INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)dx = F(x)+c \Leftrightarrow \frac{d}{dx}\{F(x)+c\} = f(x)$, where c is called the **constant of integration.**

1. Standard Results :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c ; n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\left| ax+b \right| + c$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell na} + c, (a>0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n|\sec(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \text{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b) + c$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c$

(xiii) $\int \sec x\,dx = \ell n|\sec x+\tan x|+c$

OR

$\int \sec x dx = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+c$

(xiv) $\int \operatorname{cosec} x\,dx = \ell n|\operatorname{cosec} x - \cot x|+c$

OR

$\int \operatorname{cosec} x dx = \ell n\left|\tan\frac{x}{2}\right|+c$

OR

$-\ell n\,|\operatorname{cosec} x + \cot x| + c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

(xvii) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right] + c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+c$

(xxii) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxiii) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left|x+\sqrt{x^2+a^2}\right|+c$

(xxiv) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left|x+\sqrt{x^2-a^2}\right|+c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

2. Techniques of Integration :

(a) Substitution or change of independent variable :

Integrals I = $\int f(x)\ dx$ is changed to $\int f(\phi(t))\phi'(t)\,dt$, by a suitable substitution x = $\phi(t)$

(1) $\int [f(x)^n f'(x)]\,dx$ OR $\int \frac{f'(x)}{[f(x)^n]}dx$ Substitute f(x) = t & proceed.

(2) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(3) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express px + q = A (differential coefficient of demominator) + B.

(4) $\int e^x[f(x)+f'(x)dx] = e^x.f(x)+c$

(5) $\int [f(x)+xf'(x)dx] = xf(x)+c$

(6) $\int \frac{dx}{x(x^n+1)}$, $n \in N$ take x^n common & put $1 + x^{-n} = t$.

(7) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$ take x^n common & put $1 + x^{-n} = t$

(8) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common & put $1 + x^{-n} = t$.

(9) $\int \frac{dx}{a+b\sin^2 x}$

OR

$\int \frac{dx}{a+b\cos^2 x}$

OR

$\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put tan x = t.

(10) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangent of half the angles, put $\tan\frac{x}{2} = t$

(11) $\int \frac{a.\cos x+b.\sin x+c}{p.\cos x+q.\sin x+r}dx.$

Express Numerator $\equiv$ l(denominator)+ m $\frac{d}{dx}$((denominator)) + n & proceed to find ℓ,m,n

(12) $\int \frac{x^2+1}{x^4+Kx^2+1}dx.$ OR $\int \frac{x^2-1}{x^4+Kx^2+1}dx.$
where K is any constant.

Divide Nr & Dr by x^2, then put $x-\frac{1}{x} = t$ OR $x+\frac{1}{x} = t$ respectively & proceed

(13) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$. put $px + q = t^2$.

(14) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ put $ax+b=\frac{1}{t}$.

$\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$ put $x=\frac{1}{t}$.

(15) $\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ OR $\int \sqrt{(x-\alpha)(\beta-x)}$: Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ OR $\int \sqrt{(x-\alpha)(x-\beta)}$ Put $x = \alpha\sec^2\theta + \beta\tan^2\theta$.

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$: Put $x-\alpha = t^2$ or $x-\beta = t^2$.

(16) To integrate $\int \sin^m x\cos^n x\,dx$
(i) If m is odd positive integer put cos x = t
(ii) If n is odd positive integer put sin x = t
(iii) If m + n is negative even integer then put tan x = t.

(iv) If m and n both are even positive integer then use $\sin^2 x=\frac{1-\cos 2x}{2},\cos^2 x=\frac{1+\cos 2x}{2}$.

(b) Integration by part : $\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$

where u & v are differentiable functions

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int \left[\frac{du}{dx}.\int v\,dx\right] dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

(c) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further

Note :

In competitive exams, partial fraction are generally found by inspection

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{(\alpha-\beta)}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)$$

It can be applied to the case when x^2 or any other function is there in place of x.

Example :

(1) $\frac{1}{(x^2+1)(x^2+3)} = \frac{1}{2}\left(\frac{1}{t+1}-\frac{1}{t+3}\right)$ [take $x^2 = t$]

(2) $\frac{1}{x^4(x^2+1)} = \frac{1}{x^2}\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right) = \frac{1}{x^4}-\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)$

(3) $\frac{1}{x^3(x^2+1)} = \frac{1}{x}\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right) = \frac{1}{x^3}-\frac{1}{x(x^2+1)}$

DEFINITE INTEGRATION

1. (a) The Fundamental Theorem of Calculus, Part 1:

If f is continuous on [a, b], then the function g defined by $g(x)=\int_a^x f(t)dt \quad a \le x \le b$

is continuous on [a, b] and differentiable on (a, b), and g'(x) = f(x)

(b) The Fundamental Theorem of Calculus, Part 2:

If f is continuous on [a, b], then $\int_a^b f(x)dx = F(b)-F(a)$

where F is any antiderivative of f, that is, a function such that F '= f.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation f(x) = 0 has atleast one root lying in (a, b) provided f is a continuous function in (a,b)

2. A definite integration is denoted by $\int_a^b f(x)dx$ which represents the area bounded by the curve y =f(x), the ordinates x = a, x= b and the x-axis. For example $\int_0^{2\pi} \sin x dx = 0$

3. Properties of Definite Integral :

(a) $\int_a^b f(x)dx = \int_a^b f(t)\,dt \Rightarrow \int_0^b f(x)dx$ does not depend upon x. It is a numerical quantity.

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside or outside the interval [a,b]. This property is to be used when f is piecewise continuous in (a, b).

(d) $\int_{-a}^{a} f(x)dx = \int_{0}^{a}[f(x)+f(-x)]dx = \begin{bmatrix} 0 & ;\text{if } f(x) \text{ is an odd function} \\ 2\int_{0}^{a} f(x)dx & ;\text{if } f(x) \text{ is an even function} \end{bmatrix}$

(e) $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$, In particular $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

(f) $\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(2a-x)dx = \begin{bmatrix} 2\int_{0}^{a} f(x)dx \; ; \text{ if } f(2a-x)=f(x) \\ 0 \; ; \text{ if } f(2a-x)=-f(x) \end{bmatrix}$

(g) $\int_{0}^{nT} f(x)dx = n\int_{0}^{T} f(x)dx$, $(n \in I)$; where 'T' is the period of the function i.e. f(T + x) = f(x)

Note that : $\int_{x}^{T+x} f(t)dt$ will be independent of x and equal to $\int_{0}^{T} f(t)dt$

(h) $\int_{a+nT}^{b+nT} f(x)dx = \int_{a}^{b} f(x)dx$ where f(x) is periodic with period T & n $\in$ I.

(i), $\int_{ma}^{na} f(x)dx = (n-m)\int_{0}^{a} f(x)dx$ (n, m $\in$ I) if f(x) is periodic with period 'a'.

4. Walli's Formula :

(a) $\int_{0}^{\pi/2} \sin^n x\,dx = \int_{0}^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3).....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)}K$

where $K = \begin{cases} \pi/2 & \text{if n is even} \\ 1 & \text{if n is odd} \end{cases}$

(b) $\int_{0}^{\pi/2} \sin^n x.\cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2}K$

Where $K = \begin{cases} \frac{\pi}{2} & \text{if both m and n are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$

5. Derivative of Antiderivative Function (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x then, $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$

6. Definite Integral as Limit of a Sum :

$\int_a^b f(x)dx = \lim_{n\to\infty} h$ [f(a) + f(a+h) + f(a+2h) + ... f(a + $(n-1)$ h)]

$\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh) = \int_0^1 f(x)\,dx$ where b – a = nh

if a = 0, b = 1 then, $\lim_{n\to\infty}\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where nh = 1

OR $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)$ dx.

7. Estimation of Definite Integral :

(a) If f(x) is continuous in [a, b] and it's range is [m, M], then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

(b) If f(x) $\le \phi$ (x) for a $\le$ x $\le$ b then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

(d) If $f(x) \ge 0$ on the interval [a,b], then $\int_a^b f(x)dx \ge 0$.

(e) f(x) and g(x) are two continuous function on [a,b] then

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}$$

8. Some Standard Results :

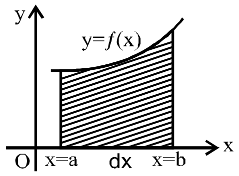
(a) $\int_0^{\pi/2} \log\sin x dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\cos x dx$

(b) $\int_a^b |x|\,dx = \frac{b-a}{2}$ **a, b $\in$ I**

(c) $\int_a^b \frac{|x|}{x}dx = |b|-|a|$

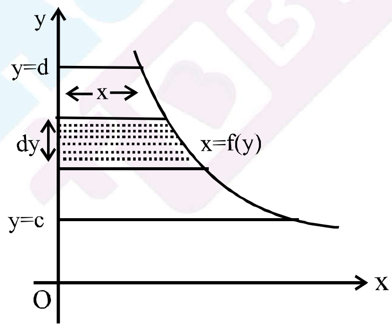
AREA UNDER THE CURVE

(1) Area bounded by the curve y = f(x), the x-axis and the ordinates at x = a and x = b is given by $A = \int_a^b f(x)dx = \int_a^b y\,dx$,



(2) If the bounded area "A" is below the x-axis then A is negative. The convention is to consider the magnitude only, i.e. $A = \left|\int_a^b y\,dx\right|$

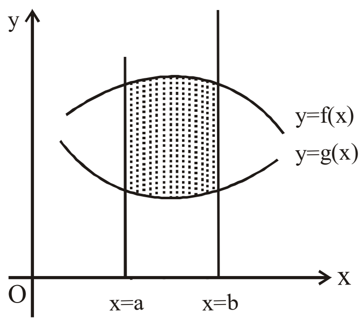
(3) The area bounded by the curve x = f(y), y-axis & abscissa y = c, y = d is given by, Area $A = \int_c^d x\,dy = \int_c^d f(y)dy$



(4) Area between the curves y = f(x) & y = g(x) between the ordinates x = a & x = b is given by.

$A = \int_a^b f(x)dx - \int_a^b g(x)dx$

$= \int_a^b [f(x) - g(x)]dx$



AREA UNDER THE CURVE

5. Average value of a function $y = f(x)$ w.r.t x over an interval $a \leq x \leq b$ is defined as :

$$y\ (av) = \frac{1}{b-a}\int_a^b f(x)dx$$

6. Curve Tracing :

The following outline procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry :The symmetry of the curve is judged as follows :
(i) If all the powers of y in the equation are even then the curve is symmetrical about the x axis.
(ii) If all the powers of x are even, the curve is symmetrical about y axis.
(iii) If powers of x & y are even, the curve is symmetrical about x and y axis.
(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.
(v) If on interchanging the signs of x & y the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x-axis, the y-axis.

(d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. Useful Results :

(a) Area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab
(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$.
(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$

DIFFERENTIAL EQUATIONS

Definition :

An equation that involves independent and dependent variables and the derivatives of the dependent variables w.r.t independent variable is called a **DIFFERENTIAL EQUATION.**

There are two types of differential equations :

1. Ordinary Differential Equation

A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2}-\frac{2dy}{dx}+\cos x=0$

2. Partial Differential Equation

A differential equation is said to be partial, if there are two or more independent variables, e.g. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation.

We are concerned with ordinary differential equations only.

Solution (Primitive) of Differential Equation

Finding the unknown function which satisfies given differential equation is called SOLVING OR INTEGRATING the differential equation. The solution of the differential equation is also called its PRIMITIVE, because the differential equation can be regarded as a relation derived from it.

Order of Differential Equation

The order of a differential equation is the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Degree of Differential Equation

The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form which is free from radicals and fractions so far as derivatives are concerned, thus the differential equation

$f(x,y)\left[\frac{d^my}{dx^m}\right]^p+f(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^4+\ldots=0$ is order m and degree P.

Note

In the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree is not defined.

Formation of Diffential Equation :

If an equation with independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

1. Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

2 .Eliminate the arbitrary constants.

The eliminant is the required differential equation.

Remark

A differential equation represents a family of curves satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

Variable Separable method to solve differential equation :

Type – 1

If the differential equation can be expressed as; $f(x)dx+g(y)dy=0$ then this is said to be variable separable type. A general solution of this is given by $\int f(x)dx + \int g(y)\ dy = c$ where C is the arbitrary constant. Consider the example $(dy/dx)=e^{(x-y)}$

TYPE - 2

$\frac{dy}{dx}=f(ax+by+c), b\neq 0$ To solve this, substitute $t=ax+by+c$ Then the equation reduces to separable type in the variable t and x which can be solved. Consider the example

$(x+y)^2\frac{dy}{dx}=a^2$

TYPE - 3

$$\frac{dy}{dx}=\frac{a_1x+b_1y_1+c_1}{a_2x+b_2y+c_2}, b_1+a_2=0$$

To solve this, simple cross multiplication and substituting $d(xy)$ for $xdy + ydx$ and integrating term by term yields the result easily.

Consider the examples $\frac{dy}{dx}=\frac{x-2y+5}{2x+y-1}; \frac{dy}{dx}=\frac{2x+3y-1}{4x+6y-5}$ and $\frac{dy}{dx}=\frac{2x-y+1}{6x-5y+4}$

TYPE - 4

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.

If $x = r\cos\theta; y = r\sin\theta$ then,

1. $x\,dx + y\,dy = r\,dr$
2. $dx^2 + dy^2 = dr^2$
3. $x\,dy - y\,dx = r^2\,d\theta$

If $x = r\sec\theta$ and $y = r\tan\theta$ then $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Homogeneous Equations

A function f(x, y) is called a homogeneous function of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

This equation may also be reduced to the form $\frac{dy}{dx} = g\frac{y}{x}$ and is solved by putting y = vx so that the dependent variable y is changed to another variable v, the differential equation is transformed to an equation with variables separable. Consider the example .

$$\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$$

Equations Reducible to the Homogeneous form :

If $\frac{dy}{dx} = \frac{a_1x + b_1y_1 + c_1}{a_2x + b_2y + c_2}$; where $a_1b_2 - a_2b_1 \neq 0$

ie. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ then the substitution $x=x+h, y=y+K$ transform this equation to a homogeneous type in the new variables x and y where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved. If

1. $a_1b_2 - a_2b_1 = 0$, then a substitution $x = a_1x + b_1y$ transforms the differential equation to an equation with variables separable form and
2. In an equation of the form: $yf(xy)dx + xg(xy)dy = 0$ the variables can be separated by the substitution xy=v.

Liner Differential Equations

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together. The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots + a_n(x) \times y = f(x)$, where $a_0(x), a_1(x) \ldots a_n(x)$ are called the coefficients of the differential equation.

Note

A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=0$ is not linear, though its degree is 1.

Linear Differential Equations of First Order

The general form of linear differential equations of the first order is $\frac{dy}{dx}+Py=Q$

where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$ (i.e. integrating factor I.F.)

Remarks

1. Then the answer will be $y.IF=\int Q.IFdx$

2. Some times a given differential equation becomes linear if we take y as the independent 0variable and x as the dependent variable. e.g. the equation ;

$(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $(y^2+3)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

Equations Reducible to Linear Form

The equation $\frac{dy}{dx}+Py=Q\cdot y^n$ where P & Q are functions, of x, is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1}=Z$. Consider the example $(x^3y^2+xy)dx=dy$.

The equation $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called **BERNOULI'S EQUATION.**

Note :

1 $xdy+ydx=d(xy)$

2. $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$

3. $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right)$

4. $\frac{xdy+ydx}{xy}=d(lnxy)$

l equations

5. $\frac{dx+dy}{x+y}=d\left(\ln(x+y)\right)$

6. $\frac{xdy\text{-}ydx}{xy}=d\left(\ln\frac{y}{x}\right)$

7. $\frac{ydx-xdy}{xy}=d\left(\ln\frac{x}{y}\right)$

8. $\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$

9. $\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right)$

10. $\frac{xdx+ydy}{x^2+y^2}=d\left[\ln\sqrt{x^2+y^2}\right]$

11. $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

12. $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

13. $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

Trajectories

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

A curve making at each of its points a right angle with the curve of the family passing through that point is called an orthogonal trajectory of that family.

Orthogonal Trajectories

We form the differential equation of the given family of curves. Let it be of the form F(x,y,y') = 0 The differential equation of the orthogonal trajectories for the above form will be

$F\left(x,y,\frac{-1}{y'}\right)=0$ The general integral of this equation $\phi_1(x, y, C) = 0$ gives the family of orthogonal trajectories.