LIMITS, CONTINUITY AND DIFFERENTIABILITY

1. Definition :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".
This implies if we can make the value of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

2. Left hand limit & right hand limit of a function:

Left hand limit LHL = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$, $h > 0$.

Right hand limit RHL = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$, $h > 0$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Important note :

In $\lim_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.
Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

3. Fundamental theorems of limits:

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\lim_{x \to a} [f(x) + g(x)] = l + m$

(b) Difference rule : $\lim_{x \to a} [f(x) - g(x)] = l - m$

(c) Product rule : $\lim_{x \to a} f(x).g(x) = \ell.m$

(d) Quotient rule : $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim_{x \to a} kf(x) = k\lim_{x \to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\lim_{x \to a} [f(x)]^{m/n} = \ell^{m/n}$ provided $\ell^{m/n}$ is a real number.

(g) $\lim_{x \to a} f[g(x)] = f\left(\lim_{x \to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim_{x \to a} \ell n(g(x)) = \ell n[\lim_{x \to a} g(x)]$

= ln (m); provided lnx is continuous at x = m, m = $\lim_{x \to a} g(x)$.

4. Indeterminate forms:

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^0, \infty^0$.

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

5. General methods to be used to evaluate limits :

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}), n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Note : $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(b) Rationalization or double rationalization :

In this method we rationalise the terms containing the irrational and simplify.

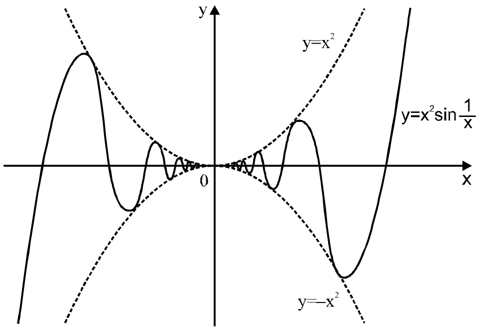
(c) Limit when x $\to \infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Subtitute x = 1/y and apply y $\to$ 0 as x $\to \infty$

(d) Squeeze play theorem (Sandwich theorem) :

If $f(x) \le g(x) \le x(x); \forall x$ in its domain , a belongs to this domain & $\lim\limits_{x \to a} f(x) = \ell = \lim\limits_{x \to a} h(x)$ then, $\lim\limits_{x \to a} g(x) = \ell$



for example: $\lim\limits_{x \to 0} x^2 \sin\frac{1}{x} = 0$, as illustrated by the graph given.

6. Limit of trigonometric functions :

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{x \to 0} \frac{\sin^{-1} x}{x}$$

[where x is measured in radians]

Note:

(a) If $\lim\limits_{x \to a} f(x) = 0$, then $\lim\limits_{x \to a} \frac{\sin f(x)}{f(x)}$ =1, e.g. $\lim\limits_{x \to 1} \frac{\sin(\ell nx)}{(\ell nx)} = 1$

(b) Using substitution

$\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a-h)$ or $\lim\limits_{h \to 0} f(a+h)$ i.e.

by substituting x by a – h or a + h

7. Limit of exponential functions :

(a) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ell na (a > 0)$ In particular $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x \to a} f(x) = 0$,then $\lim\limits_{x \to a} \frac{a^{f(x)} - 1}{f(x)} = \ell na,\ a > 0$

(b) $\lim\limits_{x \to 0} \frac{\ell n(1+x)}{x} = 1$

(c) $\lim\limits_{x \to 0} (1+x)^{1/x} = e = \lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

(Note : The base and exponent depends on the same variable)

In general, if $\lim\limits_{x\to a} f(x)=0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)}=e$

(d) If $\lim\limits_{x\to a} f(x)=1$ and $\lim\limits_{x\to a}\phi(x)=\infty$, then ; $\lim\limits_{x\to a}[f(x)]^{\phi(x)}=e^k$ where $k=\lim\limits_{x\to a}\phi(x)\,[f(x)-1]$

(e) If $\lim\limits_{x\to a} f(x)=A>0$ & $\lim\limits_{x\to a}\phi(x)=B$ (a finite quantity) then ; $\lim\limits_{x\to a}[f(x)]^{\phi(x)}=e^{B\ln A}=A^B$

8. Limit using Series Expansion :

Expansion of certain functions like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx helps in solving problems in a simpler way

(a) $a^x=1+\dfrac{x\,\ell na}{1!}+\dfrac{x^2\ell n^2 a}{2!}+\dfrac{x^3\ell n^3 a}{3!}+\text{.........}a>0$

(b) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\text{.........}$

(c) $\ell n(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\text{.......for}-1<x\le 1$

(d) $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\text{.........}$

(e) $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\text{........}$

(f) $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\text{.........}$

(g) $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\text{.......}$

(h) $\sin^{-1}x=x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+\text{.......}$

(i) $\sec^{-1}x=1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+\text{.......}$

(j) $(1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+\text{..........}\ n\in Q.$

(k) $(1+x)^{1/x}=e\left[1-\dfrac{x}{2}+\dfrac{11}{24}x^2\text{......}\right]$

9. Limit using 'L' Hospital rule :

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form, $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

provided $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists.

T

CONTINUITY

1. Continuous functions :

A function f(x) is said to be continuous at x = a, if $\lim_{x \to a} f(x) = f(a)$ Symbolically f is continuous at x = a if $\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) = f(a)$

2. Continuity of the function in an interval :

(a) A function *f* is said to be continuous in (a,b) if *f* is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

— f is continuous in the open interval (a,b)

— f is right continuous at 'a' i.e. $\lim_{x \to a^+} f(x) = f(a)$ = a finite quantity

— f is left continuous at 'b' i.e. $\lim_{x \to b^-} f(x) = f(b) =$ a finite quantity

Note :

(i) All polynomials, trigonometric functions, exponential & logarithmic functions are continuous in their domains.

(ii) If f(x) & g(x) are two functions that are continuous at x = c then the function defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any is real number

$F_3(x) = f(x).g(x)$; are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(iii) If f and g are continuous then fog and gof are also continuous.

(iv) If f and g are discontinuous at x = c, then f + g , f.g may or may not be continuous.

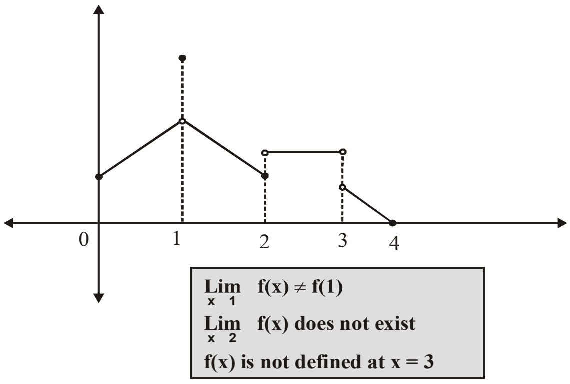
3. Reasons of discontinuity:

(a) Limit does not exist

i.e. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\lim_{x \to a} f(x) \neq f(a)$



Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1 , 2 and 3.

4. Types of discontinuities:

Type-1 : (Removable type of discontinuities) : In case $\lim_{x \to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x \to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

(a) Missing point discontinuity :

Where $\lim_{x \to a} f(x)$ exists but f(a) is not defined.

(b) Isolated point discontinuity :

Where $\lim_{x \to a} f(x)$ exists & f(a) also exists but $\lim_{x \to a} f(x) \neq f(a)$.

Type-2 : (Non-Removable type of discontinuities) :-

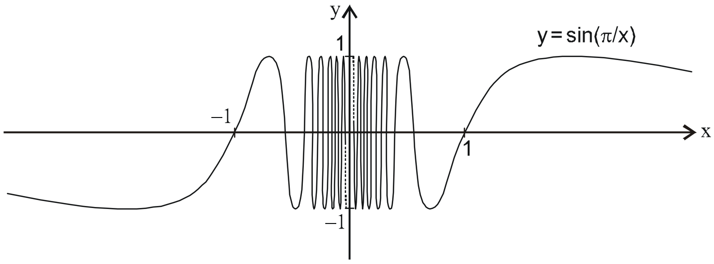
In case $\lim_{x \to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(i) Finite type discontinuity : In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(ii) Infinite type discontinuity : In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(c) Oscillatory type discontinuity :

$$f(x) = \sin\frac{\pi}{x}$$

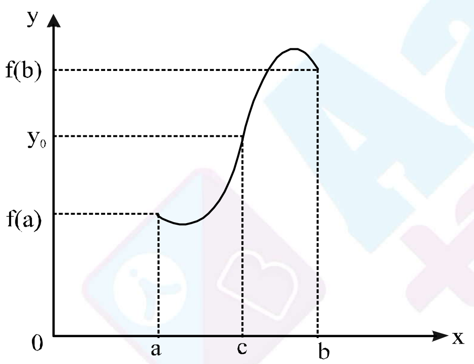


f(x) has non removable oscillatory type discontinuity at x = 0

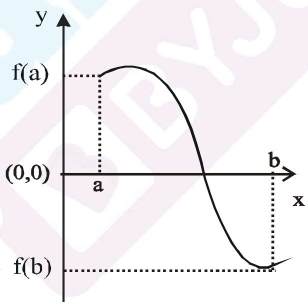
Note : In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at x = a & LHL at x = a is called THE JUMP OF DISCONTINUITY. A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

5. The intermediate value theorem

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b) then exists a number c between a and b such that f(c) = y_0



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)



Note: A function f which is continuous in [a,b] possesses the following properties:

(i) If f(a) & f(b) have opposite signs, then there exist atleast one root of the equation f(x) = 0 in the open interval (a,b).

(ii) If K is any real number between f(a) & f(b), then there exist atleast one root of the equation f(x) = K in the open interval (a,b) .

DIFFERENTIABILITY

1. Introduction

Differentiation is a method which can be used for observing the way in which functions behave. In particular, it measures how rapidly a function is changing at any given point.

2. Right hand & left hand derivatives:

(a) Right hand derivative:

The right hand derivative of f(x) at x = a denoted by $f'(a^+)$ is defined as:

$f'(a^+) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ provided the limit exists & is finite (h > 0).

(d) Left hand derivative:

The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined as :$f'(a^-) = \lim\limits_{h \to 0} \frac{f(a-h) - f(a)}{-h}$, provided the limit exists & is finite (h > 0).

(c) Derviability of function at a point :

If $f'(a^+) = f'(a^-)$ = finite quantity, then f(x) is said to be derivable or differentiable at x = a. In such case $f'(a^+) = f'(a^-) = f'(a)$ & it is called derivative or differentical coefficient of f(x) at x = a.

Note :

(i) All polynomials, trigonometric functions, inverse trigonometric functions, logarithmic and exponential functions are continuous and differentaiable in their domains, except at end points.

(ii) If f(x) & g(x) are derivable at x = a then the functions f(x)+g(x), f(x)–g(x) will also be derivable at x = a & if g(a) $\neq 0$ then the function f(x)/g(x) will also be derivable at x = a

3. Importatn note:

(a) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a

(ii) $p \neq q \Rightarrow$ f is not derivable at x = a,

It is very important to note that 'f' may be still continuous at x = a

In short, for a function 'f' :

Differentiable	$\Rightarrow$	Continuious :
Not Differentiable	$\Rightarrow$	Not Continuious
But Not Continuous	$\Rightarrow$	Not Differentiable
Continuous	$\Rightarrow$	May or may not be Differentiable

(b) Geometrical interpretation of differentiability :

(i) If the function y = f(x) is differentiable at x = a, then a unique tangent can be drawn to the curve y = f(x) at the point P(a, f(a)) & f'(a) represent the slope of the tangent at point P.

(ii) If LHD and RHD are finite but unequal then it geometrically implies a sharp corner at x = a e.g. f(x) = |x| is continuous but not differentiable at x = 0 A sharp corner is seen at x = 0 in the graph of f(x) = |x|

(iii) If a function has vertical tangent at x = a then also it is nonderivable at x = a.

(c) Vertical tangent :

If for y = f(x)

$f(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a^-) \to -\infty$

then at x = a, y = f(x) has vertical tangent but f(x) is not differentiable at x = a

4. Differentiability over an interval

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if:

(i) f(x) is differentiable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Note :

(i) If f(x) is differentiable at x = a & g(x) is not differentiable at x =a, then the product function F(x)=f(x).g(x) may or may not be differentiable at x = a.

(ii) If f(x) & g(x) both are not differentiable at x = a then the product function; F(x)=f(x).g(x) may or may not be differentiable at x = a.

(iii) If f(x) & g(x) both are non-differentiable at x=a then the sum function F(x)=f(x)+g(x) may be a differentiable function.

(iv) If f(x) is differentiable at x = a $\not\Rightarrow$ f'(x) is continuous at x = a.

METHODS OF DIFFERENTIATION

1. Derivative of F(x) from the first Principle :

Obtaining the derivative using the definition $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

2. Fundamental Theorems :

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

(b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"PRODUCT RULE"**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$

where $g \neq 0$ known as **"QUOTIENT RULE"**

(e) If y = f(u) & u = g (x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as **"CHAIN RULE"**

Note : In general if y = f(u) then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

3. Derivative of Standard Functions :

	f(x)	f'(x)
(i)	x^n	nx^{n-1}
(ii)	e^x	e^x
(iii)	a^x	$a^x \ln a$, a > 0
(iv)	$\ln x$	1/x, x > 0
(v)	$\log_a x$	$(1/x) \log_a e$, a > 0, $a \neq 1$, x > 0
(vi)	sinx	cosx
(vii)	cosx	– sinx
(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx

(x)	cosecx	– cosecx . cotx
(xi)	cotx	$-\operatorname{cosec}^2 x$
(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$
(xvi)	$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}, \|x\| > 1$
(xvii)	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\|x\|\sqrt{x^2-1}}, \|x\| > 1$
(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in R$

4. Logarithmic Differentiation :

To find the derivative of a function which is :
(a) The product or quotient of a number of functions or
(b) of the form $[f(x)]^{g(x)}$ where f & g are both derivable.
Then it is convenient to take the logarithm of the function first & then differentiate.

5. Differentiation of Implicit Functions :

(a) Let function be $\phi(x, y) = 0$ then to find dy /dx in the case of implicit functions, we differentiate each term w.r.t. x & then collect terms in dy/dx together on one side.

or $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$ where $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are partial differential coefficient of f(x,y) w.r.to x & y respectively.

(b) in answers of dy/dx in the case of implicit functions, generally , both x & y are present.

6. Parametric Differentition :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

7. Derivative of a function w.r.t. Another Function :

Let y= f (x) ; z = g (x) then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

8. Derivative of a Function and its Inverse Function :

If inverse of y = f(x)

$x = f^{-1}(y)$ is denoted by x = g(y) then g(f(x)) = x then g'(f(x))f'(x)=1

9. Higher Order Derivatives :

Let a function y = f (x) be defined on an interval (a, b). Its derivative if it exists on (a,b) is a certain function f '(x) [or (dy/dx) or y'] & it is called the first derivative of y w.r.t. x. If first derivative has a derivative on (a,b) then this derivative is called second derivative of y w.r.t. x.& is denoted by f''(x) or $[d^2y/dx^2]$ or y''. Similarly, the 3rd order derivative of y w.r.t x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by f'''(x) or y''' & so on.

10. Differentiation of Determinants :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x

then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

11. L' Hospital's Rule :

(a) This rule is applicable for the indeterminate forms in limits of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

provided the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists (L' Hospital's rule). The point 'a' may be either finite or infinite ($+\infty$ or $-\infty$).

(b) In limits indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) In limits indeterminate forms of the type 1^∞, ∞^0 or 0^∞ are reduced to forms of the type $0.\infty$ by taking logarithms or by the transformation $[f(x)]^{f(x)} = e^{f(x).\ln f(x)}$.