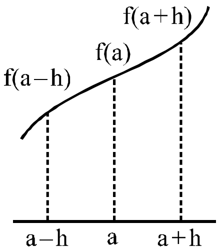
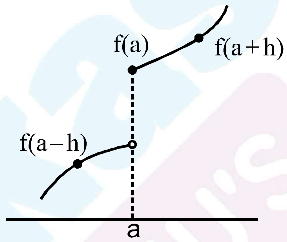
APPLICATION OF DERIVATIVES

MONOTONICITY

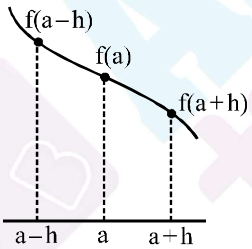
1. MonotonIcity at a Point :

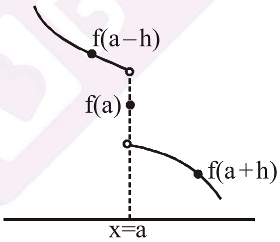
(a) A function f(x) is called an increasing function at point x = a, if in a sufficiently small neighbourhood of x = a ; f(a – h) < f(a) < f(a + h)



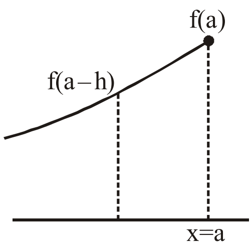


(b) A function f(x) is called a decreasing function at point x = a, if in a sufficiently small neighbourhood of x= a ; f(a – h) > f(a) > f(a + h)

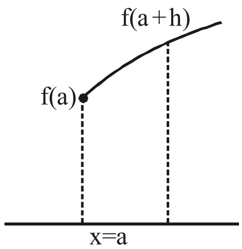




Note : If x = a is a boundary point, then use the appropriate one side inequality to test Monotonicity of f(x).



f(a) > f(a–h)
increasing at x = a



f(a+h) > f(a)
increasing at x = a

(C) Derivative test for increasing and decreasing functions at a point.

(i) If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$, then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(1) If $f'(a^+) > 0$ and $f'(a^-) > 0$, then increasing

(2) If $f'(a^+) < 0$ and $f'(a^-) < 0$, then decreasing

(3) otherwise neither increasing nor decreasing.

Note : Above rule is applicable only for functions that are differentiable at $x = a$.

2. Monotonicity Over an Interval :

(a) A function $f(x)$ is said to be monotonically increasing (MI) in (a, b) if $f'(x) \geq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not ideally become zero for $x \in (a, b)$ or any sub interval.

(b) $f(x)$ is said to be monotonically decreasing (MD) in (a, b) if $f'(x) \leq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not ideally become zero for $x \in (a, b)$ or any sub interval.

— By discrete points, we mean that points where $f'(x) = 0$ does not form an interval.

Note :

A function is said to be monotonic if it's either increasing or decreasing.

3. Special Points :

(a) Critical points : The points of domain of f for which $f'(x)$ is equal to zero or doesn't exist are called critical points.

(b) Stationary points: The stationary points are the points of domain of f where $f'(x) = 0$. Every stationary point is a critical point.

4. Rolle's Theorem :

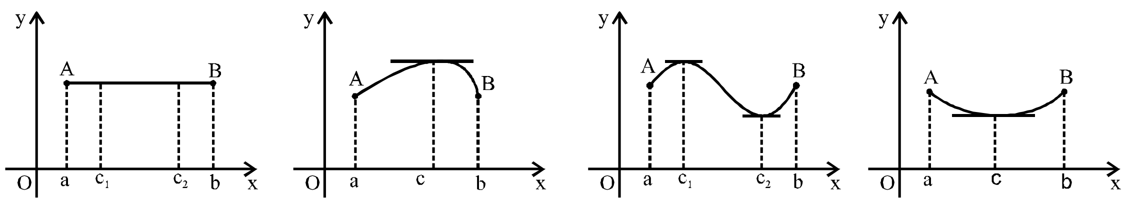
Let f be a function that satisfies the following three hypotheses :

(a) f is continuous on the closed interval $[a, b]$.

(b) f is differentiable on the open interval (a, b)

(c) $f(a) = f(b)$

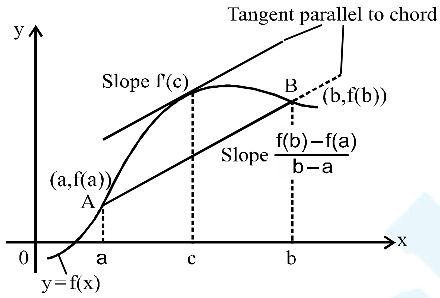
Then there exist at least one number c in (a, b) such that $f'(c) = 0$.



(a) (b) (c) (d)

Concludion: If f is differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

5. Lagrange's mean Value Theorem (LMVT) :



Let f be a function that satisfies the following hypotheses:

(i) f is continuous in [a, b]

(ii) f is differentiable in (a, b).

Then there is a number c in (a, b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

(a) Geometrical Interpretation :

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

6. Special note:

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) f(x) & its first derivative f'(x) are continuous for $a \le x \le b$.

(b) f(a) and f(b) have opposite signs.

(c) f'(x) is different from zero for all values of x between a & b.

Then there is one & only one root of the equation f(x) = 0 in (a, b).

MAXIMA & MINIMA

1. Introduction :

(a) Maxima (Local maxima) :

A function f(x) is said to have a maximum at x = a if there exist a neighbourhood (a – h, a + h) – {a} such that

$f(a) > f(x) \ \forall \ x \in (a - h, a + h) - \{a\}$

(b) Minima (Local minima):

A function f(x) is said to have a minimum at x = a if there exist a neighbourhood (a – h, a + h) – {a} such that

$f(a) < f(x) \ \forall \ x \in (a - h, a + h) - \{a\}$

(c) Absolute maximum (Global maximum) :

A function f has an absolute maximum (or global maximum) at c if f(c)≥ f(x) for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minimum (Global minimum) :

A function f has an absolute minimum at c if f(c) ≤ f (x) for all x in D and the number f(c) is called the minimum value of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Note :

(i) the maximum & minimum values of a function are also known as **local/relative maxima or local/relative minima** as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' (or extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value in the same or different intervals.

(v) local maximum & local minimum values of a continuous function occur alternately & between two consecutive local maximum values there is a local minimum value & vice versa.

(vi) Monotonic function do not have extreme points.

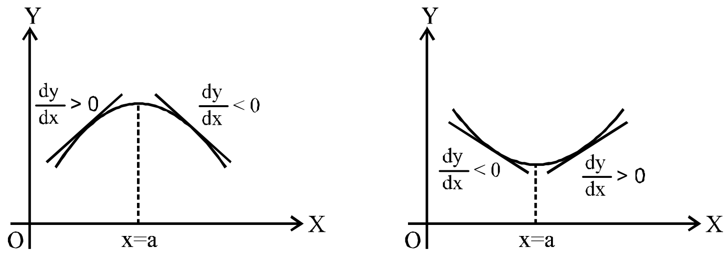
2. Derivative Test for Ascertaining Maxima/Minima :

(a) First derivative test :

If f '(x) = 0 at a point (say x = a) and

(i) If f'(x) changes sign from positive to negative while graph of the function passes through x = a then x = a is said to be a point **local maxima.**

(ii) If f'(x) changes sign from negative to positive while graph of the function passes through x = a then x = a is said to be a point **local minima.**



Note : If f '(x) does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(a) is not an extreme value of f.

(b) Second derivative test :

If f(x) is continuous and differentiable at x = a where f'(a) = 0 and f''(a) also exists then for ascertaining maxima/minima at x = a, 2nd derivative test can be used -

(i) If $f''(a) > 0$ $\Rightarrow$ x = a is a point of local minima

(ii) If $f''(a) < 0$ $\Rightarrow$ x = a is a point of local maxima

(iii) If $f''(a) = 0$ $\Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher order derivative test can be used.

(c) nth derivative test :

Let f(x) function such that $f'(a) = f''(a) = f'''(a) = \dots\dots\dots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) n = even

(1) $f^n(a) > 0$ $\Rightarrow$ Minima , $f^n(a) < 0$ $\Rightarrow$ Maxima

(ii) If n is odd ; Neither maxima nor minima at x = a

3. Useful Formulae Of Mensuration :

(a) Volume of a cuboid = lbh.
(b) Surface area of a cuboid = $2(lb + bh + hl)$.
(c) Volume of a prism = area of the base × height.
(d) Lateral surface area of prism = perimeter of the base × height.
(e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid $= \frac{1}{3}$ area of the base x height.

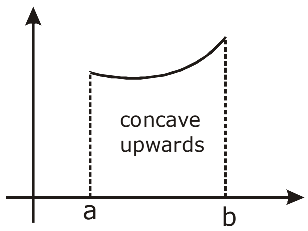
(g) Curved surface area of a pyramid $= \frac{1}{2}$ (perimeter of the base) x slant height. (Note that slant surfaces of a pyramid are triangles).

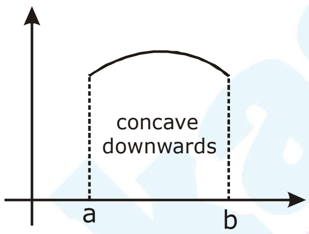
(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.
(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.
(l) Surface area of a sphere = $4\pi r^2$.
(m) Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.
(n) Perimeter of circular sector = $2r + r\theta$.

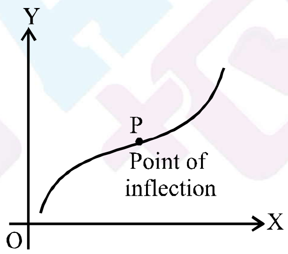
4. Significance of the Sign of 2nd Order Derivative and Point of Inflection :

The sign of the 2nd order derivative determines the concavity of the curve.
If $f''(x) > 0 \ \forall \ x \in (a,b)$ then graph of f(x) is concave upward in (a,b)
Similarly if $f''(x) < 0 \ \forall \ x \in (a,b)$ then graph of f(x) is concave downpward in (a,b).





POINT OF INFLECTION :
Point of inflection is a point on a smooth plane curve at which the graph changes from being concave upward (or concave downward) to concave downward (or concave upward).



For finding the solution of any function,

compute the solutions of $\frac{d^2y}{dx^2} = 0$ or does not exist. Let the solution be x = a, if sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

Note : If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point then it is also called point of inflection.

5. Some Standarad Results :

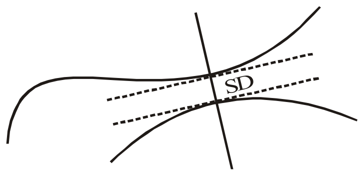
(a) Rectangle of largest area inscribed in a circle is a square.

(b) The function $y = \sin^m x \cos^n x$ attains the max value at $x = \tan^{-1}\sqrt{\frac{m}{n}}$

(c) If $0 < a < b$ then $|x-a| + |x-b| \geq b - a$ and equality hold when $x \in [a,b]$
If $0 < a < b < c$ then $|x-a| + |x-b| + |x-c| \geq c - a$ and equality hold when $x=b$
If $0 < a < b$ then $|x-a| + |x-b| + |x - c| + |x - d| \geq d - a$ and equality hold when $x \in (b,c)$.

6. Shortest Distance Between two Curves :

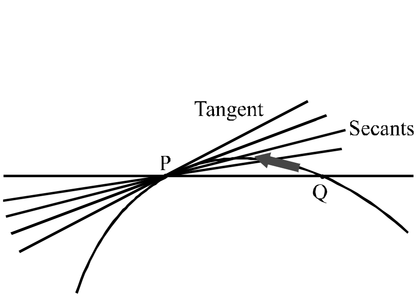
Shortest distance between two non-intersecting curves always lies along the common normal. (Wherever defined)

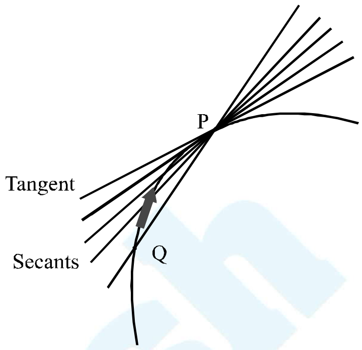


TANGENT AND NORMAL

1. Tangent to the Curve at a Point :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant slopes as $Q \to P$ from either side.





2. Normal to the Curve at a Point :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

3. Slope :

(a) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(b) Equation of tangent at (x_1, y_1) is ; $y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

(c) Equation of normal at (x_1, y_1) is ;$y - y_1 = -\dfrac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$.

Note :

(i) The point $P(x_1, y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then $dy/dx = 0$ at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or $dx/dy = 0$.

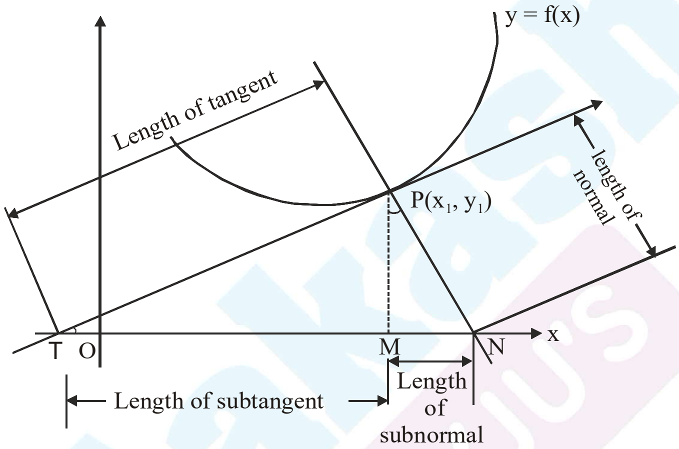
(iv) If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

4. Angle between two intersecting curves :

Angle between two intersecting curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection.
If the angle between two curves is 90° then they are called **ORTHOGONAL** curves**.**

5. Length of Tangent, Subtangent, Normal & Subnormal :



(a) Length of the tangent (PT) = $\left|\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}\right|$

(b) Length of Subtangent (MT) = $\left|\dfrac{y_1}{f'(x_1)}\right|$

(c) Length of Normal (PN) = $\left|y_1\sqrt{1+\left[f'(x_1)\right]^2}\right|$

(d) Length of Subnormal (MN) = $|y_1 f'(x_1)|$

6. Differentials :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, $y = \tan x$ then $dy = \sec^2 x\, dx$. In general $dy = f'(x)\, dx$ or $df(x) = f'(x)\, dx$

Note :

(i) d(c) = 0 where 'c' is a constant

(ii) d(u+ v) = du + dv (iii) d(uv) = udv + vdu

(iv) d (u – v) = du – dv

(v) $d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$

(vi) For the independent variable 'x', increment Δ x and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in y = f(x) is

$$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx}.\Delta x$$

(vii) The relation dy = f'(x) dx can be written as $\frac{dy}{dx}=f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.