i^2}{n} - (\overline{x})^2 = 14$$

$$\Rightarrow \frac{1^2 + 2^2 + \ldots + n^2}{n} - \left(\frac{1 + 2 + 3 + \ldots + n}{n}\right)^2 = 14$$

$$\Rightarrow \frac{\dfrac{n(n+1)(2n+1)}{6}}{n} - \left(\frac{\left(\dfrac{n(n+1)}{2}\right)}{n}\right)^2 = 14$$

$$\Rightarrow \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = 14$$

$$\Rightarrow \frac{(n+1)}{2}\left(\frac{(2n+1)}{3} - \frac{(n+1)}{2}\right) = 14$$

$$\Rightarrow (n+1)(4n+2-3n-3) = 168$$
$$\Rightarrow n^2 - 1 = 168$$
$$\therefore n = 13$$