

Subject: Mathematics

1. For each $x \in \mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x . Then

$$\lim_{x \rightarrow 0^-} \frac{x([x] + |x|) \sin[x]}{|x|} \text{ is equal to :}$$

- A. $\sin 1$
- B. 0
- C. $-\sin 1$
- D. 1

$$\lim_{x \rightarrow 0^-} \frac{x([x] + |x|) \sin[x]}{|x|}$$

We know,

As $x \rightarrow 0^-$, $[x] = -1$ and $|x| = -x$

$$\text{So, } \lim_{x \rightarrow 0^-} \frac{x(-1-x) \sin(-1)}{-x}$$

$$= \lim_{x \rightarrow 0^-} -\frac{(1+x) \sin 1}{1}$$

$$= -\frac{(1+0) \sin 1}{1}$$

$$= -\sin 1$$

2. $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3}$ is equal to :

- A. $\frac{2}{3}$
- B. 0
- C. $\frac{1}{15}$
- D. $\frac{3}{2}$

$$\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\sin |x| \cdot 2x}{3x^2} = \frac{2}{3}$$

3. For each $t \in \mathbf{R}$, let $[t]$ be the greatest integer less than or equal to t . Then,

$$\lim_{x \rightarrow 1^+} \frac{(1 - |x| + \sin |1 - x|) \sin\left(\frac{\pi}{2}[1 - x]\right)}{|1 - x|[1 - x]}$$

- A. equals 0
- B. equals 1
- C. equals -1
- D. does not exist

$$L = \lim_{x \rightarrow 1^+} \frac{(1 - |x| + \sin |1 - x|) \sin\left(\frac{\pi}{2}[1 - x]\right)}{|1 - x|[1 - x]}$$

$$\because x \rightarrow 1^+ \Rightarrow x > 1$$

$$\Rightarrow [1 - x] = -1, |1 - x| = x - 1$$

$$\begin{aligned} &\Rightarrow L = \lim_{x \rightarrow 1^+} \frac{(1 - x + \sin(x - 1)) \sin\left(\frac{\pi}{2}(-1)\right)}{(x - 1)(-1)} \\ &= \lim_{x \rightarrow 1^+} \left(1 - \frac{\sin(x - 1)}{x - 1}\right) (-1) \\ &= -1 + 1 = 0 \end{aligned}$$

4. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \max \{-|x|, -\sqrt{1-x^2}\}$.

If K be the set of all points at which f is not differentiable, then K has exactly

- A. one element
- B. two elements
- C. three elements
- D. five elements

Given

$$f : (-1, 1) \rightarrow \mathbb{R}$$

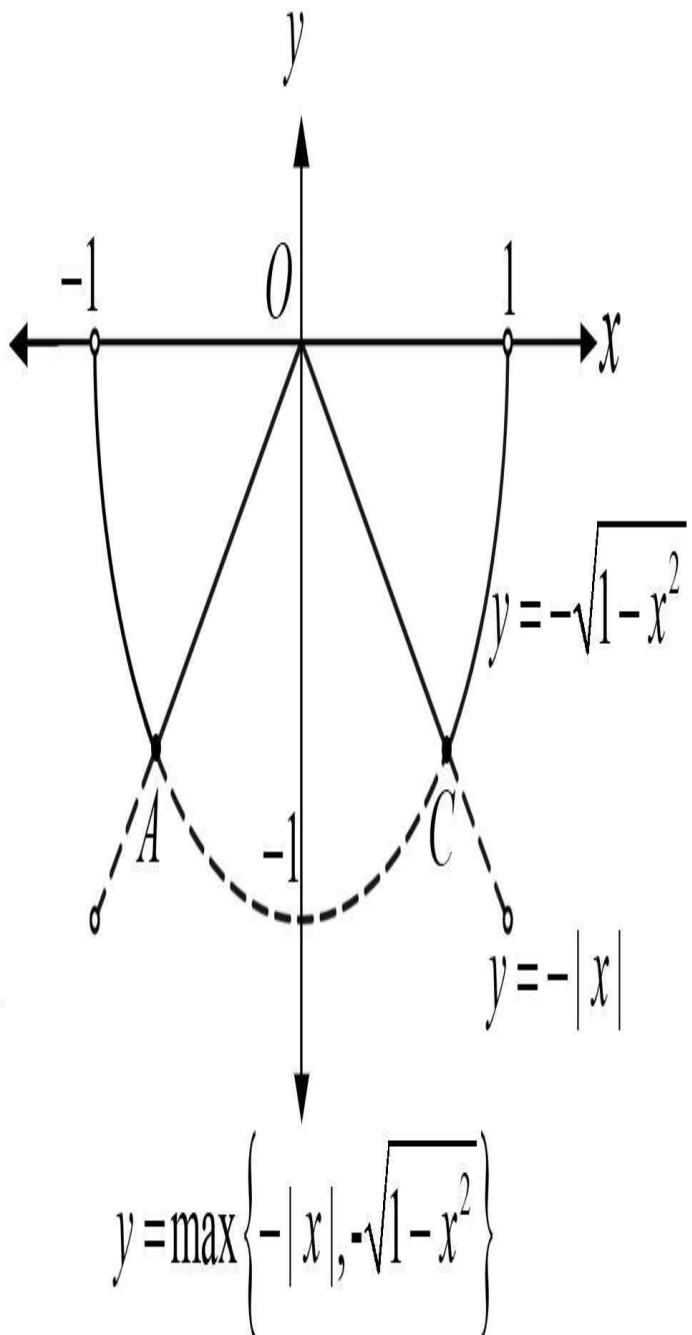
$$f(x) = \max \{-|x|, -\sqrt{1-x^2}\}$$

$$y = -|x| = \begin{cases} -x, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$$

$$\text{and } y = -\sqrt{1-x^2} \Rightarrow y^2 + x^2 = 1$$

The above equation represents a semicircle with unit radius below the x -axis.

So, plotting curves for $-|x|, -\sqrt{1-x^2}$ and representing $\max \{-|x|, -\sqrt{1-x^2}\}$ through solid line we get curve as



Clearly, from the above curve $f(x)$ is not differentiable at three points A, O, C .

5. Let $[x]$ denote the greatest integer less than or equal to x . Then :

$$\lim_{x \rightarrow 0} \frac{\tan(\pi \sin^2 x) + (|x| - \sin(x[x])))^2}{x^2}:$$

- A. equals π
- B. equals 0
- C. equals $\pi + 1$
- D. does not exist

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 0^-} \frac{\tan(\pi \sin^2 x) + (-x - \sin x(-1))^2}{x^2} \\ &= \lim_{x \rightarrow 0^-} \frac{\tan(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2} + \left(-1 + \frac{\sin x}{x} \right)^2 \\ &= 1 \times \pi + (-1 + 1)^2 \\ &= \pi \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{x \rightarrow 0^+} \frac{\tan(\pi \sin^2 x) + (x - \sin x^2)^2}{x^2} \\ &= \lim_{x \rightarrow 0^+} \frac{\tan(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2} + \left(1 - \frac{\sin x^2}{x} \right)^2 \\ &= 1 \times \pi + (1 - 0)^2 \\ &= \pi + 1 \end{aligned}$$

$\therefore \text{LHL} \neq \text{RHL}$
 \Rightarrow Limit does not exist.

6. If $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2}$, then k is :



A. $\frac{4}{3}$



B. $\frac{8}{3}$



C. $\frac{3}{8}$



D. $\frac{3}{2}$

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)x^3 + x^2 + x + 1}{(x - 1)} \\ &= \lim_{x \rightarrow 1} \frac{x^3 + x^2 + x + 1}{1} = 4\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2} &= \lim_{x \rightarrow k} \frac{x^2 + kx + k^2}{x + k} = \frac{3k^2}{2k} \\ \Rightarrow 4 &= \frac{3k^2}{2k} \Rightarrow k = \frac{8}{3}\end{aligned}$$

7. If $f(x) = \begin{cases} \frac{\sin(\alpha+2)x + \sin x}{x}, & x < 0 \\ b, & x = 0 \\ \frac{(x+3x^2)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{4}{3}}}, & x > 0 \end{cases}$ is continuous at $x = 0$ then $a + 2b$
is equal to :

- A. -2
- B. 1
- C. 0
- D. -1

$f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = b = \lim_{x \rightarrow 0^+} f(x)$$

$$\begin{aligned} b &= \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{(h+3h^2)^{\frac{1}{3}} - h^{\frac{1}{3}}}{h^{\frac{4}{3}}} \\ &\Rightarrow b = \lim_{h \rightarrow 0} \frac{(1+3h)^{\frac{1}{3}} - 1}{h} \\ &\Rightarrow b = \lim_{h \rightarrow 0} \frac{1}{3}(1+3h)^{-\frac{2}{3}} \times 3 \\ &\Rightarrow b = 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= 1 \\ \Rightarrow \lim_{h \rightarrow 0} \frac{\sin((\alpha+2)(-h)) + \sin(-h)}{-h} &= 1 \\ \Rightarrow a+3 &= 1 \Rightarrow a = -2 \\ \Rightarrow a+2b &= 0 \end{aligned}$$

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} 5, & \text{if } x \leq 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \leq x < 5 \\ 30, & \text{if } x \geq 5 \end{cases}$$

Then, which of the following is correct regarding the function f :

- A. continuous if $a = -5$ and $b = 10$
- B. continuous if $a = 5$ and $b = 5$
- C. continuous if $a = 0$ and $b = 5$
- D. Not continuous for any values of a and b

The given function will be continuous everywhere when it is continuous at $x = 1, 3$ and 5 .

For continuity at $x = 1$; $f(1^-) = f(1) = f(1^+)$

$$a + b = 5 \dots (i)$$

Similarly, at $x = 3$ and at $x = 5$

$$a + 3b = b + 15 \dots (ii)$$

$$b + 25 = 30 \dots (iii)$$

From (i) and (iii), we get $a = 0, b = 5$

From (ii) and (iii), we get $a = 5, b = 5$

Clearly, the function is not continuous for any values of a and b .

9. Let the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} x + 2, & x < 0 \\ x^2, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1 \end{cases}$$

Then, the number of points in \mathbb{R} where $(fog)(x)$ is non differentiable is equal to :

- A. 1
- B. 2
- C. 3
- D. 0

$$fog(x) = \begin{cases} x^3 + 2, & x < 0 \\ x^6, & 0 \leq x < 1 \\ (3x - 2)^2, & x \geq 1 \end{cases}$$

Clearly, $fog(x)$ is discontinuous at $x = 0$ then non - differentiable at $x = 0$

Now, at $x = 1$

R.H.D.

$$= \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{(3(1+h) - 2)^2 - 1}{h} = 6$$

L.H.D.

$$= \lim_{h \rightarrow 0^-} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0^-} \frac{(1-h)^6 - 1}{-h} = 6$$

Number of point of non - differentiability is 1.

10. If a function $f(x)$ defined by

$$f(x) = \begin{cases} ae^x + be^{-x}, & -1 \leq x < 1 \\ cx^2, & 1 \leq x \leq 3 \\ ax^2 + 2cx & 3 < x \leq 4 \end{cases}$$

be continuous for some $a, b, c \in \mathbb{R}$ and $f'(0) + f'(2) = e$, then the value of a is :

- A. $\frac{1}{e^2 - 3e + 13}$
- B. $\frac{e}{e^2 - 3e - 13}$
- C. $\frac{e}{e^2 + 3e + 13}$
- D. $\frac{e}{e^2 - 3e + 13}$

$f(x)$ is continuous

$$\text{at } x = 1 \Rightarrow c = ae + be^{-1}$$

$$\text{at } x = 3 \Rightarrow 9c = 9a + 6c \Rightarrow c = 3a$$

$$\text{Now } f'(0) + f'(2) = e$$

$$\Rightarrow a - b + 4c = e$$

$$\Rightarrow a - e(3a - ae) + 4 \cdot 3a = e$$

$$\Rightarrow a - 3ae + ae^2 + 12a = e$$

$$\Rightarrow 13a - 3ae + ae^2 = e$$

$$\Rightarrow a = \frac{e}{13 - 3e + e^2}$$

11. $\lim_{n \rightarrow \infty} \left(1 + \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n^2} \right)^n$ is equal to :

- A. $\frac{1}{2}$
- B. $\frac{1}{e}$
- C. 1
- D. 0

It is 1^∞ form

$$L = e^{\lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{n} \right)}$$

$$\Rightarrow L = e^{\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \dots + \frac{1}{n^2} \right)}$$

$$\therefore L = e^0 = 1$$

12. The value of the limit $\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$ is equal to:

- A. $-\frac{1}{2}$
- B. $-\frac{1}{4}$
- C. 0
- D. $\frac{1}{4}$

$$\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\tan(\pi - \pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \quad (\because \cos^2 \theta = 1 - \sin^2 \theta)$$

$$= \lim_{\theta \rightarrow 0} \frac{-\tan(\pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \quad (\because \tan(\pi - \theta) = \tan \theta)$$

$$= \lim_{\theta \rightarrow 0} -\frac{1}{2} \left(\frac{\tan(\pi \sin^2 \theta)}{\pi \sin^2 \theta} \right) \left(\frac{2\pi \sin^2 \theta}{\sin(2\pi \sin^2 \theta)} \right)$$

$$= -\frac{1}{2}$$

13. The value of $\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \cdot \sin^{-1}(x - [x]^2)}{x - x^3}$, where $[x]$ denotes the greatest integer $\leq x$ is:



A. 0



B. $\frac{\pi}{4}$



C. $\frac{\pi}{2}$



D. π

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \sin^{-1}(x - [x]^2)}{x - x^3} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos^{-1} x \sin^{-1} x}{x(1 - x^2)} \quad (\because [0^+] = 0) \\ &= \frac{\cos^{-1} 0^+}{1 - 0} = \frac{\pi}{2} \end{aligned}$$

14. If the function $f(x) = \begin{cases} k_1(x - \pi)^2 - 1, & x \leq \pi \\ k_2 \cos x, & x > \pi \end{cases}$ is twice differentiable,
then the ordered pair (k_1, k_2) is equal to:

- A. $(1, 1)$
- B. $(1, 0)$
- C. $\left(\frac{1}{2}, -1\right)$
- D. $\left(\frac{1}{2}, 1\right)$

$f(x)$ is continuous and differentiable

$$f(\pi^-) = f(\pi) = f(\pi^+)$$

$$\Rightarrow k_2 \cos \pi = k_1(0) - 1$$

$$\Rightarrow k_2(-1) = -1$$

$$\Rightarrow k_2 = 1$$

$$f'(x) = \begin{cases} 2k_1(x - \pi); & x < \pi \\ -k_2 \sin x; & x > \pi \end{cases}$$

$$f'(\pi^-) = f'(\pi^+)$$

$$\Rightarrow 0 = 0$$

So, differentiable at $x = 0$

$$f''(x) = \begin{cases} 2k_1; & x < \pi \\ -k_2 \cos x; & x > \pi \end{cases}$$

$$\Rightarrow f''(\pi^-) = f''(\pi^+)$$

$$\Rightarrow 2k_1 = k_2$$

$$\Rightarrow k_1 = \frac{1}{2}$$

15. If $\lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) = b$, then the ordered pair (a, b) is

- A. $\left(1, \frac{1}{2}\right)$
- B. $\left(-1, -\frac{1}{2}\right)$
- C. $\left(-1, \frac{1}{2}\right)$
- D. $\left(1, -\frac{1}{2}\right)$

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \sqrt{x^2 - x + 1} - ax \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 - x + 1) - (ax)^2}{\sqrt{x^2 - x + 1} + ax} \\ &= \lim_{x \rightarrow \infty} \frac{(1 - a^2)x^2 - x + 1}{\sqrt{x^2 - x + 1} + ax} \end{aligned}$$

For limit to exist finitely, $1 - a^2 = 0$

$$\Rightarrow L = \lim_{x \rightarrow \infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} + ax} = \lim_{x \rightarrow \infty} \frac{-1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2} + a}}$$

$$L = \frac{-1}{1 + a} = b$$

For b to be finite, $a \neq -1$

$$\therefore a = 1, b = \frac{-1}{2}$$

16. Let a function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \sin x - e^x & \text{if } x \leq 0 \\ a + [-x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \geq 1 \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x . If f is continuous on \mathbb{R} , then $(a + b)$ is equal to



A. 5



B. 3



C. 4



D. 2

$$f(x) = \begin{cases} \sin x - e^x & \text{if } x \leq 0 \\ a - 1 - [x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \geq 1 \end{cases}$$

Given: $f(x)$ is continuous everywhere

$\therefore f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

$$\Rightarrow -1 = a - 1 \Rightarrow a = 0$$

$f(x)$ is also continuous at $x = 1$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$$

$$\Rightarrow a - 1 = 2 - b \Rightarrow -1 = 2 - b$$

$$\Rightarrow b = 3$$

$$\therefore a + b = 3$$

17. The value of $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$ is equal to

- A. 4
- B. -4
- C. -1
- D. 0

Let consider

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0} \frac{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}}{x} \times \frac{\sin x}{\sin x} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}}{\sin x} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x} - 1 + 1}{\sin x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sqrt[8]{1 - \sin x} - 1}{\sin x} - \frac{\sqrt[8]{1 + \sin x} - 1}{\sin x} \right) \\
 &= \lim_{x \rightarrow 0} \left((-1) \frac{(1 - \sin x)^{1/8} - 1^{1/8}}{(1 - \sin x) - 1} - \frac{(1 + \sin x)^{1/8} - 1^{1/8}}{(1 + \sin x) - 1} \right) \\
 &= (-1) \frac{1}{8} \cdot 1 - \frac{1}{8} = -\frac{1}{4}
 \end{aligned}$$

$$\text{Now, } \frac{1}{L} = \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right) = -4$$

18. If α, β are the distinct roots of $x^2 + bx + c = 0$, then

$$\lim_{x \rightarrow \beta} \frac{e^{2(x^2+bx+c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2} \text{ is equal to}$$

- A. $b^2 - 4c$
- B. $b^2 + 4c$
- C. $2(b^2 + 4c)$
- D. $2(b^2 - 4c)$

Given: α, β are roots of $x^2 + bx + c = 0$

$$\therefore x^2 + bx + c = (x - \alpha)(x - \beta) \quad \dots (i)$$

$$\text{Also } \beta^2 + b\beta + c = 0 \quad \dots (ii)$$

$$\text{Now, } L = \lim_{x \rightarrow \beta} \frac{e^{2(x^2+bx+c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2 \times (x - \alpha)^2}$$

$$L = \lim_{x \rightarrow \beta} \frac{e^{2(x^2+bx+c)} - 1 - 2(x^2 + bx + c)}{(x^2 + bx + c)^2} \times \lim_{x \rightarrow \beta} (x - \alpha)^2 \text{ (From (i))}$$

Let $x^2 + bx + c = t$

Then, $x \rightarrow \beta \Rightarrow t \rightarrow 0$ (From (ii))

$$\therefore L = \lim_{t \rightarrow 0} \frac{e^{2t} - 1 - 2t}{t^2} \times (\beta - \alpha)^2$$

Now, using expansion, we have:

$$L = \lim_{t \rightarrow 0} \frac{\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \dots\right) - 1 - 2t}{t^2} \times (\alpha - \beta)^2$$

$$\Rightarrow L = 2(\alpha - \beta)^2$$

$$\Rightarrow L = 2[(\alpha + \beta)^2 - 4\alpha\beta]$$

$$\Rightarrow L = 2[(-b)^2 - 4c]$$

$$\therefore L = 2(b^2 - 4c)$$

19. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x + 1$, then the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + \dots + f\left(\frac{5(n-1)}{n}\right) \right], \text{ is:}$$

- A. $\frac{7}{2}$
- B. $\frac{3}{2}$
- C. $\frac{5}{2}$
- D. $\frac{1}{2}$

$$\begin{aligned}
 & f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + \dots + f\left(\frac{5(n-1)}{n}\right) \\
 \Rightarrow & 1 + 1 + \frac{5}{n} + 1 + \frac{10}{n} + \dots + 1 + \frac{5(n-1)}{n} \\
 \Rightarrow & n + \frac{\frac{5(n-1)n}{2}}{n} = \frac{2n + 5n - 5}{2} = \frac{7n - 5}{2} \\
 \lim_{n \rightarrow \infty} & \frac{1}{n} \left(\frac{7n - 5}{2} \right) = \frac{7}{2}
 \end{aligned}$$

20. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{x^3}{(1 - \cos 2x)^2} \cdot \log_e \left(\frac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2} \right) & , \quad x \neq 0 \\ \alpha & , \quad x = 0 \end{cases}$$

If f is continuous at $x = 0$, then α is equal to:



A. 0



B. 1



C. 2



D. 3

Given $f(x)$ is continuous at $x = 0$

$$\begin{aligned} \therefore \alpha &= \lim_{x \rightarrow 0} \frac{x^3}{(1 - \cos 2x)^2} \log_e \left(\frac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2} \right) \\ \alpha &= \lim_{x \rightarrow 0} \frac{x^4}{4 \sin^4 x} \cdot \frac{1}{x} \log_e \left(\frac{e^{2x} + 2x}{x^2 - 2xe^x + e^{2x}} \right) \\ &= \frac{1}{4} \lim_{x \rightarrow 0} \left\{ \frac{\ln(e^{2x} + 2x)}{x} - \frac{\ln(x^2 - 2xe^x + e^{2x})}{x} \right\} \end{aligned}$$

On applying L'hospital rule

$$\begin{aligned} &= \frac{1}{4} \lim_{x \rightarrow 0} \left\{ \frac{2e^{2x} + 2}{e^{2x} + 2x} - \frac{2x - 2e^x(x+1) + 2e^{2x}}{x^2 - 2xe^x + e^{2x}} \right\} \\ &= \frac{1}{4}(4 - 0) = 1 \end{aligned}$$

21. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{\lambda|x^2 - 5x + 6|}{\mu(5x - x^2 - 6)}, & x < 2 \\ \frac{\tan(x-2)}{e^{x-[x]}}, & x > 2 \\ \mu, & x = 2 \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x . If f is continuous at $x = 2$, then $\lambda + \mu$ is equal to

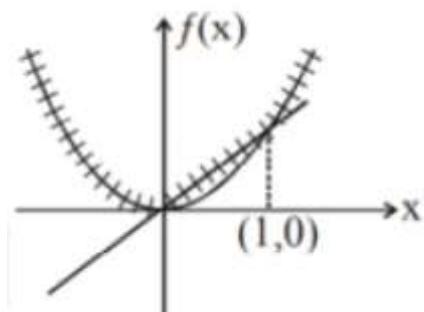
- A. 1
- B. $e(e-2)$
- C. $e(-e+1)$
- D. $2e-1$

For continuity $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 2^-} \frac{\lambda|(x-2)(x-3)|}{\mu(x-2)(3-x)} &= \lim_{x \rightarrow 2^+} e \frac{\tan(x-2)}{x-2} = \mu \\ \Rightarrow \frac{\lambda(1)}{\mu(-1)} &= e = \mu \\ \Rightarrow \mu &= e \text{ and } \lambda = -e^2 \\ \Rightarrow \lambda + \mu &= e - e^2 = e(1 - e) \end{aligned}$$

22. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \max \{x, x^2\}$. Let S denote the set of all points in \mathbb{R} , where f is not differentiable. Then

- A. $\{0, 1\}$
- B. \emptyset (an empty set)
- C. $\{1\}$
- D. $\{0\}$



Function is not differentiable at two points

$\{0, 1\}$

23. Let $S = \{t \in \mathbb{R} : f(x) = |x - \pi| \cdot (e^{|x|} - 1) \sin |x| \text{ is not differentiable at } t\}$.
 Then the set S is equal to

- A. $\{0, \pi\}$
- B. \emptyset (an empty set)
- C. $\{0\}$
- D. $\{\pi\}$

Given: $f(x) = |x - \pi| \cdot (e^{|x|} - 1) \cdot \sin |x|$

Doubtful points = $0, \pi$

Now, at $x = 0$

$$\begin{aligned}f'(0^+) &= \lim_{h \rightarrow 0^+} \left(\frac{|h - \pi|(e^{|h|} - 1) \sin |h|}{h} \right) \\&\Rightarrow f'(0^+) = \lim_{h \rightarrow 0^+} \left(\frac{|\pi - h|(e^h - 1) \sin h}{h} \right) \\&\Rightarrow f'(0^+) = 0 \\&\Rightarrow f'(0^-) = \lim_{h \rightarrow 0^+} \left(\frac{|-h - \pi|(e^{-h} - 1) \sin |-h|}{-h} \right) \\&\Rightarrow f'(0^-) = 0\end{aligned}$$

And, at $x = \pi$

$$\begin{aligned}f'(\pi^+) &= \lim_{h \rightarrow 0^+} \left(\frac{|h| \cdot (e^{|\pi+h|} - 1) \sin |\pi+h|}{h} \right) \\&\Rightarrow f'(\pi^+) = \lim_{h \rightarrow 0^+} \left(\frac{h(e^{\pi+h} - 1) \cdot (-\sin h)}{h} \right) \\&\Rightarrow f'(\pi^+) = 0 \\f'(\pi^-) &= \lim_{h \rightarrow 0^+} \left(\frac{h(e^{\pi-h} - 1) \sin h}{-h} \right) \\&\Rightarrow f'(\pi^-) = 0\end{aligned}$$

Hence, $f(x)$ is diff. for all $x \in \mathbb{R}$.

24. $\lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4}$ is equal to:

- A. 4π
- B. π^2
- C. $4\pi^2$
- D. $2\pi^2$

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2(\pi - \pi \cos^4 x)}{x^4} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2(\pi - \pi \cos^4 x)}{(\pi - \pi \cos^4 x)^2} \times \frac{(\pi - \pi \cos^4 x)^2}{x^4} \\
 &= \lim_{x \rightarrow 0} 1 \times \pi^2 \frac{(1 - \cos^4 x)^2}{x^4} \\
 &= \pi^2 \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)^2 (1 + \cos^2 x)^2}{x^4} \\
 &= \pi^2 \lim_{x \rightarrow 0} \frac{\sin^4 x (1 + \cos^2 x)^2}{x^4} \\
 &= \pi^2 \times 1 \times (1 + 1)^2 \\
 &= 4\pi^2
 \end{aligned}$$

25. Let $f : R \rightarrow R$ be a function defined as

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x}, & \text{if } x < 0 \\ b, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}, & \text{if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then the value of $a + b$ is equal

- A. -2
- B. $-\frac{5}{2}$
- C. $-\frac{3}{2}$
- D. -3

$$\text{Given : } f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x}, & \text{if } x < 0 \\ b, & \text{if } x = 0 \\ \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}, & \text{if } x > 0 \end{cases}$$

f is continuous at $x = 0$, then

$$f(0^-) = f(0) = f(0^+)$$

Now,

$$\begin{aligned} f(0^-) &= \lim_{x \rightarrow 0^-} \frac{\sin(a+1)x + \sin 2x}{2x} \\ &\Rightarrow f(0^-) = \frac{a+1}{2} + 1 = \frac{a+3}{2} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} f(0^+) &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} \\ &\Rightarrow f(0^+) = \lim_{x \rightarrow 0^+} \frac{bx^3}{bx^{5/2} \cdot (\sqrt{x+bx^3} + \sqrt{x})} \\ &\Rightarrow f(0^+) = \lim_{x \rightarrow 0^+} \frac{b}{b \cdot (\sqrt{1+bx^2} + 1)} \\ &\Rightarrow f(0^+) = \frac{1}{2} \quad \dots (2) \end{aligned}$$

From equations (1) and (2), we get

$$a = -2, \quad b = \frac{1}{2}$$

$$\therefore a + b = -\frac{3}{2}$$

26. If $f(x) = \begin{cases} \frac{1}{|x|}; & |x| \geq 1 \\ ax^2 + b; & |x| < 1 \end{cases}$ is differentiable at every point of the domain, then the values of a and b are respectively:

- A. $\frac{5}{2}, -\frac{3}{2}$
- B. $-\frac{1}{2}, \frac{3}{2}$
- C. $\frac{1}{2}, \frac{1}{2}$
- D. $\frac{1}{2}, -\frac{3}{2}$

$f(x)$ is continuous at $x = 1 \Rightarrow 1 = a + b$

$f(x)$ is differentiable at $x = 1 \Rightarrow -1 = 2a$

$$\Rightarrow a = -\frac{1}{2}$$

$$\therefore b = \frac{3}{2}$$

27. $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos 3x)}{x \tan 4x}$ is equal to :

- A. $\frac{1}{4}$
- B. $\frac{1}{2}$
- C. 1
- D. 2

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos 3x)}{x \tan 4x} \\ &= \lim_{x \rightarrow 0} \frac{(2 \sin^2 x)(3 + \cos 3x)}{x \tan 4x} \\ &= \lim_{x \rightarrow 0} \left(\frac{2}{4} \right) \left(\frac{3 + \cos 3x}{1} \right) \left(\frac{\sin^2 x}{x^2} \right) \left(\frac{4x}{\tan 4x} \right) \\ &= \frac{1}{2} \cdot (3 + 1) = 2 \end{aligned}$$

28. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by

$f(x) = [x] \cos\left(\frac{2x-1}{2}\pi\right)$, where $[x]$ denotes the greatest integer function, then f is

- A. continuous for every real x .
- B. discontinuous only at $x = 0$.
- C. discontinuous only at non-zero integral values of x .
- D. continuous only at $x = 0$.

The given function can be continuous between two integers. But when we talk at integers, we need to check the continuity by putting limits on both sides of integers for this function.

$$\begin{aligned} L.H.L. &= \lim_{x \rightarrow n^-} [x] \cos\left(\frac{2x-1}{2}\pi\right) \\ &= (n-1) \cos\left(\frac{2n-1}{2}\pi\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.L. &= \lim_{x \rightarrow n^+} [x] \cos\left(\frac{2x-1}{2}\pi\right) \\ &= n \cos\left(\frac{2n-1}{2}\pi\right) \\ &= 0 \end{aligned}$$

and $f(n) = 0$

$$\therefore f(n^-) = f(n^+) = f(0)$$

Hence, the function is continuous for every real x .

29. If $\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)}$ exists and is equal to b , then the value of $a - 2b$ is

Accepted Answers

5 5.0 5.00

Solution:

$$\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)}$$

Applying L'Hospital Rule

$$\lim_{x \rightarrow 0} \frac{a - 4e^{4x}}{a(e^{4x} - 1) + ax(4e^{4x})}$$

So for limit to exist, $a = 4$

Applying L'Hospital Rule

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{-16e^{4x}}{a(4e^{4x}) + a(4e^{4x}) + ax(16e^{4x})} \\ &= \frac{-16}{4a + 4a} = \frac{-16}{32} = -\frac{1}{2} = b \\ & \therefore a - 2b = 4 - 2 \left(\frac{-1}{2} \right) = 4 + 1 = 5 \end{aligned}$$

30.

Consider the function $f(x) = \begin{cases} \frac{P(x)}{\sin(x-2)}, & x \neq 2 \\ 7, & x = 2 \end{cases}$, where $P(x)$ is

polynomial such that $P''(x)$ is always a constant and $P(3) = 9$. If $f(x)$ is continuous at $x = 2$, then $P(5)$ is equal to

Accepted Answers

39 39.0 39.00

Solution:

$$P(x) = k(x-2)(x-b)$$

$$\lim_{x \rightarrow 2} f(x) = 7$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{k(x-2)(x-b)}{\sin(x-2)} = 7$$

$$\Rightarrow \lim_{x \rightarrow 2} k(x-b) = 7$$

$$\left(\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right)$$

$$\Rightarrow k(2-b) = 7 \quad \dots (1)$$

$$\text{and } P(3) = k(3-2)(3-b) = 9$$

$$\Rightarrow k(3-b) = 9 \quad \dots (2)$$

From (1) and (2)

$$\frac{2-b}{3-b} = \frac{7}{9}$$

$$\Rightarrow 18 - 9b = 21 - 7b$$

$$\Rightarrow -2b = 3$$

$$\Rightarrow b = -\frac{3}{2}$$

and

$$k(2-b) = 7$$

$$\Rightarrow k \left(2 + \frac{3}{2} \right) = 7$$

$$\Rightarrow k = 2$$

$$\therefore P(x) = 2(x-2) \left(x + \frac{3}{2} \right)$$

$$\Rightarrow P(5) = 3 \times 13$$

$$\therefore P(5) = 39$$