MATRICES AND DETERMINANTS

MATRICES

1. Introduction :

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns,) is called a matrix of order m by n, written as m × n matrix. In compact form, the matrix is represented by $A = [a_{ij}]_{m \times n}$.

2. Special Type of Matrices :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12} .... a_{in}]$ i.e. row matrix has exactly one now.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : ($A = O_{m \times n}$), An m×n matrix whose all entries are zeros.

(d) Horizontal Matrix : A matrix of order m×n is horizontal matrix if n > m.

(e) Vertical Matrix: A matrix of order m × n is a vertical martix if m > n.

(f) Square Matrix : (Order n) if number of rows = number of column, then matrix is a square matrix.

Note :

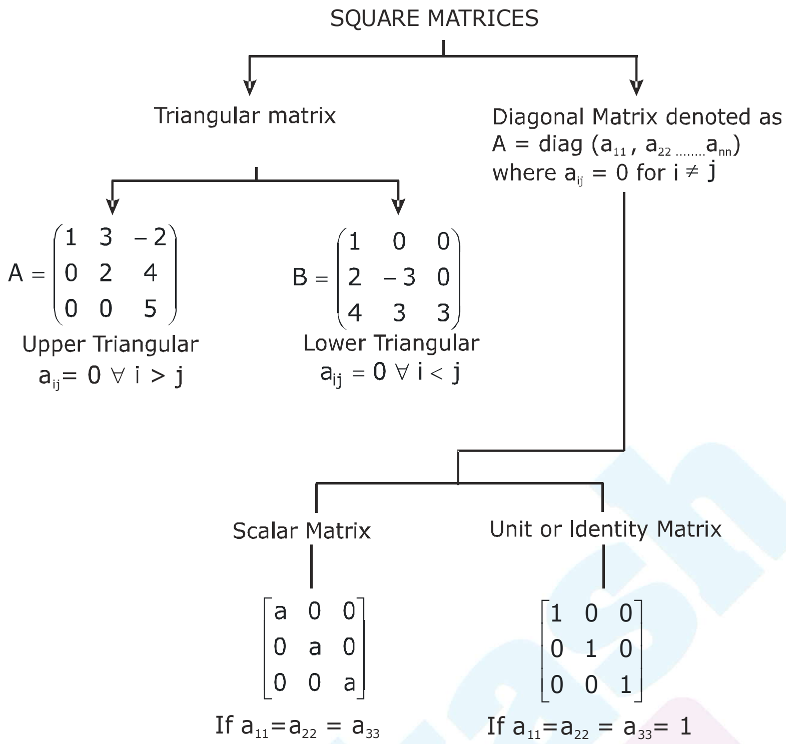
(i) The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

(ii) The elememts a_{11}, a_{22}, a_{33} ..... a_{nn} are called Diagonal Elements.The line along which the diagonal elements lie is called "Principal or Lead- ing diagonal. "The quantity $\sum_{i=1}^{n} a_{ii}$ = trace of the matrix written as , t_r (A)

(g) Singular matrix : Matrix A is said to be Singular matrix if its determinant |A| = 0, otherwise non - singular matrix i.e.

if det $|A| = 0 \Rightarrow$ Singular and det $|A| \neq 0 \Rightarrow$ non - singular

3. Square Matrices :



Note :

(i) Minimum number of zeros in triangular matrix of order n = n(n–1)/2.

(ii) Minimum number of zeros in a diagonal matrix of order n = n (n–1).

4. Equality of Matrices :

Two metirces $A = [a_{ij}]$ & $B = [b_{ij}]$ are said to be equal if,

(a) both have the same oder. (b) $a_{ij} = b_{ij}$ for each pair of i & j,

5. Algebra of Matrics :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) **Addition of matrices is commutative :** A + B = B + A

(b) **Matrix addition is associative : (**A +B**) + C =** A + (B + C)

6. Multiplication of a Matrix by a Scalar :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Properties of Scalar Multiplication of Matrices :

(i) $\lambda(A+B) = \lambda A + \lambda B$

(ii) $(\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda\mu A) = \mu(\lambda A)$

(iv) tr (kA) = k tr(A)

7. Multiplication of Matrices (Row by coloumn) :

Let 'A' be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m\times n} = [a_{ij}]$ and $B_{n\times q} = [b_{ij}]$

then order of AB is $m\times q$ & $(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

PROPERTIES OF MATRIX MULTIPLICATION :

If A , B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$	(Not commutative)
(ii) $(AB) C = A (BC)$	(Associative Law)
(iii) $A(B + C) = AB + AC$	(Distributive Law)
(iv) If $AB = AC \not\Rightarrow B = C$	(Cancellation Law is not applicable)
(v) $IA = A = AI$	(I is identity matrix)

(vi) $AB = O \not\Rightarrow A = O$ or $B = O$ (in general)

(vii) $tr (AB) = tr (BA)$

Note :

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = - BA$ then A and B are said to anticommute

8. Characteristic Equation :

Let A be a square matix and I be identity matrix, then the polynominal $|A - XI|$ is called as characteristic expression of A & the equation $|A - XI| = 0$ is called characteristic equation of A.

9. Cayley - Hamilton Theorem :

Every square matrix A satisfy its characteristic equation i.e. if $a_0x^n + a_1x^{n-1} +.... + a_{n-1}x + a_n = 0$ is the characteristic equation of matrix A, then $a_0A^n + a_1 A^{n-1} + ... + a_{n-1} A + a_n I = 0$

10. Positive Integral Powers of a Square Matrix :

(a) $A^mA^n = A^{m+n}$

(b) $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^m = I$ $m,n \in N$

(d) $A^o = I_n$ (where n is order of matrix A)

(where $m, n \in N$)

11. Transpose of a Matrix (Changing rows & columns) :

Let A be any matrix of order $m \times n$ and $A = (a_{ij})_{m\times n}$. Then A^T or $A' = (a_{ji})_{n\times m}$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$

Properties of transpose of a matrix :

If A^T & B^T denote the transpose of A and B

(a) $(A \pm B)^T = A^T \pm B^T$; note that A&B have the same order.
(b) $(AB)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB
(c) $(A^T)^T = A$
(d) $(kA)^T = kA^T$, where k is a scalar.
Note : $(A_1.A_2.......A_n)^T = A_n^T .... A_2^T . A_1^T$ (reversal law for transpose)
(e) $tr(A^T) = tr(A)$

12. Some Special Square Matrices :

(a) Orthogonal Matrix :
A square matix is said to be orthogonal matix if $AA^T = I$
Note :
(i) The determinant of orthogonal matrix is either 1 or –1 Hence orthogonal matix is always invertible
(ii) $AA^T = I = A^TA$ Hence $A^{-1} = A^T$.

(b) Idempotent Matrix : A square matix is idempotent matrix if $A^2=A$. For idempotent matrix :
(i) $A^n = A \ \forall \ n \geq 2, n \in N$.
(ii) Determinant of idempotent matix is either 0 or 1
(iii) If idempotent matrix is invertible then its inverse will be identity matrix i.e. I.

(c) Periodic Matrix :
A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K is a periodic matrix.The period of the matrix is the least value of K for which this hold true.Note that period of an idempotent matrix is 1.

(d) Nilpotent Matrix :
If $A^m = O$, $A^{m-1} \neq O$ is called nilpotent matrix where m is order of nilpotent
Note : That a nilpotent matrix will not be invertible.

(e) Involutary Matrix : If $A^2 = I$, the matrix is said to be an involutary matrix.
Note that $A = A^{-1}$ for an involutary matrix.

(f) If A and B are square matrices of same order and AB = BA
then $(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ... + {}^nC_nB^n$

13. Symmetric & Skew Symmetric Matrix :

(a) Symmertic matrix :
In square matrix if $A^t = A$ then A is called Symmetric matrix.
(b) Skew symmetric matrix :
Square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $a_{ij} = -a_{ji} \ \forall$ i & j. Hence if A is skew symmetric, then $a_{ii} = 0 \ \forall$ i.
Thus the diagonal elements of a skew square matrix are all zero, but not the converse.
For a skew symmetirc matrix $A = -A^T$.

(c) Properties of symmetric & skew symmetric matrix :

(i) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(ii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric is a skew symmetric matrix.

(iii) If A &B are symmetric matrices then

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(iv) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{skew symmetric}}$$

and $A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$

14. Adjoint of a Square Matrix :

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$ Then (adj A) = $\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

Note :

If A be a square matrix of order n, then

(i) $A(\text{adj } A) = |A|\, I_n = (\text{Adj } A) \times A$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) adj (AB) = (adj B) (adj A)

(vi) adj (KA) = K^{n-1} (adj A), where K is a scalar

15. Inverse of a Matrix (Reciprocal Matrix) :

A square matrix A said to be invertible (for non singular matrix) if there exists a matrix B such that, AB = I = BA

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, A(adj A) = |A|I = (adj A) A.

$\therefore\ A^{-1} = \dfrac{(\text{adj}A)}{|A|}$

Note :

The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$

Theorem : If A & B are invertible matrices of the same order, then $(AB)^{-1}=B^{-1}A^{-1}$.

Note

(i) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$

(ii) If A is invertible (a) $(A^{-1})^{-1} = A$

(b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$; $k \in N$

16 . System of Equation & Criteria for Consistency :

Matrix inversion method :

Example :

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow AX = B \Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow X = A^{-1}B = \dfrac{\text{adj}A}{|A|}B$

Note :

(i) If $|A| \neq 0$, system is consistent having unique solution

(ii) If $|A| \neq 0$, & (adj A) × B $\neq 0$ (Null matrix) system is consistent having uique non-trivial solution.

(iii) If $|A| \neq 0$ & (adj A) × B = 0 (Null matrix), system is consistent having trivial solution.

(iv) If |A| = 0, then **matrix method fails**

If (adj A)×B = 0 (null matrix) → Consistent (Infinite solution)

if (adj A) × B $\neq 0$ → Inconsistent (no solution)

DETERMINANTS

1. Minors :

The minor of a given element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Hence a determinant of order three will have "9 minors"

2. Cofactors :

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element : $C_{ij} = (-1)^{i+j}. M_{ij}$;

Note :

Consider $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_1 be cofactor of b_1 and so on, then,

(i) $a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \ldots\ldots\ldots = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = \ldots\ldots\ldots = 0$

3. Properties of Determinants :

(a) The value of a determinants remains unaltered, if the rows & columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

(c) If all the elements of a row (column) are zero or two rows (or columns) identical or in same proportion , then its value is zero.

(d) If all the elements of any row (or column) be multiplied by a non zero number, then the determinant is multiplied by that number.

(e) $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. They D' = D.

Note : While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when x = a, then x – a is a factor of Δ. Again, if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r : r = 1, 2,3 are three differential function,

then $\frac{d}{dx} D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$

4. Multiplication of two Determinants :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1l_2 & a_1m_1+b_1m_2 \\ a_2l_1+b_2l_2 & a_2m_1+b_2m_2 \end{vmatrix}$$

Similary two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column,

(b) If D' is the detereminant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D' = D^{n-1}$.

5. Special Determinants :

(a) Symmetric Determinant :

If the elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$.

(b) Skew Symmetric Determinants :

If $a_{ij} = -a_{ji}$ then the determinant is said to be a skew symmetric determinant. Here all the principal diagonal elements are zero.The value of a skew symmetric determinant of odd order is zero and of even order is perfect square.

e.g. $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(c) Other Important Determinants :

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

6. System of Equations :

(a) System of equations involving two variables :

$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$

- Consistent (System of equation has solution)
  - unique solution: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or $a_1b_2 - a_2b_1 \neq 0$ [Equations represents intersecting lines]
  - Infinite solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ [Equations represents coincident lines]
- Inconsistent (System of equation has no Solution): $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ [Equations represent parallel disjoint lines]

If $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then x = $\frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$

or $\quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$; $\quad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

(b) System of equations involving three variables :

$a_1x + b_1y + c_1z = d_1$.
$a_2x + b_2y + c_2z = d_2$.
$a_3x + b_3y + c_3z = d_3$.

To solve this system we first define following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is used to solve the system.

Check value of Δ

$\Delta \neq 0$ → Consistent system and has unique solution

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

$\Delta = 0$ → Check the values of Δ_1, Δ_2 and Δ_3.

- Atleast one of Δ_1, Δ_2 and Δ_3 is not zero → Inconsistent system (i.e. No. Solution)
- $\Delta_1 = \Delta_2 = \Delta_3 = 0$ → Put z = t and solve any two equations to get the

values of x& y in terms of t

↓ If these values of x, y and z in terms of t satisfy third equation

↓ Consistent system (System has infinite solutions)

↓ The values of x,y, z doesn't satisfy third equation

↓ Inconsistent system

Note :

(i) Trivial solution : In the solution st of system of equation, if all the variables assumes zero, then such a solution set is called Trivial solution otherwise the solution is called non-trivial solution.

(ii) If $d_1 = d_2 = d_3 = 0$ then system of linear eqation is known as system of Homogeneous linear equation which always posses atleast one solution (0,0,0).

(iii) If system of homogeneous linear equation posses non-zero/ nontrivial solution then $\Delta = 0$.In such case given system has infinite solutions.

PROBABILITY

1. Some Basic Terms and Concepts :

(a) An Experiment : An action or operation resulting in two or more outcomes is called an experiment.

(b) Sample space : The set of all possible outcomes of an experiment is called a sample space, denoted by S.An element of S is called a sample point.

(c) Event : Any subset of sample space is an event.

(d) Simple Event : An Event is called a simple event if it is a singleton subset of the sample space S.

(e) Compound Event : It is the joint occurrence of two or more simple events.

(f) Equally Likely events: A number of simple events are said to be equally likely if there is no reason for one event to occur in preference to any other event.

(g) Exhaustive Events : Exhaustive events are a set of events in a sample space such that one of them compulsorily occurs while performing the experiment.

(h) Mutually Exclusive or Disjoint Events: If two events cannot occur simultaneously, then they are mutually exclusive.

if A and B are mutually exclusive, then $A \cap B = \phi$

(i) **Complement of an Event :**The complement of an event A denoted by $\bar{A}$, A' or A^c or is the set of all sample points of the space other than the sample points in A.

2. Mathematical Definition of Probability :Let the outcomes of an experiment consists of n exhaustive mutaually exclusive and equally likely cases. Then the sample spaces S has n sample points. if an event A consists of m sample points,($0 \le m \le n$). then the probability of event A, denoted by P(A) is defined. to be m/n i.e. P(A) = m/n Let S = a_1, a_2, ... a_n be the sample space

(a) $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.

(b) $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.

(c) If $A_i = \{a_i\}, i = 1, \ldots\ldots n$ then A_i is the event corresponding to a single sample point a_i Then $P(A_i) = \frac{1}{n}$

(d) $0 \le P(A) \le 1$

3. Odds Against and Odd in Favour of an Event :

Let there be m + n equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition of probability of occurrences = $\frac{m}{m+n}$

The probability of non-occurrence = $\frac{n}{m+n}$

$\therefore P(A) : P(A') = m : n$

Thus the odd in favour of occurrence of the event A are defined by m:n i.e. P(A):P(A'): and the odds against the occurrence of the event A are defined by n : m i.e. P(A') : P(A).

4. Addition Theorem :

(a) If A and B are any events in S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the probability of an event is a nonnegative number, it follows that

$P(A \cup B) \le P(A) + P(B)$

For three events A, B and C in S we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

General form of addition theorem

For n events $A_1, A_2, A_3, \ldots\ldots\ldots A_n$ in S, we have

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots \cup A_n)$

$= \sum_{i=1}^{n} P(A) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots\ldots\ldots$

$+ (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots\ldots \cap A_n)$

(b) If A and B are mutually exclusive, then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B)$

5. Multiplication Theorem :

Independent event:

So if A and B are two independent events then happening of B will have no effect on A.

Difference between independent & mutually exclusive event:

(i) Mutually exclusiveness is used when events are taken from same experiment & independence when events are taken from different experiments.

(ii) Independent events are represented by word "and " but mutaually exclusive events are represented by word "OR"

(a) When events are independent:

$P(A/B) = P(A)$ and $P(B/A) = P(B)$, then

$P(A \cap B) = P(A).P(B)$ or $P(AB) = P(A). P(B)$

(b) When events are not independent

The probability of simultaneous happening of two events A and B is equal to the probability of A multiplied by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect to B) i.e

$P(A \cap B) = P(A).P(B/A)$ or $P(B).P(A/B)$

OR

$P(AB) = P(A).P(B/A)$ or $P(B).P(A/B)$

(c) Probability of at least one of the n Independent events

If $p_1, p_2, p_3, \ldots\ldots p_n$ are the probabilities of n independent event $A_1, A_2, A_3 \ldots A_n$ then the probability of happening of at least one of these event is

$1 - [(1-p_1)(1-p_2)\ldots\ldots(1-p_n)]$.

$P(A_1+A_2+A_3+\ldots\ldots+A_n) = 1 - P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3)\ldots\ldots P(\bar{A}_n)$

6. Conditional Probability :

If A and B are any event in S then the conditional probability of B relative to A is given by

$$P(B/A) = \frac{P(B \cap A)}{P(A)}, \text{ If } P(A) \neq 0$$

7. Baye's Theorem or Inverse Pbobability :

Let $A_1, A_2 \ldots\ldots A_n$ be mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then $P\left(\frac{A_1}{A}\right) = \frac{P(A_1)P(A/A_i)}{\sum_{i=1}^{n} P(A_1)P(A/A_i)}$

8. Binomial Distribution for Repeated Trials :

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment.

Outcomes of such an experiment are known as success and failure probability of success is denoted by p and probability of failure by q.

$\therefore p + q = 1$

If binomial experiment is repeated n times then

$(p+q)^n = {}^nC_0q^n + {}^nC_1pq^{n-1} + {}^nC_2p^2q^{n-2} + \ldots + {}^nC_rp^rq^{n-r} + \ldots\ldots + {}^nC_np^n = 1$

(a) Probability of exactly r successes in n trials $= {}^nC_rp^rq^{n-r}$

(b) Probability of at most r successes in n trials $= \sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$

(c) Probability of at least r successes in n trials = $\sum_{\lambda=r}^{n} {}^{n}C_{\lambda} p^{\lambda} q^{n-\lambda}$

(d) Probability of having 1^{st} success at the r^{th} trials = $p\,q^{r-1}$

(e) The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$ respectively.

9. Some Important Results :

(a) Let A and B be two events, then

(i) $P(A) + P(\bar{A}) = 1$

(ii) $P(A+B) = 1 - P(\bar{A}\,\bar{B})$

(iii) $P(A/B) = \dfrac{P(AB)}{P(B)}$

(iv) $P(A+B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$

(v) $A \subset B \Rightarrow P(A) \le P(B)$

(vi) $P(\bar{A}B) = P(B) - P(AB)$

(vii) $P(AB) \le P(A)\,P(B) \le P(A+B) \le P(A) + P(B)$

(viii) $P(AB) = P(A) + P(B) - P(A+B)$

(ix) P [Exactly one event] = $P(A\bar{B}) + P(\bar{A}B)$

$= P(A) + P(B) - 2P(AB) = P(A+B) - P(AB)$

(x) P (neither A nor B) = $P(\bar{A}\,\bar{B}) = 1 - P(A+B)$

(xi) $P(\bar{A} + \bar{B}) = 1 - P(AB)$

(b) Number of exhaustive cases of tossing n coins simultaneoulsy

(or of tossing a coin n times) = 2^n

(c) Number of exhaustive cases of tossing n dice simultaneoulsy

(or throwing one dice n times) = 6^n

(d) Playing Cards :

(i) Total Cards : 52 (26 red, 26 black)

(ii) Four suits : Heart, Diamond, Spade, Club - 13 cards each

(iii) Court Cards : 12 (4 Kings, 4 queens, 4 jacks)

(iv) Honour Cards : 16 (4 aces, 4 kings, 4 queens. 4 jacks)

(e) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envcelopes at random, then

(i) Probability that all letters are in right envelopes = $\frac{1}{n!}$

(ii) Probability that all letters are not in right envelopes =$1-\frac{1}{n!}$

(iii) Probability that no letters are in right envelopes $=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.......+(-1)^{n}\frac{1}{n!}$

(iv) Probability that exactly r letters are in right envelopes

$$=\frac{1}{r!}\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.......+(-1)^{n-r}\frac{1}{(n-r)!}\right]$$