Subject: Mathematics

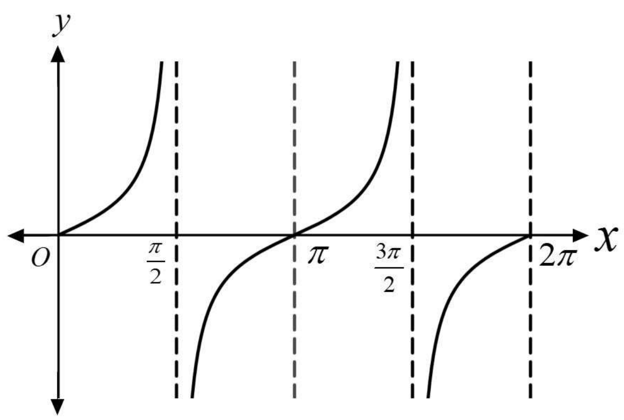
1. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in :

A. $\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

B. $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

C. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

D. $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$



$\tan 2\theta(1 + \cos 2\theta) > 0$

$\tan 2\theta > 0 \quad (\because 1 + \cos 2\theta > 0)$

$2\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(2\pi, \frac{5\pi}{2}\right) \cup \left(3\pi, \frac{7\pi}{2}\right)$

$\Rightarrow \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

2. If $0 < x, y < \pi$ and $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$, then $\sin x + \cos y$ is equal to :

A. $\frac{1+\sqrt{3}}{2}$

B. $\frac{1-\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{1}{2}$

$$\cos x + \cos y - \cos(x+y) = \frac{3}{2}$$

$$\Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - \left[2\cos^2\left(\frac{x+y}{2}\right) - 1\right] = \frac{3}{2}$$

$$\Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - 2\cos^2\left(\frac{x+y}{2}\right) = \frac{1}{2}$$

$$\Rightarrow 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - 4\cos^2\left(\frac{x+y}{2}\right) = 1 = \cos^2\frac{(x-y)}{2} + \sin^2\left(\frac{x-y}{2}\right)$$

$$\Rightarrow 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) - 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \sin^2\left(\frac{x-y}{2}\right) = 0$$

$$\Rightarrow \left(\cos\left(\frac{x-y}{2}\right) - 2\cos\left(\frac{x+y}{2}\right)\right)^2 + \sin^2\left(\frac{x-y}{2}\right) = 0$$

$$\Rightarrow \sin\frac{x-y}{2} = 0 \Rightarrow x = y$$

and $\cos\frac{x-y}{2} = 2\cos\frac{x+y}{2}$

$$\Rightarrow \cos x = \frac{1}{2} = \cos y$$

$$\therefore \sin x + \cos y = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$$

3. The number of roots of the equation, $(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$ in the interval $[0, \pi]$ is equal to:

A. 3

B. 2

C. 4

D. 8

$$(81)^{\sin^2 x} + (81)^{1-\sin^2 x} = 30$$

$$(81)^{\sin^2 x} + \frac{81}{(81)^{\sin^2 x}} = 30$$

Let $(81)^{\sin^2 x} = t$

$$t + \frac{81}{t} = 30 \Rightarrow t^2 + 81 = 30t$$

$$\Rightarrow t^2 - 30t + 81 = 0$$

$$\Rightarrow t^2 - 27t - 3t + 81 = 0$$

$$\Rightarrow (t-3)(t-27) = 0$$

$$\Rightarrow t = 3, 27$$

$$\Rightarrow (81)^{\sin^2 x} = 3, 3^3$$

$$\Rightarrow 3^{4\sin^2 x} = 3^1, 3^3$$

$$\Rightarrow 4\sin^2 x = 1, 3$$

$$\Rightarrow \sin^2 x = \frac{1}{4}, \frac{3}{4}$$

in $[0, \pi]$, $\sin x \geq 0$

$$\sin x = \frac{1}{2}, \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$$

Number of solutions $= 4$

4. The number of solutions of the equation $x+2\tan x=\frac{\pi}{2}$ in the interval $[0,\ 2\pi]$ is :

A. 5

B. 2

C. 4

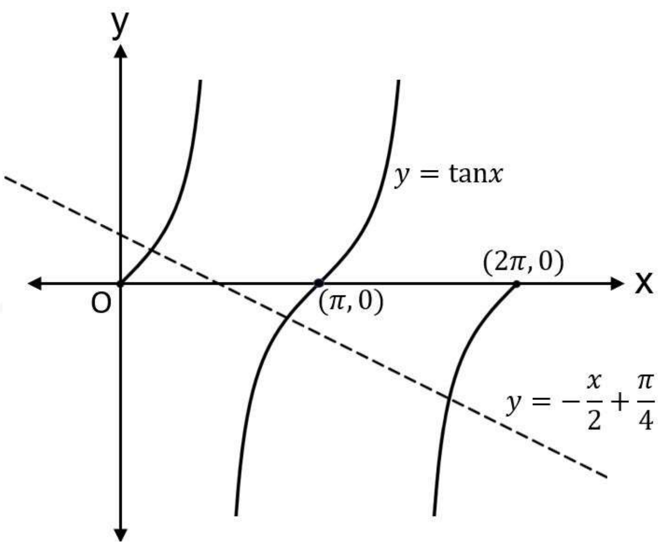
D. 3

$x+2\tan x=\frac{\pi}{2}$ in $[0,\ 2\pi]$

$2\tan x=\frac{\pi}{2}-x$

$\tan x=\frac{\pi}{4}-\frac{x}{2}$

$y=\tan x$ and $y=\frac{-x}{2}+\frac{\pi}{4}$



From the above graph it can be observed that there are 3 intersection points in $[0,\ 2\pi]$

$\therefore$ Number of solutions $=3$

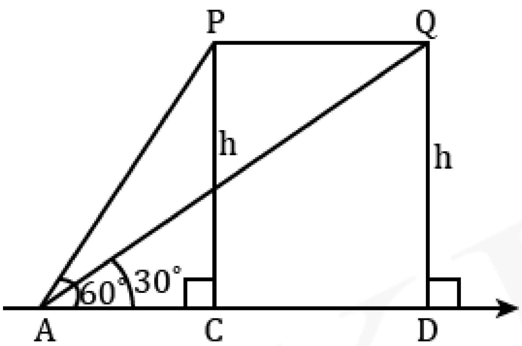
5. The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 20 seconds at the speed of $432\ \mathrm{km/hour}$, the angle of elevation changes to 30°. If the jet plane is flying at a constant height, then its height is

A. $1200\sqrt{3}$ m

B. $1800\sqrt{3}$ m

C. $3600\sqrt{3}$ m

D. $2400\sqrt{3}$ m



Velocity, $v = 432 \times \dfrac{1000}{60 \times 60} m/sec = 120\ m/sec$

Distance $PQ = v \times 20 = 2400\ m$

In ΔPAC

$\tan 60^\circ = \dfrac{h}{AC} \Rightarrow AC = \dfrac{h}{\sqrt{3}}$

In ΔAQD

$\tan 30^\circ = \dfrac{h}{AD} \Rightarrow AD = \sqrt{3}h$

Now, $AD = AC + CD$

$\Rightarrow \sqrt{3}h = \dfrac{h}{\sqrt{3}} + 2400$

$\Rightarrow \dfrac{2h}{\sqrt{3}} = 2400$

$\Rightarrow h = 1200\sqrt{3}\ m$

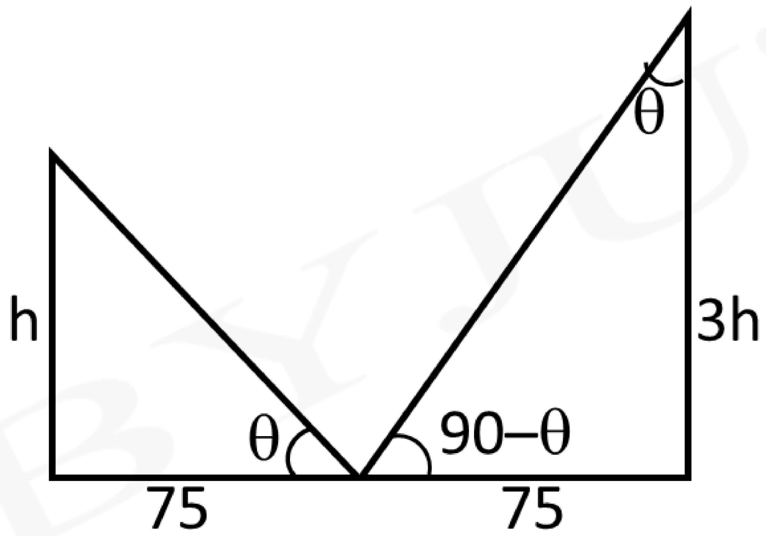
6. Two vertical poles are $150m$ apart and the height of one is three times that of the other. If from the middle point of the line joining their feet, an observer finds the angles of elevation of their tops to be complementary, then the height of the shorter pole (in meters) is:

A. 25

B. $20\sqrt{3}$

C. 30

D. $25\sqrt{3}$



$$\tan\theta = \frac{h}{75} = \frac{75}{3h}$$
$$\Rightarrow h^2 = \frac{(75)^2}{3}$$
$$h = 25\sqrt{3}m$$

7. The value of $\cot\dfrac{\pi}{24}$ is

A. $3\sqrt{2}-\sqrt{3}-\sqrt{6}$

B. $\sqrt{2}-\sqrt{3}-2+\sqrt{6}$

C. $\sqrt{2}+\sqrt{3}+2-\sqrt{6}$

D. $\sqrt{2}+\sqrt{3}+2+\sqrt{6}$

$$\cot\frac{\pi}{24}=\frac{2\cos\frac{\pi}{24}\cdot\sin\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cdot\sin\frac{\pi}{24}}$$

$$=\frac{\sin\frac{\pi}{12}}{1-\cos\frac{\pi}{12}}=\frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$=\frac{\sqrt{3}-1}{2\sqrt{2}-(\sqrt{3}+1)}\times\frac{2\sqrt{2}+(\sqrt{3}+1)}{2\sqrt{2}+(\sqrt{3}+1)}$$

$$=\frac{2\sqrt{6}+3+\sqrt{3}-2\sqrt{2}-\sqrt{3}-1}{8-(3+1+2\sqrt{3})}$$

$$=\frac{\sqrt{6}-\sqrt{2}+1}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$=2\sqrt{6}+3\sqrt{2}-2\sqrt{2}-\sqrt{6}+2+\sqrt{3}$$

$$=\sqrt{6}+\sqrt{2}+2+\sqrt{3}$$

$$=\sqrt{2}+\sqrt{3}+2+\sqrt{6}$$

8. Let $f_k(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$ for $k = 1, 2, 3, \ldots$. Then for all $x \in \mathbb{R}$, the value of $f_4(x) - f_6(x)$ is equal to:

A. $\frac{1}{12}$

B. $\frac{-1}{12}$

C. $\frac{1}{4}$

D. $\frac{5}{12}$

$f_k(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$

$f_4(x) - f_6(x) = \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$

$= \frac{1}{4}\left(1 - \frac{1}{2}\sin^2 2x\right) - \frac{1}{6}\left(1 - \frac{3}{4}\sin^2 2x\right)$

$= \frac{1}{4} - \frac{1}{6} - \frac{1}{8}\sin^2 2x + \frac{1}{8}\sin^2 2x$

$= \frac{1}{12}$

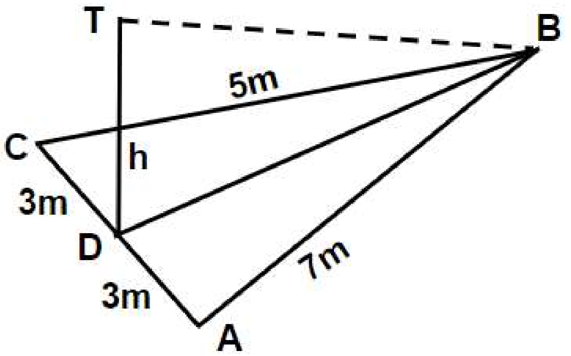
9. Consider a triangular plot ABC with sides $AB = 7m, BC = 5m$ and $CA = 6m$. A vertical lamp-post at the mid point D of AC subtends an angle 30° at B. The height (in m) of the lamp-post is :

A. $\frac{3}{2}\sqrt{21}$

B. $7\sqrt{3}$

C. $2\sqrt{21}$

D. $\frac{2}{3}\sqrt{21}$



Length of median $BD = \frac{1}{2}\sqrt{2((BC)^2 + (AB)^2) - (AC)^2}$

$= \frac{1}{2}\sqrt{2(25+49) - 36}$

$= \sqrt{28}$

$= 2\sqrt{7}$

Let h be the height of the tower

So, from ΔTBD

$\because \angle TBD = 30^\circ$

$\therefore \tan 30^\circ = \frac{h}{2\sqrt{7}}$

$h = 2\sqrt{\frac{7}{3}} = \frac{2}{3}\sqrt{21}m$

10. In a ΔPQR, if $3\sin P+4\cos Q=6$ and $4\sin Q+3\cos P=1$, then the angle R is equal to

A. $\frac{5\pi}{6}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{3\pi}{4}$

Given that, in the ΔPQR

$3\sin P+4\cos Q=6\cdots(1)$

$4\sin Q+3\cos P=1\cdots(2)$

Squaring and adding (1) and (2), we get

$9\sin^2 P\ +16\cos^2 Q+24\sin P\cos Q+16\sin^2 Q\ +9\cos^2 P+24\sin Q\cos P=36+1$
$\Rightarrow 25+24(\sin P\cos Q+\cos P\sin Q)=37$
$\Rightarrow \sin(P+Q)=\frac{1}{2}$ $[\ \because \sin(P+Q)=\sin P\cos Q+\cos P\sin Q\ ]$
$\Rightarrow P+Q=\frac{\pi}{6}$ or $\frac{5\pi}{6}$

Since,
$P+Q+R=\pi$
$\Rightarrow R=\frac{5\pi}{6}$ or $\frac{\pi}{6}$

If $R=\frac{5\pi}{6}$, then
$P>0, Q<\frac{\pi}{6}$
$\Rightarrow \sin P<\frac{1}{2}, \cos Q<1$
$\Rightarrow 3\sin P+4\cos Q<\frac{11}{2}$
But $3\sin P+4\cos Q=6$, so $R=\frac{5\pi}{6}$ is not possible.

$\therefore R=\frac{\pi}{6}$

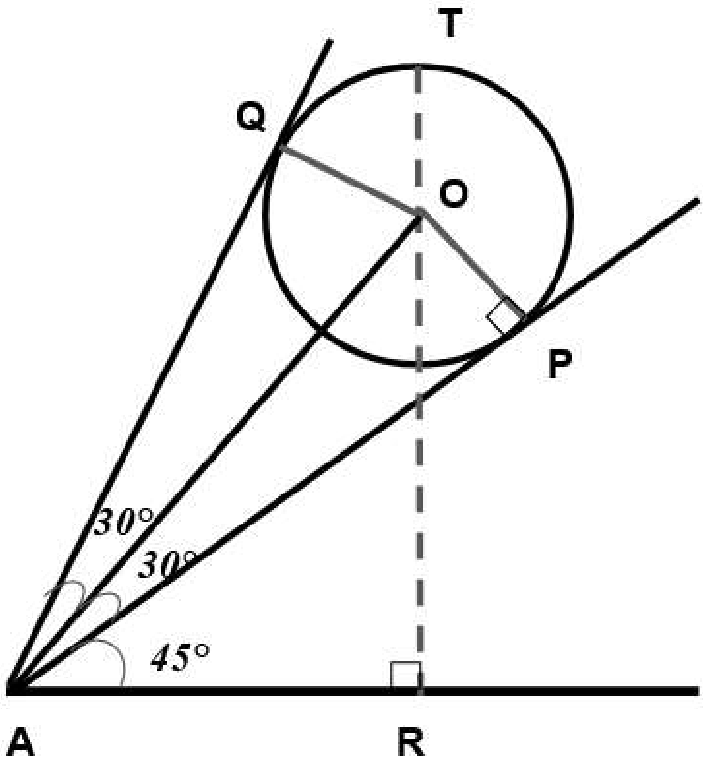
11. A spherical gas balloon of radius 16 meter subtends an angle 60° at the eye of the observer A while the angle of elevation of its center from the eye of A is 75°. Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is

A. $8(2+2\sqrt{3}+\sqrt{2})$

B. $8(\sqrt{6}-\sqrt{2}+2)$

C. $8(\sqrt{2}+2+\sqrt{3})$

D. $8(\sqrt{6}+\sqrt{2}+2)$



Let O be the centre of circle.
P, Q points of contact tangents from A
T be the topmost point on ellipse and R be the foot of perpandicular.
In ΔOAP
$OA = 16 \text{ cosec } 30^0 = 32$
In ΔABO

$$OA = OA \ \sin 75^0 = 32 \times \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$OA = 16 \times \frac{\sqrt{3}+1}{2} \times \sqrt{2} = 8\sqrt{2}(\sqrt{3}+1) = 8(\sqrt{6}+\sqrt{2})$$

So, topmost point is $= OR + OT$
$= 8(\sqrt{6}+\sqrt{2}+2)m$

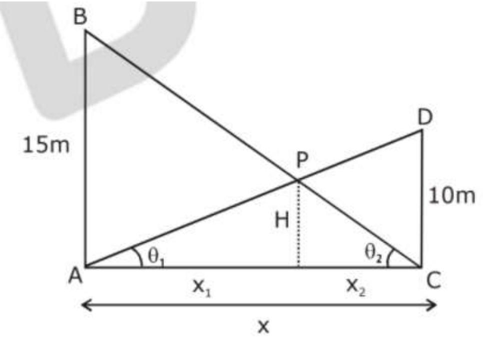
12. Two vertical poles $AB = 15$m and $CD = 10$m are standing apart on a horizontal ground with points A and C on the ground. If P is the point of intersection of BC and AD, then the height of P(in m) above the line AC is

A. 5

B. $\frac{20}{3}$

C. $\frac{10}{3}$

D. 6



Using similar triangle concept, we have

$\tan\theta_1 = \frac{10}{x} = \frac{H}{x_1}$

$\Rightarrow x_1 = \frac{Hx}{10}$

and $\tan\theta_2 = \frac{15}{x} = \frac{H}{x_2}$

$\Rightarrow x_2 = \frac{Hx}{15}$

$\because x_1 + x_2 = x$

$\Rightarrow \frac{Hx}{10} + \frac{Hx}{15} = x$

$\Rightarrow 15H + 10H = 150$

$\Rightarrow H = 6$m

13. If $\cos(\alpha+\beta)=\frac{3}{5}, \sin(\alpha-\beta)=\frac{5}{13}$ and $0<\alpha,\beta<\frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to:

A. $\frac{21}{16}$

B. $\frac{63}{16}$

C. $\frac{63}{52}$

D. $\frac{33}{52}$

$\cos(\alpha+\beta)=\frac{3}{5}\Rightarrow \tan(\alpha+\beta)=\frac{4}{3}$

$\sin(\alpha-\beta)=\frac{5}{13}\Rightarrow \tan(\alpha-\beta)=\frac{5}{12}$

Now,

$\Rightarrow \tan(2\alpha)=\tan((\alpha+\beta)+(\alpha-\beta))$

$$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\cdot\tan(\alpha-\beta)}$$

$$=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}$$

$$=\frac{63}{16}$$

14. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ \ldots\ \cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is :

A. $\frac{1}{1024}$

B. $\frac{1}{2}$

C. $\frac{1}{512}$

D. $\frac{1}{256}$

$A=\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ \ldots\ \cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$

Put $\frac{\pi}{2^{10}}=x\Rightarrow\pi=2^{10}x$

$\Rightarrow A=(\cos x\cdot\cos 2x\cdot\ \ldots\ \cdot\cos 2^8x)\sin x$

As we know

$\frac{\sin 2^n\theta}{2^n\sin\theta}=\cos\theta\cdot\cos 2\theta\cdot\ \ldots\ \cos 2^{n-1}\theta\cdot$

Here, $n-1=8\Rightarrow n=9$

$\Rightarrow A=\frac{\sin 2^9x}{2^9\sin x}\cdot\sin x$

$\Rightarrow A=\frac{\sin 2^9\frac{\pi}{2^{10}}}{2^9}$

$\Rightarrow A=\frac{\sin\frac{\pi}{2}}{2^9}$

$\Rightarrow A=\frac{1}{512}$

15. If the equation $\cos^4\theta + \sin^4\theta + \lambda = 0$ has real solutions for θ, then λ lies in the interval:

A. $\left(-\frac{1}{2}, -\frac{1}{4}\right]$

B. $\left[-1, -\frac{1}{2}\right]$

C. $\left[-\frac{3}{2}, -\frac{5}{4}\right]$

D. $\left(-\frac{5}{4}, -1\right)$

$\cos^4\theta + \sin^4\theta + \lambda = 0$

$\lambda = -\left\{1 - \frac{1}{2}\sin^2 2\theta\right\}$

$2(\lambda + 1) = \sin^2 2\theta$

$0 \le 2\ (\lambda + 1) \le 1$

$0 \le \lambda + 1 \le \frac{1}{2}$

$\Rightarrow -1 \le \lambda \le -\frac{1}{2}$

16. The maximum value of

$$3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$$

for any real value of θ is:

A. $\frac{\sqrt{79}}{2}$

B. $\sqrt{19}$

C. $\sqrt{31}$

D. $\sqrt{34}$

Let $\mu = 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$

$$= 3\cos\theta + 5\sin\theta.\cos\frac{\pi}{6} - 5\cos\theta.\sin\frac{\pi}{6}$$

$$= \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$$

We know that

$a\sin\theta + b\cos\theta \in [-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

$$\therefore \max\mu = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{75}{4}}$$

$$= \sqrt{19}$$

17. The expression $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$ can be written as :

A. $\sin A \cdot \cos A + 1$

B. $\sec A \cdot \operatorname{cosec} A + 1$

C. $\tan A + \cot A$

D. $\sec A + \operatorname{cosec} A$

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$$
$$=\frac{\sin^2 A}{\cos A(\sin A-\cos A)}+\frac{\cos^2 A}{\sin A(\cos A-\sin A)}$$
$$=\frac{1}{\sin A-\cos A}\left[\frac{\sin^2 A}{\cos A}-\frac{\cos^2 A}{\sin A}\right]$$
$$=\frac{1}{\sin A-\cos A}\cdot\frac{\sin^3 A-\cos^3 A}{\sin A\cos A}$$
$$=\frac{(\sin A-\cos A)(\sin^2 A+\cos^2 A+\sin A\cos A)}{(\sin A-\cos A)\sin A\cos A}$$
$$=\frac{1+\sin A\cos A}{\sin A\cos A}$$
$$=\sec A\cdot\operatorname{cosec} A+1$$

18. If $L = \sin^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$ and $M = \cos^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$, then:

A. $M = \frac{1}{2\sqrt{2}} + \frac{1}{2}\cos\frac{\pi}{8}$

B. $M = \frac{1}{4\sqrt{2}} + \frac{1}{4}\cos\frac{\pi}{8}$

C. $L = -\frac{1}{2\sqrt{2}} + \frac{1}{2}\cos\frac{\pi}{8}$

D. $L = \frac{1}{4\sqrt{2}} - \frac{1}{4}\cos\frac{\pi}{8}$

$$L = \sin^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$$

$$\left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$L = \left(\frac{1-\cos\frac{\pi}{8}}{2}\right) - \left(\frac{1-\cos\frac{\pi}{4}}{2}\right)$$

$$L = \frac{-1}{2}\left[\cos\frac{\pi}{8} - \cos\frac{\pi}{4}\right]$$

$$L = \frac{1}{2\sqrt{2}} - \frac{1}{2}\cos\frac{\pi}{8}$$

$$M = \cos^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{8}\right)$$

$$M = \left(\frac{1+\cos\frac{\pi}{8}}{2}\right) - \left(\frac{1-\cos\frac{\pi}{4}}{2}\right)$$

$$M = \frac{1}{2}\left[\cos\frac{\pi}{4} + \cos\frac{\pi}{8}\right]$$

$$M = \frac{1}{2\sqrt{2}} + \frac{1}{2}\cos\frac{\pi}{8}$$

19. If $\cos(\alpha+\beta)=\frac{3}{5}, \sin(\alpha-\beta)=\frac{5}{13}$ and $0<\alpha,\beta<\frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to:

A. $\frac{21}{16}$

B. $\frac{63}{16}$

C. $\frac{63}{52}$

D. $\frac{33}{52}$

$\cos(\alpha+\beta)=\frac{3}{5} \Rightarrow \tan(\alpha+\beta)=\frac{4}{3}$

$\sin(\alpha-\beta)=\frac{5}{13} \Rightarrow \tan(\alpha-\beta)=\frac{5}{12}$

Now,

$\Rightarrow \tan(2\alpha)=\tan((\alpha+\beta)+(\alpha-\beta))$

$$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\cdot\tan(\alpha-\beta)}$$

$$=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}$$

$$=\frac{63}{16}$$

20. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ \ldots\ \cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is :

A. $\frac{1}{1024}$

B. $\frac{1}{2}$

C. $\frac{1}{512}$

D. $\frac{1}{256}$

$A=\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ \ldots\ \cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$

Put $\frac{\pi}{2^{10}}=x\Rightarrow\pi=2^{10}x$

$\Rightarrow A=(\cos x\cdot\cos 2x\cdot\ \ldots\ \cdot\cos 2^8x)\sin x$

As we know

$\frac{\sin 2^n\theta}{2^n\sin\theta}=\cos\theta\cdot\cos 2\theta\cdot\ \ldots\ \cos 2^{n-1}\theta\cdot$

Here, $n-1=8\Rightarrow n=9$

$\Rightarrow A=\frac{\sin 2^9x}{2^9\sin x}\cdot\sin x$

$\Rightarrow A=\frac{\sin 2^9\frac{\pi}{2^{10}}}{2^9}$

$\Rightarrow A=\frac{\sin\frac{\pi}{2}}{2^9}$

$\Rightarrow A=\frac{1}{512}$

21. A bird is sitting on the top of a vertical pole $20\ m$ high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (in m/s) of the bird is

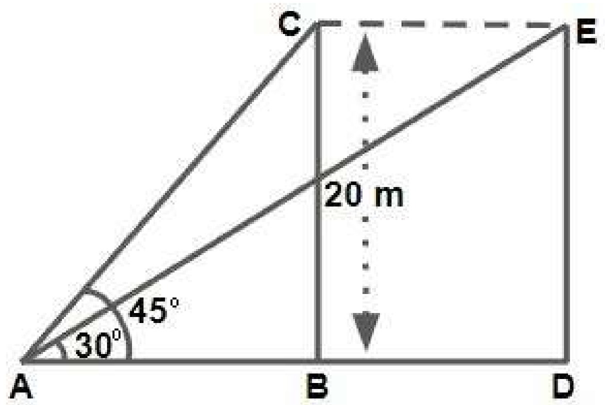
A. $40(\sqrt{2}-1)$

B. $40(\sqrt{3}-\sqrt{2})$

C. $20\sqrt{2}$

D. $20(\sqrt{3}-1)$

$\angle BAC = 45^\circ, \angle DAE = 30^\circ$ and $BC = DE = 20\ m$



$AB = 20\tan 45^\circ = 20$ and $AD = 20\cot 30^\circ = 20\sqrt{3}$ m
$BD = AD - AB = 20\sqrt{3} - 20 = 20(\sqrt{3}-1)$
$\therefore$ speed of bird $= 20(\sqrt{3}-1)\ m/s$

22. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is:

A. $\frac{-3}{5}$

B. $\frac{1}{3}$

C. $\frac{2}{9}$

D. $\frac{-7}{9}$

$5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$
$5(\sec^2 x - 1 - \cos^2 x) = 2(2\cos^2 x - 1) + 9$
Let $\cos^2 x = t$

$$5\left(\frac{1}{t} - 1 - t\right) = 2(2t - 1) + 9$$

$$5\left(1 - t - t^2\right) = 4t^2 + 7t$$

$$\Rightarrow 9t^2 + 12t - 5 = 0$$

$$t = \frac{1}{3}, \frac{-5}{3}$$

$$\Rightarrow \cos^2 x = \frac{1}{3}$$

$$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$$

$$\Rightarrow \cos 4x = -\frac{7}{9}$$

23. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to:

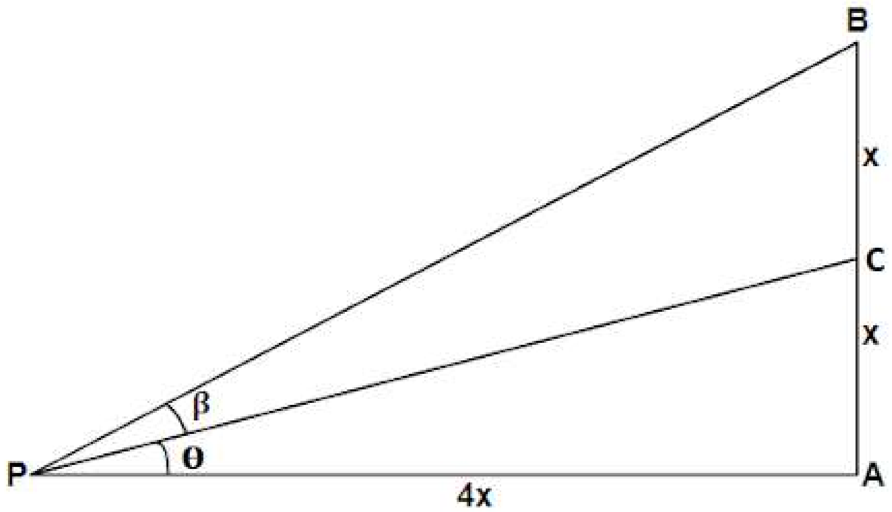
A. $\frac{6}{7}$

B. $\frac{1}{4}$

C. $\frac{2}{9}$

D. $\frac{4}{9}$

According to the question, the figure will be:



$\tan(\theta + \beta) = \frac{1}{2}$

and $\tan\theta = \frac{1}{4}$

$$\Rightarrow \tan(\theta + \beta) = \frac{\frac{1}{4} + \tan\beta}{1 - \frac{1}{4}\tan\beta} = \frac{1}{2}$$

$$\Rightarrow \tan\beta = \frac{2}{9}$$

24. PQR is a triangular park with $PQ = PR = 200$ m.
A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively $45^\circ, 30^\circ$ and 30°, then the height of the tower (in m) is:

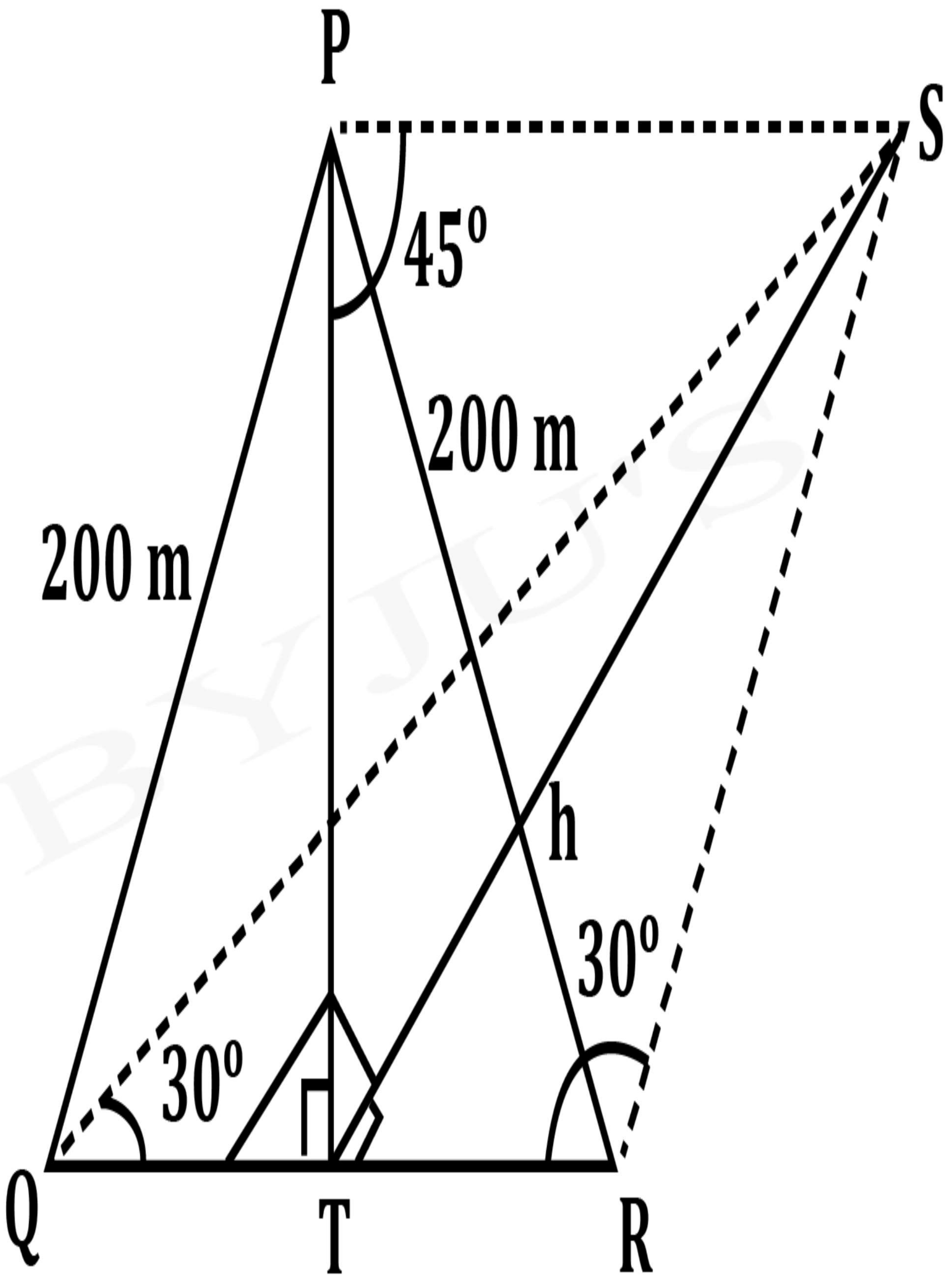
A. $50\sqrt{2}$

B. 100

C. 50

D. $100\sqrt{3}$

Let $ST = h$ (height of tower)



From figure it is clear that
In $\triangle STP$, we have

$\tan 45^\circ = \frac{ST}{PT}$

$PT = ST = h$

In $\triangle STQ$, we have

$\tan 30^\circ = \frac{ST}{QT}$

$\Rightarrow QT = h\sqrt{3}$

Now, in $\triangle PTQ$, we have

$PT^2 + QT^2 = 200^2$
$\Rightarrow 4h^2 = 200^2$
$\therefore h = 100$ m

Subject: Mathematics

1. The number of integral values of k for which the equation $3\sin x + 4\cos x = k + 1$ has a solution, $k \in \mathbb{R}$ is

   Accepted Answers

   11 11.0 11.00

   Solution:

   $3\sin x + 4\cos x = k + 1$
   The equation has a solution if
   $-\sqrt{3^2+4^2} \le k+1 \le \sqrt{3^2+4^2}$
   $\Rightarrow -5 \le k+1 \le 5$
   $\Rightarrow -6 \le k \le 4$
   $\Rightarrow k = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$
   $\therefore$ There are 11 possible integral values.

2. The number of distinct solutions of the equation, $\log_{\frac{1}{2}}|\sin x| = 2 - \log_{\frac{1}{2}}|\cos x|$ in the interval $[0, 2\pi]$, is

   Accepted Answers

   8 8.0 8.00

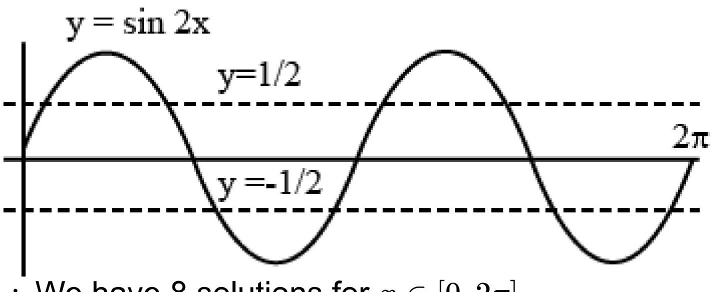
   Solution:

   $\log_{\frac{1}{2}}|\sin x| = 2 - \log_{\frac{1}{2}}|\cos x|, x \in [0, 2\pi]$

   $\Rightarrow \log_{\frac{1}{2}}|\sin x||\cos x| = 2$

   $\Rightarrow |\sin x \cos x| = \dfrac{1}{4}$

   $\therefore \sin 2x = \pm\dfrac{1}{2}$



   $\therefore$ We have 8 solutions for $x \in [0, 2\pi]$

3. The numbers of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$ in the interval $[0, 2\pi]$ is

Accepted Answers

1   1.0   1.00   01

Solution:

Case $1: x \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$

$\cot x = \cot x + \frac{1}{\sin x} \Rightarrow$ not possible

Case $2: x \in \left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

$-\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow \frac{-2\cos x}{\sin x} = \frac{1}{\sin x}$

$\Rightarrow \cos x = \frac{-1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$= 1$

4. Let S be the sum of all solutions (in radians) of the equation $\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$ in $[0, 4\pi]$. Then $\frac{8S}{\pi}$ is equal to

Accepted Answers

56   56.0   56.00

Solution:

$(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta - \sin\theta\cos\theta = 0$

Let $\sin\theta \cdot \cos\theta = t,$

$\Rightarrow 1 - 2t^2 - t = 0$

$\Rightarrow 2t^2 + t - 1 = 0$

$\Rightarrow t = \frac{1}{2}, -1$

$\Rightarrow \sin\theta\cos\theta = \frac{1}{2} \text{ or } \sin\theta\cos\theta = -1$

$\Rightarrow \sin 2\theta = 1 \text{ or } \sin 2\theta = -2 \text{ (Not Possible)}$

$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\therefore S = 7\pi$

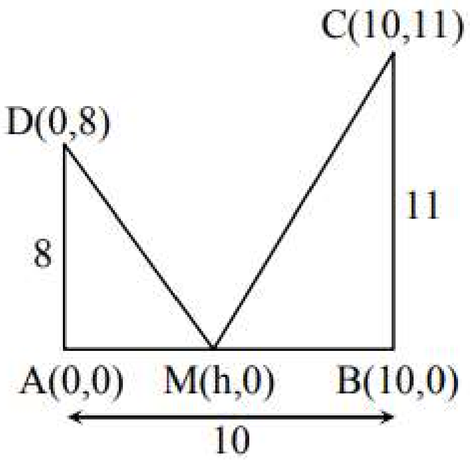
$\Rightarrow \frac{8S}{\pi} = 56$

5. Let AD and BC be two vertical poles at A and B respectively on a horizontal ground. If $AD = 8m, BC = 11m$ and $AB = 10m$; then the distance (in meters) of a point M on AB from the point A such that $MD^2 + MC^2$ is minimum is

Accepted Answers

5 5.0 5.00

Solution:



$(MD)^2 + (MC)^2 = h^2 + 64 + (h-10)^2 + 121$

$= 2h^2 - 20h + 64 + 100 + 121$

$= 2\left(h^2 - 10h\right) + 285$

$= 2(h-5)^2 + 235$

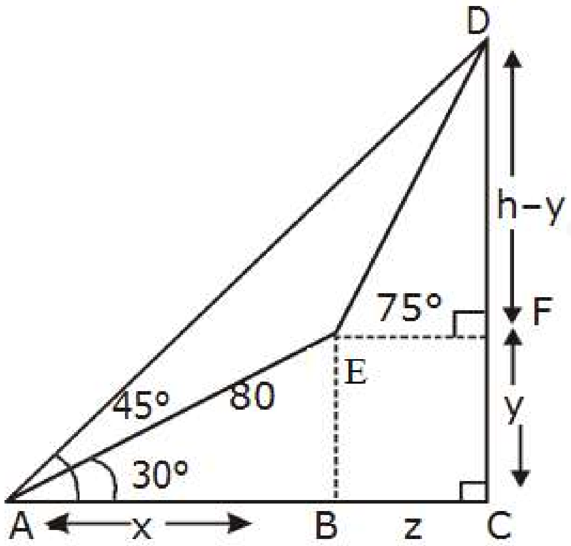
It is minimum if $h = 5$

6. The angle of elevation of the top of a hill from a point on the horizontal plane passing through the foot of the hill is found to be 45°. After walking a distance of 80 meters towards the top, up a slope inclined at an angle of 30° to the horizontal plane, the angle of elevation of the top of the hill becomes 75°. Then the height of the hill (in meters) is

Accepted Answers

80 80.0 80.00

Solution:



$x=80\cos 30^{\circ}=40\sqrt{3}$
$y=80\sin 30^{\circ}=40$
In Δ ADC

$\tan 45^{\circ}=\dfrac{h}{x+z}\Rightarrow h=x+z$

$\Rightarrow h=40\sqrt{3}+z\cdots(1)$

In Δ EDF

$\tan 75^{\circ}=\dfrac{h-y}{z}$

$2+\sqrt{3}=\dfrac{h-40}{z}\Rightarrow z=\dfrac{h-40}{2+\sqrt{3}}\cdots(2)$

Put the value of z from (1)

$h-40\sqrt{3}=\dfrac{h-40}{2+\sqrt{3}}$

$h(1+\sqrt{3})=40(2\sqrt{3}+3-1)$
$h(1+\sqrt{3})=80(1+\sqrt{3})$
$h=80$