Subject: Mathematics

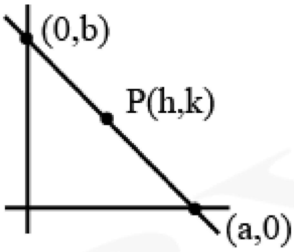
1. A man is walking on a straight line. The arithmetic mean of the reciprocals of the intercepts of this line on the coordinate axes is $\frac{1}{4}$. Three stones A, B and C are placed at the points $(1,1),(2,2)$ and $(4,4)$ respectively. Then which of these stones is/are on the path of the man?

A. B only

B. A only

C. the three

D. C only



$\frac{x}{a}+\frac{y}{b}=1$

$\frac{h}{a}+\frac{k}{b}=1 \ldots(y)$

and $\frac{\frac{1}{a}+\frac{1}{b}}{2}=\frac{1}{4}$

$\therefore \frac{1}{a}+\frac{1}{b}=\frac{1}{2} \ldots(2)$

From (1) and (2)

Line passes through fixed point $B(2,2)$

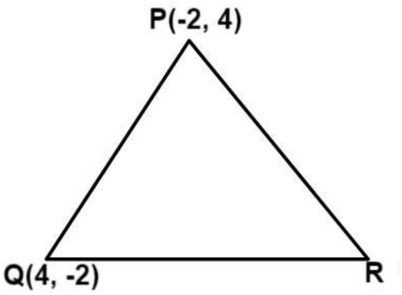
2. In a triangle PQR, the co-ordinates of the points P and Q are $(-2, 4)$ and $(4, -2)$ respectively. If the equation of the perpendicular bisector of PR is $2x - y + 2 = 0$, then the centre of the circumcircle of the ΔPQR is:

A. $(-2, -2)$

B. $(0, 2)$

C. $(-1, 0)$

D. $(1, 4)$



Perpendicular bisector of PR is
$2x - y + 2 = 0 \cdots (1)$
Mid-points of PQ is $M \equiv (1, 1)$
Equation of perpendicular bisector of PQ is

$$y - 1 = -\left(\frac{4+2}{-2-4}\right)(x - 1)$$

$$\Rightarrow x = y \cdots (2)$$

Therefore, circumcentre is point of intersection of the two perpendicular bisectors i.e., $(-2, -2)$

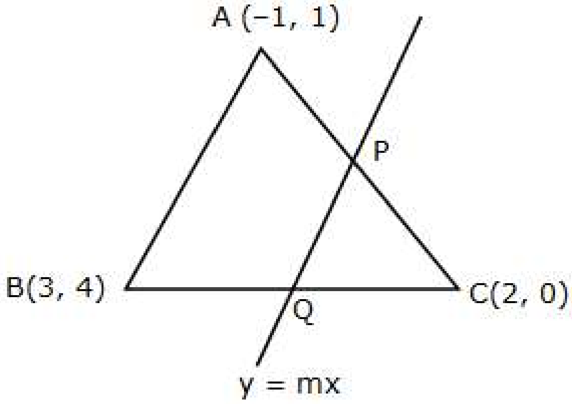
3. Let $A(-1,1), B(3,4)$ and $C(2,0)$ be given three points. A line $y = mx, m > 0$, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

A. $\frac{4}{15}$

B. 1

C. 2

D. 3



$$A_1 = \text{Area of } \Delta ABC = \frac{1}{2}\left\|\begin{matrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 4 & 1 \end{matrix}\right\|$$

$$\Rightarrow A_1 = \frac{13}{2}$$

Equation of line AC is $y - 1 = -\frac{1}{3}(x + 1)$

Solving it with line $y = mx$, we get $P\left(\frac{2}{3m+1}, \frac{2m}{3m+1}\right)$

Equation of line BC is $y - 0 = 4(x - 2)$

Solving it with line $y = mx$, we get $Q\left(\frac{-8}{m-4}, \frac{-8m}{m-4}\right)$

$$A_2 = \text{Area of } \Delta PQC = \frac{1}{2}\left\|\begin{matrix} 2 & 0 & 1 \\ \frac{2}{3m+1} & \frac{2m}{3m+1} & 1 \\ \frac{-8}{m-4} & \frac{-8m}{m-4} & 1 \end{matrix}\right\| = \frac{A_1}{3} = \frac{13}{6}$$

$$\Rightarrow \frac{26m^2}{3m^2 - 11m - 4} = \pm\frac{13}{6}$$

$\Rightarrow 12m^2 = \pm(3m^2 - 11m - 4)$

Taking $+$ve sign,

$9m^2 + 11m + 4 = 0$ (Rejected $\because$ m is imaginary)

Taking $-$ve sign,

$15m^2 - 11m - 4 = 0$

$\Rightarrow m = 1, -\frac{4}{15}$

$\Rightarrow m = 1$ as $m > 0$

4. The number of integral values of m so that the abscissa of point of intersection of lines $3x+4y=9$ and $y=mx+1$ is also an integer, is:

- ✗ **A.** 3
- ✓ **B.** 2
- ✗ **C.** 1
- ✗ **D.** 0

$3x+4(mx+1)=9$
$\Rightarrow x(3+4m)=5$
$\Rightarrow x=\dfrac{5}{3+4m}$
$\Rightarrow (3+4m)=\pm 1, \pm 5$
$\Rightarrow 4m=-3\pm 1, -3\pm 5$
$\Rightarrow 4m=-4,-2,-8,2$
$\Rightarrow m=-1,-\dfrac{1}{2},-2,\dfrac{1}{2}$
Two integral value of m.

5. If C be the centroid of the triangle having vertices $(3,-1),(1,3)$ and $(2,4)$. Let P be the point of intersection of the lines $x+3y-1=0$ and $3x-y+1=0$, then the line passing through the points C and P also passes through the point :

- ✗ **A.** $(-9,-7)$
- ✓ **B.** $(-9,-6)$
- ✗ **C.** $(7,6)$
- ✗ **D.** $(9,7)$

Coordinates of C are $\left(\dfrac{3+1+2}{3}, \dfrac{-1+3+4}{3}\right)=(2,2)$
Point of intersection of two lines
$x+3y-1=0$ and $3x-y+1=0$
is $P\left(\dfrac{-1}{5}, \dfrac{2}{5}\right)$

Equation of line CP is $8x-11y+6=0$
Point $(-9,-6)$ lies on CP.

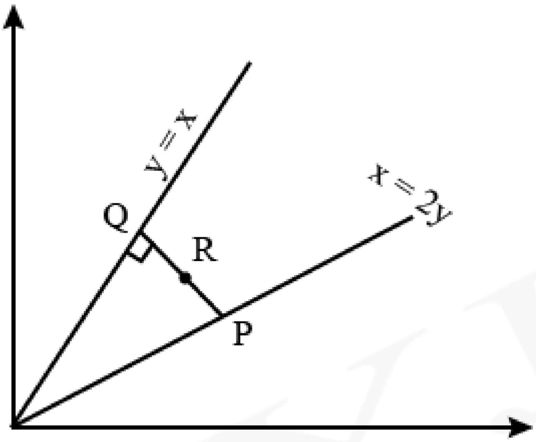
6. The locus of the mid-points of the perpendiculars drawn from points on the line, $x = 2y$ to the line $x = y$ is :

A. $2x - 3y = 0$

B. $3x - 2y = 0$

C. $5x - 7y = 0$

D. $7x - 5y = 0$



Let R be the mid-point of PQ whose locus is (h, k)
PQ is perpendicular to line $y = x$
$\therefore$ Equation of the line PQ can be written as $y = -x + c$
$y = -x + c$ intersects $y = x$ at $Q\left(\frac{c}{2}, \frac{c}{2}\right)$
$y = -x + c$ intersects $x = 2y$ at $P\left(\frac{2c}{3}, \frac{c}{3}\right)$
$\therefore$ Coordinates of midpoint is $R\left(\frac{7c}{12}, \frac{5c}{12}\right)$
Locus of $R : h = \frac{7c}{12}, k = \frac{5c}{12}$
$\Rightarrow 5h - 7k = 0$
$\therefore$ locus of required equation is $5x - 7y = 0$

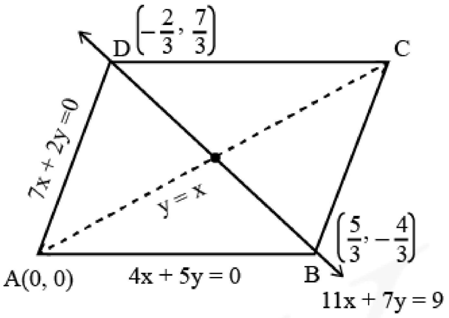
7. Two sides of a parallelogram are along the lines $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one of the diagonals of the parallelogram is $11x + 7y = 9$, then other diagonal passes through the point

A. $(2, 2)$

B. $(2, 1)$

C. $(1, 3)$

D. $(1, 2)$



On solving equation $4x + 5y = 0$, $7x + 2y = 0$ and $11x + 7y = 9$, we get

Coordinate of $D = \left(-\frac{2}{3}, \frac{7}{3}\right)$

Coordinate of $B = \left(\frac{5}{3}, -\frac{4}{3}\right)$

$\therefore$ Mid point of $BD = M = \left(\frac{1}{2}, \frac{1}{2}\right)$

Equation of other diagonal $AC : y = x$

$\therefore$ Point (2, 2) lies on other diagonal

8. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices (a, c), $(2, b)$ and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is

A. $\frac{71}{256}$

B. $-\frac{69}{256}$

C. $\frac{69}{256}$

D. $-\frac{71}{256}$

$2b = a + c$

$\frac{2a+2}{3} = \frac{10}{3}$ and $\frac{2b+c}{3} = \frac{7}{3}$

$\Rightarrow a = 4$

$\left.\begin{matrix} 2b + c = 7 \\ 2b - c = 4 \end{matrix}\right\}$, solving the two equations,

$b = \frac{11}{4}$ and $c = \frac{3}{2}$

$\therefore$ Quadratic equation is $4x^2 + \frac{11}{4}x + 1 = 0$

$\therefore$ The value of $(\alpha + \beta)^2 - 3\alpha\beta = \frac{121}{256} - \frac{3}{4} = -\frac{71}{256}$

9. Let $A(a, 0), B(b, 2b+1)$ and $C(0, b),\ b \neq 0, |b| \neq 1$, be points such that the area of triangle ABC is 1 sq. unit, then the sum of all possible values of a is

A. $\dfrac{2b}{b+1}$

B. $\dfrac{-2b^2}{b+1}$

C. $\dfrac{2b^2}{b+1}$

D. $\dfrac{-2b}{b+1}$

Area $= \left| \dfrac{1}{2} \begin{vmatrix} a & 0 & 1 \\ b & 2b+1 & 1 \\ o & b & 1 \end{vmatrix} \right| = 1$

$\Rightarrow (a(2b+1-b) - b(-b)) = \pm 2$

$\Rightarrow a(b+1) = \pm 2 - b^2$

$\Rightarrow a = \dfrac{2-b^2}{b+1} \text{or} \dfrac{-2-b^2}{b+1}$

Sum of values of $a = \dfrac{-2-b^2+2-b^2}{b+1} = \dfrac{-2b^2}{b+1}$

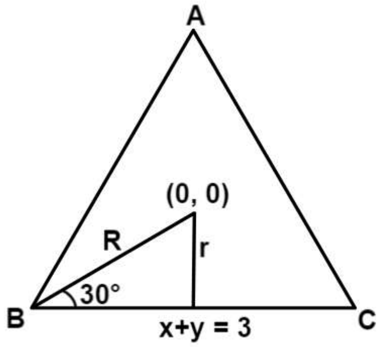
10. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x+y=3$. If R and r be the radius of circumcircle and incircle respectively of $\triangle ABC$, then $(R+r)$ is equal to

A. $2\sqrt{2}$

B. $3\sqrt{2}$

C. $7\sqrt{2}$

D. $\frac{9}{\sqrt{2}}$



$r = \frac{|0+0-3|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$\sin 30^\circ = \frac{r}{R} = \frac{1}{2}$

$\Rightarrow R = 2r$

$\Rightarrow r + R = 3r$

$\therefore r + R = \frac{9}{\sqrt{2}}$

11. The image of the point $(3,5)$ in the line $x - y + 1 = 0$, lies on :

A. $(x-2)^2 + (y-4)^2 = 4$

B. $(x-4)^2 + (y+2)^2 = 16$

C. $(x-4)^2 + (y-4)^2 = 8$

D. $(x-2)^2 + (y-2)^2 = 12$

Image of $P(3,5)$ on the line $x - y + 1 = 0$ is

$$\frac{x-3}{1} = \frac{y-5}{-1} = \frac{-2(3-5+1)}{2} = 1$$

$x = 4, y = 4$

$\therefore$ Image is $(4,4)$

Which lies on

$(x-2)^2 + (y-4)^2 = 4$

12. Let the equations of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at $(1, 1)$, then the equation of its third side is :

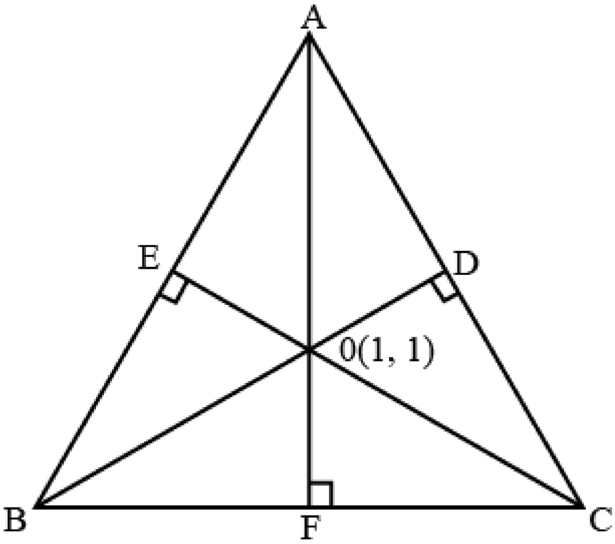
A. $122y - 26x - 1675 = 0$

B. $26x - 122y - 1675 = 0$

C. $26x + 61y + 1675 = 0$

D. $122y + 26x + 1675 = 0$

Let equation of side $AB : 3x - 2y + 6 = 0 \quad \ldots(1)$
and equation of side $BC : 4x + 5y - 20 = 0 \quad \ldots(2)$
O is the orthocentre.



Slope of line AB is $\frac{3}{2}$

Line CE is perpendicular to AB

So, slope of line CE is $-\frac{2}{3}$

$\therefore$ Equation of line CE is

$(y - 1) = -\frac{2}{3}(x - 1)$

$\Rightarrow 2x + 3y - 5 = 0 \quad \ldots(3)$

From Equations (2) and (3),

$x = \frac{35}{2}, y = -10$

Now, slope of line BC is $-\frac{4}{5}$

$\therefore$ Slope of line AF is $\frac{5}{4}$

Thus, Equation of line AF is

$y - 1 = \frac{5}{4}(x - 1)$

$\Rightarrow \ 5x - 4y - 1 = 0 \quad \ldots(4)$

From Equations (1) and (4),

$x = -13, \ y = -\frac{66}{4}$

Thus, the coordinates of A is $\left(-13, -\frac{66}{4}\right)$ and coordinates of B is $\left(\frac{35}{2}, -10\right)$

$\therefore$ Equation of AB is $\dfrac{y + 10}{x - \frac{35}{2}} = \dfrac{-10 + \frac{66}{4}}{\frac{35}{2} + 13}$

$\Rightarrow 26x - 122y - 1675 = 0$

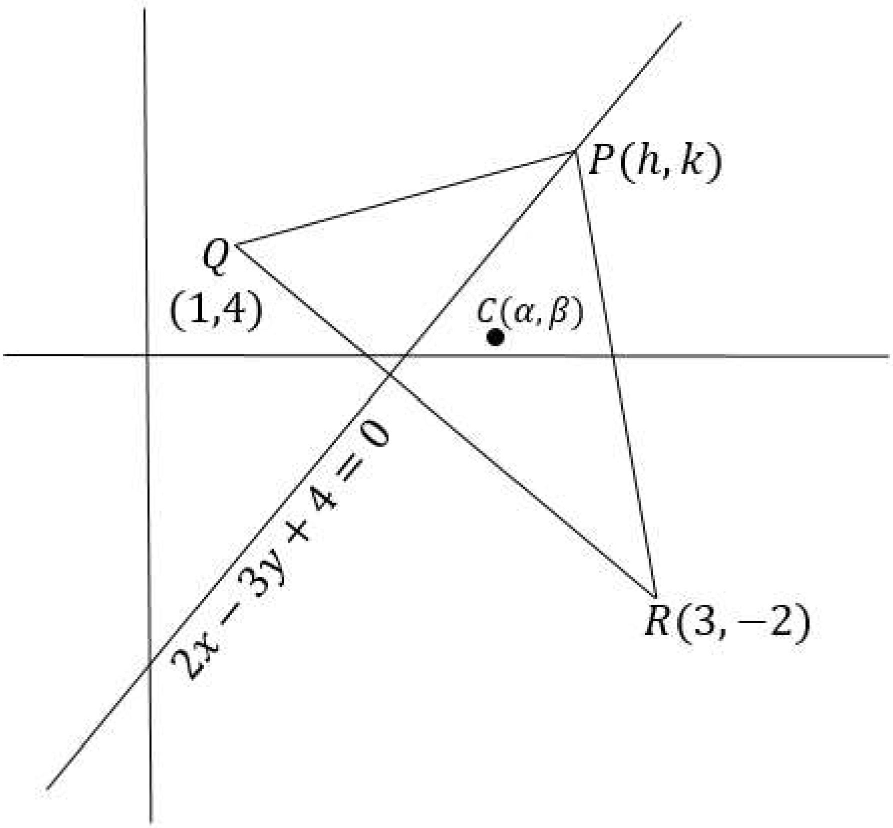
13. A point P moves on the line $2x-3y+4=0$. If $Q(1,4)$ and $R(3,-2)$ are fixed points, then the locus of the centroid of ΔPQR is a line :

A. parallel to x-axis

B. parallel to y-axis

C. with slope $\frac{3}{2}$

D. with slope $\frac{2}{3}$



Let the centroid of the ΔPQR is $C(\alpha,\beta)$

$\therefore \alpha=\frac{h+1+3}{3} \Rightarrow h=3\alpha-4$

$\beta=\frac{4+k-2}{3} \Rightarrow k=3\beta-2$

$P(h,k)$ passes through the line $2x-3y+4=0$

$\therefore\ 2(3\alpha-4)-3(3\beta-2)+4=0$

$\Rightarrow 6\alpha-9\beta+2=0$

Hence, the locus is $6x-9y+2=0$

$\therefore$ slope $=\frac{6}{9}=\frac{2}{3}$

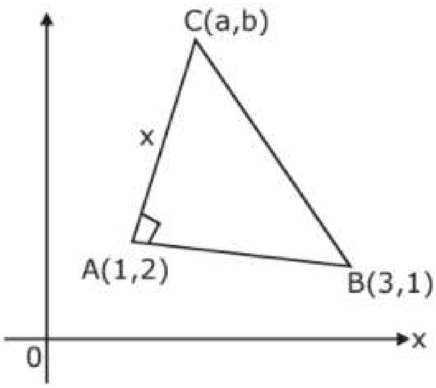
14. A triangle ABC lying in the first quadrant has two vertices as $A(1,2)$ and $B(3,1)$. If $\angle BAC = 90^\circ$, and $ar(\Delta ABC) = 5\sqrt{5}$ sq. units, then the abscissa of the vertex C is

A. $1+\sqrt{5}$

B. $1+2\sqrt{5}$

C. $2\sqrt{5}-1$

D. $2+\sqrt{5}$



Since $\angle BAC = 90^\circ$

$\therefore m_{AC} \cdot m_{AB} = -1$

$\Rightarrow \dfrac{b-2}{a-1} \cdot \dfrac{2-1}{1-3} = -1$

$\Rightarrow \dfrac{b-2}{a-1} = 2$

$\Rightarrow b = 2a \quad \cdots (1)$

Area of triangle $ABC = 5\sqrt{5}$

$\dfrac{1}{2} \times \sqrt{5} \times \sqrt{(a-1)^2 + (b-2)^2} = 5\sqrt{5}$

$\Rightarrow \sqrt{(a-1)^2 + (b-2)^2} = 10$

$\Rightarrow \sqrt{(a-1)^2 + 4(a-1)^2} = 10$

$\Rightarrow \sqrt{5}(a-1) = 10$

$\therefore a = 1 + 2\sqrt{5}$

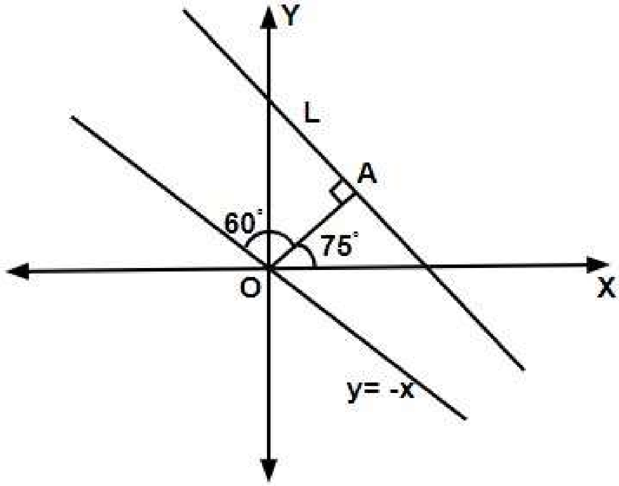
15. A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line $x+y=0$. Then an equation of the line L is :

A. $(\sqrt{3}+1)x+(\sqrt{3}-1)y=8\sqrt{2}$

B. $(\sqrt{3}-1)x+(\sqrt{3}+1)y=8\sqrt{2}$

C. $\sqrt{3}x+y=8$

D. $x+\sqrt{3}y=8$



$OA=p=4$
$x+y=0$ makes an angle of 135° with the positive x-axis.

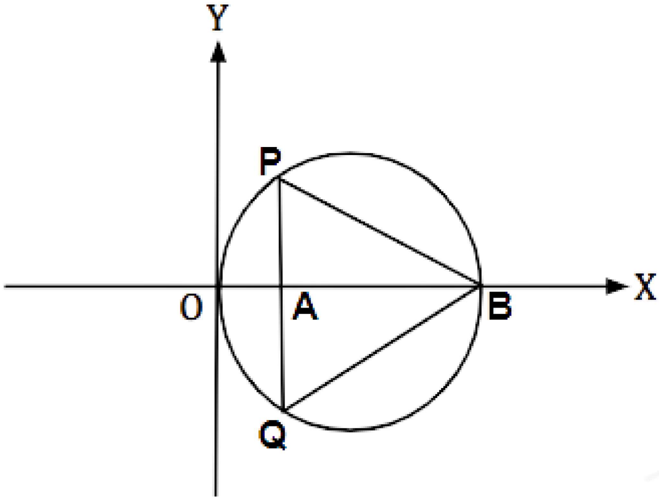
Equation of the line L in perpendicular form is
$x\cos\theta+y\sin\theta=p$

$\Rightarrow x\cos 75^\circ+y\sin 75^\circ=p$

$$\Rightarrow x\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+y\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=4$$

$\Rightarrow x(\sqrt{3}-1)+y(\sqrt{3}+1)=8\sqrt{2}$

16. In the circle given below, let $OA = 1$ unit, $OB = 13$ unit and $PQ \perp OB$. Then, the area of the triangle PQB (in square units) is :

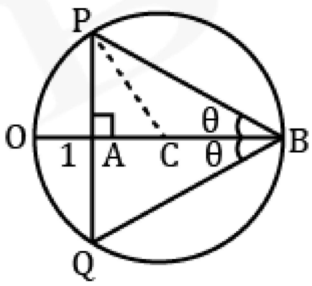


A. $26\sqrt{3}$

B. $24\sqrt{2}$

C. $24\sqrt{3}$

D. $26\sqrt{2}$



$OC = \dfrac{13}{2} = 6.5$
$AC = CO - AO$
$= 6.5 - 1 = 5.5$

In ΔPAC
$PA = \sqrt{6.5^2 - 5.5^2}$
$\Rightarrow PA = \sqrt{12}$
$\Rightarrow PQ = 2PA = 2\sqrt{12}$

Now, area of $\Delta PQB = \dfrac{1}{2} \times PQ \times AB$
$= \dfrac{1}{2} \times 2\sqrt{12} \times 12$
$= 12\sqrt{12}$
$= 24\sqrt{3}$ sq. units

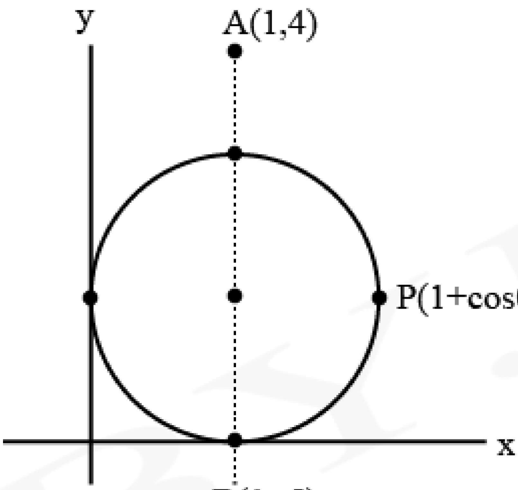
17. Let $A(1,4)$ and $B(1,-5)$ be two points. Let P be a point on the circle $(x-1)^2+(y-1)^2=1$ such that $(PA)^2+(PB)^2$ have maximum value, then the points P, A and B lie on:

A. a parabola

B. a straight line

C. a hyperbola

D. an ellipse



$\therefore PA^2=\cos^2\theta+(\sin\theta-3)^2=10-6\sin\theta$
$PB^2=\cos^2\theta+(\sin\theta+6)^2=37+12\sin\theta$
$PA^2+PB^2|_{\max}=47+6\sin\theta|_{\max}$
$\Rightarrow \theta=\frac{\pi}{2}$
$\therefore P, A, B$ lie on the line $x=1$.

18. Let r_1 and r_2 be the radii of the largest and smallest circles, respectively, which pass through the point $(-4, 1)$ and having their centres on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$. If $\frac{r_1}{r_2} = a + b\sqrt{2}$, then $a + b$ is equal to:

A. 3

B. 7

C. 11

D. 5

Given circle equation is $C \equiv (x+1)^2 + (y+2)^2 = 9$
Distance between $(-1, -2)$ and $(-4, 1)$ is $\sqrt{3^2 + 3^2} = \sqrt{18}$
Maximum radius of required circle $r_1 = \sqrt{18} + 3$
Minimum radius of required circle $r_2 = \sqrt{18} - 3$

$$\frac{r_1}{r_2} = \frac{3\sqrt{2}+3}{3\sqrt{2}-3} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\left(\sqrt{2}+1\right)^2}{1} = 3 + 2\sqrt{2}$$

19. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2+y^2+ax+2ay+c=0,\ (a<0)$ be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x+2y=0$, is equal to :

A. $\sqrt{10}$

B. $\sqrt{6}$

C. $\sqrt{11}$

D. $\sqrt{7}$

$$2\sqrt{\frac{a^2}{4}-c}=2\sqrt{2}$$
$$\Rightarrow \sqrt{a^2-4c}=2\sqrt{2}$$
$$\Rightarrow a^2-4c=8 \quad \ldots(1)$$

$$2\sqrt{a^2-c}=2\sqrt{5}$$
$$\Rightarrow a^2-c=5 \quad \ldots(2)$$
Solving (1) and (2),
$$3c=-3\Rightarrow c=-1$$
$$a^2=4\Rightarrow a=-2$$
So, the circle is $x^2+y^2-2x-4y-1=0$

Equation of tangent : $2x-y+\lambda=0$
$\therefore$ Perpendicular distance from centre to tangent $=$ radius
$$\left|\frac{2-2+\lambda}{\sqrt{5}}\right|=\sqrt{6}$$
$$\Rightarrow \lambda=\pm\sqrt{30}$$
$\therefore$ Tangents are $2x-y\pm\sqrt{30}=0$

Distance from origin $=\dfrac{\sqrt{30}}{\sqrt{5}}=\sqrt{6}$

20. Let the tangent to the circle $x^2+y^2=25$ at the point $R(3,\ 4)$ meet $x-$axis and $y-$axis at points P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to :

A. $\frac{625}{72}$

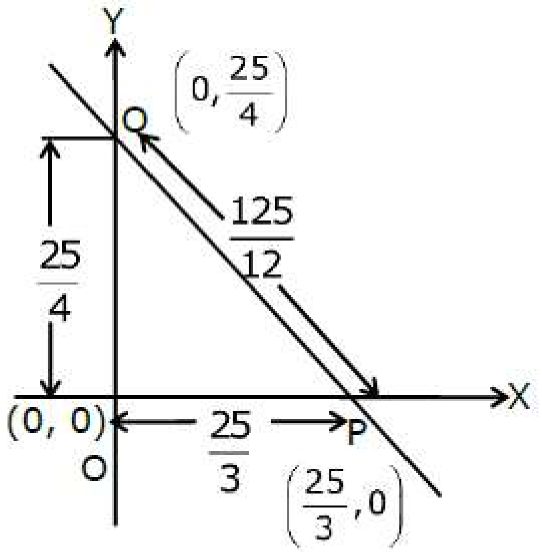
B. $\frac{585}{66}$

C. $\frac{125}{72}$

D. $\frac{529}{64}$

Given equation of circle : $x^2+y^2=25$

$\therefore$ Tangent equation at $(3,4)$, $T : 3x+4y=25$



Incentre of ΔOPQ

$$I=\left(\frac{\frac{25}{4}\times\frac{25}{3}}{\frac{25}{3}+\frac{25}{4}+\frac{125}{12}}, \frac{\frac{25}{3}\times\frac{25}{4}}{\frac{25}{3}+\frac{25}{4}+\frac{125}{12}}\right)$$

$$\therefore I=\left(\frac{625}{75+100+125}, \frac{625}{75+100+125}\right)=\left(\frac{25}{12}, \frac{25}{12}\right)$$

$\therefore$ Distance from origin to incentre is r.

$$\therefore r^2=\left(\frac{25}{12}\right)^2+\left(\frac{25}{12}\right)^2=\frac{625}{72}$$

Therefore, the correct answer is (1)

21. Choose the correct statement about two circles whose equations are given below:

$x^2 + y^2 - 10x - 10y + 41 = 0$
$x^2 + y^2 - 22x - 10y + 137 = 0$

A. circles have no meeting point

B. circles have two meeting point

C. circles have only one meeting point

D. circles have same centre

Let $S_1 : x^2 + y^2 - 10x - 10y + 41 = 0$
$\Rightarrow (x-5)^2 + (y-5)^2 = 9$
Centre $(C_1) = (5, 5)$
Radius $r_1 = 3$

$S_2 : x^2 + y^2 - 22x - 10y + 137 = 0$
$\Rightarrow (x-11)^2 + (y-5)^2 = 9$
Centre $(C_2) = (11, 5)$
Radius $r_2 = 3$

Distance $(C_1C_2) = \sqrt{(5-11)^2 + (5-5)^2}$
Distance $(C_1C_2) = 6$
$\because r_1 + r_2 = 3 + 3 = 6$
$\therefore$ Circles touch externally.
Hence, circle have only one meeting point.

22. Let $S_1 : x^2 + y^2 = 9$ and $S_2 : (x-2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points:

A. $\left(\frac{1}{2}, \pm\frac{\sqrt{5}}{2}\right)$

B. $\left(2, \pm\frac{3}{2}\right)$

C. $(1, \pm 2)$

D. $\left(0, \pm\sqrt{3}\right)$

$S_1 : x^2 + y^2 = 9$

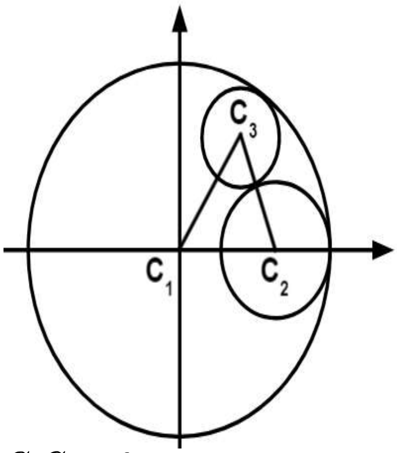
$S_2 : (x-2)^2 + y^2 = 1$

The centre and radius are

$C_1 : (0,0), r_1 = 3$

$C_2 : (2,0), r_2 = 1$

Let centre of variable circle be $C_3(h,k)$ and radius be r



$C_3C_1 = 3 - r$

$C_2C_3 = 1 + r$

$C_3C_1 + C_2C_3 = 4$

$\Rightarrow \sqrt{h^2+k^2} + \sqrt{(h-2)^2+k^2} = 4$

$\Rightarrow \sqrt{(h-2)^2+k^2} = 4 - \sqrt{h^2+k^2}$

$\Rightarrow (h-2)^2 + k^2 = 16 + h^2 + k^2 - 8\sqrt{h^2+k^2}$

$\Rightarrow -4h + 4 = 16 - 8\sqrt{h^2+k^2}$

$\Rightarrow h + 3 = 2\sqrt{h^2+k^2}$

$\Rightarrow h^2 + 6h + 9 = 4h^2 + 4k^2$

$\Rightarrow 3(h-1)^2 + 4k^2 = 12$

$\Rightarrow k = \pm\sqrt{3}\sqrt{1 - \left(\frac{h-1}{2}\right)^2}$

From the given options

$\left(2, \pm\frac{3}{2}\right)$ satisfies it.

23. Choose the incorrect statement about the two circles whose equations are given below:
$x^2+y^2-10x-10y+41=0$ and $x^2+y^2-16x-10y+80=0$

- ✗ **A.** Distance between two centres is the average of radii of both the circles.
- ✗ **B.** Circles have two intersection points.
- ✓ **C.** Both circles centres lie inside region of one another.
- ✗ **D.** Both circles pass through the centre of each other.

Given circles are
$S_1 : x^2+y^2-10x-10y+41=0$
$S_2 : x^2+y^2-16x-10y+80=0$
Center and radius of the circles
$C_1=(5,5),\ \ r_1=3$
$C_2=(8,5),\ \ r_2=3$

Distance between the centers is
$3=\dfrac{r_1+r_2}{2}$

$C_1C_2=3,\ r_1+r_2=6,\ |r_1-r_2|=0$
$\Rightarrow r_1+r_2>C_1C_2>|r_1-r_2|$
The circles intersect at two points.

Position of $C_1(5,5)$ in $S_2=0$
$25+25-80-50+80=0$
Position of $C_2(8,5)$ in $S_1=0$
$64+25-80-50+41=0$
Both circles pass through the centre of each other.

24. Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 2x^2 + 2y^2 - 2x - 2y = 1\}$,
$B = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 4x^2 + 4y^2 - 16y + 7 = 0\}$ and
$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 - 4x - 2y + 5 \leq r^2\}$.
Then the minimum value of $|r|$ such that $A \cup B \subseteq C$ is equal to:

A. $\dfrac{2+\sqrt{10}}{2}$

B. $\dfrac{3+2\sqrt{5}}{2}$

C. $1+\sqrt{5}$

D. $\dfrac{3+\sqrt{10}}{2}$

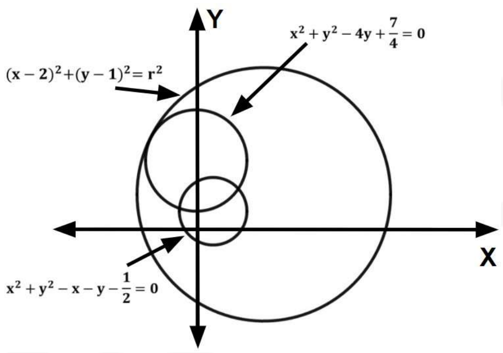
$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 2x^2 + 2y^2 - 2x - 2y = 1\}$,
Equation of circle is $x^2 + y^2 - x - y - \frac{1}{2} = 0$

Centre of circle, $C_1 \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$

and radius, $r_1 = 1$

$B = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 4x^2 + 4y^2 - 16y + 7 = 0\}$
Equation of circle is $x^2 + y^2 - 4y + \frac{7}{4} = 0$
Centre of circle, $C_2 \equiv (0, 2)$
and radius, $r_2 = \frac{3}{2}$

$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 - 4x - 2y + 5 \leq r^2\}$
Equation of circle is $(x - 2)^2 + (y - 1)^2 \leq r^2$
Centre of circle, $C_3 \equiv (2, 1)$
and radius, $r_3 = |r|$



From diagram it is clear that $C_3C_2 \leq \left|r - \frac{3}{2}\right|$

$\Rightarrow \left|r - \frac{3}{2}\right| \geq \sqrt{5}$

Case (1) :
$r - \frac{3}{2} \geq \sqrt{5}$

$\Rightarrow r \geq \frac{3 + 2\sqrt{5}}{2}$

Case (2) :
$r - \frac{3}{2} \leq -\sqrt{5}$

$\Rightarrow r \leq \frac{3 - 2\sqrt{5}}{2}$
Which is not possible.

$\therefore r_{\min} = \frac{3 + 2\sqrt{5}}{2}$

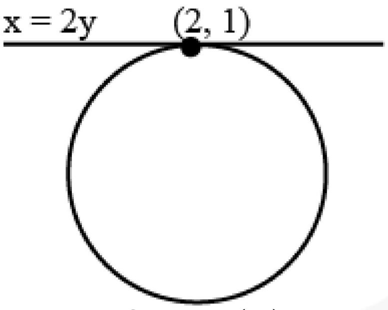
25. A circle C touches the line $x = 2y$ at the point $(2, 1)$ and intersects the circle $C_1 : x^2 + y^2 + 2y - 5 = 0$ at two points P and Q such that PQ is a diameter of C_1. Then the diameter of C is

A. $\sqrt{285}$

B. 15

C. $4\sqrt{15}$

D. $7\sqrt{5}$



Family of circles(C) touches the line $x = 2y$, at point $(2, 1)$
$C : (x-2)^2 + (y-1)^2 + \lambda(x - 2y) = 0 \cdots (i)$
$C_1 : x^2 + y^2 + 2y - 5 = 0$ has cemtre $(0, -1)$.

Common chord PQ is
$PQ : C - C_1 = 0$
$\Rightarrow PQ : (x-2)^2 + (y-1)^2 + \lambda(x-2y) - x^2 - y^2 - 2y + 5 = 0$
$\Rightarrow PQ : x(\lambda - 4) + y(-2\lambda - 4) + 10 = 0$
$\because \quad (0, -1)$ lies on PQ, then
$\Rightarrow 0 + (2\lambda + 4) + 10 = 0$
$\Rightarrow \lambda = -7$

Putting $\lambda = -7$ in equation (i)
$(x-2)^2 + (y-1)^2 - 7(x - 2y) = 0$
$\Rightarrow x^2 + y^2 - 11x + 12y + 5 = 0$

$$r = \sqrt{\frac{121}{4} + 36 - 5} = \sqrt{\frac{245}{4}} = \frac{\sqrt{245}}{2}$$

Diameter of C, $d = 2r = \sqrt{245} = 7\sqrt{5}$

26. If $y + 3x = 0$ is the equation of a chord of the circle, $x^2 + y^2 - 30x = 0$, then the equation of the circle with this chord as diameter is

- A. $x^2 + y^2 - 3x - 9y = 0$
- B. $x^2 + y^2 + 3x + 9y = 0$
- C. $x^2 + y^2 - 3x + 9y = 0$
- D. $x^2 + y^2 + 3x - 9y = 0$

$y + 3x = 0$

$x^2 + y^2 - 30x = 0$

$\Rightarrow x^2 + (-3x)^2 - 30x = 0$

$\Rightarrow 10x(x - 3) = 0$

$\Rightarrow x = 0, 3$

$\Rightarrow x = 0, y = 0 \quad x = 3, y = -9$

Equation of the circle whose end points of the diameter are $(0, 0)$ and $(3, -9)$ is

$(x - 3)(x - 0) + (y + 9)(y - 0) = 0$

$\Rightarrow x^2 + y^2 - 3x + 9y = 0$

Subject: Mathematics

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1. Let $\tan\alpha, \tan\beta, \tan\gamma; \alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}, n \in \mathbb{N}$ be the slopes of three line segment OA, OB and OC, respectively, where O is origin. If the circumcentre of $\triangle ABC$ coincides with origin and its orthocentre lies on $y-$axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$ is equal to

Accepted Answers

144 144.0 144.00

Solution:

As origin is circumcentre, so
$OA = OB = OC = R$
The coordinates of the vertices of the triangle are
$A = (R\sin\alpha, R\cos\alpha)$
$B = (R\sin\beta, R\cos\beta)$
$C = (R\sin\gamma, R\cos\gamma)$

Since orthocentre and circumcentre both lies on $y-$axis, so centroid also lies on $y-$axis
$R[\cos\alpha + \cos\beta + \cos\gamma] = 0$
$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0$
$\Rightarrow \cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\cos\beta\cos\gamma$
Now,
$$\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}$$
$$= \frac{4\left(\cos^3\alpha + \cos^3\beta + \cos^3\gamma\right) - 3\left(\cos\alpha + \cos\beta + \cos\gamma\right)}{\cos\alpha\cos\beta\cos\gamma}$$
$= 12$
$$\therefore \left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2 = 144$$

2. Let a point P be such that its distance from the point $(5,0)$ is thrice the distance of P from the point $(-5,0)$. If the locus of the point P is a circle of radius r, then $4r^2$ is equal to

Accepted Answers

56.25 56.250

Solution:

Let P be (h,k), $A(5,0)$ and $B(-5,0)$
Given $PA = 3PB$
$\Rightarrow PA^2 = 9PB^2$
$\Rightarrow (h-5)^2 + k^2 = 9\left[(h+5)^2 + k^2\right]$
$\Rightarrow 8h^2 + 8k^2 + 100h + 200 = 0$
$\therefore$ Locus of P is $x^2 + y^2 + \left(\frac{25}{2}\right)x + 25 = 0$

Centre $\equiv \left(-\frac{25}{4}, 0\right)$

$\therefore r^2 = \left(\frac{-25}{4}\right)^2 - 25$

$= \frac{625}{16} - 25 = \frac{225}{16}$

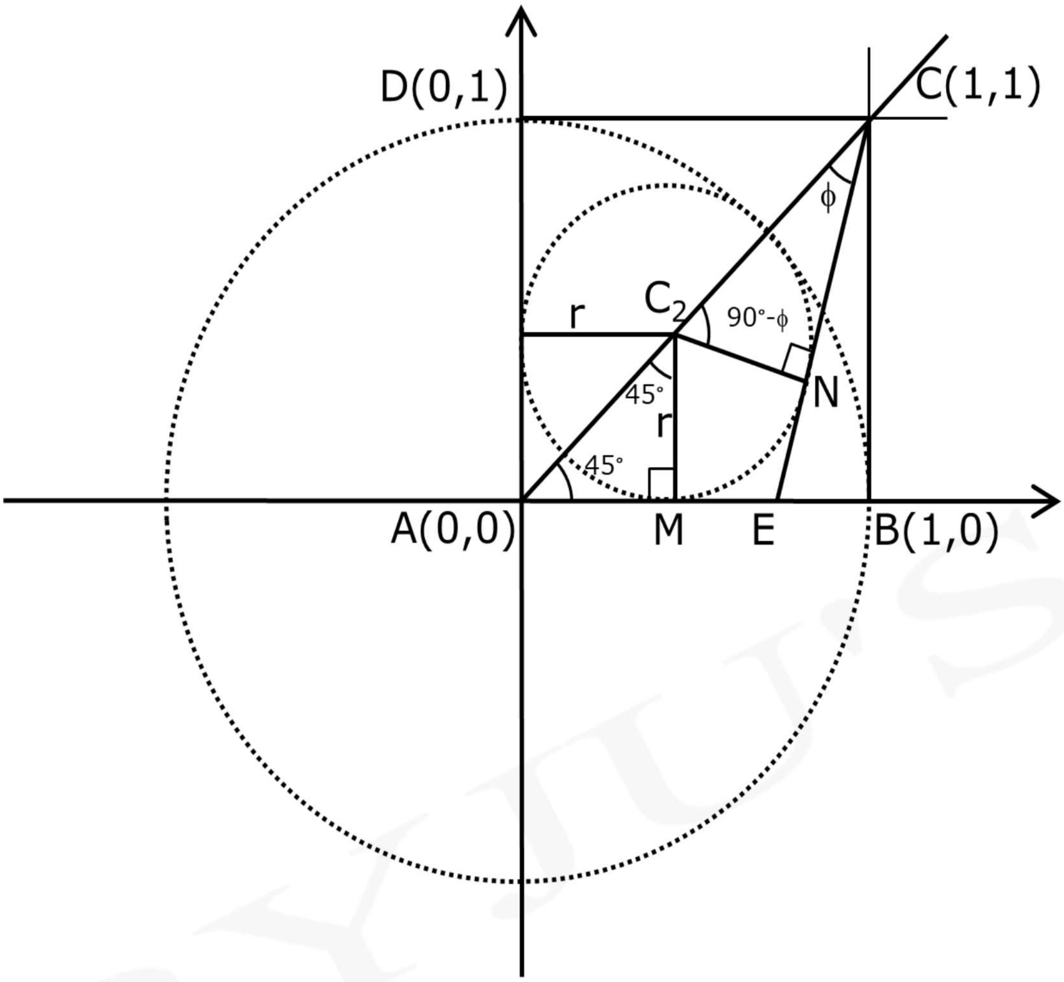
$\therefore 4r^2 = 4 \times \frac{225}{16} = \frac{225}{4} = 56.25$

3. Let $ABCD$ be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$ where α, β are integers, then $\alpha + \beta$ is equal to

Accepted Answers

1 1.0 1.00

Solution:



$(i)\ \sqrt{2}r + r = 1$

$r = \dfrac{1}{\sqrt{2}+1}$

$r = \sqrt{2} - 1$

$(ii)\ CC_2 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

From $\triangle CC_2N = \sin\phi = \dfrac{\sqrt{2}-1}{2(\sqrt{2}-1)}$

$\Rightarrow \phi = 30^\circ$

(iii) In $\triangle ACE$, from sine law,

$\dfrac{AE}{\sin\phi} = \dfrac{AC}{\sin 105^\circ}$

$\Rightarrow AE = \dfrac{1}{2} \times \dfrac{\sqrt{2}}{\sqrt{3}+1}.2\sqrt{2}$

$\Rightarrow AE = \dfrac{2}{\sqrt{3}+1} = \sqrt{3} - 1$

$\therefore EB = 1 - (\sqrt{3} - 1) = 2 - \sqrt{3}$

Hence $\alpha = 2, \beta = -1$

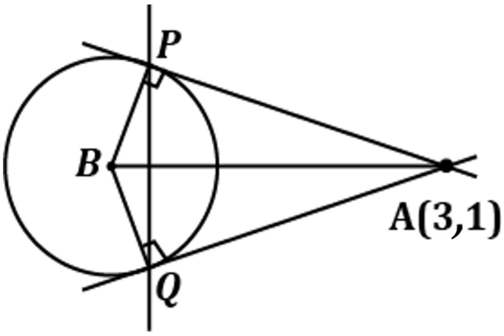
$\Rightarrow \alpha + \beta = 1$

4. Let B be the centre of the circle $x^2+y^2-2x+4y+1=0$. Let the tangents at two points P and Q on the circle intersect at the point $A(3,1)$. Then $8\cdot\left(\frac{\text{area}\triangle APQ}{\text{area}\triangle BPQ}\right)$ is equal to

Accepted Answers

18 18.0 18.00

Solution:



Let $L=$ length of tangent from A to the circle and $R=$ radius of circle
$\angle PAB=\angle BPQ=\theta$

$\Rightarrow$ area of $\triangle PAQ=2\cdot\frac{1}{2}\cdot L\sin\theta\cdot Lcos\theta=L^2\sin\theta\cos\ \theta$

area of $\triangle PBQ=2\cdot\frac{1}{2}\cdot R\sin\theta\cdot R\cos\theta=R^2\sin\theta\cos\theta$

Hence, $\frac{\text{area of }\triangle APQ}{\text{area of}\triangle BPQ}=\frac{L^2}{R^2}$

Now, $L=\sqrt{S_1}=\sqrt{3^2+1^2-2\times3+4\times1+1}=3$
and $R=2$

$\Rightarrow 8\times\left(\frac{\text{area of }\triangle APQ}{\text{area of}\triangle BPQ}\right)=8\times\left(\frac{3}{2}\right)^2=18$