

Mathematics JEE Solutions 2022

Mathematics

1. $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$, then (a, b, c) are the sides of $\triangle ABC$, then find $\frac{R}{r}$ where R is circumradius and r is inradius

- (1) $\frac{5}{2}$ (2) 3
(3) 1 (4) $\frac{1}{2}$

Sol. Answer (1)

$$\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9} = k \text{ (say)}$$

$$\Rightarrow a = 4k, b = 3k, c = 5k \Rightarrow s = 6k$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = 6k^2$$

$$r = \frac{\Delta}{s} = \frac{6k^2}{6k} = k \text{ and } R = \frac{abc}{4\Delta} = \frac{60k^3}{24k^2} = \frac{5}{2}k$$

$$\text{Here } \frac{R}{r} = \frac{5}{2}$$

2. Circle C touch the lines $L_1 = 4x - 3y + k_1 = 0$, $L_2 = 4x - 3y + k_2 = 0$, $k_1, k_2 \in R$. If a lines passing the centre of circle intersect at L_1 at $(-1, 2)$ and L_2 at $(3, -6)$ then equation of circle is
- (1) $x^2 + y^2 - 2x + 4y - 11 = 0$
(2) $x^2 + y^2 + 2x - 4y - 11 = 0$
(3) $x^2 + y^2 - 2x + 6y - 11 = 0$
(4) $x^2 + y^2 - 2x - 4y + 11 = 0$

Sol. Answer (1)

$$\because (-1, 2) \text{ lies on line } L_1 = 0$$

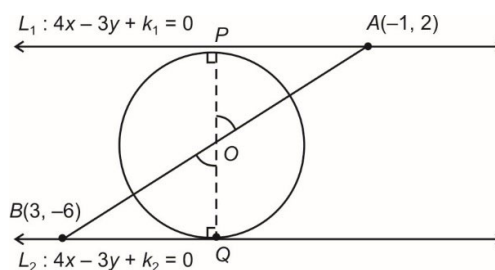
$$\therefore K_1 = 10$$

$$\text{and } (3, -6) \text{ lies on line } L_2 = 0$$

$$\therefore K_2 = -30$$

$$\text{Distance between lines } L_1 \text{ and } L_2$$

$$= \frac{|K_1 - K_2|}{\sqrt{4^2 + (-3)^2}} = 8$$



\therefore radius of circle = 4 units

Centre of circle = mid point of $AB = (1, -2)$

\therefore equation of circle is $(x-1)^2 + (y+2)^2 = 4^2$

$\therefore x^2 + y^2 - 2x + 4y - 11 = 0$

3. If $g : (0, \infty) \rightarrow R$ is differentiable function

$$\int \left[x \frac{\cos x - \sin x}{e^x + 1} + g(x) \frac{e^x + 1 - xe^x}{(e^x + 1)^2} \right] dx = \frac{xg(x)}{e^x + 1} + c$$

for all $x > 0$, here $c = \text{constant}$

(1) g is decreasing in $\left(0, \frac{\pi}{4}\right)$

(2) g is increasing in $\left(0, \frac{\pi}{4}\right)$

(3) $g + g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

(4) $g - g'$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

Sol. Answer (2)

$g : (0, \infty) \rightarrow R$ is differentiable

$$\int \left[x \frac{\cos x - \sin x}{e^x + 1} + g(x) \frac{e^x + 1 - xe^x}{(e^x + 1)^2} \right] dx = \frac{xg(x)}{e^x + 1} + c$$

Differentiating to x :

$$x \cdot \frac{\cos x - \sin x}{e^x + 1} + (x) \cdot \frac{e^x + 1 - xe^x}{(e^x + 1)^2} = \frac{(g(x) + xg'(x)(e^x + 1) - x(x)e^x)}{(e^x + 1)^2}$$

$$\Rightarrow x \cdot \frac{(\cos x - \sin x)}{e^x + 1} = \frac{xg'(x)}{(e^x + 1)}$$

$$\Rightarrow g'(x) = \cos x - \sin x$$

$$\Rightarrow g'(x) > 0 \quad \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow g(x) \text{ increase in } \left(0, \frac{\pi}{4}\right)$$

4. $f(x)$ be a polynomial function such that $f(x) +$

$$f'(x) + f''(x) = x^5 + 64, \text{ value of } \lim_{x \rightarrow 1} \frac{f(x)}{x-1}$$

(1) -15

(2) 15

(3) 60

(4) -60

Sol. Answer (1)

$$\text{Let } p(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

$$p'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

$$p''(x) = 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2$$

$$\Rightarrow p(x) + p'(x) + p''(x) = a_5x^5 + x^4(a_4 + 5a_5) + x^3(a_3 + 4a_4 + 20a_5) + x^2(a_2 + 3a_3 + 12a_4) + x(a_1 + 2a_2 + 6a_3) + (a_0 + a_1 + 2a_2)$$

But this is equal to $x^5 + 64$

$$\Rightarrow a_5 = 1 \Rightarrow a_4 = -5 \Rightarrow a_3 = 0 \Rightarrow a_2 = 60$$

$$\Rightarrow a_1 = -120 \text{ and } a_0 = 64$$

$$\Rightarrow p(x) = x^5 - 5x^4 + 60x^2 - 120x + 64$$

$$\lim_{x \rightarrow 1} \frac{p(x)}{x-1} = p'(1) = 5(1)^4 - 20(1) + 120 - 120 = -15$$

5. If $y = y(x)$ be the solution of given equation $y^2 dx + (x^2 - xy + y^2) dy = 0$ and this curve also passes through $(1, 1)$. Line intersect at $y = \sqrt{3}x$ at $(\alpha, \sqrt{3}\alpha)$ then find the value of

$$\log_e(\sqrt{3}x)$$

(1) $\frac{\pi}{2}$

(2) $\frac{\pi}{4}$

(3) $\frac{\pi}{6}$

(4) $\frac{\pi}{12}$

Sol. Answer (4)

$$\frac{dy}{dx} = \frac{-y^2}{x^2 + y^2 - xy}$$

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + \frac{xdv}{dx}$$

$$\Rightarrow v + \frac{xdv}{dx} = \frac{-v^2}{1+v^2-v}$$

$$\frac{xdv}{dx} = \frac{-v^2}{1+v^2-v} - v = \frac{-v^2 - v - v^3 + v^2}{1+v^2-v}$$

$$\Rightarrow \frac{v^2 - v + 1}{-v(v^2 + 1)} dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{-1}{v} + \frac{1}{v^2 + 1}\right) dv = \frac{dx}{x}$$

$$\Rightarrow -\log x + \tan^{-1} v = \log x + C$$

$$\Rightarrow -\log\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) = \log x + C$$

$$\text{Put } \begin{matrix} x=1 \\ y=1 \end{matrix} \Rightarrow \frac{\pi}{4} = C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{y}{x}\right) = \log x + \frac{\pi}{4}$$

$$\text{Put } y = \sqrt{3}x \Rightarrow \frac{\pi}{3} - \log\sqrt{3} = \log x + \frac{\pi}{4}$$

$$\Rightarrow \log x = \frac{\pi}{12} - \log\sqrt{3} \Rightarrow \log\sqrt{3}x = \frac{\pi}{12}$$

6. $f: R \rightarrow R, g: R \rightarrow R$ be two function defined

$$\text{by } f(x) = \log_e(x^2 + 1) - e^{-x} + 1 \text{ and } g(x) = \frac{1 - 2e^{2x}}{x}$$

range of α , if the inequality

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$$

(1) $\alpha \in (0, 2)$

(2) $\alpha \in (2, 3)$

(3) $\alpha \in (3, 4)$

(4) $\alpha \in R$

Sol. Answer (2)

$$f(x) = \ln(x^2 + 1) - e^{-x} + 1 \Rightarrow f'(x) = \frac{2x}{1+x^2} + e^{-x} > 0$$

$$g(x) = \frac{1 - e^{2x}}{e^x} = e^{-x} - e^x \Rightarrow g'(x) = -e^{-x} - e^{-x} < 0$$

$$\Rightarrow f \text{ is } \uparrow \text{ and } g(x) \text{ is } \downarrow$$

$$\Rightarrow f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$$

$$\Rightarrow g\left(\frac{(\alpha-1)^2}{3}\right) > g\left(\alpha - \frac{5}{3}\right)$$

$$\Rightarrow \frac{(\alpha-1)^2}{3} < \alpha - \frac{5}{3}$$

$$\Rightarrow a^2 - 5a + b < 0$$

$$\Rightarrow (\alpha-2)(\alpha-3) < 0$$

Here $\alpha \in (2, 3)$

7. If E_1 and E_2 are two conditional probability event such that $P\left(\frac{E_1}{E_2}\right) = \frac{1}{2}$, $P\left(\frac{E_2}{E_1}\right) = \frac{3}{4}$,

$$P(E_1 \cap E_2) = \frac{1}{8}, \text{ then}$$

$$(1) P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

$$(2) P(E_1 \cap E_2) = P(E_1') \cdot P(E_2)$$

$$(3) P(E_1' \cap E_2) = P(E_1) \cdot P(E_2)$$

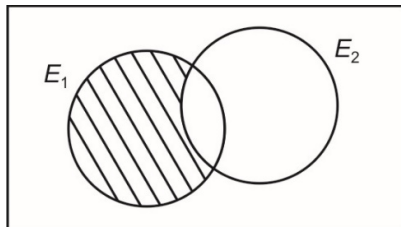
$$(4) P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Sol. Answer (4)

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} \Rightarrow P(E_2) = \frac{1}{4}$$

$$P\left(\frac{E_1}{E_2}\right) = \frac{1}{8}, P\left(\frac{E_2}{E_1}\right) = \frac{3}{4} \text{ and } P(E_1) = \frac{1}{6}$$

$$\Rightarrow P(E_1 \cup E_2) = \frac{7}{24} \text{ \{by inclusion-exclusion\}}$$



$$P(E_1 \cap E_2')$$

$$= P(E_1) - P(E_1 \cap E_2)$$

$$= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

$$\text{Thus } P(E_1 \cap E_2) = \frac{1}{4} \times \frac{1}{6} = P(E_1)P(E_2)$$

8. $\int_0^{\pi} \frac{\sin x \cdot e^{\cos x}}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$

$$(1) \frac{\pi}{2}$$

$$(2) \pi$$

$$(3) \frac{\pi}{4}$$

(4) None of these

Sol. Answer (3)

$$I = \int_0^{\pi} \frac{\sin x \cdot e^{\cos x}}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx \dots (1)$$

$$\text{Use } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi} \frac{\sin x \cdot e^{-\cos x}}{(1 + \cos^2 x)(e^{-\cos x} + e^{\cos x})} dx \dots (2)$$

$$(1) + (2) \Rightarrow 2I = \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

$$2I = -\left[\frac{-\pi}{4} - \frac{\pi}{4}\right] \Rightarrow I = \frac{\pi}{4}$$

9. $f(x) = x^3 + x - 5$

$$f(g(x)) = x, \text{ find } g'(63)$$

$$(1) \frac{1}{49} \quad (2) \frac{1}{48}$$

$$(3) \frac{1}{17} \quad (4) \frac{1}{16}$$

Sol. Answer (1)

$$f(x) = x^3 + x - 5, f(g(x)) = x$$

$$\Rightarrow g \text{ is } f^{-1} \Rightarrow g(f(x)) = x$$

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

$$f'(x) = 3x^2 + 1 \Rightarrow g'(f(x)) = \frac{1}{3x^2 + 1}$$

$$\text{Now put } x = 4 \Rightarrow f(4) = 4^3 + 4 - 5 = 63$$

$$\Rightarrow g'(63) = \frac{1}{3(4)^2 + 1} = \frac{1}{49}$$

10. $y = y(x)$ be the solution $\rightarrow (x+1)y' - y = e^{3x}$

$$(x+1)^2 y(0) = \frac{1}{3}, \text{ then } x = \frac{-4}{3} \text{ for the curve } y =$$

$y(x)$ is

$$(1) \frac{1}{9e^4} \quad (2) \frac{-1}{9e^4}$$

$$(3) \frac{e^4}{9} \quad (4) \frac{-e^4}{9}$$

Sol. Answer (2)

$$\frac{dy}{dx} - \left(\frac{1}{x+1}\right)y = e^{3x}(x+1)$$

$$\text{I.F. } e^{\int \frac{1}{(x+1)} dx} = \frac{1}{(x+1)}$$

$$\text{Solution is } y \cdot \frac{1}{(x+1)} = \int e^{3x} dx + c$$

$$\frac{y}{(x+1)} = \frac{e^{3x}}{3} + c, \quad y(0) = \frac{1}{3} \Rightarrow c = 0$$

$$\Rightarrow \frac{y}{(x+1)} = \frac{e^{3x}}{3} \Rightarrow y = \frac{(x+1) \cdot e^{3x}}{3}$$

$$y\left(-\frac{4}{3}\right) = \frac{-1}{9e^4}$$

11. If \vec{a} and \vec{b} are unit vector and acute angle between them is θ , then

$$(1) |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \tan\left(\frac{\theta}{2}\right)$$

$$(2) |\vec{a} - \vec{b}| = |\vec{a} + \vec{b}| \tan\frac{\theta}{2}$$

$$(3) |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

$$(4) |\vec{a} - \vec{b}| = |\vec{a} + \vec{b}| \tan\theta$$

Sol. Answer (2)

$$|\vec{a}| = |\vec{b}| = 1$$

$$\therefore |\vec{a} + \vec{b}|^2 = 1 + 1 + 2\cos\theta$$

$$\text{and } |\vec{a} - \vec{b}|^2 = 2 - 2\cos\theta$$

$$\therefore \frac{|\vec{a} - \vec{b}|}{|\vec{a} + \vec{b}|} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \tan\frac{\theta}{2}$$

12. $\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \dots$ 100 terms and S is the sum of this series, then find the values of $[s]$ where $[\cdot]$ is greatest integer function.

Sol. Answer (98)

$$S = \frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \dots 100 \text{ terms}$$

$$= \left(\frac{3-2}{3}\right) + \left(\frac{3^2-2^2}{3^2}\right) + \left(\frac{3^3-2^3}{3^3}\right) + \dots 100 \text{ terms}$$

$$= \left(1 - \frac{2}{3}\right) + \left(1 - \left(\frac{2}{3}\right)^2\right) + \left(1 - \left(\frac{2}{3}\right)^3\right) + \dots 100 \text{ terms}$$

$$= 100 - \left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots 100 \text{ terms}\right]$$

$$= 100 - \frac{2}{3} \left[1 - \left(\frac{2}{3}\right)^{100}\right]$$

$$= 100 - 2 \left[1 - \left(\frac{2}{3}\right)^{100}\right]$$

$$= 98 + 2 \times \left(\frac{2}{3}\right)^{100}$$

$$\Rightarrow [S] = 98$$

13. $a_n = 19^n - 12^n$. Find $\frac{(31a_9 - a_{10})}{57a_8}$

Sol. Answer (4)

$$\frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57(19^8 - 12^8)}$$

$$= \frac{19^9(31-19) - 12^9(31-12)}{57(19^8 - 12^8)}$$

$$= \frac{19^9 \times 12 - 12^9 \times 19}{57(19^8 - 12^8)}$$

$$= \frac{12 \times 19(19^8 - 12^8)}{57(19^8 - 12^8)} = 4$$

14. $f: N \rightarrow R$ be a function, $f(x+y) = 2f(x) \cdot f(y)$ for natural number x and y if $f(1) = 2$ and $\sum_{k=1}^{10} f(a+k) = \frac{512}{3}(2^{20} - 1)$, then the value of 'a' is _____

Sol. Answer (a = 4)

$$f(x+y) = 2f(x)f(y), \quad x, y \in N \quad \text{---- (i)}$$

and

$$f(1) = 2$$

$$f(2) = 2^3$$

$$f(3) = 2^5$$

$$\text{Now, } \sum_{k=1}^{10} f(a+k) = 2f(a) \sum_{k=1}^{10} f(k) = \frac{512}{3}(2^{20} - 1)$$

$$= 2f(a) \left[2 \frac{(2^{20} - 1)}{3}\right] = \frac{512}{3}(2^{20} - 1)$$

$$= f(a) = 128$$

$$\therefore \boxed{a = 4}$$

15. $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{k}{2^{10} \cdot 3^{10}}$, then find the remainder when k is divisible by 6.

Sol. Answer (5)

$$\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots + \frac{1}{2^{10} \cdot 3} = \frac{k}{2^{10} \cdot 3^{10}}$$

$$\Rightarrow \text{G.P. with } a = \frac{1}{2 \cdot 3^{10}}$$

$$= \frac{1}{2 \cdot 3^{10}} \left(\left(\frac{3}{2} \right)^{10} - 1 \right) = 2 \left(\frac{1}{2^{11}} - \frac{1}{2 \cdot 3^{10}} \right) = \left(\frac{3^{10} - 2^{10}}{2^{10} \cdot 3^{10}} \right)$$

$$\Rightarrow k = 3^{10} - 2^{10} = (3^5 + 2^5)(3^5 - 2^5) = (275) (211)$$

$\downarrow \qquad \qquad \downarrow$
 $(6k_1 + 5) \quad (6k_2 + 1)$

$$k = (6k_1 + 5)(6k_2 + 1) = 6(6k_1k_2 + k_1 + 5k_2) + 5$$

$$\Rightarrow \text{Hence remainder} = 5$$

16. $f(x) = x^3 + 3x^2 + 2x + 9$, find point of inflection

$$P(\alpha, \beta) \text{ and calculate } \frac{(\alpha^2 + \beta)}{5}$$

Sol. Answer (2)

$$f(x) = x^3 + 3x^2 + 2x + 9$$

$$f'(x) = 3x^2 + 6x + 2$$

$$f''(x) = 6x + 6 = 0 \Rightarrow x = -1 \Rightarrow \alpha = -1$$

$$\text{So } \beta = -1 + 3 - 2 + 9 = 9$$

$$\text{So } \frac{(\alpha^2 + \beta)}{5} = 2$$

17. $y = m_1x + c_1$, $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of $x^2 + y^2 = 2$ and parabola $y^2 = x$ then find the value of $|8m_1m_2|$

Sol. Answer (0.24)

$$\text{tangent to } y^2 = x \text{ is } y = mx + \frac{1}{4m} \quad \dots(1)$$

$$(1) \text{ is tangent to } x^2 + y^2 = 2$$

$$\text{then } c^2 = r^2(1 + m^2)$$

$$\frac{1}{16m^2} = 2(1 + m^2)$$

$$\Rightarrow m^4 + m^2 - \frac{1}{32} = 0$$

$$\Rightarrow \left(m^2 + \frac{1}{2} \right)^2 = \frac{1}{32} + \frac{1}{4} = \frac{9}{32}$$

$$m^2 + \frac{1}{2} = \pm \frac{3}{4\sqrt{2}} \Rightarrow m^2 = \frac{3}{4\sqrt{2}} - \frac{1}{2} - \frac{1}{2} \text{ only}$$

$$\therefore |8m_1m_2| = 8 \left| \frac{3}{4\sqrt{2}} - \frac{1}{2} \right|$$

$$= 8 \times \frac{|3 - 2\sqrt{2}|}{4\sqrt{2}} = \sqrt{2}(3 - 2\sqrt{2}) = 3\sqrt{2} - 4$$

18. If A is 3×3 Matrix having the elements from the set $\{-1, 0, 1\}$. Find the number of matrices A for which the sum of the elements is 5.

Sol. Answer (414)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \{-1, 0, 1\}$

$$\text{Now } a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3 = 5$$

Possible cases:

(i) five 1, No -1, four 0

$$\Rightarrow \frac{9!}{5!4!} \text{ matrices}$$

(ii) six 1, one -1, two 0

$$\Rightarrow \frac{9!}{6!2!} \text{ matrices}$$

(iii) seven 1, two -1, No 0

$$\text{So ANS} = \frac{9!}{5!4!} + \frac{9!}{6!2!} + \frac{9!}{7!2!} = 126 + 252 + 36 = 414$$

19. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and θ lies between $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{4}$

$$\text{the find } \left| (\vec{b} - \vec{a}) \times (\vec{b} + \vec{a}) \right|^2 + 4(\vec{a} \cdot \vec{b})^2$$

Sol. Answer (576)

$$|\vec{a}| = 3, |\vec{b}| = 4$$

$$(\vec{b} - \vec{a}) \times (\vec{b} + \vec{a}) = 2(\vec{b} \times \vec{a})$$

$$\left| (\vec{b} - \vec{a}) \times (\vec{b} + \vec{a}) \right|^2 + 4(\vec{a} \cdot \vec{b})^2$$

$$= 4(\vec{b} \times \vec{a})^2 + 4(\vec{a} \cdot \vec{b})^2$$

$$= 4 \left[|\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) \right]$$

$$= 4 \times 9 \times 16 = 576$$

20. Consider:

$$P_1 : \sim(P \rightarrow \sim q)$$

$$P_2 : (P \wedge \sim q) \wedge ((\sim P) \vee q)$$

If $P \rightarrow ((\sim P) \vee q)$ is false then

(1) P_1 is true, P_2 is false

-
- (2) P_1 is false, P_2 is true
(3) Both P_1 and P_2 are true
(4) Both P_1 and P_2 are false

Sol. Answer (4)

$$P_1 : \sim(P \rightarrow \sim q)$$

$$P_2 : (P \wedge \sim q) \wedge ((\sim P) \vee q)$$

If $P \rightarrow ((\sim P) \vee q)$ is false

then P is true and

$\sim P \vee q$ is false

$\therefore q$ is false

$\therefore P_1$ is false and P_2 is also false

