STRAIGHT LINES

1. Relation between cartesian co-ordinate & Polar co-ordinate system:

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$ and

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2. Distance formula and its Applications:

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Note :

(i) Three given points A,B and C are collinear, if sum of any two distances from AB,BC, CA is equal to the remaining third else the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; $AC \neq BD$; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

3. Section Formula:

The co-ordinates of a point dividing a line joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m:n is given by :

(a) For internal division : $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

(b)For external division : $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

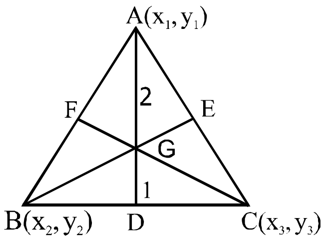
(c) Line ax + by + c = 0 divides line joining points $P(x_1y_1)$ & $Q(x_2y_2)$ in ratio $= -\frac{(ax_1+by_1+c)}{(ax_2+by_2+c)}$

(d) Coordinates of mid point of AB are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

4. Co-ordinates of some particular points :

Let $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

(a) Centroid :



(i) Centroid is the point of concurrence of the medians (line segment joining the mid point of sides to opposite vertices).

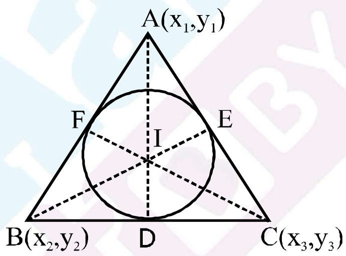
(ii) Centroid divides the median in the ratio of 2 :1.

(iii) Co-ordinates of centroid $G\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

(iv) If P is any internal point of triangle such that area of ΔAPB, ΔAPC and ΔBPC are same then P must be centroid.

(b) Incenter :

Incenter is the point of concurrence of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle



Co-ordinates of incenter $I\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC, opposite to A, B, C respectively.

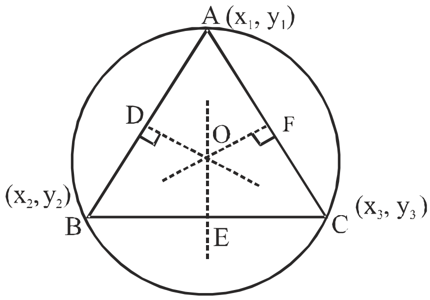
Note:

(i) Angle bisector divides the opposite sides in the ratio of their corresponding sides. e.g. $\dfrac{BD}{DC}=\dfrac{AB}{AC}=\dfrac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a,\ (c+a):b,\ (a+b):c$

(c) Circumcenter :

It is the point of concurrence of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2=OB^2=OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

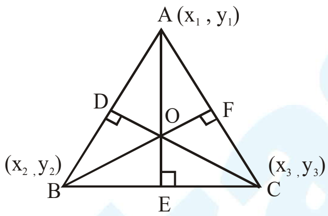


(i) If a triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(ii) Find perpendicular bisector of any two sides and solve them to find circumcentre.

(d) Orthocenter :

It is the point of concurrence of perpendiculars drawn from vertices to opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.



Note :

If a triangle is right angled triangle, then orthocenter is the point where right angle is formed.

Remarks :

(i) If the triangle is equilateral, then centroid, incenter, orthocenter, circumcenter coincide.

(ii) Orthocenter, centroid and circumcenter are always collinear and centroid divides the line joining orthocenter and circumcenter in the ratio 2 : 1 for non equilateral triangles.

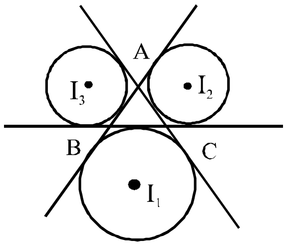
(iii) In an isosceles triangle centroid, orthocenter, incenter & circumcenter lie on the same line.

(e) Ex-centers :

Point of concurrence of two external angle bisectors and one internal angle bisector of a triangle is called Ex-center.

The center of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-center of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$



Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),\ I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

5. Area of triangle :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

To remember the above formula, take the help of the following method:

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.
(ii) Area of Equilateral triangle : If altitude of any equilateral triangle is

P, then its area = $\frac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area = $\left(\frac{a^2\sqrt{3}}{4}\right)$

(iii) Area of quadrilateral whose consecutive vertices are :

(x_1, y_1), (x_2, y_2), (x_3, y_3) & (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

6. Conditions of collinearity For three points :

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if any one of the given point lies on the line passing through the remaining two points Thus the required condition is-

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1} \text{ or } \frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

7. Equation of straight Line :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, every linear equation in variable x and y always represents a straight line i.e. $\mathbf{ax + by + c = 0}$ **;** $\mathbf{a \ \& \ b \neq 0}$ simultaneously.

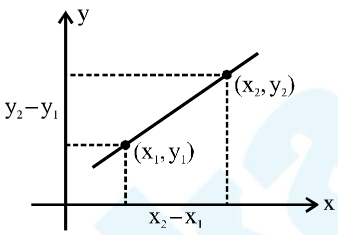
(a) Equation of a line parallel to x-axis at a distance 'a' from x-axis is $\mathbf{y = a}$ or $\mathbf{y = -a}$

(b) Equation of x-axis is $\mathbf{y = 0}$

(c) Equation of a line parallel to y-axis at a distance 'b' from y -axis is $\mathbf{x = b}$ or $\mathbf{x = -b}$

(d) Equation of y-axis is $\mathbf{x = 0}$

8. Slope of line :



If a given line makes an angle θ $\mathbf{(0° \leq \theta < 180°, \theta \neq 90°)}$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $\mathbf{m = \tan\theta}$. Obviously the slope of the x-axis and line parallel to it is zero and y-axis and line parallel to it does not exist.

If $A(x_1,y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB = $\dfrac{y_2 - y_1}{x_2 - x_1}$

9. Standard forms of a straight line :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $\mathbf{y = mx + c}$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $\mathbf{y - y_1 = m(x - x_1)}$

(c) Two point form : Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

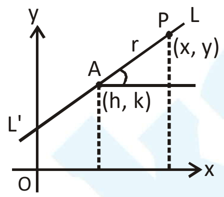
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) Intercept form : If a and b are the intercepts made by a line on the axes of x and y respectively, then its equation is written as : $\frac{x}{a}+\frac{y}{b}=1$

(e) Normal form : If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive direction of x-axis, then the equation of this line is written as : **$x\cos\alpha + y\sin\alpha = p$** (p is always positive) where $0 \le \alpha < 2\pi$.

(f) Parametric form : The equation of a straight line which passes through a point A(h, k) and makes an angle θ with the positive direction of the x-axis is $\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r$

Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).



(g) General form : We know that a first degree equation in x and y, **$ax + by + c = 0$** always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

10. Angle between two lines :

(a) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(iii) Intersecting $\Leftrightarrow$ $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

11. Length of perpendicular from a point on a line :

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P=\frac{|c|}{\sqrt{a^2+b^2}}$

12. Distance between two parallel lines :

(a) The distance between two parallel lines $ax + by + c_1= 0$ and $ax + by + c_2= 0$ is $=\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $=\frac{p_1p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y=m_1x+c_1$, $y=m_1x+c_2$ and $y=m_2x+d_1$, $y=m_2x+d_2$ is given by $\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$

13. Equation of lines parallel and perpendicular to a given line:

(a) Equation of line parallel to line $ax + by + c = 0$

$\mathbf{ax + by + \lambda = 0}$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$\mathbf{bx - ay + k = 0}$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

14. Straight line making a given angle with a line :

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line y= mx+c is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

15. Position of two points with respect to a given line :

Let the given line be ax + by + c = 0 and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line ax + by + c = 0. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

16. Concurrency of lines :

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if Δ

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Note :
If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}

17. Reflection of a point :

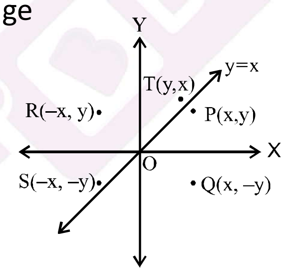
Let P(x, y) be any point, then its image with respect to

(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)



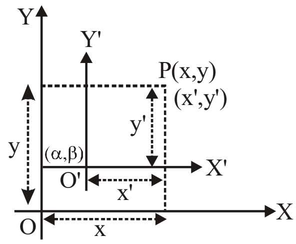
18. Transformation of axes:

(a) Shifting of origin without rotation of axes :

If coordinates of any point P (x, y) with respect to new origin (α,β)will be (x'y') then $x = x' + \alpha$, $y = y' + \beta$

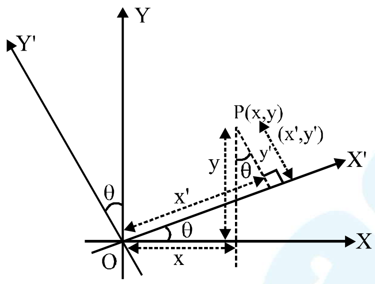
or $x' = x - \alpha$, $y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by substituting $x + \alpha$ in place of x and $y + \beta$ in place of y.



(b) Rotation of axes without shifting the origin :

Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$,



then $\quad x = x' \cos\theta - y' \sin\theta$

$\quad\quad\; y = x' \sin\theta + y' \cos\theta$

and $\quad x' = x \cos\theta + y \sin\theta$

$\quad\quad\; y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	$x\downarrow$	$y\downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

19. Equation of Bisectors of angles between two lines :

If lines $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$ are intersecting, then equation of angle bisectors between these lines are :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \quad\quad \text{..........(1)}$$

(a) Equation of angle bisector containing origin :

If the equation of the lines are written with constant terms $\mathbf{c_1}$ and $\mathbf{c_2}$ positive, then the equation of the angle bisectors containing the origin is obtained by taking positive sign in (1)

(b) Equation of bisector of acute/obtuse angles :
See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by-1 to make the constant terms positive.
Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2+b_1b_2$	For obtuse angle bisector	For acute angle bisector
$+$	use $+$ sign in eq. (1)	use $-$ sign in eq. (1)
$-$	use $-$ sign in eq. (1)	use $+$ sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

20. Family of lines :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2=0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

21. General Equation and Homogeneous Equation of Second Degree :

(a) The general equation of second degree
$\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represents a pair of straight lines,

if $\Delta=abc+2fgh - af^2 - bg^2 - ch^2 = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$, $h^2 \geq ab$, $g^2 \geq ca$ and $f^2 \geq bc$

(b) If θ be the angle between the lines, then $\tan\theta=\pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are
(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$
(ii) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.
(c) Homogeneous equation of 2nd degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y=\left(\frac{-h\pm\sqrt{h^2-ab}}{b}\right)x \equiv y = m_1x \ \& \ y = m_2x$$

and $m_1 + m_2 = -\frac{2h}{b}$: $m_1m_2 = \frac{a}{b}$

These straight lines pass through the origin and to find the angle between these lines same formula as given for general equation is used.
The condition that these lines are:
(i) At right angle to each other is $a + b = 0$. i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$.
(ii) Coincident if $h^2 = ab$.

(iii) Equally inclined to the axis of x if h = 0. i.e. coeff. of xy = 0.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, $a \neq b$, $h \neq 0$.

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$

and through origin are given by $bx^2 – 2hxy + ay^2 = 0$.

(f) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is =

$2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $\sqrt{2}\left[\frac{(f^2 - bc)}{b(a+b)}\right]$

22. Equations of lines joining the points of intersection of a line and a curve to the origin :

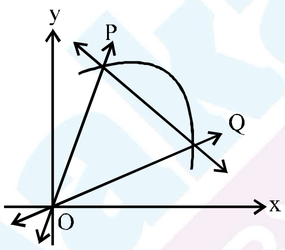
(a) Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$lx + my + n = 0$ ...(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation



(i) homogenous with the help of equation of the line. Thus required equation is given by-

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x + my}{-n}\right)+c\left(\frac{\ell x + my}{-n}\right)^2 = 0$$

23. Standard Results:

(a) Area of rhombus formed by lines $a|x| - b|y| + c = 0$ or $\pm ax \pm by + c = 0$ is $\frac{2c^2}{|ab|}$

(b) Area of triangle formed by line $ax + by + c = 0$ and axes is $\frac{c^2}{2|ab|}$.

(c) Co-ordinate of foot of perpendicular (h,k) from (x_1,y_1) to the line $ax + by + c = 0$ is given by $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

(d) Image of point (x_1, y_1) w.r.t the line $ax + by + c = 0$ is given by

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

CIRCLES

1. Definition :

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

2. Standard Equations of the Circle :

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is $\mathbf{(x-h)^2 + (y-k)^2 = r^2}$

(b) General equation of circle: $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$. where g,f,c are constants then centre is $(-g,-f)$

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Note :

The general quadratic equation in x and y.

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :

(i) coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$

(ii) coefficient of $xy = 0$ or $h = 0$

(iii) $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

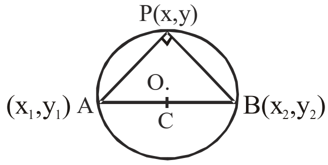
(i) x-axis $= 2\sqrt{g^2 - c}$ (ii) y-axis $= 2\sqrt{f^2 - c}$

Note :

Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where 'a' is the radius and 'P' is the length of perpendicular from the centre to the chord.

(d) Diameter form of circle :

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle then the equation of the circle is given by $\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0}$



(e) The parametric forms of the circle :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $\mathbf{x = r\cos\theta, y = r\sin\theta}$; $\theta \in$ **[0, 2π)**

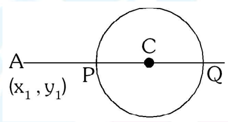
(ii) The parametric equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ are
$\mathbf{x = h + r\cos\theta, y = k + r\sin\theta}$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $\mathbf{x = -g + \sqrt{g^2+f^2-c}\cos\theta}$,
$\mathbf{y = -f + \sqrt{g^2+f^2-c}\sin\theta}$ where θ is parameter.

Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$$

3. Position of a point w.r.t circle: :



(a) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

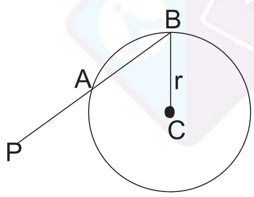
and the point be (x_1,y_1) then-

Point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.

(c) If a line drawn from a point $p(x_1,y_1)$ intersects the circle S = 0 in two distinct points A and B then $PA.PB = S_1$ and $PA.PB = CP^2-r^2$ is called the power of the point P w.r.t. circle S = 0



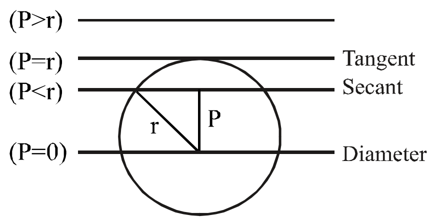
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

4. Tangent Line of Circle :

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :



The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.

(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1,y_1) on the circle $x^2 + y^2 = a^2$ is $\mathbf{xx_1 + yy_1 = a^2}$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is **$x\cos t + y\sin t = a$**

(2) The point of intersection of the tangents at the points

P (α) and Q (β) is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is T = 0

$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$

(iv) If line y = mx + c is a straight line touching the circle

$x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact **points are**

$\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{a}{\sqrt{1+m^2}}\right)$ **or** $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is

$$\mathbf{y = mx \pm a\sqrt{1+m^2}}$$

(v) The equation of tangent with slope m of the circle

$(x - h)^2 + (y - k)^2 = a^2$ is

$$\mathbf{(y - k) = m(x - h) \pm a\sqrt{1+m^2}}$$

Note :

To get the equation of tangent at the point $(x_1\ y_1)$ on any curve we replace x^2 with xx_1, y^2 with yy_1, x with $\dfrac{x + x_1}{2}$, y with $\dfrac{y + y_1}{2}$, xy with $\dfrac{xy_1 + yx_1}{2}$ and c in place of c.

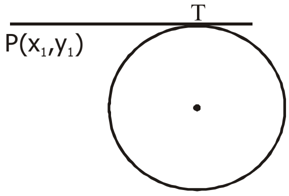
(c) Length of tangent ($\sqrt{S_1}$) :

The length of tangent drawn from point

(x_1,y_1) out side the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$



(d) Equation of Pair of tangents ($SS_1 = T^2$) :

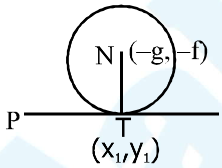
Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2 \text{ or } SS_1 = T^2$$

5. Normal of circle :

Normal at a point of the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

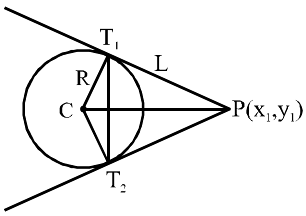
(a) Equation of normal NT at point $T(x_1,y_1)$ of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$

6. Chord of Contact:



If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$,

then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$).

7. Equation Of The Chord With A Given Middle Point ($T = S_1$) :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $\mathbf{y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)}$.

This on simplification can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $\mathbf{T = S_1}$.

8. Director Circle :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let the circle $x^2 + y^2 = a^2$. Then the equation of the director circle is $x^2 + y^2 = 2a^2$.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

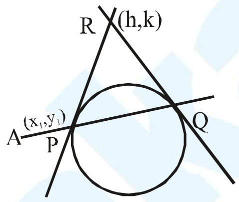
Note :

The director circle of

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

9. Pole and Polar :



Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.

The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Pole of a given line with respect to a circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $lx + my + n = 0$ w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

10. Family Of Circles :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0 \quad (K \neq -1)$.

(b) The equation of the family of circles passing through the point of Intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.

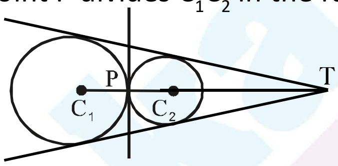
(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

11. Direct and Transverse common tangents :

Let two circles having centre C_1, C_2, radii r_1, r_2 and C_1C_2 is the distance between their centres then :

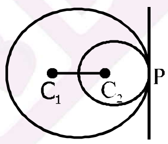
(a) Circles touch each other:

(i) Externally if $C_1C_2 = r_1 + r_2$ point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).

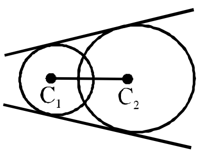


In this case there are **three common tangents.**

(ii) Internally if $C_1C_2 = |r_1 - r_2|$ point P divides C_1C_2 in the ratio $r_1 : r_2$ externally and in this case there will be only one common tangent.

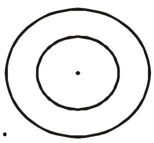


(b) The circles will intersect : when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**

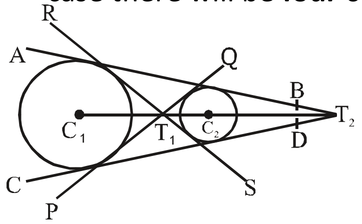


(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.



(ii) Circles neither touch nor intersect then $C_1C_2 > r_1+r_2$ and in this case there will be **four common tangents.**



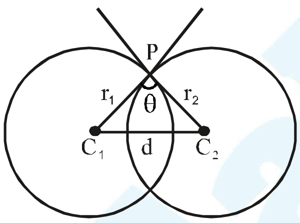
Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines intersect line C_1C_2 at T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines intersect C_1C_2 produced at T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.

Note : Length of direct common tangent = $\sqrt{(C_1C_2)^2-(r_1-r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2-(r_1+r_2)^2}$

12. The angle of intersection of two circles :

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles.



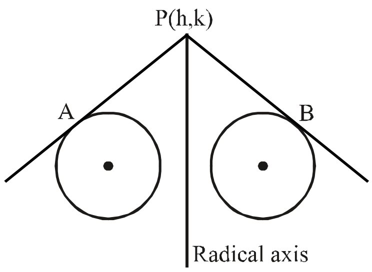
then $\cos\theta=\dfrac{2g_1g_2+2f_1f_2-c_1-c_2}{2\sqrt{g_1^2+f_1^2-c_1}\ \sqrt{g_2^2+f_2^2-c_2}}$ or $\cos\theta=\left(\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.
If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and condition for the circles to be orthogonal is -

$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$ **or** $\mathbf{r_1^2 + r_2^2 = d^2}$

13. Radical axis of the two circles ($S_1 - S_2 = 0$) :

Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal is called the radical axis. If two circles are -



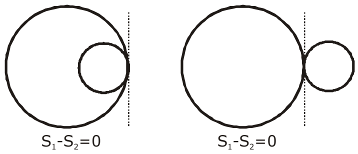
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1 y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

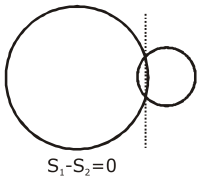
Then the equation of radical axix is given by $\mathbf{S_1 - S_2 = 0}$

Note :

(i) If two circles touches each other then common tangent radical axis



(ii) If two circles intersect each other then common chord is radical axis



(iii) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

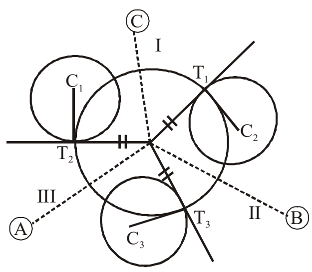
(iv) The radical axis of the two circles is perpendicular to the line joining the centres of two circles but not always pass through mid point of it.

14. Radical centre :

The point of concurrence of radical axes of three circles whose centres are non collinear, taken in pairs is called their radical centre.

Note :

(i) The circle with centre as radical centre and radius equal to the length of tangent from radical cent any of the circle, will cut the three circles orthogonally.



(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

(iii) The radical centre of three circles is the point from which length of tangents on three circles are equal.