

Subject: Mathematics

1. The solution of the differential equation

$x \frac{dy}{dx} + 2y = x^2$ ($x \neq 0$) with $y(1) = 1$ is

A. $y = \frac{3}{4}x^2 + \frac{1}{4x^2}$

B. $y = \frac{4}{5}x^3 + \frac{1}{5x^2}$

C. $y = \frac{x^2}{4} + \frac{3}{4x^2}$

D. $y = \frac{x^3}{5} + \frac{1}{5x^2}$

$x \frac{dy}{dx} + 2y = x^2$ and $y(1) = 1$,

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = x$$

$$\therefore I.F. = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$$

$$\therefore y \cdot x^2 = \int x \cdot x^2 dx = \frac{x^4}{4} + C$$

But, $y(1) = 1$

$$\Rightarrow 1 = \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$$

$$So, yx^2 = \frac{x^4}{4} + \frac{3}{4}$$

$$i.e., y = \frac{x^2}{4} + \frac{3}{4x^2}$$

2. The value of the integral

$\int \frac{\sin \theta \cdot \sin 2\theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta) \sqrt{2 \sin^4 \theta + 3 \sin^2 \theta + 6}}{1 - \cos 2\theta} d\theta$ is
 (where c is a constant of integration)

A. $\frac{1}{18}[9 - 2 \sin^6 \theta - 3 \sin^4 \theta - 6 \sin^2 \theta]^{\frac{3}{2}} + c$

B. $\frac{1}{18}[11 - 18 \sin^2 \theta + 9 \sin^4 \theta - 2 \sin^6 \theta]^{\frac{3}{2}} + c$

C. $\frac{1}{18}[11 - 18 \cos^2 \theta + 9 \cos^4 \theta - 2 \cos^6 \theta]^{\frac{3}{2}} + c$

D. $\frac{1}{18}[9 - 2 \cos^6 \theta - 3 \cos^4 \theta - 6 \cos^2 \theta]^{\frac{3}{2}} + c$

$$\int \frac{2 \sin^2 \theta \cos \theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta) \sqrt{2 \sin^4 \theta + 3 \sin^2 \theta + 6}}{2 \sin^2 \theta} d\theta$$

Let $\sin \theta = t, \cos \theta d\theta = dt$

$$= \int (t^6 + t^4 + t^2) \sqrt{2t^4 + 3t^2 + 6} dt$$

$$= \int (t^5 + t^3 + t) \sqrt{2t^6 + 3t^4 + 6t^2} dt$$

$$\text{Let } 2t^6 + 3t^4 + 6t^2 = z$$

$$12(t^5 + t^3 + t)dt = dz$$

$$= \frac{1}{12} \int \sqrt{z} dz = \frac{1}{18} z^{3/2} + c$$

$$= \frac{1}{18} [(2 \sin^6 \theta + 3 \sin^4 \theta + 6 \sin^2 \theta)^{\frac{3}{2}} + c]$$

$$= \frac{1}{18} [(1 - \cos^2 \theta) \{2(1 - \cos^2 \theta)^2 + 3 - 3 \cos^2 \theta + 6\}]^{\frac{3}{2}} + c$$

$$= \frac{1}{18} [(1 - \cos^2 \theta)(2 \cos^4 \theta - 7 \cos^2 \theta + 11)]^{\frac{3}{2}} + c$$

$$= \frac{1}{18} [-2 \cos^6 \theta + 9 \cos^4 \theta - 18 \cos^2 \theta + 11]^{\frac{3}{2}} + c$$

3. The general solution of the differential equation $(y^2 - x^3)dx - xydy = 0$ ($x \neq 0$) is : (where c is a constant of integration)

A. $y^2 + 2x^2 + cx^3 = 0$

B. $y^2 - 2x^3 + cx^2 = 0$

C. $y^2 + 2x^3 + cx^2 = 0$

D. $y^2 - 2x^2 + cx^3 = 0$

$$(y^2 - x^3)dx - xydy = 0$$

$$\Rightarrow y(ydx - xdy) = x^3dx$$

$$\Rightarrow \frac{y}{x} \left(\frac{ydx - xdy}{x^2} \right) = dx$$

$$\Rightarrow -\frac{y}{x} d\left(\frac{y}{x}\right) = dx$$

$$\Rightarrow -\frac{1}{2} \left(\frac{y}{x}\right)^2 = x + C$$

$$\Rightarrow y^2 + 2x^3 + cx^2 = 0, \text{ where } c = 2C$$

4.

$$\text{Given } f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}$$

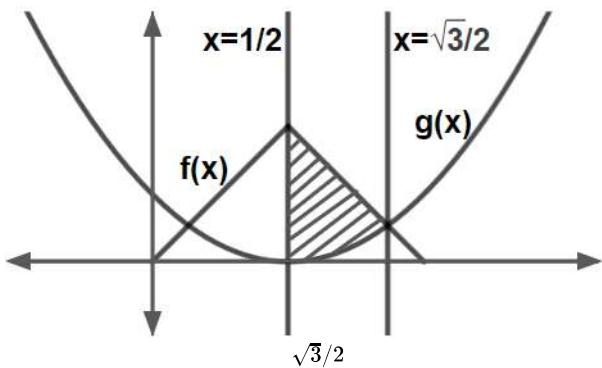
and $g(x) = \left(x - \frac{1}{2}\right)^2$, $x \in R$. Then the area (in sq. units) of the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $2x = 1$ to $2x = \sqrt{3}$ is:

- A. $\frac{\sqrt{3}}{4} - \frac{1}{3}$
- B. $\frac{1}{3} + \frac{\sqrt{3}}{4}$
- C. $\frac{1}{2} + \frac{\sqrt{3}}{4}$
- D. $\frac{1}{2} - \frac{\sqrt{3}}{4}$

$$\text{Given } f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}$$

$$g(x) = \left(x - \frac{1}{2}\right)^2$$

The area between $f(x)$ and $g(x)$ from $x = \frac{1}{2}$ to $x = \frac{\sqrt{3}}{2}$:



$$\therefore \text{Required area} = \int_{1/2}^{\sqrt{3}/2} (f(x) - g(x)) dx$$

$$= \int_{1/2}^{\sqrt{3}/2} \left((1-x) - \left(x - \frac{1}{2}\right)^2 \right) dx$$

$$= \left[x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x}{4} + \frac{x^2}{2} \right]_{1/2}^{\sqrt{3}/2}$$

$$= \left(\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{24} \right) - \left(\frac{3}{8} - \frac{1}{24} \right)$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{3} \text{ sq.units}$$

5. The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals :

- A. $\frac{1}{10} \left[\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right]$
- B. $\frac{1}{5} \left[\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{3\sqrt{3}} \right) \right]$
- C. $\frac{\pi}{40}$
- D. $\frac{1}{20} \left[\tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right]$

$$I = \int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/4} \frac{(1 + \tan^2 x)dx}{2 \tan x \left(\tan^5 x + \frac{1}{\tan^5 x} \right)}$$

Put $\tan x = t \Rightarrow \begin{cases} t = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \\ t = \tan \frac{\pi}{4} = 1 \end{cases}$

$$\Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow (1 + \tan^2 x)dx = dt$$

$$\therefore I = \int_{1/\sqrt{3}}^1 \frac{1}{2t \left(t^5 + \frac{1}{t^5} \right)} dt$$

$$\Rightarrow I = \int_{1/\sqrt{3}}^1 \frac{t^4}{2(t^{10} + 1)} dt$$

Put $t^5 = u \Rightarrow 5t^4 dt = du$

$$\Rightarrow I = \frac{1}{10} \int_{3^{-5/2}}^1 \frac{1}{u^2 + 1} du$$

$$\Rightarrow I = \left[\frac{1}{10} \tan^{-1} u \right]_{3^{-5/2}}^1$$

$$= \frac{1}{10} \left[\frac{\pi}{4} - \tan^{-1} 3^{-5/2} \right]$$

$$= \frac{1}{10} \left[\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right]$$

6. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$, where $[t]$ denotes the greatest integer less than or equal to t , is :

A. $\frac{3}{20}(4\pi - 3)$

B. $\frac{3}{10}(4\pi - 3)$

C. $\frac{1}{12}(7\pi + 5)$

D. $\frac{1}{12}(7\pi - 5)$

$$\begin{aligned}
 I &= \int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4} \\
 &= \int_{-\pi/2}^{-1} \frac{dx}{-2 - 1 + 4} + \int_{-1}^0 \frac{dx}{-1 - 1 + 4} + \int_0^1 \frac{dx}{0 + 0 + 4} + \int_1^{\pi/2} \frac{dx}{1 + 0 + 4} \\
 &= [x]_{-\pi/2}^{-1} + \frac{1}{2}[x]_{-1}^0 + \frac{1}{4}[x]_0^1 + \frac{1}{5}[x]_1^{\pi/2} \\
 &= -1 + \frac{\pi}{2} + \frac{1}{2}(0 + 1) + \frac{1}{4}(1 - 0) + \frac{1}{5}\left(\frac{\pi}{2} - 1\right) \\
 &= -\frac{9}{20} + \frac{3\pi}{5} \\
 &= \frac{3}{20}(4\pi - 3)
 \end{aligned}$$

7. The value of the integral $\int_{-2}^2 \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$

(where $[x]$ denotes the greatest integer less than or equal to x) is:

- A. $4 - \sin 4$
- B. 0
- C. 4
- D. $\sin 4$

$$I = \int_{-2}^2 \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$$

When $x \in [-2, 0]$

$$-1 < \frac{x}{\pi} < 0 \Rightarrow \left[\frac{x}{\pi}\right] = -1$$

When $x \in [0, 2]$

$$0 \leq \frac{x}{\pi} < 1 \Rightarrow \left[\frac{x}{\pi}\right] = 0$$

$$\Rightarrow I = \int_{-2}^0 \frac{\sin^2 x}{-1} dx + \int_0^2 \frac{\sin^2 x}{1} dx$$

$$\Rightarrow I = -2 \int_{-2}^0 \sin^2 x dx + 2 \int_0^2 \sin^2 x dx$$

Replacing x by $-x$ in the first integral, we get

$$I = -2 \int_0^2 \sin^2 x dx + 2 \int_0^2 \sin^2 x dx$$

or, $I = 0$

8. If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a \sin^{-1} \left(\frac{\sin x + \cos x}{b} \right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to:

A. $(1, -3)$

B. $(1, 3)$

C. $(-1, 3)$

D. $(3, 1)$

Let $\sin x + \cos x = t$

then $1 + \sin 2x = t^2$

$$\Rightarrow (\cos x - \sin x)dx = dt$$

$$\Rightarrow I = \int \frac{dt}{\sqrt{8 - (t^2 - 1)}}$$

$$= \int \frac{dt}{\sqrt{9 - t^2}}$$

$$= \sin^{-1} \left(\frac{t}{3} \right) + c$$

$$= \left(\frac{\sin x + \cos x}{3} \right) + c$$

$$\Rightarrow a = 1 \text{ and } b = 3$$

9. Let $f : R \rightarrow R$ be a continuously differentiable function such that $f(2) = 6$ and $f'(2) = \frac{1}{48}$.

If $\int_6^{f(x)} 4t^3 dt = (x - 2)g(x)$, then $\lim_{x \rightarrow 2} g(x)$ is equal to

- A. 12
- B. 18
- C. 24
- D. 36

$$\int_6^{f(x)} 4t^3 dt = (x - 2)g(x)$$

$$\int_6^{f(x)} 4t^3 dt$$

$$\Rightarrow g(x) = \frac{\int_6^{f(x)} 4t^3 dt}{x - 2}; \quad x \neq 2$$

$$\Rightarrow \lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{x - 2}$$

Apply L'Hospital Rule,

$$\Rightarrow \lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} \frac{4(f(x))^3 \cdot f'(x)}{1}$$

$$\Rightarrow \lim_{x \rightarrow 2} g(x) = 4(f(2))^3 \cdot f'(2)$$

$$= 4 \cdot 6^3 \cdot \frac{1}{48} = 18$$

10. If $(2 + \sin x) \frac{dy}{dx} + (y + 1) \cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to:

- A. $\frac{1}{3}$
- B. $-\frac{2}{3}$
- C. $-\frac{1}{3}$
- D. $\frac{4}{3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-(y+1)\cos x}{2+\sin x} \\ \Rightarrow \int \frac{dy}{y+1} &= - \int \frac{\cos x}{2+\sin x} dx \\ &= \ln(y+1) = -\ln(2+\sin x) + c \\ &= (y+1)(2+\sin x) = C; \\ \text{for } x=0, y=1 &\Rightarrow C=4 \\ (y+1)(2+\sin x) &= 4 \\ y\left(\frac{\pi}{2}\right) &= \frac{1}{3} \end{aligned}$$

11. $\int f(x)dx = \psi(x)$, then $\int x^5 f(x^3)dx$ is equal to

- A. $\frac{1}{3} \left[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx \right] + c$
- B. $\frac{1}{3} x^3 \psi(x^3) - \int x^3 \psi(x^3) dx + c$
- C. $\frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$
- D. $\frac{1}{3} \left[x^3 \psi(x^3) - \int x^3 \psi(x^3) dx \right] + c$

Given, $\int f(x)dx = \psi(x)$

$$I = \int x^5 f(x^3)dx$$

$$\text{put } x^3 = t \Rightarrow x^2 dx = \frac{1}{3} dt$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{3} \int t f(t) dt \\ &= \frac{1}{3} \left[t \int f(t) dt - \int \left(\int f(t) dt \right) dt \right] + c \\ &= \frac{1}{3} \left[t \psi(t) - \int \left(\int f(t) dt \right) dt \right] + c \\ &= \frac{1}{3} \left[x^3 \psi(x^3) - \int 3x^2 \psi(x^3) dx \right] + c \\ &= \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c \end{aligned}$$

12. If $\cos x \frac{dy}{dx} - y \sin x = 6x$, $\left(0 < x < \frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{3}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to :

- A. $-\frac{\pi^2}{2}$
- B. $-\frac{\pi^2}{2\sqrt{3}}$
- C. $-\frac{\pi^2}{4\sqrt{3}}$
- D. $\frac{\pi^2}{2\sqrt{3}}$

$$\begin{aligned} \cos x \frac{dy}{dx} - y \sin x &= 6x \\ \Rightarrow \frac{dy}{dx} - (\tan x)y &= 6x \sec x \\ \text{I.F.} &= e^{-\int \tan x \, dx} = e^{\ln(\cos x)} = \cos x \end{aligned}$$

$$\begin{aligned} \therefore \cos x \times y &= \int 6x \, dx + C \\ \Rightarrow y \cos x &= 3x^2 + C \\ \Rightarrow y &= 3x^2 \sec x + C \sec x \\ \text{As } y\left(\frac{\pi}{3}\right) &= 0 \Rightarrow C = -\frac{\pi^2}{3} \\ \therefore y &= \left(3x^2 - \frac{\pi^2}{3}\right) \sec x \\ y\left(\frac{\pi}{6}\right) &= \left(\frac{3\pi^2}{36} - \frac{\pi^2}{3}\right) \times \frac{2}{\sqrt{3}} = -\frac{\pi^2}{2\sqrt{3}} \end{aligned}$$

13. If $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$; $y(1) = 1$; then a value of x satisfying $y(x) = e$ is:

- A. $\sqrt{3}e$
- B. $\frac{1}{2}\sqrt{3}e$
- C. $\sqrt{2}e$
- D. $\frac{e}{\sqrt{2}}$

Let $x = vy$

$$\frac{dx}{dy} = v + y \frac{dv}{dy}$$

$$\Rightarrow v + y \frac{dv}{dy} = \frac{y^2(1+v^2)}{vy^2} = \frac{1+v^2}{v}$$

$$\Rightarrow y \frac{dv}{dy} = \frac{1}{v}$$

$$\Rightarrow \frac{1}{y} dy = v dv$$

$$\Rightarrow \log y = \frac{v^2}{2} + \log c$$

$$\Rightarrow \log y = \frac{x^2}{2y^2} + \log c$$

$$\Rightarrow \log c + \frac{x^2}{2y^2} - \log y = 0$$

$$y(1) = 1 \Rightarrow \log c + \frac{1}{2} - 0 = 0$$

$$\log c = -\frac{1}{2}$$

$$y(x) = e$$

$$\Rightarrow -\frac{1}{2} + \frac{x^2}{2e^2} - 1 = 0$$

$$\Rightarrow \frac{x^2}{e^2} = 3$$

$$\Rightarrow x = \pm\sqrt{3e}$$

14. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} = (\tan x - y) \sec^2 x$,
 $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $y(0) = 0$, then $y\left(-\frac{\pi}{4}\right)$ is equal to :

A. $\frac{1}{2} - e$

B. $e - 2$

C. $2 + \frac{1}{e}$

D. $\frac{1}{e} - 2$

$$\frac{dy}{dx} + y(\sec^2 x) = \tan x \sec^2 x$$

$$\Rightarrow I.F. = e^{\int \sec^2 x} = e^{\tan x}$$

$$\Rightarrow ye^{\tan x} = \int e^{\tan x} \tan x \sec^2 x dx$$

$$I = \int e^{\tan x} \tan x \sec^2 x dx$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow I = \int te^t dt$$

$$= te^t - e^t + c$$

$$= e^t(t - 1) + c$$

$$= e^{\tan x}(\tan x - 1) + c$$

$$\Rightarrow ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

$$\because y(0) = 0 \Rightarrow c = 1$$

$$\Rightarrow ye^{\tan x} = e^{\tan x}(\tan x - 1) + 1$$

$$\therefore y\left(-\frac{\pi}{4}\right) = e - 2$$

15. Let f and g be continuous function on $[0, a]$ such that $f(x) = f(a - x)$ and

$g(x) + g(a - x) = 4$, then $\int_0^a f(x)g(x)dx$ is equal to :



A. $\int_0^a f(x)dx$



B. $2 \int_0^a f(x)dx$



C. $4 \int_0^a f(x)dx$



D. $-3 \int_0^a f(x)dx$

Given $f(x) = f(a - x)$

$g(x) + g(a - x) = 4$

$\Rightarrow g(a - x) = 4 - g(x)$

$$I = \int_0^a f(x) \cdot g(x) dx$$

$$= \int_0^a f(a - x) \cdot g(a - x) dx$$

$$= \int_0^a f(x) \cdot (4 - g(x)) dx$$

$$\Rightarrow I = \int_0^a 4f(x)dx - \int_0^a f(x) \cdot g(x) dx$$

$$\Rightarrow 2I = \int_0^a 4f(x)dx$$

$$\Rightarrow I = 2 \int_0^a f(x)dx$$

16. The integral $\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx, x > 0$, is equal to :
 (where c is a constant of integration)



A. $\log_e |x^2 + 5x - 7| + c$



B. $\frac{1}{4}\log_e |x^2 + 5x - 7| + c$



C. $4\log_e |x^2 + 5x - 7| + c$



D. $\log_e \sqrt{x^2 + 5x - 7} + c$

$$\int \frac{e^{3\log_e 2x+5e^{2\log_e 2x}}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx$$

$$= \int \frac{8x^3 + 5(4x^2)}{x^4 + 5x^3 - 7x^2} dx$$

$$= \int \frac{8x + 20}{x^2 + 5x - 7} dx$$

$$= \int \frac{4(2x + 5)}{x^2 + 5x - 7} dx \quad \left\{ \begin{array}{l} \text{Let } x^2 + 5x - 7 = t \\ \Rightarrow (2x + 5)dx = dt \end{array} \right\}$$

$$= \int \frac{4dt}{t}$$

$$= 4\log_e |t| + C$$

$$= 4\log_e |(x^2 + 5x - 7)| + c$$

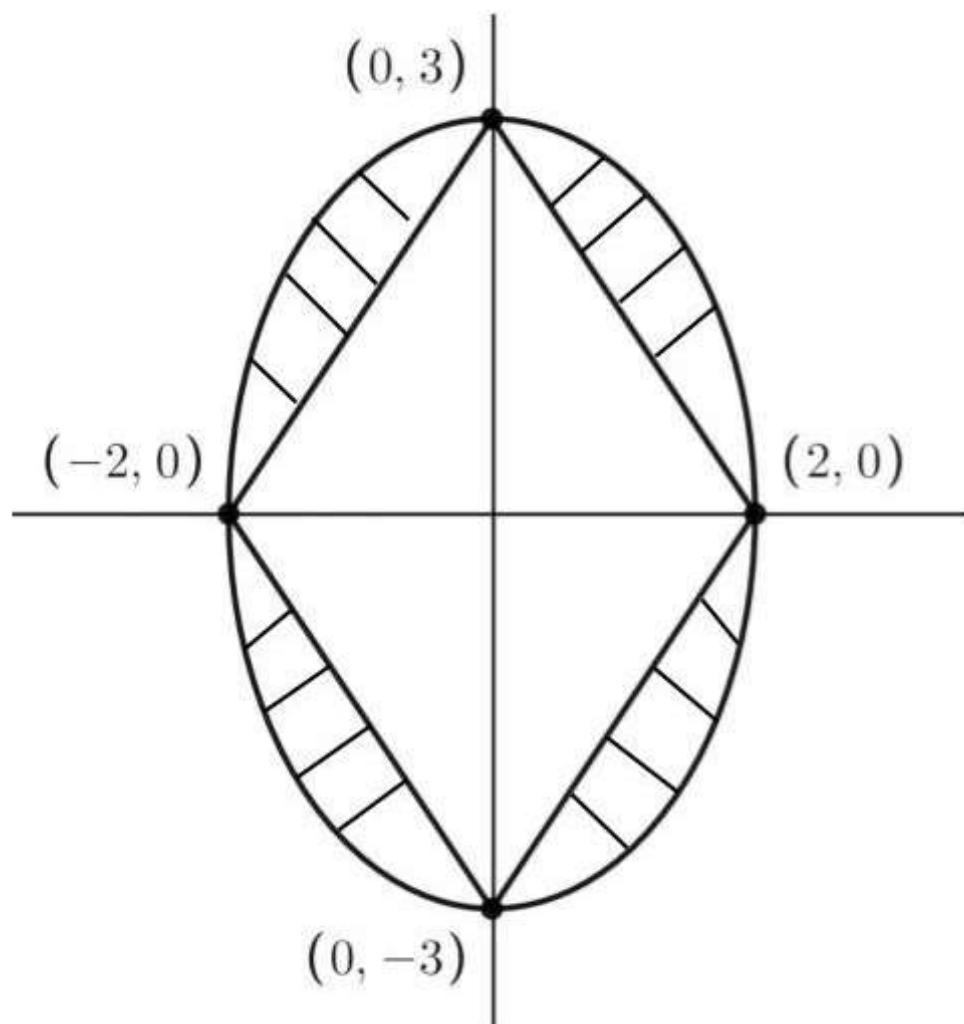
17. Area (in sq. units) of the region outside $\frac{|x|}{2} + \frac{|y|}{3} = 1$ and inside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is:

A. $3(\pi - 2)$

B. $6(\pi - 2)$

C. $6(4 - \pi)$

D. $3(4 - \pi)$



$$A = 4 \left(\frac{\pi ab}{4} - \frac{1}{2} \cdot 2 \cdot 3 \right)$$

$$A = \pi \cdot 2 \cdot 3 - 12$$

$$A = 6(\pi - 2)$$

18. Let $g(x) = \int_0^x f(t) dt$, where f is continuous function in $[0, 3]$ such that

$\frac{1}{3} \leq f(t) \leq 1$ for all $t \in [0, 1]$ and $0 \leq f(t) \leq \frac{1}{2}$ for all $t \in (1, 3]$. The largest possible interval in which $g(3)$ lies is :

- A. $[1, 3]$
- B. $\left[1, -\frac{1}{2}\right]$
- C. $\left[-\frac{3}{2}, -1\right]$
- D. $\left[\frac{1}{3}, 2\right]$

$$\text{Given : } g(x) = \int_0^x f(t) dt$$

Now,

$$g(3) = \int_0^3 f(t) dt$$

$$\Rightarrow g(3) = \int_0^1 f(t) dt + \int_1^3 f(t) dt$$

We know

$$\frac{1}{3} \leq f(t) \leq 1 \text{ for all } t \in [0, 1]$$

$$\int_0^1 \frac{1}{3} dt \leq \int_0^1 f(t) dt \leq \int_0^1 1 dt$$

$$\Rightarrow \frac{1}{3} \leq \int_0^1 f(t) dt \leq 1$$

Similarly,

$$0 \leq \int_1^3 f(t) dt \leq 1$$

Therefore,

$$g(3) \in \left[\frac{1}{3}, 2\right]$$

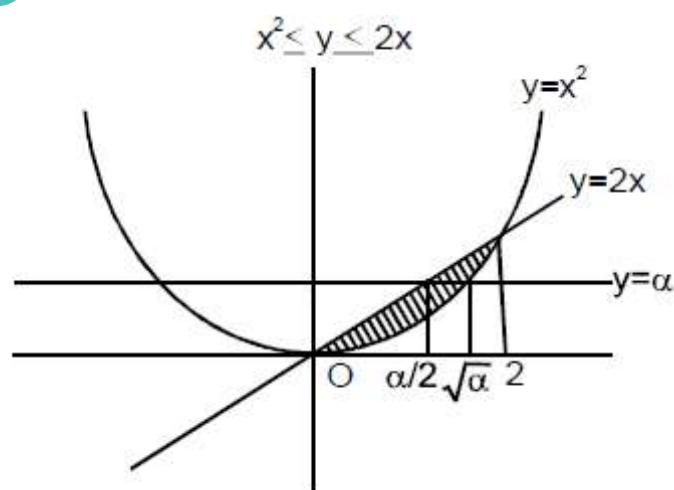
19. Consider a region $R = \{(x, y) \in R^2 : x^2 \leq y \leq 2x\}$. If a line $y = \alpha$ divides the area of region R into two equal parts, then which of the following is true

A. $\alpha^3 - 6\alpha^2 + 16 = 0$

B. $3\alpha^2 - 8\alpha^{\frac{3}{2}} + 8 = 0$

C. $\alpha^3 - 6\alpha^{\frac{3}{2}} - 16 = 0$

D. $3\alpha^2 - 8\alpha + 8 = 0$



$$\text{Area} = \int_0^2 (2x - x^2) dx = x^2 - \frac{x^3}{3} \Big|_0^2 = \frac{4}{3}$$

$$\therefore \int_0^a \left(\sqrt{y} - \frac{y}{2} \right) dy = \frac{2}{3}$$

$$\Rightarrow \frac{2}{3}y^{\frac{3}{2}} - \frac{y^2}{4} \Big|_0^a = \frac{2}{3}$$

$$\Rightarrow 8 \cdot \alpha^{\frac{3}{2}} - 3\alpha^2 = 8$$

20. The value of $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2}$ is

A. $\frac{1}{2}\tan^{-1}(2)$

B. $\frac{1}{2}\tan^{-1}(4)$

C. $\tan^{-1}(4)$

D. $\frac{1}{4}\tan^{-1}(4)$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{1}{1 + 4\left(\frac{r}{n}\right)^2} \\ &= \int_0^2 \frac{dx}{1 + 4x^2} = \frac{1}{2}\tan^{-1}(2x)|_0^2 = \frac{1}{2}\tan^{-1} 4 \end{aligned}$$

21. The differential equation of the family of curves, $x^2 = 4b(y + b)$, $b \in R$, is:

A. $xy'' = y'$

B. $x(y')^2 = x + 2yy'$

C. $x(y')^2 = x - 2yy'$

D. $x(y')^2 = 2yy' - x$

$$x^2 = 4b(y + b)$$

Differentiating both the sides w.r.t. x , we get

$$\Rightarrow 2x = 4by'$$

$$\Rightarrow b = \frac{x}{2y'}$$

Putting the value of b in (1), we get

$$\Rightarrow x^2 = \frac{2x}{y'} \left(y + \frac{x}{2y'} \right)$$

$$\Rightarrow x^2 = \frac{2xy}{y'} + \frac{x^2}{y'^2}$$

$$\Rightarrow x(y')^2 = 2yy' + x$$

22. If

$$\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln |\sin x - 2 \cos x| + k$$

then a is equal to

- A. -1
- B. -2
- C. 1
- D. 2

$$\begin{aligned}
 \text{L.H.S} &= \int \frac{5 \tan x}{\tan x - 2} dx \\
 &= \int \frac{5 \cdot \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} - 2} dx \\
 &= \int \frac{5 \sin x}{\sin x - 2 \cos x} dx \\
 &= \int \frac{(\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)}{\sin x - 2 \cos x} dx \\
 &= \int 1 dx + 2 \int \frac{\cos x + 2 \sin x}{\sin x - 2 \cos x} dx \\
 &= x + 2 \ln |\sin x - 2 \cos x| + k
 \end{aligned}$$

Comparing to given R.H.S

Hence the value of $a = 2$

23. Let $y = y(x)$ be the solution of the differential equation

$\sin x \frac{dy}{dx} + y \cos x = 4x, x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

- A. $-\frac{4}{9}\pi^2$
- B. $\frac{4}{9\sqrt{3}}\pi^2$
- C. $\frac{-8}{9\sqrt{3}}\pi^2$
- D. $-\frac{8}{9}\pi^2$

$$\begin{aligned}\frac{d}{dx}(y \sin x) &= 4x \\ \Rightarrow y \sin x &= 4 \frac{x^2}{2} + c \\ \Rightarrow y \sin x &= 2x^2 + c\end{aligned}$$

$$\text{Given } y(\pi/2) = 0$$

$$c = -\frac{\pi^2}{2}$$

$$\text{Thus, } y \sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\text{Now } y\left(\frac{\pi}{6}\right) = -\frac{8}{9}\pi^2$$

24. The difference between degree and order of a differential equation that represents the family of curves given by $y^2 = a \left(x + \frac{\sqrt{a}}{2} \right)$, $a > 0$ is

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2 2.0 2.00

Solution:

$$y^2 = a \left(x + \frac{\sqrt{a}}{2} \right)$$

Differentiating with respect to x ,

$$2yy' = a$$

$$\Rightarrow y^2 = 2yy' \left(x + \frac{\sqrt{2yy'}}{2} \right)$$

$$\Rightarrow y = 2y' \left(x + \frac{\sqrt{yy'}}{\sqrt{2}} \right)$$

$$\Rightarrow y - 2xy' = \sqrt{2}y' \sqrt{yy'}$$

$$\Rightarrow \left(y - 2x \frac{dy}{dx} \right)^2 = 2y \left(\frac{dy}{dx} \right)^3$$

Degree = 3 and Order = 1

Degree – Order = 3 – 1 = 2

25. The integral $\int_0^2 |x - 1| - x dx$ is equal to

Accepted Answers

1.5 1.50

Solution:

$$\begin{aligned}
 & \int_0^2 |x - 1| - x dx \\
 &= \int_0^1 |1 - x - x| dx + \int_1^2 |x - 1 - x| dx \\
 &= \int_0^1 |2x - 1| dx + \int_1^2 1 dx \\
 &= \int_0^{\frac{1}{2}} (1 - 2x) dx + \int_{\frac{1}{2}}^1 (2x - 1) dx + \int_1^2 1 dx \\
 &= \left[\left(\frac{1}{2} - 0 \right) - \left(\frac{1}{4} - 0 \right) \right] + \left(1 - \frac{1}{4} \right) - \left(1 - \frac{1}{2} \right) + 1 \\
 &= \frac{1}{2} - \frac{1}{4} + \frac{3}{4} - \frac{1}{2} + 1 = \frac{3}{2}
 \end{aligned}$$

26. Let $f(x)$ and $g(x)$ be two functions satisfying $f(x^2) + g(4-x) = 4x^3$ and

$$g(4-x) + g(x) = 0, \text{ then the value of } \int_{-4}^4 f(x^2) dx$$

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512 512.0 512.00

Solution:

$$I = 2 \int_0^2 f(x^2) dx \quad \dots (1)$$

$$I = 2 \int_0^2 f((4-x)^2) dx \quad \dots (2)$$

Adding equation (1) and (2), we get

$$2I = 2 \int_0^2 [f(x^2) + f((4-x)^2)] dx \quad \dots (3)$$

Now, using $f(x^2) + g(4-x) = 4x^3 \quad \dots (4)$

$$x \rightarrow 4-x$$

$$f((4-x)^2) + g(x) = 4(4-x)^3 \quad \dots (5)$$

Adding equation (4) and (5)

$$\begin{aligned} f(x^2) + f(4-x^2) + g(x) + g(4-x) &= 4(x^3 + (4-x)^3) \\ \Rightarrow f(x^2) + f(4-x^2) &= 4(x^3 + (4-x)^3) \end{aligned}$$

$$\text{Now, } I = 4 \int_0^4 x^3 + (4-x)^3 dx = 512$$

27. Let $[T]$ denote the greatest integer less than or equal to T . Then the value of $\int_1^2 |2x - [3x]| dx$ is

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1 1.0 1.00

Solution:

$$\int_1^2 |2x - [3x]| dx$$

Substituting $3x = t$

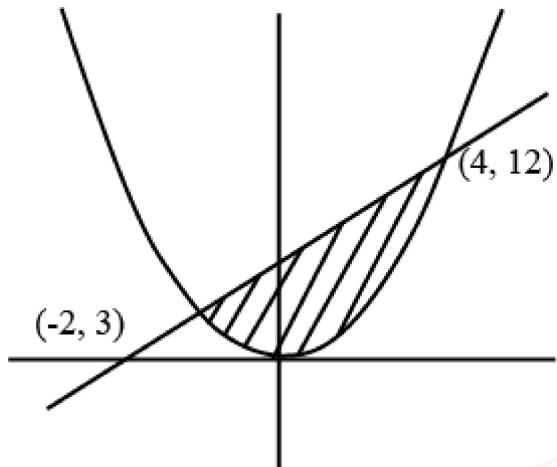
$$\begin{aligned}
 &= \frac{1}{3} \int_3^6 \left| \frac{2t}{3} - [t] \right| dt \\
 &= \frac{1}{9} \left[\int_3^6 |2t - 3[t]| \right] dt \\
 &= \frac{1}{9} \left[\int_3^4 |2t - 9| + \int_4^5 |2t - 12| + \int_5^6 |2t - 15| \right] dt \\
 &= \frac{1}{9} \left[\int_3^4 (9 - 2t) + \int_4^5 (12 - 2t) + \int_5^6 (15 - 2t) \right] dt \\
 &= \frac{1}{9} [9.1 + 12.1 + 15.1 - [4^2 - 3^2] - [5^2 - 4^2] - [6^2 - 5^2]] \\
 &= \frac{1}{9} [36 - [4^2 - 3^2 + 5^2 - 4^2 + 6^2 - 5^2]] \\
 &= \frac{1}{9} [36 - 36 + 9] = 1
 \end{aligned}$$

28. The area of the region $S = \{(x, y) : 3x^2 \leq 4y \leq 6x + 24\}$ is

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27 27.0 27.00

Solution:



$$\frac{3x^2}{4} \leq y \leq \frac{3x}{2} + 6$$

Point of intersection for two curves

$$y = \frac{3x^2}{4} \text{ and } y = \frac{3x}{2} + 6 \text{ are } (-2, 3) \text{ and } (4, 12)$$

Area of shaded region

$$\begin{aligned} \int_{-2}^4 \left(\frac{3x}{2} + 6 - \frac{3x^2}{4} \right) dx &= \frac{3x^2}{4} + 6x - \frac{x^3}{4} \Big|_{-2}^4 \\ &= 20 - (-7) = 27 \end{aligned}$$

29. Let a and b respectively be the points of local maximum and local minimum of the function $f(x) = 2x^3 - 3x^2 - 12x$.

If A is the total area of the region bounded by $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$, then $4A$ is equal to .

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114 114.0 114.00

Solution:

$$f(x) = 2x^3 - 3x^2 - 12x$$

$$\Rightarrow f'(x) = 6x^2 - 6x - 12$$

$$= 6(x - 2)(x + 1)$$

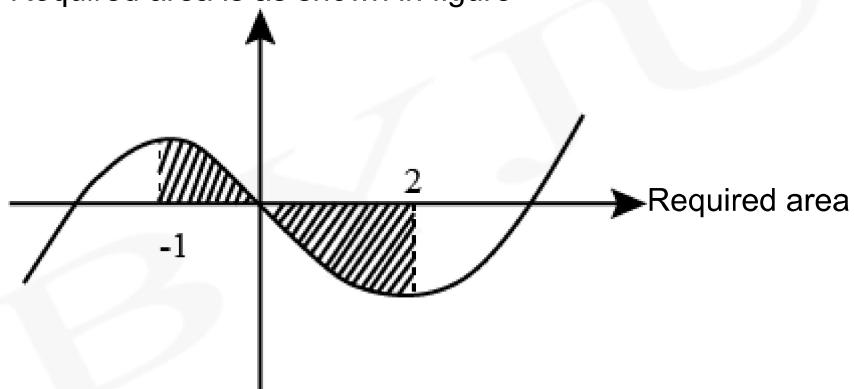
$$\Rightarrow f'(x) = 0 \Rightarrow x = -1, 2$$

$x = -1$ is point of local maximum $\Rightarrow a = -1$

$x = 2$ is point of local minimum $\Rightarrow b = 2$

$$f(-1) = 8 \text{ and } f(2) = -20$$

Required area is as shown in figure



$$= \int_{-1}^0 (f(x) - 0)dx + \int_0^2 (0 - f(x))dx$$

$$= \left(\frac{1}{2}x^4 - x^3 - 6x^2 \right)_{-1}^0 - \left(\frac{x^4}{2} - x^3 - 6x^2 \right)_0^2 = \frac{57}{2} = A$$

$$\Rightarrow 4A = 114$$

30. Let $f : (0, 2) \rightarrow R$ be defined as $f(x) = \log_2 \left(1 + \tan \left(\frac{\pi x}{4} \right) \right)$. Then,
 $\lim_{n \rightarrow \infty} \frac{2}{n} \left(f \left(\frac{1}{n} \right) + f \left(\frac{2}{n} \right) + \cdots + f(1) \right)$ is equal to

Accepted Answers

1 1.0 1.00

Solution:

$$E = 2 \lim_{n \rightarrow \infty} \sum \frac{1}{n} f \left(\frac{r}{n} \right)$$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi x}{4} \right) dx \dots (i)$$

replacing $x \rightarrow 1 - x$

$$E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \frac{\pi}{4}(1-x) \right) dx$$

$$\Rightarrow E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \tan \left(\frac{\pi}{4} - \frac{\pi}{4}x \right) \right) dx$$

$$\Rightarrow E = \frac{2}{\ln 2} \int_0^1 \ln \left(1 + \frac{1 - \tan \frac{\pi x}{4}}{1 + \tan \frac{\pi x}{4}} \right) dx$$

$$\Rightarrow E = \frac{2}{\ln 2} \int_0^1 \ln \left(\frac{2}{1 + \tan \frac{\pi x}{4}} \right) dx$$

$$\Rightarrow E = \frac{2}{\ln 2} \int_0^1 \left(\ln 2 - \ln \left(1 + \tan \frac{\pi x}{4} \right) \right) dx \dots (ii)$$

equation (i) + (ii)

$$E = 1$$