

Subject: Mathematics

1. Let the random variable X have a binomial distribution with mean 8 and variance

4. If $P(X \leq 2) = \frac{k}{2^{16}}$, then k is equal to:

- ☐ A. 1
- ☐ B. 17
- ☐ C. 121
- ☒ D. 137

Given that binomial distribution with mean $np = 8$ and variance $npq = 4$
therefore, $q = \frac{1}{2} \Rightarrow n = 16, p = \frac{1}{2}$,

$$\begin{aligned} \text{Hence } P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}^{16}C_0 \left(\frac{1}{2}\right)^{16} + {}^{16}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{15} + {}^{16}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{14} \\ &= \frac{1 + 16 + 120}{2^{16}} = \frac{137}{2^{16}} = \frac{k}{2^{16}} \end{aligned}$$

2. Minimum number of times a fair coin must be tossed so that the probability of getting at least one head is more than 99% is :

☐ A. 5

☐ B. 6

☒ C. 7

☐ D. 8

Let n be the minimum number of toss required to get at least one head, then required probability = $1 - \text{probability that on all } n \text{ toss we are getting tail.}$

So, the probability to get at least one head

$$= 1 - P(\text{no head}) > \frac{99}{100}$$

$$\Rightarrow 1 - \left(\frac{1}{2}\right)^n > \frac{99}{100}$$

$$\Rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{100}$$

$$\Rightarrow 2^n > 100$$

$$\Rightarrow n \geq 7$$

Minimum value of n is 7.

3. The probability that two randomly selected subsets of the set $\{1, 2, 3, 4, 5\}$ have exactly two elements in their intersection, is

☐ A. $\frac{65}{2^7}$

☒ B. $\frac{135}{2^9}$

☐ C. $\frac{65}{2^8}$

☐ D. $\frac{35}{2^7}$

Let A and B be two subsets.

For each $x \in \{1, 2, 3, 4, 5\}$, there are four possibilities :

$x \in A \cap B, x \in A' \cap B, x \in A \cap B', x \in A' \cap B'$

So, the number of elements in sample space = 4^5

Required probability

$$\begin{aligned} &= \frac{{}^5C_2 \times 3^3}{4^5} \\ &= \frac{10 \times 27}{2^{10}} = \frac{135}{2^9} \end{aligned}$$

4. Let A and B be two non-null events such that $A \subset B$. Then, which of the following statements is always correct?

- ☒ A. $P(A|B) = 1$
☒ B. $P(A|B) \leq P(A)$
☒ C. $P(A|B) \geq P(A)$
☒ D. $P(A|B) = P(B) - P(A)$

Given,

$$A \subset B \Rightarrow A \cap B = A$$

$$\Rightarrow P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$\Rightarrow P(A/B) = \frac{P(A)}{P(B)} \geq P(A)$$

$$(\because 0 \leq P(B) \leq 1)$$

5. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where \overline{A} stands for the complement of the event A . Then the events A and B are

- ☒ A. mutually exclusive and independent
- ☐ B. equally likely but not independent
- ☒ C. independent but not equally likely
- ☐ D. independent and equally likely

Given: $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$

Since, $P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$

and, $P(A) = 1 - P(\overline{A}) = \frac{3}{4}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \frac{5}{6} = \frac{3}{4} + P(B) - \frac{1}{4}$$

$$\Rightarrow P(B) = \frac{1}{3}$$

$P(A) \neq P(B)$, So not equally likely

Now, $P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{3}{4}$

$$\Rightarrow P(A) \cdot P(B) = \frac{1}{4} = P(A \cap B)$$

Hence, events A and B are independent but not equally likely

6. A person throws two fair dice. He wins Rs. 15 for throwing a doublet (same numbers on the two dice), wins Rs. 12 when the throw results in the sum of 9, and loses Rs. 6 for any other outcome on the throw. Then the expected gain/loss (in Rs.) of the person is :

- ☒ A. 2 gain
☒ B. $\frac{1}{2}$ loss
☒ C. $\frac{1}{2}$ gain
☒ D. $\frac{1}{4}$ loss

X	+15	+12	-6
$P(X)$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{13}{18}$

$$E(X) = 15 \times \frac{1}{6} + 12 \times \frac{1}{9} - 6 \times \frac{13}{18} = -\frac{1}{2}$$

So, expectation is $\frac{1}{2}$ with a loss

7. Box I contains 30 cards numbered 1 to 30 and Box II contains 20 cards numbered 31 to 50. A box is selected at random and a card is drawn from it. The number on the card is found to be a non-prime number. The probability that the card was drawn from Box I is:

☒ A. $\frac{4}{17}$

☒ B. $\frac{8}{17}$

☒ C. $\frac{2}{5}$

☒ D. $\frac{2}{3}$

Box I : 1 to 30 numbered cards

Prime on box I

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29}

Box II : 31 to 50 numbered cards

Prime on box II

{31, 37, 41, 43, 47}

A : selected number on card is non - prime

$$P(A) = P(I) \cdot P\left(\frac{A}{I}\right) + P(II) \cdot P\left(\frac{A}{II}\right)$$

$$= \frac{1}{2} \times \frac{20}{30} + \frac{1}{2} \cdot \frac{15}{20}$$

$$\text{Now, } P\left(\frac{I}{A}\right) = \frac{P(II) \cdot P\left(\frac{A}{II}\right)}{P(A)}$$

$$= \frac{\frac{1}{2} \cdot \frac{20}{30}}{\frac{1}{2} \cdot \frac{20}{30} + \frac{1}{2} \cdot \frac{15}{20}} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{3}{4}} = \frac{8}{17}$$

8. A seven-digit number is formed using digit 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is:

☐ A. $\frac{6}{7}$

☐ B. $\frac{4}{7}$

☒ C. $\frac{3}{7}$

☐ D. $\frac{1}{7}$

Number of ways in which the seven digit number formed is

$$n(s) = \frac{7!}{2!3!2!}$$

Number of ways in which the seven digit number is divisible by 2 is

$$n(E) = \frac{6!}{2!2!2!}$$

Hence required probability is

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!} = \frac{3}{7}$$

9. The coefficients a, b and c of the quadratic equation, $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is :

- ☐ A. $\frac{1}{54}$
- ☐ B. $\frac{1}{72}$
- ☐ C. $\frac{1}{36}$
- ☒ D. $\frac{5}{216}$

$$ax^2 + bx + c = 0$$

$$a, b, c \in \{1, 2, 3, 4, 5, 6\}$$

$$n(S) = 6 \times 6 \times 6 = 216$$

$$D = 0 \Rightarrow b^2 = 4ac$$

$$\Rightarrow ac = \frac{b^2}{4}$$

$$\text{If } b = 2, ac = 1 \Rightarrow a = 1, c = 1$$

$$\text{If } b = 4, ac = 4$$

$$\Rightarrow a = 1, c = 4$$

$$\text{or } a = 4, c = 1$$

$$\text{or } a = 2, c = 2$$

$$\text{If } b = 6, ac = 9 \Rightarrow a = 3, c = 3$$

$$\therefore \text{Probability} = \frac{5}{216}$$

10. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is :

- ☐ A. $\frac{3}{4}$
- ☐ B. $\frac{3}{10}$
- ☒ C. $\frac{2}{5}$
- ☐ D. $\frac{1}{5}$

Initially 4 Red balls and 6 Black balls

$$\text{Required probability} = \frac{4}{10} \times \frac{6}{12} + \frac{6}{10} \times \frac{4}{12} = \frac{2}{5}$$

11. Let α, β be the roots of the equation $x^2 + x + 1 = 0$. Then for $y \neq 0$ in \mathbb{R} ,

$$\begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

is equal to:

☒ A. $y(y^2 - 3)$

☒ B. $y^3 - 1$

☒ C. $y(y^2 - 1)$

☒ D. y^3

Roots of the equation $x^2 + x + 1 = 0$ will be $\alpha = w$ and $\beta = w^2$, where w and w^2 are the cube roots of unity.

$$\Rightarrow \Delta = \begin{vmatrix} y+1 & w & w^2 \\ w & y+w^2 & 1 \\ w^2 & 1 & y+w \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$\Rightarrow \Delta = y \begin{vmatrix} 1 & 1 & 1 \\ w & y+w^2 & 1 \\ w^2 & 1 & y+w \end{vmatrix}$$

$$[\because 1 + w + w^2 = 0]$$

On expanding along R_1 , we get

$$\Delta = y(y^2) = y^3$$

12. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point:

☐ A. $\left(2, -\frac{1}{2}\right)$

☐ B. $\left(1, \frac{3}{4}\right)$

☐ C. $\left(1, -\frac{3}{4}\right)$

☒ D. $\left(2, \frac{1}{2}\right)$

Given vertices of the triangle are $(k, -3k)$, $(5, k)$ and $(-k, 2)$

Now, the area of triangle is

$$\frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = 28$$

$$\Rightarrow \frac{1}{2} \begin{vmatrix} k & 5 & -k & k \\ -3k & k & 2 & -3k \end{vmatrix} = 28$$

$$\Rightarrow |(k^2 + 15k) + (10 + k^2) + (3k^2 - 2k)| = 56$$

$$\Rightarrow |5k^2 + 13k + 10| = 56$$

$$\Rightarrow 5k^2 + 13k + 10 = 56 \quad (\because 5k^2 + 13k + 10 > 0)$$

$$\Rightarrow 5k^2 + 13k - 46 = 0$$

$$\Rightarrow (5k + 23)(k - 2) = 0$$

$$\Rightarrow k = 2 \quad (\because k \text{ is integer})$$

Now vertices are $A(2, -6)$, $B(5, 2)$, $C(-2, 2)$

Equation of the altitude dropped from vertex A is $x = 2$

Equation of the altitude dropped from vertex C is $3x + 8y - 10 = 0$

On solving the equations of altitude we get, Orthocentre = $\left(2, \frac{1}{2}\right)$

13. If the system of equations $2x + 3y - z = 0$, $x + ky - 2z = 0$ and $2x - y + z = 0$ has a non-trivial solution (x, y, z) , then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to :

☐ A. $\frac{3}{4}$

☐ B. -4

☐ C. $-\frac{1}{4}$

☒ D. $\frac{1}{2}$

$$\begin{array}{lcl} 2x + 3y - z = 0 & \dots & [1] \\ x + ky - 2z = 0 & \dots & [2] \\ 2x - y + z = 0 & \dots & [3] \end{array}$$

Since, the given system of equations has a non-trivial solution

$$\therefore \begin{vmatrix} 2 & 3 & -1 \\ 1 & k & -2 \\ 2 & -1 & 1 \end{vmatrix} = 0 \Rightarrow k = \frac{9}{2}$$

From [1] and [3], we get

$$4x + 2y = 0 \Rightarrow \frac{x}{y} = -\frac{1}{2}$$

From [1] and [3], we get

$$4y - 2z = 0 \Rightarrow \frac{y}{z} = \frac{1}{2}$$

From [1] and [3], we get

$$8x + 2z = 0 \Rightarrow \frac{z}{x} = -4$$

$$\begin{aligned} \therefore \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k &= -\frac{1}{2} + \frac{1}{2} - 4 + \frac{9}{2} \\ &= \frac{1}{2} \end{aligned}$$

14. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to :

☐ A. 4

☒ B. 11

☐ C. 5

☐ D. 0

$$\text{adj}(A) = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$$

As we know,

$$\begin{aligned} |\text{adj}(A)| &= |A|^2 \\ \Rightarrow 1(12 - 12) - \alpha(4 - 6) + 3(4 - 6) &= (4)^2 \\ \Rightarrow \alpha &= 11 \end{aligned}$$

15. Let A and B be two invertible matrices of order 3×3 . If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to :

☐ A. 1

☐ B. $\frac{1}{4}$

☐ C. 16

☒ D. $\frac{1}{16}$

$$\det(ABA^T) = 8$$

$$\Rightarrow |A| \cdot |B| \cdot |A^T| = 8$$

$$\Rightarrow |A|^2 |B| = 8 \quad \dots (1) \quad [\because |A| = |A^T|]$$

$$\det(AB^{-1}) = 8$$

$$\Rightarrow |A| \cdot |B^{-1}| = 8$$

$$\Rightarrow \frac{|A|}{|B|} = 8$$

$$\Rightarrow |A| = 8|B| \quad \dots (2)$$

Put $|A|$ from equation (2) in equation (1), we get

$$64|B|^2 |B| = 8$$

$$\Rightarrow |B| = \frac{1}{2}$$

$$\therefore |A| = 8|B| = 4$$

$$\text{Now, } \det(BA^{-1}B^T) = |B| \cdot \frac{1}{|A|} \cdot |B^T|$$

$$= \frac{|B|^2}{|A|} = \frac{1}{16}$$

16. If $\begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$; then for all

$\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, $\det(A)$ lies in the interval :

☐ A. $\left(0, \frac{3}{2}\right]$

☐ B. $\left(1, \frac{5}{2}\right]$

☐ C. $\left(\frac{3}{2}, 3\right]$

☒ D. $\left[\frac{5}{2}, 4\right)$

$$|A| = \begin{vmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_3$$

$$\Rightarrow |A| = \begin{vmatrix} 0 & 0 & 2 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{vmatrix}$$

$$\Rightarrow |A| = 2(\sin^2 \theta + 1)$$

$$\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \Rightarrow \sin \theta \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\therefore 0 \leq \sin^2 \theta < \frac{1}{2}$$

$$\Rightarrow 2 \leq 2(1 + \sin^2 \theta) < 3$$

$$|A| \in [2, 3)$$

17. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2, \text{ then } K \text{ is equal to:}$$

☒ A. $\alpha\beta$

☒ B. $\frac{1}{\alpha\beta}$

☒ C. 1

☒ D. -1

$$f(1) = \alpha + \beta$$

$$f(2) = \alpha^2 + \beta^2$$

$$f(3) = \alpha^3 + \beta^3$$

$$f(4) = \alpha^4 + \beta^4$$

$$\text{So, } \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

Splitting it as a product of two determinants.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 1-\alpha & \alpha-\beta & \beta \\ 1-\alpha^2 & \alpha^2-\beta^2 & \beta^2 \end{vmatrix}^2$$

$$= [(1-\alpha)(\alpha-\beta)]^2 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \beta \\ 1+\alpha & \alpha+\beta & \beta^2 \end{vmatrix}^2$$

$$= [(1-\alpha)(\alpha-\beta)]^2 (\beta-1)^2$$

$$= (1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$$

Hence, $K = 1$

18. If the system of linear equations

$$x + y + z = 5$$

$$x + 2y + 3z = 9$$

$$x + 3y + \alpha z = \beta$$

has infinitely many solutions, then $\beta - \alpha$ equals:

☒ A. 5

☒ B. 18

☒ C. 8

☒ D. 21

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \alpha \end{vmatrix}$$

$$= 1(2\alpha - 9) - 1(\alpha - 3) + 1(3 - 2)$$

$$= 2\alpha - 9 - \alpha + 3 + 1$$

$$= \alpha - 5$$

For infinitely many solutions,

$$\Delta = 0$$

$$\Rightarrow \alpha - 5 = 0$$

$$\Rightarrow \alpha = 5$$

Now,

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 9 \\ 1 & 3 & \beta \end{vmatrix}$$

$$= 1(2\beta - 27) - 1(\beta - 9) + 5(3 - 2)$$

$$= 2\beta - 27 - \beta + 9 + 5$$

$$= \beta - 13$$

For infinitely many solutions,

$$\Delta_1 = 0$$

$$\Rightarrow \beta - 13 = 0$$

$$\Rightarrow \beta = 13$$

$$\text{Hence, } \beta - \alpha = 13 - 5 = 8$$

19. Let the numbers $2, b, c$ be in an A.P. and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$. If $\det(A) \in [2, 16]$, then c lies in the interval :

☐ A. $[3, 2 + 2^{3/4}]$

☒ B. $[4, 6]$

☐ C. $[2, 3]$

☐ D. $[2 + 2^{3/4}, 4]$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$$

$$\Rightarrow |A| = (b-2)(c-b)(c-2)$$

$2, b, c$ are in A.P.

Let the common difference be k .

$$\text{Then } |A| = k \cdot k \cdot 2k = 2k^3$$

$$\Rightarrow 2 \leq 2k^3 \leq 16$$

$$\Rightarrow 1 \leq k \leq 2$$

$$\Rightarrow 1 \leq \frac{c-2}{2} \leq 2$$

$$\Rightarrow 4 \leq c \leq 6$$

20. Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{bmatrix}$ where $b > 0$. Then the minimum value of $\frac{\det(A)}{b}$ is :

☐ A. $-2\sqrt{3}$

☐ B. $-\sqrt{3}$

☐ C. $\sqrt{3}$

☒ D. $2\sqrt{3}$

$$A = \begin{bmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{bmatrix} \text{ where } b > 0$$

$$\Rightarrow |A| = 2[2(b^2 + 1) - b^2] - b[2b - b] + 1[b^2 - b^2 - 1]$$

$$\Rightarrow |A| = 2b^2 + 4 - b^2 - 1$$

$$\Rightarrow |A| = b^2 + 3$$

We have to calculate, $\min \left(\frac{\det(A)}{b} \right)$

$$\Rightarrow \frac{b^2 + 3}{b} = b + \frac{3}{b}$$

Let $f(b) = b + \frac{3}{b}$

$$\Rightarrow f'(b) = 1 - \frac{3}{b^2}$$

$$\Rightarrow f''(b) = \frac{6}{b^3}$$

For maxima and minima, $f'(b) = 0$

$$\therefore b = \sqrt{3} \quad (\because b > 0)$$

and $f''(\sqrt{3}) > 0$

Therefore, $f(b)$ is minimum at $b = \sqrt{3}$

\therefore Minimum value of $f(b) = \frac{\det(A)}{b}$ is

$$f(b) = f(\sqrt{3}) = \sqrt{3} + \frac{3}{\sqrt{3}}$$

$$\Rightarrow f(\sqrt{3}) = 2\sqrt{3}$$

Minimum value of $\frac{\det(A)}{b}$ is $2\sqrt{3}$

21. Let $\theta \in \left(0, \frac{\pi}{2}\right)$. If the system of linear equations,

$$(1 + \cos^2 \theta) x + \sin^2 \theta y + 4 \sin 3\theta z = 0$$

$$\cos^2 \theta x + (1 + \sin^2 \theta) y + 4 \sin 3\theta z = 0$$

$$\cos^2 \theta x + \sin^2 \theta y + (1 + 4 \sin 3\theta) z = 0$$

has a non-trivial solution, then the value of θ is

☒ A. $\frac{7\pi}{18}$

☐ B. $\frac{\pi}{18}$

☐ C. $\frac{4\pi}{9}$

☐ D. $\frac{5\pi}{18}$

$$\Delta = \begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \sin 3\theta \\ \cos^2 \theta & 1 + \sin^2 \theta & 4 \sin 3\theta \\ \cos^2 \theta & \sin^2 \theta & 1 + 4 \sin 3\theta \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\Rightarrow \Delta = \begin{vmatrix} 2 + 4 \sin 3\theta & \sin^2 \theta & 4 \sin 3\theta \\ 2 + 4 \sin 3\theta & 1 + \sin^2 \theta & 4 \sin 3\theta \\ 2 + 4 \sin 3\theta & \sin^2 \theta & 1 + 4 \sin 3\theta \end{vmatrix}$$

$$\Rightarrow \Delta = (2 + 4 \sin 3\theta) \begin{vmatrix} 1 & \sin^2 \theta & 4 \sin 3\theta \\ 1 & 1 + \sin^2 \theta & 4 \sin 3\theta \\ 1 & \sin^2 \theta & 1 + 4 \sin 3\theta \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & \sin^2 \theta & 4 \sin 3\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times (2 + 4 \sin 3\theta)$$

$$\Rightarrow \Delta = (2 + 4 \sin 3\theta)$$

For non-trivial solution

$$\Delta = 0$$

$$\Rightarrow \sin 3\theta = \frac{-1}{2}$$

$$\Rightarrow \theta = \frac{7\pi}{18}$$

22. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then $A^{2025} - A^{2020}$ is equal to

☒ A. $A^6 - A$

☐ B. A^5

☐ C. $A^5 - A$

☐ D. A^6

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$$

$$A^3 - 2A^2 + A = 0$$

$$\Rightarrow A^2 - A = A^3 - A^2 = A^4 - A^3 = A^5 - A^4 = A^6 - A^5 = A^7 - A^6$$

$$\text{So, } A^7 - A^2 = A^6 - A$$

$$\Rightarrow A^8 - A^3 = A^7 - A^2 = A^6 - A$$

And so on.

$$\therefore A^{2025} - A^{2020} = A^6 - A$$

23. Two fair dice are thrown. The numbers on them are taken as λ and μ , and a system of linear equations

$$x + y + z = 5$$

$$x + 2y + 3z = \mu$$

$$x + 3y + \lambda z = 1$$

is constructed. If p is the probability that the system has a unique solution and q is the probability that the system has no solution, then

☐ A. $p = \frac{1}{6}$ and $q = \frac{1}{36}$

☒ B. $p = \frac{5}{6}$ and $q = \frac{5}{36}$

☐ C. $p = \frac{1}{6}$ and $q = \frac{5}{36}$

☐ D. $p = \frac{5}{6}$ and $q = \frac{1}{36}$

$$\begin{vmatrix} 1 & 3 & \lambda \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 1 - 3(2) + \lambda(1) = 0$$

$$\Rightarrow \lambda = 5$$

For $\lambda \neq 5$ there will be unique solution

$$p = 1 - \frac{1}{6} = \frac{5}{6}$$

For $\lambda = 5$ and $\mu = 3$ there will be infinitely many solutions and for $\lambda = 5$ and $\mu \neq 3$ there will be no solution.

$$q = \frac{1}{6} \cdot \left(1 - \frac{1}{6}\right) = \frac{5}{36}$$

24. Let $[\lambda]$ be the greatest integer less than or equal to λ . The set of all values of λ for which the system of linear equations

$$x + y + z = 4, \quad 3x + 2y + 5z = 3, \quad 9x + 4y + (28 + [\lambda])z = [\lambda] \text{ has a solution is}$$

☐ A. $(-\infty, -9) \cup [-8, \infty)$

☐ B. $[-9, -8)$

☒ C. \mathbb{R}

☐ D. $(-\infty, -9) \cup (-9, \infty)$

$$x + y + z = 4$$

$$3x + 2y + 5z = 3$$

$$9x + 4y + (28 + [\lambda])z = [\lambda]$$

For unique solution $\Delta \neq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 5 \\ 9 & 4 & 28 + [\lambda] \end{vmatrix} \neq 0$$

$$\Rightarrow (56 + 2[\lambda] - 20) - 1(84 + 3[\lambda] - 45) + 1(-6) \neq 0$$

$$\Rightarrow 36 + 2[\lambda] - 39 - 3[\lambda] - 6 \neq 0$$

$$\Rightarrow [\lambda] \neq -9$$

$$\Rightarrow \lambda \in (-\infty, -9) \cup [-8, \infty)$$

and if $[\lambda] = -9$, $\Delta_x = \Delta_y = \Delta_z = 0$ gives infinite solution.

\therefore For $\lambda \in \mathbb{R}$ set of equations have solution.

25. If the system of linear equations

$$2x + 2ay + az = 0$$

$$2x + 3by + bz = 0$$

$$2x + 4cy + cz = 0,$$

where $a, b, c \in R$ are non-zero and distinct; has non-zero solution, then



A. $a + b + c = 0$



B. a, b, c are in A.P.



C. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.



D. a, b, c are in G.P.

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0$$

$$R_2 \rightarrow R_2 - R_1, R_3 - R_1$$

$$\Rightarrow \begin{vmatrix} 2 & 2a & a \\ 0 & 3b - 2a & b - a \\ 0 & 4c - 2a & c - a \end{vmatrix} = 0$$

$$\Rightarrow 3bc - 2ac - 3ab + 2a^2 - [4bc - 4ac - 2ab + 2a^2] = 0$$

$$\Rightarrow -bc + 2ac - ab = 0$$

$$\Rightarrow ab + bc = 2ac$$

$$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b}$$

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.}$$

26. If $A = \begin{bmatrix} 0 & -\tan \frac{\theta}{2} \\ \tan \frac{\theta}{2} & 0 \end{bmatrix}$ and $(I_2 + A)(I_2 - A)^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then $13(a^2 + b^2)$ is equal to

Accepted Answers

13 13.0 13.00

Solution:

$$A = \begin{bmatrix} 0 & -\tan \frac{\theta}{2} \\ \tan \frac{\theta}{2} & 0 \end{bmatrix}$$

$$\begin{aligned} (I_2 + A)(I_2 - A)^{-1} &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ \Rightarrow |(I_2 + A)(I_2 - A)^{-1}| &= a^2 + b^2 \\ \Rightarrow a^2 + b^2 &= \frac{|I_2 + A|}{|I_2 - A|} \dots (1) \end{aligned}$$

$$I_2 + A = \begin{bmatrix} 1 & 1 - \tan \frac{\theta}{2} \\ 1 + \tan \frac{\theta}{2} & 1 \end{bmatrix}$$

$$I_2 - A = \begin{bmatrix} 1 & 1 + \tan \frac{\theta}{2} \\ 1 - \tan \frac{\theta}{2} & 0 \end{bmatrix}$$

$$\Rightarrow |I_2 + A| = |I_2 - A|$$

From equation (1)

$$\therefore a^2 + b^2 = 1$$

$$\Rightarrow 13(a^2 + b^2) = 13$$

27. If the system of equations

$$kx + y + 2z = 1$$

$$3x - y - 2z = 2$$

$$-2x - 2y - 4z = 3$$

has infinitely many solutions, then k is equal to

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21 21.0 21.00

Solution:

$$D = 0$$

$$\Rightarrow \begin{vmatrix} k & 1 & 2 \\ 3 & -1 & -2 \\ -2 & -2 & -4 \end{vmatrix}$$

$$\Rightarrow k(4 - 4) - 1(-12 - 4) + 2(-6 - 2)$$

$$\Rightarrow 16 - 16 = 0$$

$$\text{Also, } D_1 = D_2 = D_3 = 0$$

$$\Rightarrow D_2 = \begin{vmatrix} k & 1 & 2 \\ 3 & 2 & -2 \\ -2 & 3 & -4 \end{vmatrix}$$

$$\Rightarrow k(-8 + 6) - 1(-12 - 4) + 2(9 + 4) = 0$$

$$\Rightarrow -2k + 16 + 26 = 0$$

$$\Rightarrow 2k = 42$$

$$\Rightarrow k = 21$$

28. Let $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$, $x \in \mathbb{R}$ and $A^4 = [a_{ij}]$. If $a_{11} = 109$, then a_{22} is equal to

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10 10.0 10.00

Solution:

$$A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} x^2 + 1 & x \\ x & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} x^2 + 1 & x \\ x & 1 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} x^3 + 2x & x^2 + 1 \\ x^2 + 1 & x \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} x^3 + 2x & x^2 + 1 \\ x^2 + 1 & x \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} x^4 + 2x^2 + x^2 + 1 & x^3 + 2x \\ x^3 + x + x & x^2 + 1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} x^4 + 3x^2 + 1 & x^3 + 2x \\ x^3 + 2x & x^2 + 1 \end{bmatrix}$$

$$a_{11} = 109 \text{ (Given)}$$

$$\Rightarrow x^4 + 3x^2 + 1 = 109$$

$$\Rightarrow x^4 + 3x^2 - 108 = 0$$

$$\Rightarrow (x^2 + 12)(x^2 - 9) = 0$$

$$\Rightarrow x = \pm 3$$

$$\therefore a_{22} = x^2 + 1 = 10$$

29. If the system of linear equations

$$2x + y - z = 3$$

$$x - y - z = \alpha$$

$$3x + 3y + \beta z = 3$$

has infinitely many solutions, then $(\alpha + \beta - \alpha\beta)$ is equal to

Accepted Answers

5 5.0 5.00 05

Solution:

For infinite solutions :

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & -1 \\ 3 & 3 & \beta \end{vmatrix} = 0$$

$$\Rightarrow 2(-\beta + 3) - 1(\beta + 3) - 1(3 + 3) = 0$$

$$\Rightarrow -2\beta + 6 - \beta - 3 - 6 = 0$$

$$\therefore \beta = -1$$

and

$$\Delta_1 = \begin{vmatrix} 3 & 1 & -1 \\ \alpha & -1 & -1 \\ 3 & 3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 3(1 + 3) - 1(-\alpha + 3) - 1(3\alpha + 3) = 0$$

$$\Rightarrow 12 + \alpha - 3 - 3\alpha - 3 = 0$$

$$\therefore \alpha = 3$$

$$\text{Hence, } \alpha + \beta - \alpha\beta = 5$$

30. Let $A = \{a_{ij}\}$ be a 3×3 matrix, where $a_{ij} = \begin{cases} (-1)^{j-i} & \text{if } i < j, \\ 2 & \text{if } i = j, \\ (-1)^{i+j} & \text{if } i > j, \end{cases}$
- then $\det(3 \operatorname{Adj}(2A^{-1}))$ is equal to

Accepted Answers

108 108.0 108.00

Solution:

$$\operatorname{adj}(2A^{-1}) = |2A^{-1}|(2A^{-1})^{-1} = \frac{8}{|A|} \cdot \frac{1}{2}A = \frac{4A}{|A|}$$

$$\text{So, } |3\operatorname{adj}(2A^{-1})| = \left| 12 \frac{A}{|A|} \right| = \left(\frac{12}{|A|} \right)^3 \cdot |A| = \frac{12^3}{|A|^2}$$

$$\because A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow |A| = 4$$

$$\text{Hence, } |3\operatorname{adj}(2A^{-1})| = \frac{12^3}{4^2} = 108$$