JEE Main Part Test 1

Subject: Mathematics — Class: Standard XII

1. Let a_n denote the n^{th} term of a geometric progression with common ratio less than 1. If $a_1+a_2+a_3=13$ and $a_1^2+a_2^2+a_3^2=91$, then the value of a_{10} is

   - A. 3^{10}
   - B. 3^{11}
   - C. $\frac{1}{3^{10}}$
   - D. $\frac{1}{3^7}$ ✔

Let $a_1=\frac{a}{r}, a_2=a, a_3=ar$

$$a_1+a_2+a_3=13$$
$$\Rightarrow \frac{a}{r}+a+ar=13$$
$$\Rightarrow a\left(\frac{1}{r}+r\right)=13-a$$
$$\Rightarrow \frac{1}{r}+r=\frac{13-a}{a} \quad \cdots(1)$$

$$a_1^2+a_2^2+a_3^2=91$$
$$\Rightarrow a^2\left(\frac{1}{r^2}+1+r^2\right)=91$$
$$\Rightarrow a^2\left[\left(\frac{1}{r}+r\right)^2-1\right]=91$$
$$\Rightarrow a^2\left[\left(\frac{13-a}{a}\right)^2-1\right]=91 \quad \text{[From (1)]}$$
$$\Rightarrow (13-a)^2-a^2=91$$
$$\Rightarrow a=3$$

So, $\frac{1}{r}+r=\frac{10}{3}$
$$\Rightarrow 3r^2-10r+3=0$$
$$\Rightarrow r=\frac{1}{3} \quad [\because r<1]$$

$$a_1=9$$
$$\therefore a_{10}=9\left(\frac{1}{3}\right)^9=\left(\frac{1}{3}\right)^7$$

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2. The complete set of values of x for which the inequality $\log_x\left(\dfrac{4x+5}{6-5x}\right) < -1$ holds good, is

A. $\left(1, \dfrac{6}{5}\right)$

B. $(0, 1)$

C. $\left(\dfrac{1}{2}, 1\right)$

D. $(0, 1) \cup \left(1, \dfrac{6}{5}\right)$

For $\log_x\left(\frac{4x+5}{6-5x}\right) < -1$ to be defined,

$\frac{4x+5}{6-5x} > 0,\ x > 0,\ x \neq 1$

$\Rightarrow x \in \left(-\frac{5}{4}, \frac{6}{5}\right),\ x > 0,\ x \neq 1$

$\therefore x \in \left(0, \frac{6}{5}\right) - \{1\}$

Case 1 : When $x \in (0,1)$

$\log_x \frac{4x+5}{6-5x} < -1$

$\Rightarrow \frac{4x+5}{6-5x} > x^{-1}$

$\Rightarrow \frac{4x+5}{6-5x} - \frac{1}{x} > 0$

$\Rightarrow \frac{4x^2+10x-6}{x(6-5x)} > 0$

$\Rightarrow \frac{2(2x-1)(x+3)}{x(5x-6)} < 0$

$\Rightarrow x \in (-3, 0) \cup \left(\frac{1}{2}, \frac{6}{5}\right)$

$\Rightarrow x \in \left(\frac{1}{2}, 1\right) \quad \cdots (1)$ as $x \in (0,1)$

Case 2 : When $x \in \left(1, \frac{6}{5}\right)$

$\log_x \frac{4x+5}{6-5x} < -1$

$\Rightarrow \frac{4x+5}{6-5x} < x^{-1}$

$\Rightarrow \frac{4x+5}{6-5x} - \frac{1}{x} < 0$

$\Rightarrow \frac{4x^2+10x-6}{x(6-5x)} < 0$

$\Rightarrow \frac{2(2x-1)(x+3)}{x(5x-6)} > 0$

$\Rightarrow x \in (-\infty, -3) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{6}{5}, \infty\right)$

$\Rightarrow x \in \phi \quad \cdots (2)$ as $x \in \left(1, \frac{6}{5}\right)$

From (1) and (2),

$x \in \left(\frac{1}{2}, 1\right)$

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3. A survey conducted in a city reveals that 48% children like cricket while 77% children like football. Then the percentage of children who like both cricket and football can be

A. 23

B. 31

C. 51

D. 65

Let us assume that the number of children in city be 100.
Let C denote the set of children who like cricket,
and F denote the set of children who like football.
$n(C) = 48$ and $n(F) = 77$
$n(C \cup F) \leq 100$
$\Rightarrow n(C) + n(F) - n(C \cap F) \leq 100$
$\Rightarrow 48 + 77 - n(C \cap F) \leq 100$
$\Rightarrow n(C \cap F) \geq 25 \quad \cdots (1)$

Also, $n(C \cap F) \leq \min\{n(C), n(F)\}$
$\Rightarrow n(C \cap F) \leq 48 \quad \cdots (2)$

From (1) and (2),
$25 \leq n(C \cap F) \leq 48$

4. Given that α, β, a, b are in A.P. ; α, β, c, d are in G.P. and α, β, e, f are in H.P. If b, d, f are in G.P., then the value of $\dfrac{\beta^6-\alpha^6}{\alpha\beta(\beta^4-\alpha^4)}$ is

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. $\frac{4}{3}$

D. $\frac{3}{4}$

α, β, a, b are in A.P.
$\Rightarrow b=\alpha+3(\beta-\alpha)=3\beta-2\alpha$

α, β, c, d are in G.P.
$$\Rightarrow d=\alpha\cdot\left(\frac{\beta}{\alpha}\right)^3=\frac{\beta^3}{\alpha^2}$$

α, β, e, f are in H.P.
$$\Rightarrow \frac{1}{f}=\frac{1}{\alpha}+3\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)$$
$$\Rightarrow f=\frac{\alpha\beta}{3\alpha-2\beta}$$

Now, since b, d, f are in G.P.,
$d^2=bf$
$$\Rightarrow \left(\frac{\beta^3}{\alpha^2}\right)^2=(3\beta-2\alpha)\frac{\alpha\beta}{(3\alpha-2\beta)}$$
$$\Rightarrow \beta^5(3\alpha-2\beta)=\alpha^5(3\beta-2\alpha)$$
$$\Rightarrow 3\alpha\beta^5-2\beta^6=3\beta\alpha^5-2\alpha^6$$
$$\Rightarrow 3\alpha\beta(\beta^4-\alpha^4)=2(\beta^6-\alpha^6)$$
$$\Rightarrow \frac{\beta^6-\alpha^6}{\alpha\beta(\beta^4-\alpha^4)}=\frac{3}{2}$$

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5. If there are 12 points in a plane out of which only 5 are collinear, then the number of quadrilaterals that can be formed using these points is

A. 210

B. 280

C. 350

D. 420

Given : 12 points in a plane out of which only 5 are collinear.
Number of non-collinear points $= 7$
Number of quadrilaterals
$=$ (select 4 points from 7 points) $+$ (select 3 points from 7 points and select 1 point from 5 collinear points)
$+$ (select 2 points from 7 points and select 2 points from 5 collinear points)

Number of quadrilaterals
$= {}^7C_4 + {}^7C_3 \times {}^5C_1 + {}^7C_2 \times {}^5C_2$
$= {}^7C_4[1+5] + \dfrac{7 \times 6 \times 10}{2}$
$= \dfrac{7 \times 6 \times 5 \times 6}{6} + 210$
$= 210 + 210 = 420$

6. If the function $f(x) = \lambda\,|\sin x| + \lambda^2\,|\cos x| + g(\lambda),\ \lambda \in \mathbb{R}$ is periodic with fundamental period $\frac{\pi}{2}$, then

A. $\lambda = 0, 1$

B. $\lambda = 1$

C. $\lambda = 0$

D. $\lambda = -1$

As, period of $f(x)$ is $\dfrac{\pi}{2}$
$\Rightarrow f\left(\dfrac{\pi}{2} + x\right) = f(x)\ \forall\, x \in \mathbb{R}$
$\Rightarrow \lambda\,|\cos x| + \lambda^2\,|\sin x| + g(\lambda) = \lambda\,|\sin x| + \lambda^2\,|\cos x| + g(\lambda)$
$\Rightarrow \left(\lambda - \lambda^2\right)|\cos x| + \left(\lambda^2 - \lambda\right)|\sin x| = 0\ \forall\, x\ \in \mathbb{R}$
$\Rightarrow \lambda - \lambda^2 = 0\ \text{ or }\ |\sin x| = |\cos x|$
$\Rightarrow \lambda = 0, 1\ \text{ or }\ x = \dfrac{(2n+1)\pi}{4}, n \in \mathbb{Z}$
But at $\lambda = 0$, $f(x)$ becomes a constant function, so $\lambda \neq 0$
$\Rightarrow \lambda = 1\ \text{ or }\ x = \dfrac{(2n+1)\pi}{4}, n \in \mathbb{Z}$

7. If t lies between real roots of the equation $2x^2 - 2\left(2t+1\right)x + t\left(t+1\right) = 0$, then t cannot be

A. 1

B. -2

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

For real roots, $D > 0$
$\Rightarrow 4(2t+1)^2 - 4 \times 2t(t+1) > 0$
$\Rightarrow 2t^2 + 2t + 1 > 0$, which is true $\forall\, t \in \mathbb{R}$

Now, if t lies between real roots of given quadratic, then
$f(t) < 0$
$\Rightarrow 2t^2 - 2\left(2t+1\right)t + t\left(t+1\right) < 0$
$\Rightarrow 2t^2 - 4t^2 - 2t + t^2 + t < 0$
$\Rightarrow -t^2 - t < 0$
$\Rightarrow t^2 + t > 0$
$\Rightarrow t \in (-\infty, -1) \cup (0, \infty)$
$\therefore t$ cannot be $-\frac{1}{2}$

8. Set of all real values of x satisfying the inequation $\dfrac{\log_2(x^2-5x+4)}{\log_2(x^2+1)} > 1$ is

A. $\left(-\infty, \dfrac{3}{5}\right) - \{0\}$

B. $(-\infty, 1) - \{0\}$

C. $\left(\dfrac{3}{5}, \infty\right)$

D. $\left(-\infty, \dfrac{3}{5}\right)$

Clearly, $x^2 - 5x + 4 > 0$
$\Rightarrow (x-4)(x-1) > 0$
$\Rightarrow x \in (-\infty, 1) \cup (4, \infty) \quad \cdots(1)$
$x^2 + 1 > 0$ which is true $\forall\, x \in \mathbb{R}$
Also, $x^2 + 1 \neq 1 \Rightarrow x \neq 0 \quad \cdots(2)$

$$\frac{\log_2(x^2-5x+4)}{\log_2(x^2+1)} > 1$$
$\Rightarrow \log_2(x^2-5x+4) > \log_2(x^2+1)$
$\Rightarrow x^2 - 5x + 4 > x^2 + 1$
$\Rightarrow x < \dfrac{3}{5} \quad \cdots(3)$

From $(1) \cap (2) \cap (3)$, we get
$x \in \left(-\infty, \dfrac{3}{5}\right) - \{0\}$

9. If $A=\{\theta : 2\cos^2\theta+\sin\theta\leq 2\}$ and $B=\left\{\theta : \frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}\right\}$, then $A\cap B$ is equal to

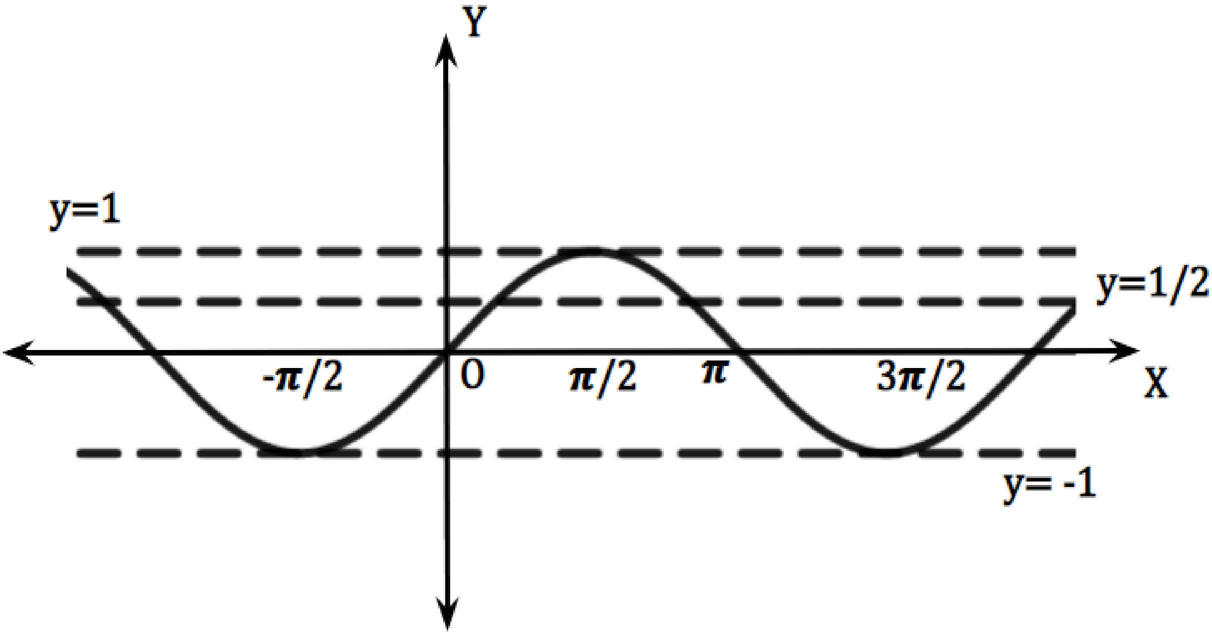
A. $\left\{\theta : \frac{\pi}{2}\leq\theta\leq\frac{5\pi}{6}\right\}$

B. $\left\{\theta : \pi\leq\theta\leq\frac{3\pi}{2}\right\}$

C. $\left\{\theta : \frac{\pi}{2}\leq\theta\leq\frac{5\pi}{6}\right\}\cup\left\{\theta : \pi\leq\theta\leq\frac{3\pi}{2}\right\}$

D. $\left\{\theta : \frac{\pi}{2}\leq\theta\leq\frac{5\pi}{6}\right\}\cap\left\{\theta : \pi\leq\theta\leq\frac{3\pi}{2}\right\}$

$2\cos^2\theta+\sin\theta\leq 2$
$\Rightarrow 2(1-\sin^2\theta)+\sin\theta\leq 2$
$\Rightarrow 2\sin^2\theta-\sin\theta\geq 0$
$\Rightarrow \sin\theta(2\sin\theta-1)\geq 0$
$\Rightarrow \sin\theta\in[-1,0]\cup\left[\frac{1}{2},1\right]$



From the diagram, we can conclude
$A\cap B=\left\{\theta : \frac{\pi}{2}\leq\theta\leq\frac{5\pi}{6}\right\}\cup\left\{\theta : \pi\leq\theta\leq\frac{3\pi}{2}\right\}$

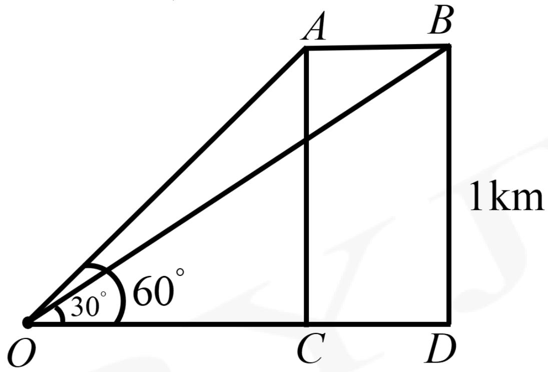
10. An aeroplane flying with uniform speed horizontally 1 km above the ground is observed at an elevation of 60° from a point on the ground. After 10 seconds, if the elevation is observed to be 30°, then the speed of the plane (in km/hr) is

A. $\frac{240}{\sqrt{3}}$

B. $200\sqrt{3}$

C. $240\sqrt{3}$

D. $\frac{120}{\sqrt{3}}$



Let O be the point of observation and A be the position of the aeroplane such that $\angle AOC = 60^\circ$ and $AC = 1$ km.
After 10 seconds, let B be the position of the aeroplane such that $\angle BOD = 30^\circ$ and $BD = 1$ km

In right angled triangle AOC,

$$\tan 60^\circ = \frac{AC}{OC}$$
$$\Rightarrow \sqrt{3} = \frac{1}{OC}$$
$$\Rightarrow OC = \frac{1}{\sqrt{3}}$$

In right angled triangle BOD,

$$\tan 30^\circ = \frac{BD}{OD}$$
$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{1}{OD}$$
$$\Rightarrow OD = \sqrt{3}$$

Now, $CD = OD - OC$
$$= \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Distance covered by the aeroplane in 10 seconds $= \frac{2}{\sqrt{3}}$ km

Time taken $= 10$ sec $= \frac{10}{3600} = \frac{1}{360}$ hr

Speed of the aeroplane
$$= \frac{\text{distance}}{\text{time}} = \frac{2/\sqrt{3}}{1/360} = \frac{720\sqrt{3}}{3}$$
$$= 240\sqrt{3} \text{ km/hr}$$

11. The number of ways in which 20 letters $a_1, a_2, a_3, \ldots, a_{10}, b_1, b_2, b_3, \ldots, b_{10}$ can be arranged in a line so that suffixes of the letters a and also those of b are respectively in ascending order of magnitude is

A. $\frac{20!}{10!}$

B. $\frac{20!}{(10!)^2}$

C. 2^{20}

D. $20! - 10! \cdot 10!$

Order of a's is fixed that is
$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}$
These 10 a's can have any 10 places out of 20 available places in ${}^{20}C_{10}$ ways.
Now, 10 places are empty where we can insert b's in such a way that all b's are in ascending order of their suffixes.
Insertion of b's can be done in one way only.

So, required number of ways $= {}^{20}C_{10} \times 1 = \frac{20!}{(10!)^2}$

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12. If $\operatorname{sgn}(y)$ denotes the signum function of y, then the number of solution(s) of the equation $||x+2|-3| = \operatorname{sgn}\left(1-\left|\frac{(x-2)\left(x^2+10x+24\right)}{\left(x^2+1\right)(x+4)\left(x^2+4x-12\right)}\right|\right)$ is

A. 0

B. 1

C. 3

D. 4

$||x+2|-3| = \operatorname{sgn}\left(1-\left|\frac{(x-2)(x+6)(x+4)}{\left(x^2+1\right)(x+4)(x+6)(x-2)}\right|\right)$

$\Rightarrow ||x+2|-3| = \operatorname{sgn}\left(1-\left|\frac{1}{x^2+1}\right|\right)$; $x \neq 2, -4, -6$

$\Rightarrow ||x+2|-3| = \operatorname{sgn}\left(\frac{x^2}{x^2+1}\right) \quad \ldots(1)$

We know $\frac{x^2}{x^2+1} \geq 0$ for all $x \in \mathbb{R}$

Case 1 :
At $x=0$,
RHS of equation (1) is 0 but LHS $\neq 0$
Hence, $x=0$ is not the solution.

Case 2 :
For $x \neq 0$
$||x+2|-3| = 1$
$\Rightarrow |x+2|-3 = \pm 1$
$\Rightarrow |x+2| = 4, 2$
$\Rightarrow x+2 = \pm 4, \pm 2$
$\Rightarrow x = 2, -4, 0, -6$
But $x \neq -6, -4, 0, 2$
Hence, there is no solution.

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13. The equation $\left(x^2-5x+1\right)\left(x^2+x+1\right)+8x^2=0$ has

A. four real and distinct roots

B. three real and distinct roots

C. two real and distinct roots

D. only one real root

$\left(x^2-5x+1\right)\left(x^2+x+1\right)+8x^2=0$
$\Rightarrow \left(x+\frac{1}{x}-5\right)\left(x+\frac{1}{x}+1\right)+8=0$
Let $x+\frac{1}{x}=t$
Then, $(t-5)(t+1)+8=0$
$\Rightarrow t^2-4t-5+8=0$
$\Rightarrow t^2-4t+3=0$
$\Rightarrow t=1$ or $t=3$
But $x+\frac{1}{x}=1$ rejected because $x+\frac{1}{x}\in(-\infty,-2]\cup[2,\infty)$
So, $x+\frac{1}{x}=3$
$\Rightarrow x^2-3x+1=0$
$D=9-4=5>0$
$\Rightarrow$ Two real and distinct roots.

14. If 5^{40} is divided by 11, then remainder is α and if 2^{2003} is divided by 17, then remainder is β. Then the value of $(\beta-\alpha)$ is

A. 3

B. 5

C. 7

D. 8

$5^{40}=\left(5^2\right)^{20}=(22+3)^{20}=22\lambda+3^{20},\lambda\in\mathbb{N}$
Also, $3^{20}=\left(3^2\right)^{10}=(11-2)^{10}=11\mu+2^{10},\mu\in\mathbb{N}$
Now, $2^{10}=1024=11\times 93+1$
$\therefore$ Remainder $=1$ i.e., $\alpha=1$

$2^{2003}=2^3\cdot 2^{2000}=8\left(2^4\right)^{500}=8(16)^{500}$
$=8(17-1)^{500}=8\left(17v+1\right),v\in\mathbb{N}$
$=8\times 17v+8$
$\therefore$ Remainder $=8$ i.e., $\beta=8$
$\Rightarrow \beta-\alpha=8-1=7$

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15. The number of solution(s) of the equation $3\tan\left(x-\frac{\pi}{12}\right)=\tan\left(x+\frac{\pi}{12}\right)$ in $A=\left\{x\in\mathbb{R}: x^2-6x\le 0\right\}$ is

A. 2

B. 3

C. 1

D. 4

$x^2-6x\le 0$
$\Rightarrow x(x-6)\le 0$
$\Rightarrow x\in[0,6]$

$$3\tan\left(x-\frac{\pi}{12}\right)=\tan\left(x+\frac{\pi}{12}\right)$$
$$\Rightarrow 3\sin\left(x-\frac{\pi}{12}\right)\cos\left(x+\frac{\pi}{12}\right)=\sin\left(x+\frac{\pi}{12}\right)\cos\left(x-\frac{\pi}{12}\right)$$
$$\Rightarrow 3\left(\sin 2x-\sin\frac{\pi}{6}\right)=\sin 2x+\sin\frac{\pi}{6}$$
$$\Rightarrow 2\sin 2x=2$$
$$\Rightarrow \sin 2x=1$$
$$\Rightarrow 2x=2n\pi+\frac{\pi}{2}, n\in\mathbb{Z}$$
$$\Rightarrow x=n\pi+\frac{\pi}{4}, n\in\mathbb{Z}$$
$$n=0;\ x=\frac{\pi}{4}\in A$$
$$n=1;\ x=\pi+\frac{\pi}{4}=\frac{5\pi}{4}\in A$$
$$n=3;\ x=2\pi+\frac{\pi}{4}\notin A$$

Hence, two solutions are there.

16. The value of $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \cdot 16^{\frac{1}{32}} \ldots\ldots$ is

A. 2

B. $\frac{3}{2}$

C. 1

D. $\frac{2}{3}$

$2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \cdot 16^{\frac{1}{32}} \ldots\ldots$
$= 2^{\frac{1}{4}} \cdot (2^2)^{\frac{1}{8}} \cdot (2^3)^{\frac{1}{16}} \cdot (2^4)^{\frac{1}{32}} \ldots\ldots$

$= 2^{\left(\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots\right)} = 2^S$ (Let)

$$S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \cdots \quad \rightarrow (1)$$
$$\frac{S}{2} = \quad \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{4}{64} + \cdots \quad \rightarrow (2)$$

Subtracting equation (2) from equation (1), we get

$$\frac{S}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$
$$\Rightarrow \frac{S}{2} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$$
$$\Rightarrow \frac{S}{2} = \frac{1/4}{1 - 1/2} = \frac{1}{2}$$
$$\Rightarrow S = 1$$

$\therefore$ Required value $= 2^S = 2^1 = 2$

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17. If $\log_{10}\sin x+\log_{10}\cos x=-1;\ x\in\left(0,\frac{\pi}{2}\right)$ and $\log_{10}(\sin x+\cos x)=\frac{(\log_{10}n)-1}{2}$, then the value of n is

A. 7

B. 15

C. 10

D. 12

Given : $\log_{10}\sin x+\log_{10}\cos x=-1$
$\Rightarrow \log_{10}(\sin x\cos x)=-1$
$\Rightarrow \log_{10}\left(\frac{\sin 2x}{2}\right)=-1$
$\Rightarrow \frac{\sin 2x}{2}=\frac{1}{10}$
$\Rightarrow \sin 2x=\frac{1}{5}\quad\cdots(1)$

And $\log_{10}(\sin x+\cos x)=\frac{(\log_{10}n)-1}{2}$
$\Rightarrow 2\log_{10}(\sin x+\cos x)=\log_{10}n-\log_{10}10$
$\Rightarrow \log_{10}(\sin x+\cos x)^2=\log_{10}\left(\frac{n}{10}\right)$
$\Rightarrow (\sin x+\cos x)^2=\frac{n}{10}$
$\Rightarrow 1+\sin 2x=\frac{n}{10}$
Using equation (1), we get
$1+\frac{1}{5}=\frac{n}{10}$
$\therefore n=12$

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18. Let $\alpha = 3^{\log_4 5} - 5^{\log_4 3} + 2$. If p and q are the roots of the equation $\log_\alpha x + \log_x \alpha = \frac{10}{3}$, then the value of $p^3 + q^3$ is

- A. 10
- B. 514
- C. 66
- D. 564

$\alpha = 3^{\log_4 5} - 5^{\log_4 3} + 2$
$= 5^{\log_4 3} - 5^{\log_4 3} + 2$
$= 2$

Now, $\log_2 x + \log_x 2 = \frac{10}{3}$
Let $\log_2 x = t$
$\Rightarrow t + \frac{1}{t} = \frac{10}{3}$
$\Rightarrow 3t^2 - 10t + 3 = 0$
$\Rightarrow t = 3, \frac{1}{3}$
$\Rightarrow \log_2 x = 3, \frac{1}{3}$
$\Rightarrow x = 2^3, 2^{1/3}$
$\therefore p = 8,\ q = 2^{1/3}$
Hence, $p^3 + q^3 = 512 + 2 = 514$

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19. If $A_1, A_2; G_1, G_2$ and H_1, H_2 are arithmetic mean, geometric mean and harmonic mean between two numbers, then the value of $\dfrac{G_1G_2}{H_1H_2} \times \dfrac{H_1+H_2}{A_1+A_2}$ is

A. 1

B. 0

C. 2

D. 3

Let a and b be two numbers.
Sum of n A.M.'s $= n \times$ single A.M.

$$\Rightarrow A_1 + A_2 = 2 \times \left(\frac{a+b}{2}\right) = a + b \quad \cdots (1)$$

Product of n G.M.'s $=$ (single G.M.)^n

$$\Rightarrow G_1G_2 = (\sqrt{ab})^2 = ab \quad \cdots (2)$$

Dividing equation (1) by (2), we get

$$\frac{A_1+A_2}{G_1G_2} = \frac{a+b}{ab} \quad \cdots (3)$$

Since a, H_1, H_2, b are in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = 2\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)$$

$$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{H_1+H_2}{H_1H_2} = \frac{a+b}{ab} \quad \cdots (4)$$

From (3) and (4), we get

$$\frac{A_1+A_2}{G_1G_2} = \frac{H_1+H_2}{H_1H_2}$$

$$\Rightarrow \frac{H_1H_2}{H_1+H_2} = \frac{G_1G_2}{A_1+A_2}$$

$$\Rightarrow \frac{G_1G_2}{H_1H_2} \times \frac{H_1+H_2}{A_1+A_2} = 1$$

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20. The number of value(s) of $\theta \in [0, 2\pi]$ satisfying the equation $\left(\log_{\sqrt{5}} \tan\theta\right)\sqrt{\log_{\tan\theta} 5\sqrt{5} + \log_{\sqrt{5}} 5\sqrt{5}} = -\sqrt{6}$ is

A. 0

B. 4

C. 2 ✔

D. 5

$\left(\log_{\sqrt{5}} \tan\theta\right)\sqrt{\log_{\tan\theta} 5\sqrt{5} + \log_{\sqrt{5}} 5\sqrt{5}} = -\sqrt{6}$

$\Rightarrow \log_{\sqrt{5}} \tan\theta \sqrt{\dfrac{3}{\log_{\sqrt{5}} \tan\theta} + 3} = -\sqrt{6}$

Put $x = \log_{\sqrt{5}} \tan\theta$

Then, $x\sqrt{\dfrac{3}{x} + 3} = -\sqrt{6} \quad \ldots(1)$

$\Rightarrow \dfrac{x\sqrt{x+1}}{\sqrt{x}} = -\sqrt{2}$

$\Rightarrow x^2 + x - 2 = 0,\ x \neq 0$

$\Rightarrow (x+2)(x-1) = 0$

$\therefore x = 1, -2$

But from (1), we can conclude $x < 0$

$\therefore x = -2$

$\Rightarrow \log_{\sqrt{5}} \tan\theta = -2$

$\Rightarrow \tan\theta = \dfrac{1}{5}$

As $\theta \in [0, 2\pi]$, there are two solutions.

21. The number of integral terms in the expansion of $\left(\sqrt{3} + \sqrt[8]{5}\right)^{256}$ is

Accepted Answers

33 33.0 33.00

Solution:

General term,

$T_{r+1} = {}^{256}C_r (\sqrt{3})^{256-r} (\sqrt[8]{5})^r$

$= {}^{256}C_r (3)^{(256-r)/2} (5)^{r/8}$

Now this term is an integer if $\dfrac{256-r}{2}, \dfrac{r}{8}$ will be an integer, for which $r = 0, 8, 16, \ldots, 256$

For A.P., $0, 8, 16, 24, \ldots, 256$ number of terms will be

$= \dfrac{256}{8} + 1 = 33$

Hence, there are 33 integral terms.

22. If the sum of the solutions of the equation $\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)-\frac{\sec\theta}{4}=0$ in $[0,10\pi]$ is $k\pi$, then the value of k is

Accepted Answers

80 80.0 80.00

Solution:

$$\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)-\frac{\sec\theta}{4}=0$$
$$\Rightarrow \cos\theta\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)-\frac{1}{4}=0,\ \ \cos\theta\neq 0$$
$$\Rightarrow \frac{1}{4}\cos 3\theta-\frac{1}{4}=0$$
$$\Rightarrow \cos 3\theta=1$$
$$\Rightarrow 3\theta=2n\pi,\ n\in\mathbb{Z}$$
$$\Rightarrow \theta=\frac{2n\pi}{3}$$
Since, $\theta\in[0,10\pi]$, n should be less than or equal to 15.
Required sum of solutions
$$=\frac{2\pi}{3}\sum_{n=1}^{15}n=\frac{2\pi}{3}\times\frac{15\times 16}{2}=80\pi$$

23. If $\left(1+x+x^2\right)^8=a_0+a_1x+a_2x^2+\cdots+a_{16}x^{16}$ for all real x, then a_5 is equal to

Accepted Answers

504 504.0 504.00

Solution:

$T_r=\frac{8!}{a!\,b!\,c!}(1)^a(x)^b(x^2)^c$, where $a+b+c=8$
For coefficient of x^5, $b+2c=5$
$(b,c)=(1,2),(3,1),(5,0)$

$$\therefore a_5=\frac{8!}{5!\cdot 1!\cdot 2!}+\frac{8!}{4!\cdot 3!\cdot 1!}+\frac{8!}{3!\cdot 5!\cdot 0!}$$
$$\Rightarrow a_5=168+280+56=504$$

Alternate Solution :
$$[(1+x)+x^2]^8={}^8C_0(1+x)^8+{}^8C_1(x^2)^1(1+x)^7+{}^8C_2(x^2)^2(1+x)^6+{}^8C_3(x^2)^3(1+x)^5+\cdots$$
First three terms consist x^5.
So, $a_5={}^8C_0\cdot{}^8C_5+{}^8C_1\cdot{}^7C_3+{}^8C_2\cdot{}^6C_1$
$$\Rightarrow a_5=56+280+168=504$$

24. If $f : [-2, 2] \rightarrow \mathbb{R}$ defined by $f(x) = x^3 + \tan x + \left[\frac{x^2+1}{p}\right]$ is an odd function, then the least value of $[p]$ is
([.] represents the greatest integer function)

Accepted Answers

5 5.0 5.00

Solution:

$f(x) = x^3 + \tan x + \left[\frac{x^2+1}{p}\right]$
f is an odd function.
$\therefore f(-x) = -f(x)$
$\Rightarrow -x^3 - \tan x + \left[\frac{x^2+1}{p}\right] = -x^3 - \tan x - \left[\frac{x^2+1}{p}\right]$
$\Rightarrow \left[\frac{x^2+1}{p}\right] = 0$

Now, $x \in [-2, 2]$
$\therefore x^2 + 1 \in [1, 5]$
So, for f to be an odd function, $p \in (5, \infty)$
So, the least value of $[p]$ is 5.

25. If α, β are the roots of $\lambda\left(x^2+x\right)+x+5=0$ and λ_1, λ_2 are two values of λ for which α, β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=4$, then the value of $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}$ is equal to

Accepted Answers

782 782.0 782.00

Solution:

Given equation is
$\lambda\left(x^2+x\right)+x+5=0$
$\Rightarrow \lambda x^2+(\lambda+1) x+5=0$
Roots are α, β.
Here, $\alpha+\beta=-\frac{\lambda+1}{\lambda}, \alpha\beta=\frac{5}{\lambda}$

$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=4$
$\Rightarrow \alpha^2+\beta^2=4\alpha\beta$
$\Rightarrow (\alpha+\beta)^2=6\alpha\beta$
$\Rightarrow \left(\frac{1+\lambda}{\lambda}\right)^2=6\times\frac{5}{\lambda}$
$\Rightarrow \frac{1+\lambda^2+2\lambda}{\lambda^2}=\frac{30}{\lambda}$
$\Rightarrow 1+\lambda^2+2\lambda=30\lambda$
$\Rightarrow \lambda^2-28\lambda+1=0$
$\Rightarrow \lambda_1+\lambda_2=28, \lambda_1\lambda_2=1$

Now, $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}=\frac{(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2}{\lambda_1\lambda_2}$
$=\frac{(28)^2-2\times 1}{1}=782$

JEE Main Part Test 1

26. If the number of ways in which four distinct balls can be put into two identical boxes so that no box remains empty is equal to k, then k is

Accepted Answers

7 7.0 7.00 07

Solution:

Every ball has two options
$\Rightarrow 4$ balls can be put in 2^4 ways.
As given that boxes are identical so,
arrangement is $= \frac{1}{2!} \times 2^4 = 2^3$
But, above count also includes the one case in which all the balls are put in one box.
$\Rightarrow$ Number of ways $= 2^3 - 1 = 7$

Alternate Solution :
We can divide the balls into two groups : $(1,3)$ and $(2,2)$
In $(2,2)$, both groups have equal number of balls.
Now, number of ways
$= \frac{4!}{1!\,3!} + \frac{4!}{2!\,2!} \times \frac{1}{2!}$
$= 4 + 3 = 7$

27. Let f be a real function defined as $f(x) = \frac{2^x+1}{2^x-1}$. The number of integer(s) which are not in the range of f is

Accepted Answers

3 3.0 3.00 03

Solution:

$f(x) = \frac{2^x+1}{2^x-1}$
Domain of f is $\mathbb{R} - \{0\}$
Let $y = \frac{2^x+1}{2^x-1}$
$\Rightarrow y\left(2^x - 1\right) = 2^x + 1$
$\Rightarrow 2^x(y-1) = y+1$
$\Rightarrow 2^x = \frac{y+1}{y-1}$
Since $2^x > 0$,
$\frac{y+1}{y-1} > 0$
$\Rightarrow y \in (-\infty, -1) \cup (1, \infty)$
Range of f is $(-\infty, -1) \cup (1, \infty)$
Integers which are not there in range are $-1, 0, 1$
Hence, required answer is 3

28. Let A, B, C be finite sets. Suppose that $n(A)=10, n(B)=15, n(C)=20, n(A\cap B)=8$ and $n(B\cap C)=9$. Then the maximum possible value of $n(A\cup B\cup C)$ is

Accepted Answers

28 28.0 28.00

Solution:

$$\begin{aligned} n(A\cup B\cup C) &= n(A)+n(B)+n(C)-n(A\cap B) \\ &\quad -n(B\cap C)-n(C\cap A)+n(A\cap B\cap C) \\ &= 28-[n(C\cap A)-n(A\cap B\cap C)] \end{aligned}$$

We know that $n(C\cap A)\geq n(A\cap B\cap C)$

$$\Rightarrow n(C\cap A)-n(A\cap B\cap C)\geq 0$$

$\therefore$ Maximum possible value of $n(A\cup B\cup C)$ is 28

29. The number of integral values of x satisfying $||x-\pi|-|\pi x-1||=(x-1)(1+\pi)$, is

Accepted Answers

3 3.0 3.00

Solution:

Let $x-\pi=a$ and $\pi x-1=b$.

Then, $a+b=(x-1)(1+\pi)$

So, the equation is of the form $||a|-|b||=|a+b|$, which is possible only if $ab\leq 0$

So, $(x-\pi)(\pi x-1)\leq 0$

$$\Rightarrow \pi(x-\pi)\left(x-\frac{1}{\pi}\right)\leq 0$$

$$\Rightarrow x\in\left[\frac{1}{\pi},\pi\right]$$

Possible integral values of x are $1,2,3$

So, number of integral values of x is 3

JEE Main Part Test 1

30. Number of integer values of x satisfying the inequality
$|x-3|+|2x+4|+|x| \leq 11$ is

Accepted Answers

6 6.0 6.00

Solution:

$|x-3|+|2x+4|+|x| \leq 11$
If $x \geq 3$,
$x-3+2x+4+x \leq 11$
$\Rightarrow 4x \leq 10$
$\Rightarrow x \leq 2.5$ No solution

If $0 \leq x < 3$,
$-x+3+2x+4+x \leq 11$
$\Rightarrow x \leq 2 \Rightarrow x \in [0,2]$

If $-2 \leq x < 0$,
$-x+3+2x+4-x \leq 11$
$\Rightarrow 7 \leq 11$ which is true
$\Rightarrow x \in [-2,0)$

If $x < -2$,
$-x+3-2x-4-x \leq 11$
$\Rightarrow -4x-1 \leq 11$
$\Rightarrow x \geq -3 \Rightarrow x \in [-3,-2)$

Hence, $x \in [0,2] \cup [-2,0) \cup [-3,-2)$
i.e., $x \in [-3,2]$