Subject: Mathematics

1. Let f and g be differentiable funcitons on R, such that fog is the identity funciton. If for some $a, b \in R, g'(a) = 5$ and $g(a) = b$, then $f'(b)$ is equal to :

A. $\frac{2}{5}$

B. 5

C. 1

D. $\frac{1}{5}$

$f(g(x)) = x$
$f'(g(x))g'(x) = 1$
Put $x = a$
$f'(g(a))g'(a) = 1 \Rightarrow f'(b) \times 5 = 1$
$\Rightarrow f'(b) = \frac{1}{5}$

2. Let $y = y(x)$ be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$ is equal to :

A. $-\frac{\sqrt{5}}{2}$

B. $\frac{\sqrt{5}}{2}$

C. $-\frac{\sqrt{5}}{4}$

D. $\frac{2}{\sqrt{5}}$

$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$

Differentiating w.r.t. x on both the sides, we get

$$y'\sqrt{1-x^2} + y \times \frac{1}{2\sqrt{1-x^2}} \times (-2x)$$

$$= -\sqrt{1-y^2} - x \times \frac{1}{2\sqrt{1-y^2}} \times (-2y)y'$$

$$\Rightarrow y'\sqrt{1-x^2} - \frac{xy}{\sqrt{1-x^2}} = \frac{xy}{\sqrt{1-y^2}}y' - \sqrt{1-y^2}$$

Putting $x = \frac{1}{2}$, $y = -\frac{1}{4}$, we get

$$y'\left[\frac{\sqrt{3}}{2} - \frac{\frac{1}{8}}{\frac{\sqrt{15}}{4}}\right] = \frac{\frac{1}{8}}{\frac{\sqrt{3}}{2}} - \frac{\sqrt{15}}{4}$$

$$\Rightarrow y'\left[\frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{15}}\right] = \frac{1}{4\sqrt{3}} - \frac{\sqrt{15}}{4}$$

$$\Rightarrow y'\left[\frac{\sqrt{45}-1}{2\sqrt{15}}\right] = \frac{1-\sqrt{45}}{4\sqrt{3}}$$

$$\Rightarrow y'\Big|_{x=1/2} = -\frac{\sqrt{5}}{2}$$

3. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx, x \in [-1, 1]$,at the point $x = \frac{1}{2}$, then $2b + c$ equals :

A. 1

B. 2

C. -1

D. -3

By Rolle's theorem
$f(1) = f(-1)$
$2 + b + c = -2 + b - c$
$\Rightarrow c = -2$
$f'(x) = 6x^2 + 2bx + c$
$f'(\frac{1}{2}) = \frac{3}{2} + b + c = 0$
$\Rightarrow b = \frac{1}{2}$
So,$(2b + c) = -1$

4. If f and g are differentiable functions in $[0, 1]$ satisfying $f(0) = 2 = g(1),\ g(0) = 0$ and $f(1) = 6$, then for some $c \in [0, 1]$:

A. $2f'(c) = g'(c)$

B. $2f'(c) = 3g'(c)$

C. $f'(c) = g'(c)$

D. $f'(c) = 2g'(c)$

Let $h(x) = f(x) - 2g(x) \ldots (1)$
$\therefore h(0) = f(0) - 2g(0) = 2 - 0 = 2$
and $h(1) = f(1) - 2g(1) = 6 - 2(2) = 2$
Thus, $h(0) = h(1) = 2$
Now apply Rolle's theorem on equation (1),
$h'(c) = 0$ where $c \in (0, 1)$
Differentiating equation (1) w.r.t x
$\therefore h'(x) = f'(x) - 2g'(x)$
At $x = c,\ h'(c) = f'(c) - 2g'(c)$
Hence, $0 = f'(c) - 2g'(c)$
$\therefore f'(c) = 2g'(c)$

5. The derivative of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x\in\left(0,\frac{\pi}{2}\right)\right)$ is:

A. 1

B. $\frac{1}{2}$

C. 2

D. $\frac{2}{3}$

$$y=\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{\tan x-1}{\tan x+1}\right)$$

$$\Rightarrow y=-\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$\Rightarrow y=-\tan^{-1}\left[\tan\left(\frac{\pi}{4}-x\right)\right]$$

$$\because 0<x<\frac{\pi}{2}\Rightarrow -\frac{\pi}{2}<-x<0$$

$$\Rightarrow -\frac{\pi}{4}<\frac{\pi}{4}-x<\frac{\pi}{4}$$

$$\Rightarrow y=-\left(\frac{\pi}{4}-x\right)$$

$$\Rightarrow y=-\frac{\pi}{4}+x$$

$$\frac{dy}{d(x/2)}=\frac{1}{(1/2)}=2$$

6. If P is a point on the parabola $y = x^2 + 4$ which is closest to the straight line $y = 4x - 1$, then the co-ordinates of P are

A. $(-2, 8)$

B. $(1, 5)$

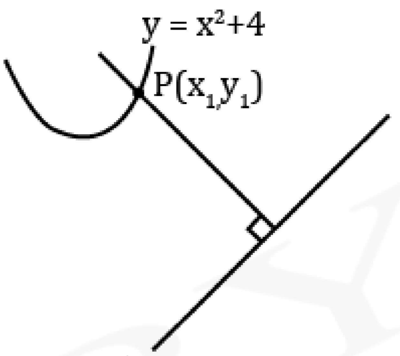
C. $(3, 13)$

D. $(2, 8)$

Tangent at P is parallel to the given line.

$$\left.\frac{dy}{dx}\right|_P = 4$$

$\Rightarrow 2x_1 = 4$



$\Rightarrow x_1 = 2$

Required point is $(2, 8)$

7. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is:

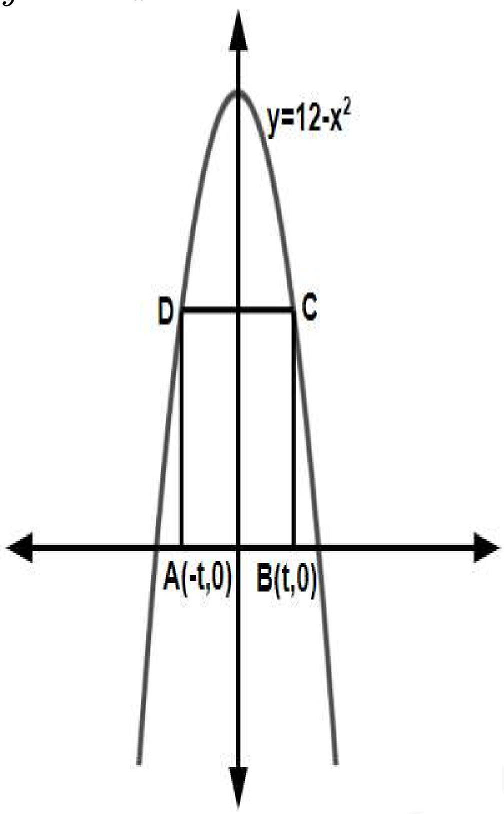
A. 36

B. 32

C. $20\sqrt{2}$

D. $18\sqrt{3}$

$y = 12 - x^2$



$\therefore AB = 2t$

$AD = 12 - t^2$

$\therefore$

area of rectangle $ABCD$

$(A_r) = 2t(12 - t^2)$

$\Rightarrow A_r = 24t - 2t^3$

To find maximum area -

$$\frac{dA_r}{dt} = 24 - 6t^2 = 0$$

$\Rightarrow 24 - 6t^2 = 0$

$\Rightarrow t = \pm 2$

$$\frac{d^2A_r}{dt^2} = -12t$$

at $t = 2, \dfrac{d^2A_r}{dt^2} < 0$

$$\begin{aligned}\therefore A_r &= |24(2) - 2(2)^3| \\ &= |48 - 16| \\ &= |32|\end{aligned}$$

$\Rightarrow A_r = 32 \text{ sq. units}$

So, maximum area = 32 sq. units

8. The range of $a \in \mathbb{R}$ for which the function

$f(x) = (4a-3)(x+\log_e 5) + 2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi, n \in \mathbb{N}$ has critical points, is:

A. $\left[-\frac{4}{3}, 2\right]$

B. $(-\infty, -1]$

C. $[1, \infty)$

D. $(-3, 1)$

$$f(x) = (4a-3)(x+\ln 5) + 2(a-7)\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} . \sin^2\frac{x}{2}\right)$$

$$f(x) = (4a-3)(x+\ln 5) + (a-7)\sin x$$

$$\Rightarrow f'(x) = (4a-3) + (a-7)\cos x = 0$$

$$\Rightarrow \cos x = -\frac{(4a-3)}{a-7}$$

$$\Rightarrow -1 \leq -\frac{(4-3)}{a-7} < 1 \quad (\because 1 \leq \cos x \leq 1)$$

$$\Rightarrow -1 < \frac{4a-3}{a-7} \leq 1$$

$$\Rightarrow \frac{4a-3}{a-7} - 1 \leq 0 \text{ and } \frac{4a-3}{a-7} + 1 > 0$$

$$\Rightarrow a \in \left[-\frac{4}{3}, 7\right) \text{ and } a \in (-\infty, 2) \cup (7, \infty)$$

$$\Rightarrow \frac{-4}{3} \leq a < 2$$

9. Let $P(h, k)$ be a point on the curve $y = x^2 + 7x + 2$, nearest to the line, $y = 3x - 3$. Then the equation of the normal to the curve at P is:

A. $x + 3y - 62 = 0$

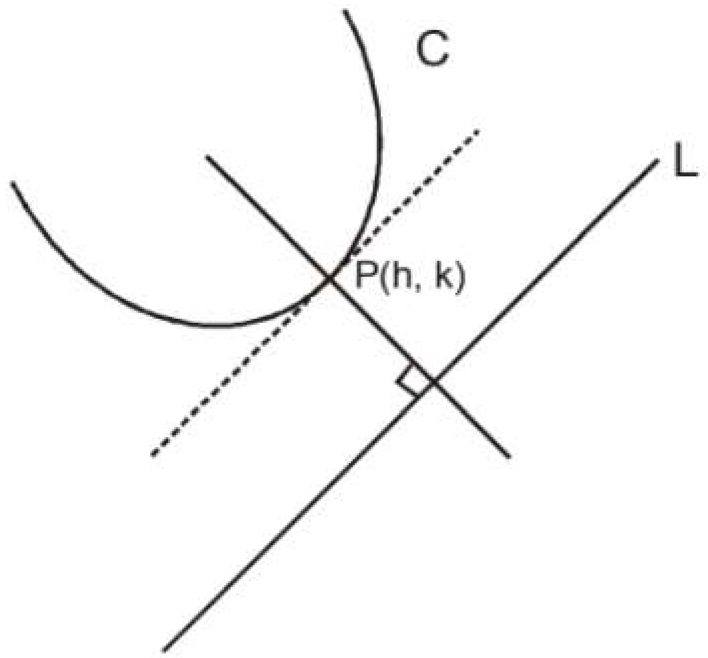
B. $x - 3y - 11 = 0$

C. $x - 3y + 22 = 0$

D. $x + 3y + 26 = 0$

$C : y = x^2 + 7x + 2$

Let P : (h, k) lies on Curve$k = h^2 + 7h + 2 \qquad \cdots (1)$



Now for the shortest distance

Slope of tangent line at point $P =$ slope of line L

$\left.\frac{dy}{dx}\right|_{\text{at P}(h,k)}$

$\Rightarrow \left.\frac{d}{dx}(x^2 + 7x + 2)\right|_{\text{at P}(h,k)} = 3$

$\Rightarrow (2x + 7)|_{\text{at P}(h,k)} = 32h + 7 = 3$

$h = -2$ from equation (1)

$k = -8$

$P : (-2, -8)$ equation of normal to the curve is perpendicular to

L : $3x - y = 3$

$N : x + 3y = \lambda$ pass through $(-2, -8)$

$\Rightarrow \lambda = -26$

$\therefore N : x + 3y + 26 = 0$

10. If the tangent to the curve $y = x + \sin y$ at a point (a, b) is parallel to the line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$ then:

A. $b = \frac{\pi}{2} + a$

B. $|a + b| = 1$

C. $|b - a| = 1$

D. $b = a$

$$\left.\frac{dy}{dx}\right|_{p(a,b)}^{c} = \frac{2 - \frac{3}{2}}{\frac{1}{2} - 0}$$

$$\left.\begin{aligned} 1 + \cos \mathrm{b} &= 1 \\ \cos \mathrm{b} &= 0 \end{aligned}\right|_{b=a+\sin b}^{p:(a,b)\text{lies on curve}}$$

$$b = a \pm 1$$

$$|b - a| = 1$$

11. The shortest distance between the line $x - y = 1$ and the curve $x^2 = 2y$ is :

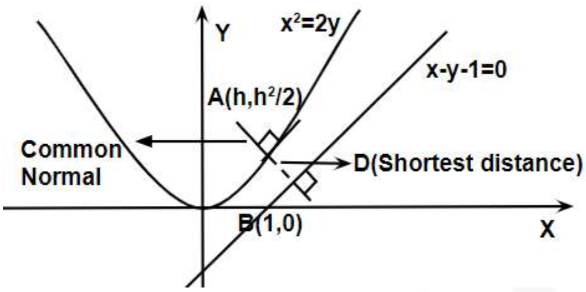
A. $\frac{1}{2}$

B. 0

C. $\frac{1}{2\sqrt{2}}$

D. $\frac{1}{\sqrt{2}}$

Shortest distance must be along common normal



$m_1 \text{ (slope of line } x - y = 1) = 1$

$\Rightarrow$ slope of perpendicular line $= -1$

Slope of tangent line at $\left(h, \frac{h^2}{2}\right)$ is

$m_2 = \frac{2x}{2} = x \Rightarrow m_2 = h$

$\Rightarrow$slope of normal $= -\frac{1}{h}$

So, $-\frac{1}{h} = -1 \Rightarrow h = 1$

so point is $\left(1, \frac{1}{2}\right)$

$$\Rightarrow D = \left|\frac{1 - \frac{1}{2} - 1}{\sqrt{1+1}}\right| = \frac{1}{2\sqrt{2}}$$

12. The function $f(x)=\dfrac{4x^3-3x^2}{6}-2\sin x+(2-1)\cos x$:

A. increases in $\left[\frac{1}{2},\infty\right)$

B. decreases $\left(-\infty,\frac{1}{2}\right]$

C. increases in $\left(-\infty,\frac{1}{2}\right]$

D. decreases $\left[\frac{1}{2},\infty\right)$

$f'(x)=(2x-1)(x-\sin x)$

$\Rightarrow f'(x)\geq 0 \text{ in } x\in(-\infty,0]\cup\left[\frac{1}{2},\infty\right)$

and $f'(x)<0 \text{ in } x\in\left(0,\frac{1}{2}\right)$

13. Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x\neq 0\\ 0, & x=0\end{cases}$$. Then f is

A. monotonic on $(0,\ \infty)$ only

B. Not monotonic on $(-\infty,\ 0)$ and $(0,\ \infty)$

C. monotonic on $(-\infty,\ 0)$ only

D. monotonic on $(-\infty,\ 0)\cup(0,\ \infty)$

$$f(x)=\begin{cases}\left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\Rightarrow f(x)=\begin{cases}-\left(2-\sin\left(\frac{1}{x}\right)\right)x, & x<0\\ 0 & ,x=0\\ \left(2-\sin\left(\frac{1}{x}\right)\right)x, & x>0\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}-x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)-\left(2-\sin\frac{1}{x}\right), & x<0\\ x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+\left(2-\sin\frac{1}{x}\right), & x>0\end{cases}$$

$$=\begin{cases}-\frac{1}{x}\cos\frac{1}{x}+\sin\frac{1}{x}-2, & x<0\\ \frac{1}{x}\cos\frac{1}{x}-\sin\frac{1}{x}+2, & x>0\end{cases}$$

For $x>0$,

$f'(x)=\left(2+\frac{1}{x}\cos\frac{1}{x}\right)-\sin\frac{1}{x}$

here for $x\geq 1, \frac{1}{x}\in(0,1]$

$\left(2+\frac{1}{x}\cos\frac{1}{x}\right)>\sin\frac{1}{x}\Rightarrow f'(x)>0$

But for, $x\in(0,1)$

$f'(x)=\left(2+\frac{1}{x}\cos\frac{1}{x}\right)-\sin\frac{1}{x}$ need not be greater than zero, as there exists some values for $x\in(0,1)$ where $\left(2+\frac{1}{x}\cos\frac{1}{x}\right)<\sin\frac{1}{x}$

So ,$f(x)$ is not monotonic in $x\in(0,\infty)$

Similarly for $x\in(-\infty,0),\ f(x)$ is not monotonic.

14. The value of $\lim\limits_{x \to 0^+} \dfrac{\cos^{-1}(x-[x]^2) \cdot \sin^{-1}(x-[x]^2)}{x-x^3}$, where $[x]$ denotes the greatest integer $\leq x$ is:

A. 0

B. $\dfrac{\pi}{4}$

C. $\dfrac{\pi}{2}$

D. π

$$\lim_{x \to 0^+} \frac{\cos^{-1}(x-[x]^2) \sin^{-1}(x-[x]^2)}{x-x^3}$$

$$= \lim_{x \to 0^+} \frac{\cos^{-1} x \sin^{-1} x}{x(1-x^2)} \quad (\because [0^+] = 0)$$

$$= \frac{\cos^{-1} 0^+}{1-0} = \frac{\pi}{2}$$

15. The equation of the normal to the curve $y=(1+x)^{2y}+\cos^2\left(\sin^{-1}x\right)$ at $x=0$ is

A. $y+4x=2$

B. $2y+x=4$

C. $x+4y=8$

D. $y=4x+2$

Given curve is $y=(1+x)^{2y}+\cos^2\left(\sin^{-1}x\right)$
At $x=0 \rightarrow y=1+\cos^2(0)=2$
Thus, the point is $(0,2)$

$$y=(1+x)^{2y}+\cos^2\left(\sin^{-1}x\right)$$
$$\Rightarrow y=(1+x)^{2y}+\cos^2\left(\cos^{-1}\sqrt{1-x^2}\right)$$
$$=(1+x)^{2y}+\left(\cos\left(\cos^{-1}\sqrt{1-x^2}\right)\right)^2$$
$$=(1+x)^{2y}+\left(\sqrt{1-x^2}\right)^2$$

$$\Rightarrow y=(1+x)^{2y}+1-x^2$$

Differentiating with respect to $'x'$
$$y'=(1+x)^{2y}\left\{\frac{2y}{1+x}+\ln(1+x).2y'\right\}-2x$$

$$y'\big|_{(0,2)}=4-0$$

Equation of normal at $(0,2) is: y-2=-\frac{1}{4}(x-0)$
$$\Rightarrow 4y-8=-x$$
$$\Rightarrow x+4y=8$$

16. If $y^2 + \log_e(\cos^2 x) = y,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then

- **A.** $|y'(0)| + |y''(0)| = 1$
- **B.** $y''(0) = 0$
- **C.** $|y'(0)| + |y''(0)| = 3$
- **D.** $|y''(0)| = 2$

Given:
$y^2 + \log_e(\cos^2 x) = y \quad \cdots (i)$
At $x = 0$
$\Rightarrow y^2(0) + 0 = y(0) \Rightarrow y(0) = 0, 1$
$(\because y = y(x))$
differentiating (i) w.r.t. x
$\Rightarrow 2yy' + 2\ (-\tan x) = y' \quad \cdots (ii)$
At $(x, y) = (0, 0) \Rightarrow y'(0) = 0$
At $(x, y) = (0, 1) \Rightarrow y'(0) = 0$

differentiating (ii) w.r.t. x
$2yy'' + 2(y')^2 - 2\sec^2 x = y''$
At $(x, y) = (0, 0) \Rightarrow y''(0) = -2$
At $(x, y) = (0, 1) \Rightarrow y''(0) = 2$

$\therefore |y''(0)| = 2$ and
$\therefore |y'(0)| + |y''(0)| = 2$

17. Let f be a twice differentiable function on $(1,6)$. If $f(2)=8, f'(2)=5, f'(x)\geq 1$ and $f''(x)\geq 4$, for all $x\in(1,6)$, then

A. $f(5)+f'(5)\geq 28$

B. $f'(5)+f''(5)\leq 20$

C. $f(5)\leq 10$

D. $f(5)+f'(5)\leq 26$

$f(2)=8, f'(2)=5, f'(x)\geq 1, f''(x)\geq 4\ \forall\ x\in(1,6)$

Using LMVT,

$$f''(x)=\frac{f'(5)-f'(2)}{5-2}\geq 4\ \forall\ x\in(1,6)$$

$$\Rightarrow f'(5)\geq 17 \quad \cdots(1)$$

and

$$f'(x)=\frac{f(5)-f(2)}{5-2}\geq 1\ \forall\ x\in(1,6)$$

$$\Rightarrow f(5)\geq 11 \quad \cdots(2)$$

Adding (1) and (2), we get

$$f(5)+f'(5)\geq 28$$

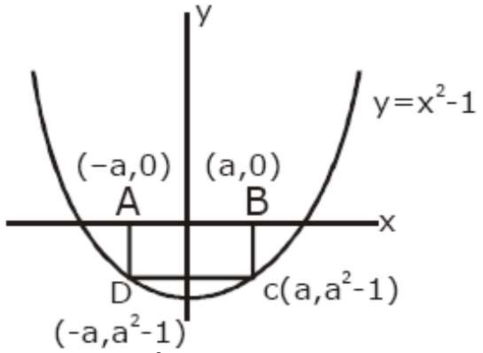
18. The area (in sq. units) of the largest rectangle $ABCD$ whose vertices A and B lie on the $x-$axis and vertices C and D lie on the parabola, $y = x^2 - 1$ below the $x-$axis, is

A. $\frac{2}{3\sqrt{3}}$

B. $\frac{4}{3}$

C. $\frac{1}{3\sqrt{3}}$

D. $\frac{4}{3\sqrt{3}}$



Area $= 2a(a^2-1)$

(Say) $A = 2a^3 - 2a$

$\Rightarrow \dfrac{dA}{da} = 6a^2 - 2$

For maximum and minimum, $\dfrac{dA}{da} = 0$

$\therefore 6a^2 - 2 = 0$

$\Rightarrow a = \pm \dfrac{1}{\sqrt{3}}$

Now, $\dfrac{d^2A}{da^2} = 12a$

At $a = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{d^2A}{da^2} = 4\sqrt{3} > 0$

$\therefore a = \dfrac{1}{\sqrt{3}}$(rejected)

At $a = -\dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{d^2A}{da^2} = -4\sqrt{3} < 0$

$\therefore$ Maximum area

$= (2a^3 - 2a) = \left(-\dfrac{2}{3\sqrt{3}} + \dfrac{2}{\sqrt{3}}\right)$

$= \dfrac{4}{3\sqrt{3}}$ sq. units

19. Let $f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right), 0<x<1$. Then

A. $(1-x)^2 f'(x)-2(f(x))^2=0$

B. $(1-x)^2 f'(x)+2(f(x))^2=0$

C. $(1+x)^2 f'(x)-2(f(x))^2=0$

D. $(1+x)^2 f'(x)+2(f(x))^2=0$

$f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$

Let $\frac{\sqrt{1-x}}{\sqrt{x}}=\cot\theta$

$\Rightarrow f(x)=\cos(2\tan^{-1}\sin\theta)=\cos(2\tan^{-1}(\sqrt{x}))$

Let $\sqrt{x}=\tan\theta$

$\Rightarrow f(x)=\cos 2\theta=2\cos^2\theta-1=\frac{2}{1+x}-1=\frac{1-x}{1+x}$

$\Rightarrow f'(x)=-\frac{2}{(1+x)^2}$

$\Rightarrow (1-x)^2 f'(x)=-2\left(\frac{1-x}{1+x}\right)^2$

$\Rightarrow (1-x)^2 f'(x)+2f^2(x)=0$

20. Let f be a real valued function, defined on $\mathbb{R}-\{-1,1\}$ and given by $f(x)=3\log_e\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}$. Then in which of the following intervals, function $f(x)$ is increasing ?

A. $(-\infty,-1)\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

B. $\left(-1,\frac{1}{2}\right]$

C. $(-\infty,\infty)-\{-1,1\}$

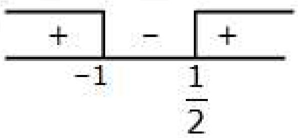
D. $\left(-\infty,\frac{1}{2}\right]-\{-1\}$

$$f(x)=3\log_e\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}$$

$$f'(x)=3\left(\frac{x+1}{x-1}\times\frac{x+1-(x-1)}{(x+1)^2}\right)+\frac{2}{(x-1)^2}$$

$$=\frac{6}{(x-1)(x+1)}+\frac{2}{(x-1)^2}$$

$$=\frac{4(2x-1)}{(x+1)(x-1)^2}$$



$$\therefore x\in(-\infty,-1)\cup\left[\frac{1}{2},\infty\right)-\{1\}$$

21. The maximum slope of the curve $y=\frac{1}{2}x^4-5x^3+18x^2-19x$ occurs at the point :

A. $(2,9)$

B. $(2,2)$

C. $\left(3,\frac{21}{2}\right)$

D. $(0,0)$

$y=\frac{1}{2}x^4-5x^3+18x^2-19x$
$\frac{dy}{dx}=2x^3-15x^2+36x-19$
Let $f(x)=2x^3-15x^2+36x-19$
$f'(x)=6x^2-30x+36=0$
For extrema, $x^2-5x+6=0$
$\Rightarrow x=2,3$

$f''(x)=12x-30$
$f''(x)<0$ for $x=2$
So, at $x=2$, slope is maximum.
$y=8-40+72-38$
$=72-70=2$
Hence, maximum slope occurs at $(2,2)$.

22. If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

A. $\frac{9}{2}$

B. 6

C. $\frac{7}{2}$

D. 4

Let the point of intersection be (x_1, y_1) finding slope of both the curves at point of intersection for $y^2 = 6x, 9x^2 + by^2 = 16$

$2y\frac{dy}{dx} = 6, m_1 = \frac{6}{2y_1}$

And $18x + 2by\frac{dy}{dx} = 0, m_2 = -\frac{18x_1}{2by_1}$

$m_1 m_2 = -1$

$\Rightarrow \left(\frac{6}{2y_1}\right)\left(\frac{-18x_1}{2by_1}\right) = -1$

$\Rightarrow \frac{(6)(-18)x_1}{(4b)(6x_1)} = -1 \Rightarrow b = \frac{9}{2}$

23. If $y(\alpha)=\sqrt{2\left(\frac{\tan\ \alpha+\cot\ \alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}$ where $\alpha\in\left(\frac{3\pi}{4},\pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha=\frac{5\pi}{6}$ is

A. $-\frac{1}{4}$

B. $\frac{4}{3}$

C. 4

D. -4

$$y(\alpha)=\sqrt{2\left(\frac{\tan\ \alpha+\cot\ \alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}$$

$$\Rightarrow y(\alpha)=\sqrt{\frac{2}{\sin\alpha\ \cos\alpha\times\frac{1}{\cos^2\alpha}}+\frac{1}{\sin^2\alpha}}$$

$$\Rightarrow y(\alpha)=\sqrt{2\cot\alpha+\operatorname{cosec}^2\alpha}$$

$$\Rightarrow y(\alpha)=\sqrt{(1+\cot\alpha)^2}$$

$$\Rightarrow y(\alpha)=-1-\cot\alpha\qquad\left[\because\alpha\in\left(\frac{3\pi}{4},\pi\right)\right]$$

$$\frac{dy}{d\alpha}=0+\operatorname{cosec}^2\alpha$$

$$\Rightarrow\left.\frac{dy}{d\alpha}\right|_{x=\frac{5\pi}{6}}=\operatorname{cosec}^2\frac{5\pi}{6}$$

$$\Rightarrow\left.\frac{dy}{d\alpha}\right|_{x=\frac{5\pi}{6}}=4$$

24. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that
$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3), x \in \mathbf{R}$. Then $f(2)$ equals :

A. -4

B. 30

C. 8

D. -2

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$
$\Rightarrow f'(x) = 3x^2 + 2x f'(1) + f''(2) \quad \cdots (i)$
$\Rightarrow f''(x) = 6x + 2f'(1) \quad \cdots (ii)$
$\Rightarrow f'''(x) = 6 \quad \cdots (iii)$
$\Rightarrow f'''(3) = 6$
Now from eqn (i)
$f'(1) = 3 + 2f'(1) + f''(2)$
$\Rightarrow f'(1) + f''(2) + 3 = 0 \quad \cdots (iv)$
from eqn (ii)
$f''(2) = 12 + 2f'(1)$
$\Rightarrow 2f'(1) - f''(2) + 12 = 0 \quad \cdots (v)$
from eqn. (iv) and (v)
$3f'(1) + 15 = 0$
$\Rightarrow f'(1) = -5$ and $f''(2) = 2$
So,
$f(x) = x^3 - 5x^2 + 2x + 6$
$\Rightarrow f(2) = 8 - 20 + 4 + 6 = -2$

25. Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersects the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, them $|m|$ is equal to

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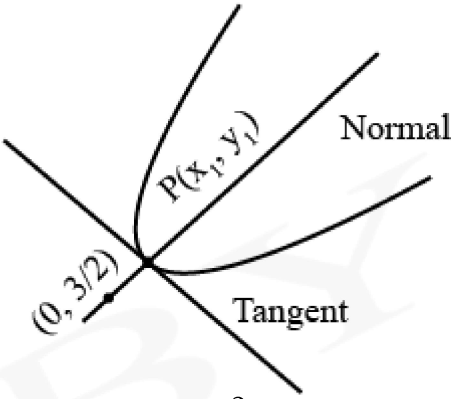
4 4.0 4.00

Solution:

Let co-ordinates of P be (x_1, y_1)
Differentiating the curve w.r.t. x
$2yy' - 6x + y' = 0$
Slope of tangent at $P(x_1, y_1)$ is
$y' = \dfrac{6x_1}{1 + 2y_1}$



$m_{\text{normal}} = \dfrac{y_1 - \frac{3}{2}}{x_1 - 0}$
$\because m_{\text{normal}} \times m = -1$
$\Rightarrow \dfrac{y_1 - \frac{3}{2}}{x_1} \times \dfrac{6x_1}{1 + 2y_1} = -1$
$\Rightarrow y_1 = 1$
$\Rightarrow x_1 = \pm 2$
Therefore, slope of tangent $= \pm \dfrac{12}{3} = \pm 4$
$\Rightarrow |m| = 4$

26. Let $f:[-1,1] \rightarrow \mathbb{R}$ be defined as $f(x)=ax^2+bx+c$ for all $x \in [-1,1]$, where $a,b,c \in \mathbb{R}$ such that $f(-1)=2, f'(-1)=1$ and for $x \in (-1,1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \leq \alpha, x \in [-1,1]$, then the least value of α is equal to

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5 5.0 5.00 05

Solution:

$f(x)=ax^2+bx+c$

$f'(x)=2ax+b$

$f''(x)=2a$

We know

$f(-1)=2$

$\Rightarrow a-b+c=2 \cdots(1)$

$f'(-1)=1$

$\Rightarrow b-2a=1 \cdots(2)$

$f''(x) \leq \frac{1}{2}$

$\Rightarrow a \leq \frac{1}{4} \cdots(3)$

From equations (1) and (2), we get

$b=1+2a, \quad c=3+a$

$\Rightarrow f(x)=ax^2+(1+2a)x+(3+a)$

$\Rightarrow f(x)=a(x+1)^2+(x+3)$

To get the maximum value of $f(x)$, a should be maximum, so

$a=\frac{1}{4}$

$f(x)=\frac{(x+1)^2}{4}+(x+3)$

$\Rightarrow f(x) \in [2,5], \; x \in [-1,1]$

As $f(x) \leq \alpha, x \in [-1,1]$

$\therefore \alpha=5$

27. Suppose a differentiable function $f(x)$ satisfies the identity

$f(x+y)=f(x)+f(y)+xy^2+x^2y$ for all real x and y. If $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=1$, then $f'(3)$ is equal to

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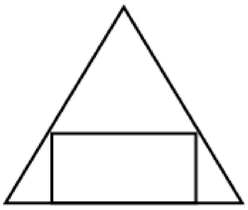
10 10.0 10.00

Solution:

Given: $f(x+y)=f(x)+f(y)+xy^2+x^2y$
At $x=y=0$
$f(0)=2f(0)\Rightarrow f(0)=0$
Now,
$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$
(take $y=h$)
$f'(x)=\lim\limits_{h\to 0}\dfrac{f(h)}{h}+\lim\limits_{h\to 0}(xh)+x^2$
$f'(x)=1+0+x^2$
$f'(x)=1+x^2$
$f'(3)=10$

28. If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is



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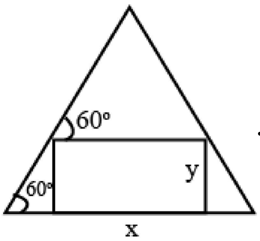
Solution:

Let the sides of rectangle be x and y.



$$\therefore x+\frac{2y}{\sqrt{3}}=2\sqrt{2}$$

and area $\Delta=xy$

$$\Rightarrow \Delta=y\left(2\sqrt{2}-\frac{2y}{\sqrt{3}}\right)$$

$$\Rightarrow \Delta=\left(2\sqrt{2}y-\frac{2y^2}{\sqrt{3}}\right)$$

Differentiation w.r.t. x, we get

$$\frac{d\Delta}{dx}=\left(2\sqrt{2}-\frac{4y}{\sqrt{3}}\right)$$

For max/min $\frac{d\Delta}{dx}=0$

$$\Rightarrow \left(2\sqrt{2}-\frac{4y}{\sqrt{3}}\right)=0$$

$$\Rightarrow y=\frac{\sqrt{6}}{2}$$

Now, $\frac{d^2\Delta}{dx^2}=\left(0-\frac{4}{\sqrt{3}}\right)<0$

$\therefore$ area is maximum at $y=\frac{\sqrt{6}}{2}$

So, mximum area $\Delta_{max}=\sqrt{2}\times\frac{\sqrt{6}}{2}=\sqrt{3}$

Hence, $\Delta_{max}^2=3$

29. Let $f(x)$ be a cubic polynomial with $f(1)=-10$, $f(-1)=6$, and has a local minima at $x=1$, and $f'(x)$ has a local minima at $x=-1$. Then $f(3)$ is equal to

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22 22.0 22.00

Solution:

$f(1)=-10,\ f(-1)=6$
$f'(1)=0$ and $f''(-1)=0$ as given
$f(x)$ has minima at $x=1$
and $f'(x)$ has minima at $x=-1$
So, $f''(x)=a(x+1)$
Integrating both sides
$f'(x)=\frac{a}{2}(x+1)^2+c$
$f'(1)=0=2a+c$
$\Rightarrow c=-2a$
$f'(x)=\frac{a}{2}(x+1)^2-2a$
Intergrating both side
$f(x)=\frac{a}{6}(x+1)^3-2xa+c'$
$f(1)=-10=\frac{8a}{6}-2a+c'$
$f(-1)=6=2a+c'$
$2a+c'=6$
$4a-6c'=60$
$\Rightarrow a=6, c'=-6$
$f(x)=(x+1)^3-12x-6$
$f(3)=(4)^3-36-6$
$f(3)=22$

30. If the point on the curve $y^2 = 6x$, nearest to the point $\left(3, \frac{3}{2}\right)$ is (α, β) then $2(\alpha + \beta)$ is equal to

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9 9.0 9.00 09

Solution:

Let a point on $y^2 = 6x$ is $P\left(\frac{3}{2}t^2, 3t\right)$

The distance between P and $\left(3, \frac{3}{2}\right)$ is D.

$$\therefore D^2 = \left(\frac{3t^2}{2} - 3\right)^2 + \left(3t - \frac{3}{2}\right)^2$$

$$= 9\left(\frac{t^4}{4} - t^2 + 1 + t^2 - t + \frac{1}{4}\right)$$

$$= \frac{9}{4}\left(t^4 - 4t + 5\right)$$

$$\therefore 2D.\frac{dD}{dt} = \frac{9}{4}\left(4t^3 - 4\right) = 9(t-1)\left(t^2 + t + 1\right)$$

$$\therefore \text{ For } t = 1, D^2 \text{ will be minimum.}$$

$$\therefore P = \left(\frac{3}{2}, 3\right) = (\alpha, \beta)$$

$$\therefore 2(\alpha + \beta) = 9$$