JEE Main Part Test 3

Subject: Mathematics

1. The value of $\lim\limits_{x \to 1} \left(\frac{4}{\pi}\tan^{-1} x\right)^{\frac{1}{x^2-1}}$ is equal to

A. $\frac{1}{\pi}$

B. $-\frac{1}{\pi}$

C. $e^{1/\pi}$

D. $e^{-1/\pi}$

$$\lim_{x \to 1} \left(\frac{4}{\pi}\tan^{-1} x\right)^{\frac{1}{x^2-1}}$$

$$= e^{\lim\limits_{x \to 1} \frac{\left(\frac{4}{\pi}\tan^{-1} x - 1\right)}{x^2-1}}$$

$= e^L$ (say)

Now, $L = \lim\limits_{x \to 1} \frac{\left(\frac{4}{\pi}\tan^{-1} x - 1\right)}{x^2-1}$

Using L'Hospital's rule,

$$L = \lim_{x \to 1} \frac{\left(\frac{4}{\pi(1+x^2)}\right)}{2x}$$

$$\Rightarrow L = \frac{4}{4\pi} = \frac{1}{\pi}$$

Therefore, $\lim\limits_{x \to 1} \left(\frac{4}{\pi}\tan^{-1} x\right)^{\frac{1}{x^2-1}} = e^{1/\pi}$

2. Let $f(x) = x^2 - 6x + 5$ and m is the number of points of non-derivability of $y = |f(|x|)|$. If $|f(|x|)| = k, k \in \mathbb{R}$ has at least m distinct solution(s), then the number of integral values of k is

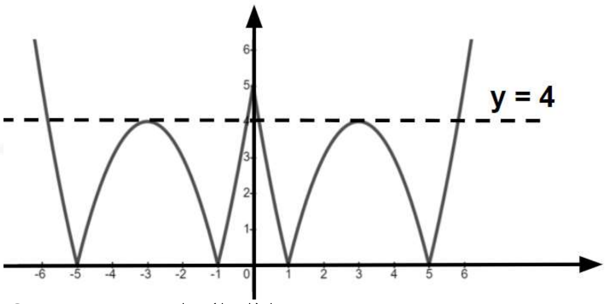
A. 2

B. 3

C. 4

D. 5

Given : $f(x) = x^2 - 6x + 5 = (x-1)(x-5)$
Vertex of the parabola
$= \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$
$= (3, -4)$
The graph of $y = |f(|x|)| = |(|x|-1)(|x|-5)|$ is



Clearly, $y = |f(|x|)|$ is non-derivable at 5 points.
For $|f(|x|)| = k$ to have at least 5 solutions,
$0 < k \leq 4$

Hence, the number of integral values of k is 4.

JEE Main Part Test 3

3. Let $f(x)=\begin{cases}\dfrac{e^{\frac{-x^2}{2}}-\cos x}{x\ln(1+x)\sin x(e^x-1)}, & x\neq 0\\ k\quad , & x=0\end{cases}$.

If $f(x)$ is continuous at $x=0$, then k equals

A. $\frac{1}{4}$

B. $\frac{1}{6}$

C. $\frac{1}{12}$

D. $\frac{1}{8}$

$$k=\lim_{x\to 0}\frac{e^{\frac{-x^2}{2}}-\cos x}{x\ln(1+x)\sin x(e^x-1)}$$

$$k=\lim_{x\to 0}\frac{e^{\frac{-x^2}{2}}-\cos x}{x^4}$$

$$=\lim_{x\to 0}\frac{\left(1-\frac{x^2}{2}+\frac{x^4}{8}-\ldots\right)-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\ldots\right)}{x^4}$$

$$=\frac{1}{8}-\frac{1}{24}=\frac{1}{12}$$

4. $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f. Then $g'(2\pi)$ is equal to

A. $\frac{7}{3}$

B. $\frac{3}{7}$

C. $\frac{30\pi^4 + 4}{3}$

D. $\frac{3}{30\pi^4 + 4}$

Since, g is the inverse function of f,
$\Rightarrow g(f(x)) = x$
Differentiate w.r.t. x, we get
$g'(f(x)) \cdot f'(x) = 1$
$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

Now, $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$

$f'(x) = 5(2x - 3\pi)^4 + \frac{4}{3} - \sin x$

$\therefore g'(f(x)) = \frac{1}{5(2x - 3\pi)^4 + \frac{4}{3} - \sin x}$

When $f(x) = 2\pi$,
$2\pi = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$
Since, in L.H.S., the power of π is 1 and in R.H.S. the power of $(2x - 3\pi)$ is 5, so solution exists only when $x = \frac{3\pi}{2}$

$\therefore g'(2\pi) = \frac{1}{\frac{4}{3} + 1} = \frac{3}{7}$

5. The radius of a right circular cylinder increases at the rate of $0.1\ \mathrm{cm/min}$, and the height decreases at the rate of $0.2\ \mathrm{cm/min}$. The rate of change of the volume of the cylinder, in $\mathrm{cm}^3/\mathrm{min}$, when the radius is $2\ \mathrm{cm}$ and the height is $3\ \mathrm{cm}$ is
(The negative sign(-) indicates that volume decreases)

- A. $-\frac{2\pi}{5}$
- B. $\frac{8\pi}{5}$
- C. $-\frac{3\pi}{5}$
- D. $\frac{2\pi}{5}$ ✔

Let r and h be the radius and height of the cylinder respectively.
Given, $r = 2\ \mathrm{cm},\ h = 3\ \mathrm{cm}$

$$\frac{dr}{dt} = 0.1\ \mathrm{cm/min}$$
$$\frac{dh}{dt} = -0.2\ \mathrm{cm/min}$$

$V = \pi r^2 h$
Differentiating both sides

$$\begin{aligned}\frac{dV}{dt} &= \pi\left(r^2\frac{dh}{dt} + 2r\frac{dr}{dt}h\right)\\ &= \pi r\left(r\frac{dh}{dt} + 2\frac{dr}{dt}h\right)\\ &= \pi r\left(r(-0.2) + 2h(0.1)\right)\\ &= \frac{\pi r}{5}(-r+h)\end{aligned}$$

Thus, when $r = 2$ and $h = 3$,

$$\frac{dV}{dt} = \frac{\pi(2)}{5}(-2+3) = \frac{2\pi}{5}\mathrm{cm}^3/\mathrm{min}$$

6. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and $y-$ axis respectively, then the value of a^2b is

A. $27c^3$

B. $\frac{4}{27}c^3$

C. $\frac{27}{4}c^3$

D. $\frac{4}{9}c^3$

$x^2y = c^3$

$\Rightarrow x^2\frac{dy}{dx} + 2xy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$

Equation of tangent at (x, y)

$Y - y = -\frac{2y}{x}(X - x)$

$Y = 0$, gives, $X = \frac{3x}{2} = a$

and $X = 0$, gives, $Y = 3y = b$

Now,

$a^2b = \frac{9x^2}{4} \times 3y$

$= \frac{27}{4}x^2y = \frac{27}{4}c^3$

7. The absolute difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

A. 2

B. e

C. e^2

D. 1

$f(x) = x(\ln x - 2)$
$\Rightarrow f'(x) = x\left(\frac{1}{x}\right) + (\ln x - 2) = \ln x - 1$
$\Rightarrow f'(x) = \ln x - 1$
$f'(x) = 0 \Rightarrow x = e$
For $x \in [1, e),\ f'(x) < 0$ and for $x \in (e, e^2],\ f'(x) > 0$
So, the minimum value of $f(x)$ occurs at $x = e$

Now, $f(1) = -2$
$\Rightarrow f(e) = -e$ (least)
$\Rightarrow f(e^2) = 0$ (greatest)

Hence, $|f(e^2) - f(e)| = |0 - (-e)| = e$

JEE Main Part Test 3

8. In which of the following functions is Rolle's theorem applicable?

A. $f(x)=\begin{cases} x, & 0\le x<1 \\ 0, & x=1 \end{cases}$ on $[0,1]$

B. $f(x)=\begin{cases} \dfrac{\sin x}{x}, & -\pi\le x<0 \\ 0, & x=0 \end{cases}$ on $[-\pi,0]$

C. $f(x)=\dfrac{x^2-x-6}{x-1}$ on $[-2,3]$

D. $f(x)=\begin{cases} \dfrac{x^3-2x^2-5x+6}{x-1}, & \text{if } x\ne 1, \\ -6, & \text{if } x=1 \end{cases}$ on $[-2,3]$

$f(x)=\begin{cases} x, & 0\le x<1 \\ 0, & x=1 \end{cases}$ on $[0,1]$

Discontinuous at $x=1$, so Rolle's theorem is not applicable.

$f(x)=\begin{cases} \dfrac{\sin x}{x}, & -\pi\le x<0 \\ 0, & x=0 \end{cases}$ on $[-\pi,0]$

$f(x)$ is not continuous at $x=0$

So, Rolle's theorem is not applicable.

$f(x)=\dfrac{x^2-x-6}{x-1}$ on $[-2,3]$

Discontinuity at $x=1$

So, Rolle's theorem is not applicable.

$f(x)=\begin{cases} \dfrac{x^3-2x^2-5x+6}{x-1}, & \text{if } x\ne 1, \\ -6, & \text{if } x=1 \end{cases}$ on $[-2,3]$

$$\lim_{x\to 1}\frac{x^3-2x^2-5x+6}{x-1}$$
$$=\lim_{x\to 1}\frac{(x-1)(x^2-x-6)}{x-1}$$
$$=\lim_{x\to 1}(x^2-x-6)$$
$$=-6$$

$\Rightarrow$ f is continuous at $x=1$ and differentiable in $(-2,3)$

Also, $f(-2)=f(3)=0$

By Roll's theorem,

$f'(x)=2x-1=0\Rightarrow x=-\dfrac{1}{2}$

Thus, $\dfrac{1}{2}$ lies between -2 and 3.

So, Rolle's theorem is applicable.

9. Let the function f be defined by $f(x) = x \ln x$, for all $x > 0$. Then

A. f is increasing on $(0, e^{-1})$

B. f is decreasing on $(0, 1)$

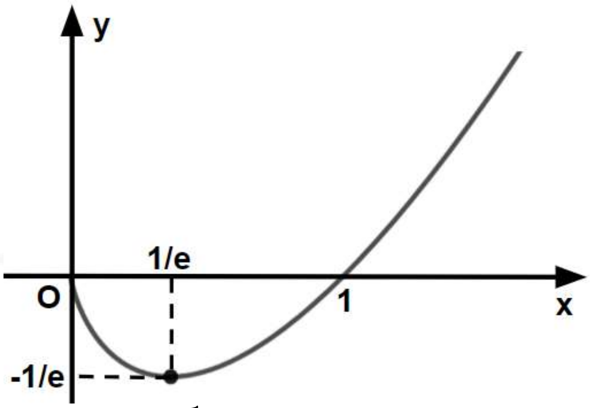
C. The graph of f is concave down for all x

D. The graph of f is concave up for all x

$\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = 0 \Rightarrow x = e^{-1} = \frac{1}{e}$

$\Rightarrow$ It is increasing on $\left(\frac{1}{e}, \infty\right)$ and decreasing on $\left(0, \frac{1}{e}\right)$



Clearly, $x = \frac{1}{e}$ gives local minima

and $f\left(\frac{1}{e}\right) = -\frac{1}{e}$

Also, $\frac{d^2y}{dx^2} = \frac{1}{x} > 0 \quad (\because x > 0)$

$\Rightarrow$ Concave up for all $x > 0$

10. If $x = 3\cos\theta - \cos 3\theta$ and $y = 3\sin\theta - \sin 3\theta$, then $\dfrac{dy}{dx}$ is

A. $\tan 2\theta$

B. $\sin 2\theta$

C. $-\tan 2\theta$

D. $\cot 2\theta$

We have,

$$\frac{dx}{d\theta} = -3\sin\theta + 3\sin 3\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\cos 3\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{3\cos\theta - 3\cos 3\theta}{-3\sin\theta + 3\sin 3\theta}$$

$$= \frac{\cos\theta - \cos 3\theta}{\sin 3\theta - \sin\theta} \qquad \cdots (1)$$

Since, $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

and, $\sin C - \sin D = 2\sin\left(\dfrac{C-D}{2}\right)\cos\left(\dfrac{C+D}{2}\right)$

equation (1) can be written as $\dfrac{\sin 2\theta\ \sin\theta}{\cos 2\theta\ \sin\theta} = \tan 2\theta$

11. Let $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$, where $x>0$. If x satisfies the cubic equation $ax^3+bx^2+cx-1=0$, then $a+b+c$ has the value equal to

A. 24

B. 25

C. 26

D. 28

We have,
$\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$
$\Rightarrow \cos^{-1}(2x)+\cos^{-1}(3x)=\pi-\cos^{-1}(x)$
$\Rightarrow \cos^{-1}\left[(2x)(3x)-\sqrt{1-4x^2}\sqrt{1-9x^2}\right]=\cos^{-1}(-x)$
$\Rightarrow 6x^2-\sqrt{1-4x^2}\cdot\sqrt{1-9x^2}=-x$
$\Rightarrow (6x^2+x)^2=(1-4x^2)(1-9x^2)$
$\Rightarrow x^2+12x^3=1-13x^2$
$\Rightarrow 12x^3+14x^2-1=0 \quad \cdots(1)$
x satisfies the equation $ax^3+bx^2+cx-1=0$
Comparing this equation with equation (1), we get
$a=12, b=14, c=0$
$\therefore a+b+c=12+14+0=26$

JEE Main Part Test 3

12. If a, b, c are the sides opposite to angles A, B, C of a triangle ABC, respectively and $\angle A = \frac{\pi}{3}$, $b : c = \sqrt{3}+1 : 2$, then the value of $\angle B - \angle C$ is

A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

$$\frac{b}{c} = \frac{\sqrt{3}+1}{2}$$
$$\Rightarrow \frac{b-c}{b+c} = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}$$
$$\Rightarrow \frac{b-c}{b+c} = \frac{\sqrt{3}-1}{\sqrt{3}+3}$$

Now, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

$$\Rightarrow \tan\frac{B-C}{2} = \frac{\sqrt{3}-1}{\sqrt{3}+3}\cdot\sqrt{3}$$
$$\Rightarrow \tan\frac{B-C}{2} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$$
$$\Rightarrow \frac{B-C}{2} = \frac{\pi}{12}$$
$$\Rightarrow B - C = \frac{\pi}{6}$$

13. The value of $\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{4}{19}+\tan^{-1}\frac{4}{39}+\tan^{-1}\frac{4}{67}+\cdots\infty$ equals

A. $\tan^{-1}1+\tan^{-1}\frac{1}{2}$

B. $\frac{\pi}{2}-\cot^{-1}2$

C. $\frac{\pi}{2}-\cot^{-1}1$

D. $\cot^{-1}1+\tan^{-1}3$

Let $S=7+19+39+67+\cdots+T_n$
$S=\ 0+7+19+39+67+\cdots+T_{n-1}+T_n$
Subtracting the above equations, we get
$T_n=7+12+20+28+\cdots+(T_n-T_{n-1})$
$=7+\frac{n-1}{2}[24+8(n-2)]$
$=4n^2+3$

Now, $T_n'=\tan^{-1}\left(\frac{4}{4n^2+3}\right)=\tan^{-1}\left(\frac{1}{n^2+\frac{3}{4}}\right)$

$$=\tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)}$$

$$=\tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$$

$$=\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

Hence, $S_\infty=\sum_{n=1}^{\infty}T_n'=\frac{\pi}{2}-\tan^{-1}\frac{1}{2}$
$=\frac{\pi}{2}-\cot^{-1}2$

14. The solution set of the inequality

$$(\cot^{-1}x)(\tan^{-1}x) + \left(2-\frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2-\frac{\pi}{2}\right) > 0, \text{ is}$$

A. $x \in (\tan 2, \tan 3)$

B. $x \in (\cot 3, \cot 2)$

C. $x \in (-\infty, \tan 2) \cup (\tan 3, \infty)$

D. $x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$

Given, $(\cot^{-1}x)(\tan^{-1}x) + \left(2-\frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2-\frac{\pi}{2}\right) > 0$

$\Rightarrow \cot^{-1}x\left(\tan^{-1}x + 2 - \frac{\pi}{2}\right) - 3\left(\tan^{-1}x + 2 - \frac{\pi}{2}\right) > 0$

As $\tan^{-1}x - \frac{\pi}{2} = -\cot^{-1}x$, we get

$(\cot^{-1}x - 3)(2 - \cot^{-1}x) > 0$

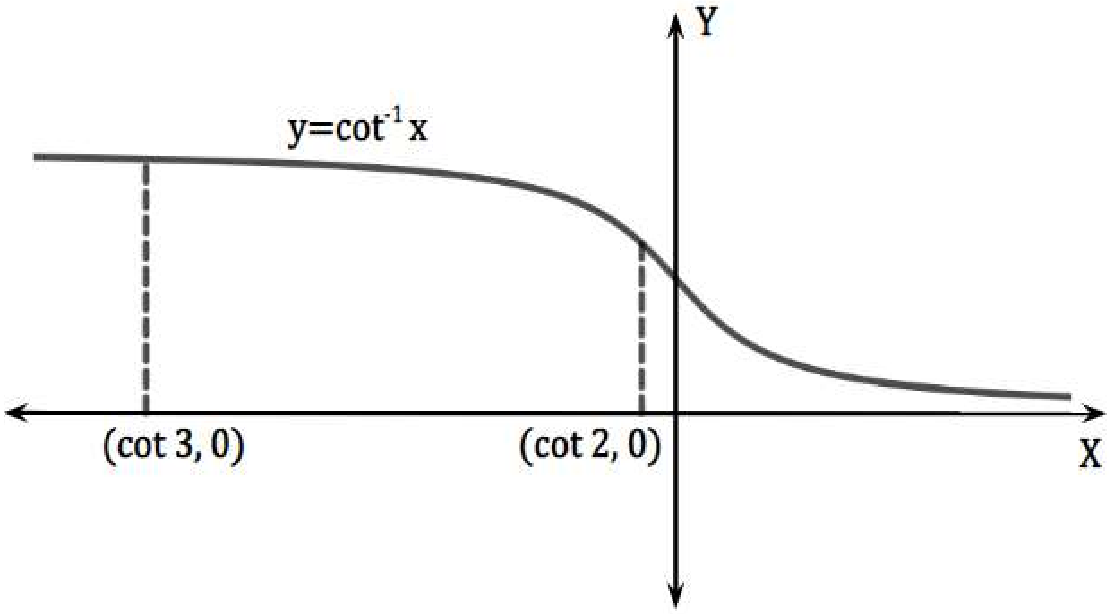
$\Rightarrow (\cot^{-1}x - 3)(\cot^{-1}x - 2) < 0$

$\Rightarrow 2 < \cot^{-1}x < 3$

$\Rightarrow \cot 3 < x < \cot 2$

($\because \cot^{-1}x$ is a decreasing function)

Hence, $x \in (\cot 3, \cot 2)$



JEE Main Part Test 3

15. If two sides of a triangle are the roots of $x^2 - 7x + 8 = 0$ and the angle between these sides is $\frac{\pi}{3}$, then the product of inradius and circumradius of the triangle is

A. $\frac{8}{7}$

B. $\frac{5}{3}$

C. $\frac{5\sqrt{2}}{3}$

D. 8

Let a, b, c be the sides of the triangle.
a and b are roots of $x^2 - 7x + 8 = 0$
$\Rightarrow a + b = 7$ and $ab = 8$
$C = \frac{\pi}{3}$

$$\Rightarrow \cos C = \frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$$

$\Rightarrow ab = a^2 + b^2 - c^2$
$\Rightarrow c^2 = (a + b)^2 - 3ab = 49 - 24 = 25$
$\Rightarrow c = 5$

$$\therefore rR = \frac{\Delta}{s} \times \frac{abc}{4\Delta}$$

$$= \frac{abc}{2(a + b + c)} = \frac{5}{3}$$

16. The value of

$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$, where $a, b, c > 0$, is

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. π

D. 0

JEE Main Part Test 3

$$S=\underbrace{\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}}_{x}+\underbrace{\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}}_{y}+\underbrace{\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}}_{z}$$

Let $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}=x$

$\Rightarrow \tan x=\sqrt{\frac{a(a+b+c)}{bc}}$

$\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}=y$

$\Rightarrow \tan y=\sqrt{\frac{b(a+b+c)}{ca}}$

and $\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=z$

$\Rightarrow \tan z=\sqrt{\frac{c(a+b+c)}{ab}}$

Now, $\tan x+\tan y+\tan z$

$=\sqrt{a+b+c}\left(\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ca}}+\sqrt{\frac{c}{ab}}\right)$

$=\sqrt{a+b+c}\left(\frac{a+b+c}{\sqrt{abc}}\right)$

$=\frac{(a+b+c)^{3/2}}{(abc)^{1/2}}$

$\tan x\cdot\tan y\cdot\tan z$

$=\sqrt{\frac{a(a+b+c)}{bc}}\cdot\sqrt{\frac{b(a+b+c)}{ac}}\cdot\sqrt{\frac{c(a+b+c)}{ab}}$

$=\frac{(a+b+c)^{3/2}}{(abc)^{1/2}}$

So, $\tan x+\tan y+\tan z=\tan x\cdot\tan y\cdot\tan z$

As x,y,z are positive,

$\Rightarrow x+y+z=\pi$

JEE Main Part Test 3

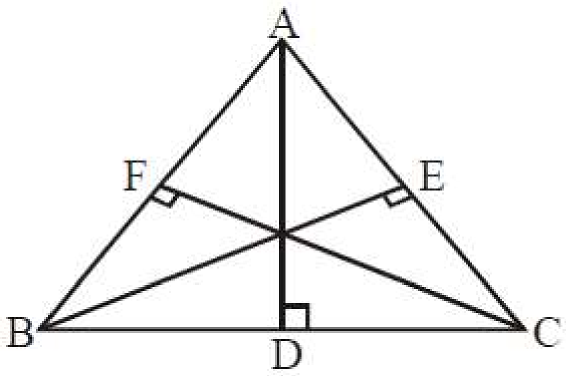
17. In a triangle ABC, altitudes from vertices A and B have lengths 3 and 6 respectively. Then the exhaustive set of values of the length of altitude from vertex C is

A. $(3,7)$

B. $(2,6)$

C. $(3,4)$

D. $(1,5)$



Let altitudes from vertices A, B and C have lengths h_a, h_b and h_c respectively.
Then $h_a = 3$ and $h_b = 6$
We know that
$|a-b| < c < a+b$
$\Rightarrow \left|\frac{2\Delta}{h_a} - \frac{2\Delta}{h_b}\right| < \frac{2\Delta}{h_c} < \frac{2\Delta}{h_a} + \frac{2\Delta}{h_b}$,
where Δ denotes the area of triangle ABC.
$\Rightarrow \left|\frac{1}{3} - \frac{1}{6}\right| < \frac{1}{h_c} < \frac{1}{3} + \frac{1}{6}$
$\Rightarrow \frac{1}{6} < \frac{1}{h_c} < \frac{1}{2}$
$\Rightarrow h_c \in (2,6)$

18. The value of $\lim\limits_{n \to \infty} \dfrac{\{x\} + \{2x\} + \ldots + \{nx\}}{n^2}$ is

(where $\{x\}$ denotes the fractional part of x)

A. 0

B. 1

C. ∞

D. It does not exist

We know,

$0 \leq \{f(x)\} < 1$

$$\Rightarrow \frac{0}{n^2} \leq \frac{\{x\} + \{2x\} + \ldots\ldots + \{nx\}}{n^2} < \frac{n}{n^2}$$

$$\Rightarrow \lim_{n \to \infty} \frac{0}{n^2} \leq \lim_{n \to \infty} \frac{\{x\} + \{2x\} + \ldots\ldots + \{nx\}}{n^2} < \lim_{n \to \infty} \frac{1}{n}$$

$$\Rightarrow 0 \leq \lim_{n \to \infty} \frac{\{x\} + \{2x\} + \ldots\ldots + \{nx\}}{n^2} < 0$$

$\therefore$ Using sandwich theorem, we have:

$$\lim_{n \to \infty} \frac{\{x\} + \{2x\} + \ldots\ldots + \{nx\}}{n^2} = 0$$

19. The derivative of $\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ with respect to $\cos(\ln x)$ is

A. $\frac{2x}{\sqrt{(1+x^2)}\sin(\ln x)}$

B. $-\frac{2x}{\sqrt{(1+x^2)}\sin(\ln x)}$

C. 1

D. $-\frac{4x}{\sqrt{(1+x^2)}\sin(\ln x)}$

Let $y=\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$

$\Rightarrow y=\ln\left(\sqrt{1+x^2}+x\right)-\ln\left(\sqrt{1+x^2}-x\right)$

$\Rightarrow \frac{dy}{dx}=\frac{\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)}{\sqrt{1+x^2}+x}-\frac{\left(\frac{2x}{2\sqrt{1+x^2}}-1\right)}{\sqrt{1+x^2}-x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1+x^2}}$

Let $z=\cos(\ln x)$

$\Rightarrow \frac{dz}{dx}=-\frac{\sin(\ln x)}{x}$

Therefore, $\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}$

$\Rightarrow \frac{dy}{dz}=-\frac{2x}{\sqrt{1+x^2}\sin(\ln x)}$

JEE Main Part Test 3

20. In $\triangle ABC$, sides opposite to angles A, B, C are denoted by a, b, c respectively. If $\angle A = 60^\circ, a = 5, b = 2\sqrt{3}$, then $\angle B =$

- **A.** $\sin^{-1}\frac{3}{5}$ ✔
- **B.** $\sin^{-1}\frac{4}{5}$ ✘
- **C.** $180^\circ - \sin^{-1}\frac{3}{5}$ ✘
- **D.** $180^\circ - \sin^{-1}\frac{4}{5}$ ✘

Given :
$\angle A = 60^\circ, a = 5, b = 2\sqrt{3}$
Using sine rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$
$\Rightarrow \dfrac{5}{\sin 60^\circ} = \dfrac{2\sqrt{3}}{\sin B}$
$\Rightarrow \sin B = \dfrac{3}{5}$
$\therefore B = \sin^{-1}\dfrac{3}{5}$
or $180^\circ - \sin^{-1}\dfrac{3}{5}$(obtuse angle)
As, $a > b$
$\Rightarrow \angle A > \angle B$
So, $\angle B$ is acute angle.
$\therefore \angle B = \sin^{-1}\dfrac{3}{5}$

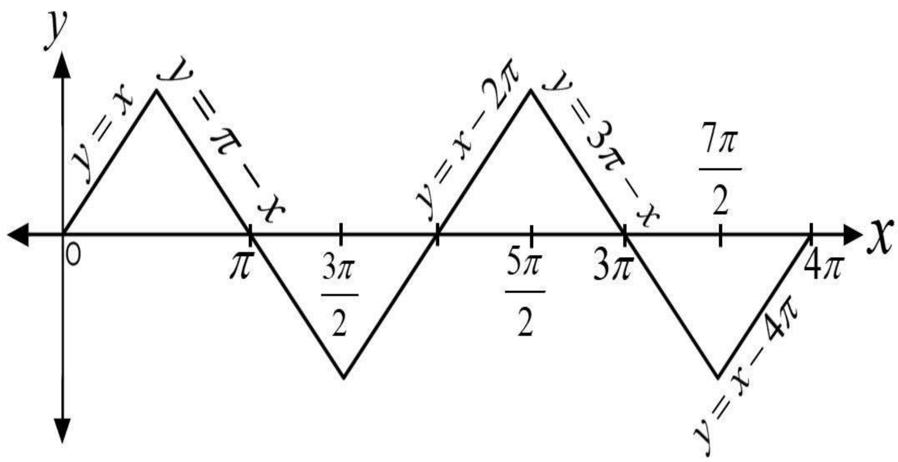
21. If $\sin^{-1}(\sin p) = 3\pi - p$ and the point of intersection of the lines $x + y = 6$ and $px - y = 3$ will have integral co-ordinates (both abscissa and ordinate), then the number of values of p is

Accepted Answers

1 1.0 1.00

Solution:

Given : $\sin^{-1}(\sin p) = 3\pi - p$



We know that $\sin^{-1}(\sin x) = 3\pi - x$ iff $\frac{5\pi}{2} \le x \le \frac{7\pi}{2}$

$\Rightarrow p \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$

Now given lines are : $x + y = 6$ and $px - y = 3$ (solving both)

$\Rightarrow$ Point of intersection of lines is $\left(\frac{9}{1+p}, \frac{6p-3}{1+p}\right)$

$\because$ Co-ordinates are integral,

$\Rightarrow \frac{9}{1+p} \in \mathbb{Z}$

$\Rightarrow 1 + p = \pm 1$ (or) $1 + p = \pm 3$ (or) $1 + p = \pm 9$

$\Rightarrow p \in \{-10, -4, -2, 0, 2, 8\}$

Also, $\frac{6p-3}{1+p} \in \mathbb{Z}$ for $p \in \{-10, -4, -2, 0, 2, 8\}$

But $p \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$

$\therefore p = 8$

$\therefore$ Number of possible value of p is 1.

22. Let $L_1 = \lim_{x \to 0} \frac{\cos(\pi x)(e^{\lambda x} - 1)}{\pi \sin x}$ and $L_2 = \lim_{x \to 0} \frac{\ln(1-x) + \sin 2x}{x}$. If $L_1 = L_2$, then the value of $[\lambda]$ is

(Note: $[\lambda]$ denotes the largest integer less than or equal to λ.)

Accepted Answers

3 3.0 3.00

Solution:

$$L_1 = \lim_{x \to 0} \frac{\cos(\pi x)(e^{\lambda x} - 1)}{\pi \sin x}$$

$$\Rightarrow L_1 = \lim_{x \to 0} \frac{\cos(\pi x)(e^{\lambda x} - 1)\lambda}{\pi(\lambda x)\frac{\sin x}{x}}$$

$$\Rightarrow L_1 = \frac{\lambda}{\pi} \quad \cdots (1)$$

$$L_2 = \lim_{x \to 0} \frac{\ln(1-x) + \sin 2x}{x}$$

$$\Rightarrow L_2 = \lim_{x \to 0} \frac{\ln(1-x)}{x} + \lim_{x \to 0} \frac{2 \sin 2x}{2x}$$

$$\Rightarrow L_2 = -1 + 2 = 1 \quad \cdots (2)$$

We know that,

$L_1 = L_2$

Using equation (1) and (2),

$$\Rightarrow \frac{\lambda}{\pi} = 1$$

$$\Rightarrow \lambda = \pi$$

$$\Rightarrow [\lambda] = 3$$

23. If the value of $\lim_{x\to 0}\left(\frac{x^n \sin^n x}{x^n - \sin^n x}\right)$ is non-zero finite, then n is equal to

Accepted Answers

2 2.0 2.00

Solution:

$$\lim_{x\to 0}\frac{x^n \sin^n x}{x^n - \sin^n x} = \lim_{x\to 0}\frac{x^{2n}\frac{\sin^n x}{x^n}}{x^n - \sin^n x}$$
$$= \lim_{x\to 0}\frac{x^{2n}}{x^n - \sin^n x}$$

Now, using expansion we have:

$$= \lim_{x\to 0}\frac{x^{2n}}{x^n - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^n}$$
$$= \lim_{x\to 0}\frac{x^n}{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^n}$$

Now, using binomial expansion, we have:

$$= \lim_{x\to 0}\frac{x^n}{n\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right) - \frac{n(n-1)}{2}\cdot\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)^2 \cdots}$$

$\therefore$ For $n = 2$ limit exist.

24. Let $f(x) = x^2 + px + 3$ and $g(x) = x + q$, where $p, q \in \mathbb{R}$. If $F(x) = \lim_{n\to\infty} \dfrac{f(x) + x^n g(x)}{1 + x^n}$ is derivable at $x = 1$, then the value of $p^2 + q^2$ is

Accepted Answers

5 5.0 5.00

Solution:

$$F(x) = \lim_{n\to\infty} \frac{f(x) + x^n g(x)}{1 + x^n}$$

$$\Rightarrow F(x) = \begin{cases} f(x)\,; & 0 \le x < 1 \\ \dfrac{f(1) + g(1)}{2}; & x = 1 \\ g(x)\,; & x > 1 \end{cases}$$

$$\Rightarrow F(x) = \begin{cases} x^2 + px + 3\,; & 0 \le x < 1 \\ \dfrac{p + q + 5}{2}; & x = 1 \\ x + q\,; & x > 1 \end{cases}$$

Differentiating w.r.t x,

$$\Rightarrow F'(x) = \begin{cases} 2x + p\,; & 0 < x < 1 \\ 1\,; & x > 1 \end{cases}$$

As the function is derivable, L.H.D = R.H.D,

$2 + p = 1 \Rightarrow p = -1$

$\because F(x)$ is derivable, therefore continuous too.

Checking continuity at $x = 1$,

$1 + p + 3 = 1 + q$

$\Rightarrow p + 3 = q$

$\Rightarrow -1 + 3 = q$

$\Rightarrow q = 2$

$\therefore p^2 + q^2 = 5$

25. The number of point(s) of non-differentiability for $f(x)=[e^x]+|x^2-3x+2|$ in $(-1,3)$ is (where [.] denotes greatest integer function, $e^3=20.1$)

Accepted Answers

22 22.0 22.00

Solution:

Given : $f(x)=[e^x]+|x^2-3x+2|$ & $x\in(-1,3)$

$\because e^x$ is increasing function $\Rightarrow e^x\in\left(\frac{1}{e},e^3\right)$ for $x\in(-1,3)$

$\Rightarrow e^x\in(0.36,20.1)$

$\Rightarrow [e^x]$ is not differentiable at 20 integral points.

and $|x^2-3x+2|=|(x-2)(x-1)|$ is non differentiale at $x=1,2$

Clearly, e^x is not integer at $x=1,2$

$\therefore$ Total 22 points of non-differentiability.

26. If $f(x)=\begin{cases} a+\sin^{-1}(x+b), & x\geq 1 \\ x, & x<1 \end{cases}$ is differentiable function, then value of $a+b$ is

Accepted Answers

0 0.0 0.00

Solution:

$$f(x)=\begin{cases} a+\sin^{-1}(x+b), & x\geq 1 \\ x, & x<1 \end{cases}$$

$$\Rightarrow f'(x)=\begin{cases} \dfrac{1}{\sqrt{1-(x+b)^2}}, & x>1 \\ 1, & x<1 \end{cases}$$

For $f(x)$ to be continuous at $x=1$, $f\left(1^+\right)=f\left(1^-\right)=f(1)$

$\Rightarrow a+\sin^{-1}(1+b)=1\cdots(1)$

For differentiability at $x=1$ $f'\left(1^+\right)=f'\left(1^-\right)$

$$\Rightarrow \frac{1}{\sqrt{1-(1+b)^2}}=1$$

$$\Rightarrow b=-1$$

From equation (1),

$a=1$

$\therefore a+b=0$

JEE Main Part Test 3

27. If $f(x) = \tan^{-1} \dfrac{x}{1+\sqrt{(1-x^2)}} + \sin\left\{2\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}, x \in (0,1)$, then the value of $f'\left(\dfrac{1}{2}\right)$ is

Accepted Answers

0 0.0 0.00 00

Solution:

Putting $x = \cos\theta$

$$y = \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) + \sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right\}$$

$$= \tan^{-1}\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} + \sin\left(2\tan^{-1}\tan\left(\frac{\theta}{2}\right)\right)$$

$$\left(\because x \in (0,1) \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)\right)$$

$$= \tan^{-1}\frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} + \sin\left(2\cdot\frac{\theta}{2}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + \sin\theta$$

$$= \frac{\pi}{4} - \frac{\theta}{2} + \sqrt{1-\cos^2\theta}$$

$$= \frac{\pi}{4} - \frac{\cos^{-1}x}{2} + \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{(1-x^2)}} + \frac{-2x}{2\sqrt{(1-x^2)}}$$

$$= \frac{1-2x}{2\sqrt{(1-x^2)}}$$

$$f'\left(\frac{1}{2}\right) = 0$$

28. The normal at the point $P\left(2, \frac{1}{2}\right)$ on the curve $xy = 1$ meets the curve again at Q. If m is the slope of the curve at Q, then the value of $|m|$ is

Accepted Answers

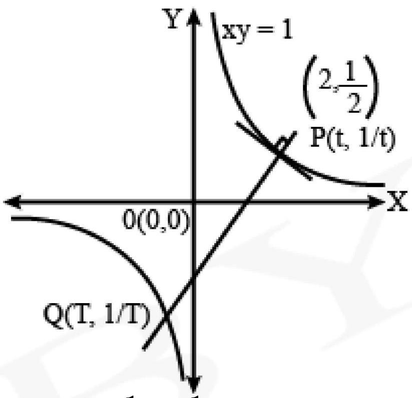
64 64.0 64.00

Solution:

We have $y = \frac{1}{x}$

Slope of tangent, $\frac{dy}{dx} = \frac{-1}{x^2}$

Slope of normal at P is t^2



$$\therefore t^2 = \frac{\frac{1}{t} - \frac{1}{T}}{t - T} = \frac{T - t}{tT} \cdot \frac{1}{t - T} = \frac{-1}{tT}$$

$$\Rightarrow T = \frac{-1}{t^3}$$

Slope of curve at Q is, $m = \frac{-1}{x^2}$

$$m = \frac{-1}{T^2} = \left(\frac{-1}{\left(\frac{-1}{t^3}\right)^2}\right) = -t^6$$

Given that $t = 2$

$m = -2^6 = -64$

Hence, $|m| = 64$.

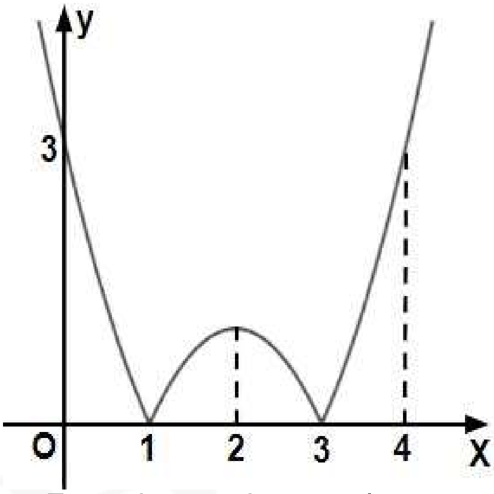
29. Let $f(x) = |x^2 - 4x + 3|$ be a function defined on $x \in [0, 4]$ and α, β, γ are the abscissas of the critical points of $f(x)$. If m and M are the local and absolute maximum values of $f(x)$ respectively, then the value of $\alpha^2 + \beta^2 + \gamma^2 + m^2 + M^2$ is

Accepted Answers

24 24.0 24.00

Solution:

$f(x) = |x^2 - 4x + 3|, \;\; x \in [0, 4]$



$\rightarrow$ From the graph, critical points are $(1, 0), (2, 1), (3, 0)$
$\alpha = 1, \; \beta = 2, \; \gamma = 3$

$\rightarrow$ Local maximum occurs at $x = 2$
$\therefore$ Local maximum value, $m = f(2) = 1$

$\rightarrow$ Absolute maximum value, $M = f(0) = f(4) = 3$

$\therefore \alpha^2 + \beta^2 + \gamma^2 + m^2 + M^2 = 24$

30. The minimum value of α for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=\alpha$ has at least one solution in $\left(0,\frac{\pi}{2}\right)$ is

Accepted Answers

9 9.0 9.00

Solution:

Let $f(x)=\frac{4}{\sin x}+\frac{1}{1-\sin x}$ and $\sin x=t$

$\because x\in\left(0,\frac{\pi}{2}\right)\Rightarrow 0<t<1$

$f(x)=\frac{4}{t}+\frac{1}{1-t}$

$f'(x)=\frac{-4}{t^2}+\frac{1}{(1-t)^2}=0$

$\Rightarrow \frac{t^2-4(1-t)^2}{t^2(1-t)^2}=0$

$\Rightarrow t=\frac{2}{3}$

$f_{\min}$ at $t=\frac{2}{3}$

$$\alpha_{\min}=f\left(\frac{2}{3}\right)=\frac{4}{2/3}+\frac{1}{1-2/3}$$
$$=6+3$$
$$=9$$