VECTOR ALGEBRA

1. Physical quantities are broadly divided in two categories viz (a) Vector Quantities & (b) Scalar quantities.

(a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction is called a vector quantity.

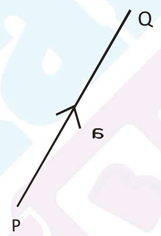
For Vector quantity addition should be meaningfully defined.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

2. Representation :

Vectors are represented by directed straight line segment say $\vec{a}$ and magnitude of $\vec{a} = |\vec{a}| =$ length PQ direction of $\vec{a}$ = P to Q.



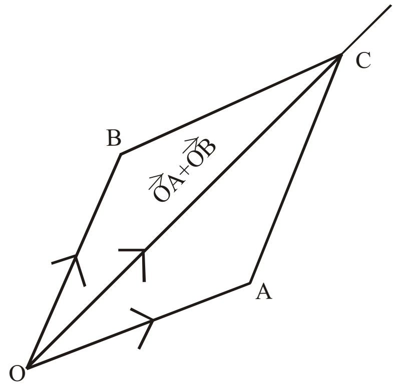
3. Addition of Vectors :

(a) It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(i) If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\vec{OA}$ & $\vec{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overline{OC}$, where OC is the diagonal of the parallelogram OACB.

(ii) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative law)

(iii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative law)



(d) Multiplication of vector by scalars :

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$ (ii) $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ (iv) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

4. (A) Zero Vector or Null Vector :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a zero vector. It is denoted by $\vec{O}$. It can have any arbitrary direction and any line as its line of support.

(B) UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

(C) COLLINEAR VECTORS :

Two vectors are said to be collinear if their supports are parallel irrespective to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.** Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

(D) COPLANAR VECTORS :

A given number of vectors are called coplanar if their supports are all parallel to the same plane. Note that "TWO VECTORS ARE ALWAYS COPLANAR ".

(E) EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(i) the same length,

(ii) the same or parallel supports and

(iii) the same sense

(F) Free vectors : If a vector can be translated anywhere in space without changing its magnitude & direction, then such a vector is called **free vector**. In other words, the initial point of free vector can be taken anywhere in space keeping its magnitude & direction same.

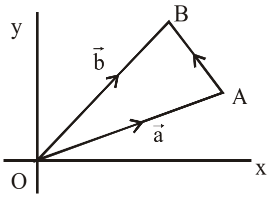
(G) Localised vectors : For a vector of given magnitude and direction, if its initial point is fixed in space, then such a vector is called localised vector. Unless & untill stated, vectors are treated as free vectors.

5. Position Vector :

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

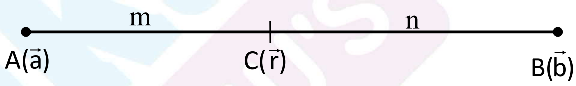
$= \overrightarrow{OB} - \overrightarrow{OA}$



6. Section Formula :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point $C(\vec{r})$ which divides AB in the ratio m : n is given by :

$$\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n}$$



7. Vector Equation of a Line :

Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{r} = \vec{a} + t\vec{b}$.

8. Test of Collinear points :

(a) Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = \vec{O}$, where **x + y + z = 0**

(b) Three points A, B, C are collinear, if any two vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ $\overrightarrow{CA}$ are parallel.

9. Scalar product of two vectors (Dot Product) :

(a) $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)$, θ is angle between $\vec{a}$ & $\vec{b}$.

Note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

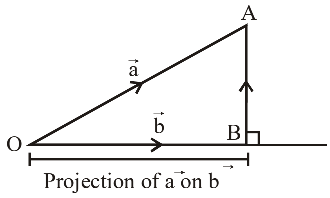
(b) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(c) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$; $(\vec{a}, \vec{b} \neq 0)$

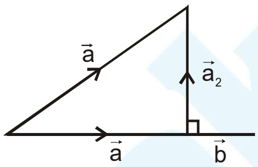
(d) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(e) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.



Note :

(i) The vector component of $\vec{a}$ along $\vec{b}$. i.e. $\vec{a}_1 = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular



to $\vec{b}$ i.e. $\vec{a}_2 = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ $(\vec{a} = \vec{a}_1 + \vec{a}_2)$

(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \leq \phi \leq \pi$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(iv) $-|\vec{a}||\vec{b}| \leq \vec{a}.\vec{b} \leq |\vec{a}||\vec{b}|$

(v) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

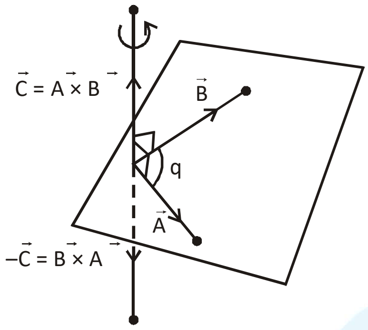
(vi) A vector in the direction of the internal angle bisector between the two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} - \hat{b})$, $\lambda \in R^+$

(vii) $|\vec{a}\pm\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2\pm 2\vec{a}.\vec{b}$

(viii) $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}.+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

10. Vector Product of Two Vectors (Cross Product) :

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed system.



(b) Lagranges Identity **:** For any two vectors $\vec{a}$ & $\vec{b}$ **;** $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}|=\sqrt{\vec{a}^2\vec{b}^2-(\vec{a}.\vec{b})^2}$ (ii) $\vec{c}.\vec{a}=0;\ \vec{c}.\vec{b}=0$ and (iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(d) $\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0,\ \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar

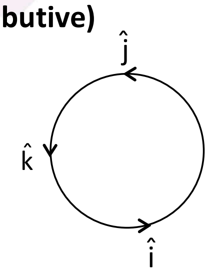
(i) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ **(not commutative)**

(ii) $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ where m is a scalar.

(iii) $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ **(distributive)**

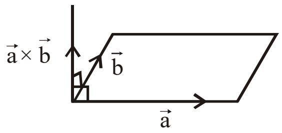
(iv) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$



(e) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

(f) Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) (i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(h) Vector area :

(i) If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle .

$ABC = \dfrac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

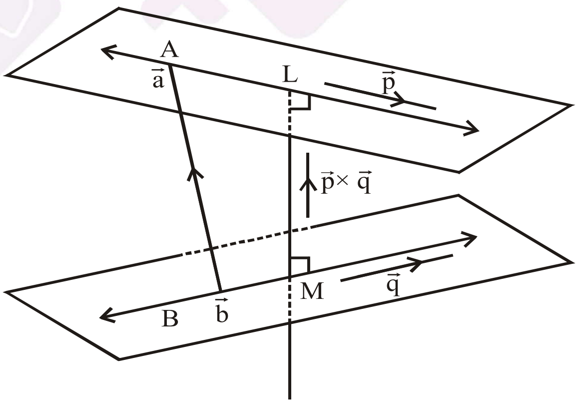
The points A, B & C are collinear if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{O}$

(ii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\Delta=\dfrac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|$

Area of triangle = $\dfrac{1}{2}|\vec{a}\times\vec{b}|$

11. Shortest Distance Between Two Lines :

Lines which do not intersect & are also not parallel are called skew lines. In other words the lines which are not coplanar are skew lines. For Skew lines shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p}\times\vec{q}$



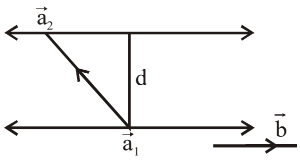
i.e. $\overrightarrow{LM} = |\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}| =$

$$|\text{Projection of } \vec{AB} \text{ on } \vec{p}\times\vec{q}| = \left|\frac{\vec{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(a) The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only **if** shortest distance = 0 i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$ $\Rightarrow \left[\left(\vec{b}-\vec{a}\right)\ \vec{p}\ \ \vec{q}\right] = 0$

(b) If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then,

$$d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$$



12. Scalar Triple Product / Box Product / Mixed Product :

(a) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as : $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.

(b) In a Scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = [\vec{b}\ \ \vec{c}\ \ \vec{a}] = [\vec{c}\ \ \vec{a}\ \ \vec{b}]$

(c) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(d) if $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0 \Rightarrow \vec{a}\,\vec{b}\,\vec{c}$ are linearly dependent.

(e) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

(f) $[\hat{i}\,\hat{j}\,\hat{k}] = 1; [K\vec{a}\ \ \vec{b}\ \ \vec{c}] = K[\vec{a}\ \ \vec{b}\ \ \vec{c}]; [(\vec{a}+\vec{b})\,\vec{c}\,\vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}]$

(g) (i) The Volume of the tetrahedron OABC with O as origin & the pv's of, A, B, and C being $\vec{a}, \vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

(ii) Volume of parallelopiped whose co-terminus edges are $\vec{a}, \vec{b}$ & $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$

(h) Remember that :

(i) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

(ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\,\vec{b}\,\vec{c}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}\vec{a} & \vec{a}.\vec{b} & \vec{a}\vec{c} \\ \vec{b}\vec{a} & \vec{b}\vec{b} & \vec{b}\vec{c} \\ \vec{c}\vec{a} & \vec{c}\vec{b} & \vec{c}\vec{c} \end{vmatrix}$

13. Vector Triple Product :

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called a vector triple product.

(a) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(b) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(c) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

14. Linear Independence and Dependence of Vectors :

Linear combination of vectors:

Given a finite set of non-zero vector $\vec{a},\vec{b},\vec{c}$, ... then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$ is called a linear combination of $\vec{a},\vec{b},\vec{c}$,.......... for any x,y,z, ...$\in$R.

(a) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are n non zero vectors, & $k_1, k_2, ...k_n$ are n scalars & if the linear combination

$k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n = \vec{O}$ and $k_1 = 0, k_2 = 0....k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are **linearly independent vectors.**

(b) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n$ **= 0** & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are said to be **linearly dependent.**

(c) Fundamental theorem in plane:

Let $\vec{a}$, $\vec{b}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}$, $\vec{b}$ i.e. There exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.

(D) Fundamental theorem in space :

Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

15. Coplanarity of Four Points :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{O}$ where, x + y + z + w = 0

16. Reciprocal System of Vectors :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, then the two systems are called Reciprocal System of vectors.

Where : $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

(i) Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point of concurrency is called the centre of the tetrahedron.

(ii) In a tetrahedron, straight lines joining the mid points of each pair of opposite edge are also concurrent at the centre of the tetrahedron.

(iii) The angle between any two plane faces of regular tetrahedron is $\cos^{-1}\frac{1}{3}$

THREE DIMENSIONAL GEOMETRY

1. Distance Formula :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by $AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

2. Section Formulae :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$.

Then R is (x, y, z) = $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

Mid-Point : Mid point of PQ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

3. Centroid of a Triangle :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by

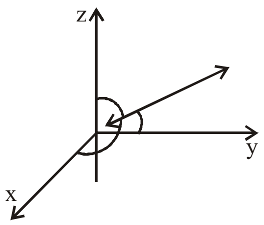
$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

4. Direction cosines of line :

If α, β, γ be the angles made by a line with positive x-axis, y-axis & z-axis respectively then $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are called direction cosines of a line, denoted by l, m & n respectively and the relation between l, m,, n, is given by $l^2 + m^2 + n^2 = 1$

Direction cosines of x-axis, y a-xis, & z-axis are respectively

1, 0, 0 ; 0, 1, 0 ; 0, 0,1



5. Direction Ratios :

Any three numbers a, b, c proportional to direction cosines l, m, n are called direction ratios of the line. i.e. $\frac{\ell}{a}=\frac{m}{b}=\frac{n}{c}$

6. Relation between D.C's & D.R's :

$$\frac{\ell}{a}=\frac{m}{b}=\frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2}=\frac{m^2}{b^2}=\frac{n^2}{c^2}=\frac{\ell^2+m^2+n^2}{a^2+b^2+c^2}$$

$$\therefore \ \ell=\frac{\pm a}{\sqrt{a^2+b^2+c^2}} \quad ; \quad m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}} \quad ; \quad n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

7. Direction Cosine of a Line :

Direction ratios and Direction cosines of the joining two points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the d.c.'s of AB are

$\frac{1}{r}(x_2 - x_1), \frac{1}{r}(y_2 - y_1), \frac{1}{r}(z_2 - z_1)$ where $r=\sqrt{[\Sigma(x_2-x_1)^2]} = |\overrightarrow{AB}|$

8 Projection of a Line on Another Line :

Let PQ be a line segment with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let L. be a straight line whose d.c.'s are l,m,n. Then the length of projection of PQ on the line L is

$|l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)|$

9. Angle Between Two Lines :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = 1_1 l_2 + m_1m_2 + n_1n_2$ .If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by $\cos\theta=\frac{(a_1a_2+b_1b_2+c_1c_2)}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$

10. Perpendicular and Parallel Lines :

Let the two lines have their d.c.'s given by l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $l_1 l_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note:

If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

11. Equation of Straight Line in Symmetrical Form :

(A) One point form : Let A (x_1,y_1,z_1) be a given point on the straight line and l,m,n the d.c's of the line, then its equation is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)}$$

it should be noted that $P(x_1+lr, y_1+mr, z_1+nr)$ is a general point on this line at a distance r from the point $A(x_1,y_1,z_1)$ i.e. AP = r. One should note that for AP = r;l,m n must be d.c.'s not d.r.'s. if a, b, c are direction ratios of the line then $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r_1$ but here $AP \neq r_1$

(b) Equation of the line passing through two points $A(x_1,y_1,z_1)$ and

$B(x_2 . y_2 . z_2)$ is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

12. Foot, Length and Equation of Perpendiular Form a Point to a Line :

Let equation of the line be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)} \qquad \text{.......(i)}$$

and $A(\alpha,\beta,\lambda)$ be any point on the plane and P be foot of perpendicular and

$P = (lr + x_1, mn + y_1, nr + z_1)$, then AP is $\perp$ to line, so

$l(lr+x_1-\alpha) + m(mr+y_1-\beta) + n(nr+z_1-\gamma) = 0$

i.e. $r = (\alpha - x_1)\,l + (\beta - y_1)\,m + (\gamma - z_1)n$

since $l^2 + m^2 + n^2 = 1$

putting this value of r in (ii) we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(lr+x_1-\alpha)^2 + (mr+y_1-\beta)^2 + (nr+z_1-\gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r + x_1 - \alpha} = \frac{y-\beta}{mr + y_1 - \beta} = \frac{z-\gamma}{nr + z_1 - \gamma}$$

13. Equations of a Plane :

The equation of every plane is in the first degree and is of the form $ax + by + cz + d = 0$, in which a, b, c are dr's normal to the plane, where $a^2 + b^2 + c^2 \neq 0$ (i.e.a,b,c, $\neq 0$ simultaneously).

(a) Vector form of equation of plane

If $\vec{a}$ be the position vector of a point on the plane and $\vec{n}$ be a vector normal to the plane then it's vectorial equation is given by

$(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$, where $d = \vec{a}.\vec{n} =$ constant.

(b) Plane Parallel to the Coordinate Planes :

(i) Equation of yz plane is $x = 0$.

(ii) Equation of zx plane is $y = 0$.

(iii) Equation of xy plane is $z = 0$.

(iv) Equation of the plane parallel to x-y plane at a distance c from origin is $z = c$.

(d) Equation of a Plane in Intercept Form :

Equation of plane which cuts off intercepts a, b, c on the axes x, y, z respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(e) Equation of a Plane in Normal Form :

It the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l,m,n,) then the equation of the plane is $lx + my + nz = p$.

(f) Vectorial form of Normal equaion of plane :

If $\hat{n}$ is a unit vector normal to the plane from the origin and d be the perpendicular distance of plane from origin then its vector eqauation is $\hat{r}.\hat{n} = d$.

(g) Equation of a Plane through three points :

The equation of the plane through three non-collinear points

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

14. Angle Between two Planes :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if $aa' + bb' + cc' = 0$ and they are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

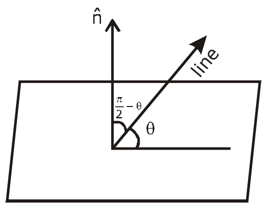
Planes parallel to a given Plane :

Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$. d' is to be found by other given condition.

15. Angle Between a Line and a Plan :

Let equation of the line and plane be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and ax + by + cz + d = 0 respectively and θ be the angle which line makes with the plane. Then $(\pi/2-\theta)$ is the angle between the line and the normal to the plane.

So $\sin\theta = \frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(l^2+m^2+n^2)}}$



Line is parallel to plane if $\theta = 0$
i.e. if al + bm + cn = 0.

Line is $\perp$ to the plane if line is parallel to the normal of the plane i.e. if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$

16. Condition in Order that Line May Lie on the Given Plane :

The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ will be on the plane Ax + By + Cz + D = 0 if (a) $Al + Bm + Cn = 0$ (b) $Ax_1 + by_1 + Cz_1 + D = 0$

17. Position of two Points W.R.T. A Plane :

Two points $P(x_1,y_1,z_1)$ & $Q(x_2,y_2,z_2)$ are on the same or opposite sides of a plane ax + by + cz + d = 0 according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs.

18. Image of a Point in the Plane :

Let the image of a point $P(x_1,y_1,z_1)$ in a plane ax + by + cz + d = 0 is $Q(x_2,y_2,z_2)$ and foot of perpendicular of point P on plane is $R(x_3,y_3,z_3)$ then

(a) $\frac{x_3-x_1}{a}=\frac{y_3-y_1}{b}=\frac{z_3-z_1}{c}=-\left(\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}\right)$

(b) $\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{z_2-z_1}{c}=-2\left(\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}\right)$

19. Condition for Coplanarity of Two Lines :

Let the two lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad \text{...... (i)}$$

and $\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ ....(ii)

These lines will be coplaner if $\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

the plane containing the two lines is $\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

20. Perpendicular Distance of a Point from the Plane :

(A) Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane ax + by + cz + d = 0 is given by $p=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{(a^2+b^2+c^2)}}$

(B) Distance between two parallel planes $ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

21. A Plane Through the Line of Intersection of Two given Planes :

Consider two planes

$u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0$, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

22. Bisectors of Angles Between two Planes :

Let the equations of the two planes be ax + by + cz + d = 0 and $a_1x + b_1y + c_1z + d_1 = 0$.
Then equations of bisectors of angles between them are given by

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} \quad \text{...(i)}$$

(a) Equation of bisector of the angle containing origin : First make both constant terms positive. Then positive sign in (i) gives the bisector of the angle which contains the origin.

(b) Bisector of acute/obtuse angle : First make both constant terms positive, is $aa_1 + bb_1 + cc_1 > 0 \equiv$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \quad \Rightarrow$ origin lies in acute angle