MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$. Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

Answer (B)

Sol. Given $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$

So, $(x^2 + 1) * 1 = x * 2$

$\Rightarrow (x^2 + 1)^2 + 1 = x^2 + 8$

$\Rightarrow x^4 + 2x^2 + 2 = x^2 + 8$

$\Rightarrow (x^2)^2 + x^2 - 6 = 0$

$\therefore (x^2 + 3)(x^2 - 2) = 0$

$\therefore \boxed{x^2 = 2}$

Now, $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right) = 2\sin^{-1}\left(\frac{4}{8}\right)$

$= 2\cdot\frac{\pi}{6} = \frac{\pi}{3}$

2. The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is

(A) $\log_e 3$ (B) $-\log_e 3$

(C) $\log_e 6$ (D) $-\log_e 6$

Answer (B)

Sol. Given equation : $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$

$\Rightarrow e^{2x} - 4 = 0$ or $6e^{2x} - 5e^x + 1 = 0$

$\Rightarrow e^{2x} = 4$ or $6(e^x)^2 - 3e^x - 2e^x + 1 = 0$

$\Rightarrow 2x = \ln 4$ or $(3e^x - 1)(2e^x - 1) = 0$

$\Rightarrow \boxed{x = \ln 2}$ or $e^x = \frac{1}{3}$ or $e^x = \frac{1}{2}$

or $x = \ln\left(\frac{1}{3}\right), -\ln 2$

Sum of all real roots = ln2 – ln3 – ln2

= –ln3

3. Let the system of linear equations

$x + y + az = 2$

$3x + y + z = 4$

$x + 2z = 1$

have a unique solution (x^*, y^*, z^*). If (α, x^*), (y^*, α) and $(x^*, -y^*)$ are collinear points, then the sum of absolute values of all possible values of α is

(A) 4 (B) 3

(C) 2 (D) 1

Answer (C)

Sol. Given system of equations

$x + y + az = 2$ …(i)

$3x + y + z = 4$ …(ii)

$x + 2z = 1$ …(iii)

Solving (i), (ii) and (iii), we get

$x = 1, y = 1, z = 0$ (and for unique solution $a \neq -3$)

Now, $(\alpha, 1)$, $(1, \alpha)$ and $(1, -1)$ are collinear

$\therefore \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow \alpha(\alpha + 1) - 1(0) + 1(-1 - \alpha) = 0$

$\Rightarrow \alpha^2 - 1 = 0$

$\therefore \alpha = \pm 1$

$\therefore$ Sum of absolute values of $\alpha = 1 + 1 = 2$

4. Let $x, y > 0$. If $x^3y^2 = 2^{15}$, then the least value of $3x + 2y$ is

(A) 30 (B) 32

(C) 36 (D) 40

Answer (D)

Sol. $x, y > 0$ and $x^3y^2 = 2^{15}$

Now, $3x + 2y = (x + x + x) + (y + y)$

So, by A.M $\geq$ G.M inequality

$\frac{3x+2y}{5} \geq \sqrt[5]{x^3 \cdot y^2}$

$\therefore 3x + 2y \geq 5\sqrt[5]{2^{15}}$

≥ 40

$\therefore$ Least value of $3x + 4y = 40$

5. Let $f(x)=\begin{cases}\dfrac{\sin(x-[x])}{x-[x]}, & x\in(-2,-1)\\ \max\{2x,\ 3[|x|]\}, & |x|<1\\ 1, & \text{otherwise}\end{cases}$

Where $[t]$ denotes greatest integer $\le t$. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is

(A) (3, 3) (B) (2, 4)

(C) (2, 3) (D) (3, 4)

Answer (C)

Sol. $f(x)=\begin{cases}\dfrac{\sin(x-[x])}{x[x]} & , \quad x\in(-2,-1)\\ \max\{2x, 3[|x|]\} & , \quad |x|<1\\ 1 & , \quad \text{otherwise}\end{cases}$

$$f(x)=\begin{cases}\dfrac{\sin(x+2)}{x+2} & , \quad x\in(-2,-1)\\ 0 & , \quad x\in(-1,\ 0]\\ 2x & , \quad x\in(0,\ 1)\\ 1 & , \quad \text{othersiwe}\end{cases}$$

It clearly shows that $f(x)$ is discontinuous

At $x = -1$, 1 also non differentiable

and at $x = 0$, L.H.D = $\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}=0$

R.H.D = $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=2$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

$\therefore$ $m = 2, n = 3$

6. The value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\left(1+e^{x}\right)\left(\sin^6 x+\cos^6 x\right)}$$ is equal to

(A) 2π (B) 0

(C) π (D) $\dfrac{\pi}{2}$

Answer (C)

Sol. $I=\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\left(1+e^{x}\right)\left(\sin^6 x+\cos^6 x\right)}$ ...(i)

$I=\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\left(1+e^{-x}\right)\left(\sin^6 x+\cos^6 x\right)}$ ...(ii)

(i) and (ii)

From equation (i) & (ii)

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\sin^6 x+\cos^6 x}$$

$$\Rightarrow \quad I=\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^6 x+\cos^6 x}=\int_0^{\frac{\pi}{2}}\frac{dx}{1-\frac{3}{4}\sin^2 2x}$$

$$\Rightarrow \quad I=\int_0^{\frac{\pi}{2}}\frac{4\sec^2 2x dx}{4+\tan^2 2x}=2\int_0^{\frac{\pi}{4}}\frac{4\sec^2 2x}{4+\tan^2 2x}dx$$

Now, $\tan 2x = t$ when $x = 0$, $t = 0$; when, $x=\dfrac{\pi}{4}$, $t\to\infty$

$2\sec^2 2x\,dx = dt$

$$\therefore \quad I=2\int_0^{\infty}\frac{2dt}{4+t^2}=2\left(\tan^{-1}\frac{t}{2}\right)_0^{\infty}$$

$$=2\frac{\pi}{2}=\pi$$

7. $\lim\limits_{n\to\infty}\left(\dfrac{n^2}{\left(n^2+1\right)(n+1)}+\dfrac{n^2}{\left(n^2+4\right)(n+2)}+\dfrac{n^2}{\left(n^2+9\right)(n+3)}+\ldots+\dfrac{n^2}{\left(n^2+n^2\right)(n+n)}\right)$

is equal to

(A) $\dfrac{\pi}{8}+\dfrac{1}{4}\log_e 2$ (B) $\dfrac{\pi}{4}+\dfrac{1}{8}\log_e 2$

(C) $\dfrac{\pi}{4}-\dfrac{1}{8}\log_e 2$ (D) $\dfrac{\pi}{8}+\dfrac{1}{8}\log_e\sqrt{2}$

Answer (A)

Sol. $\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)(n+1)}+\frac{n^2}{(n^2+4)(n+2)}+....+\frac{n^2}{(n^2+n^2)(n+n)}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)(n+r)}$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]\left[1+\left(\frac{r}{n}\right)\right]}$

$= \int_0^1\frac{1}{(1+x^2)(1+x)}dx$

$= \frac{1}{2}\int_0^1\left[\frac{1}{1+x}-\frac{(x-1)}{(1+x^2)}\right]dx$

$= \frac{1}{2}\left[\ln(1+x)-\frac{1}{2}\ln(1+x^2)+\tan^{-1}x\right]_0^1$

$= \frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{2}\ln 2\right]=\frac{\pi}{8}+\frac{1}{4}\ln 2$

8. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ, then C is a parabola with

(A) Length of latus rectum 3

(B) Length of latus rectum 6

(C) Focus$\left(\frac{4}{3},0\right)$

(D) Focus$\left(0,\frac{3}{4}\right)$

Answer (A)

Sol. According to the question $\left(\text{Let } P(x,y)\right)$

$2x-y\frac{dx}{dy}=0$ $\left(\because \text{ equation of tangent at } P: y-y=\frac{dy}{dx}(y-x)\right)$

$\therefore\quad 2\frac{dy}{y}=\frac{dx}{x}$

$\Rightarrow\quad 2\ln y=\ln x+\ln c$

$\Rightarrow\quad y^2=cx$ $\quad\because$ this curve passes through (3, 3) $\therefore$ $\boxed{c=3}$ $\quad\therefore$ required parabola $y^2 = 3x$ and L.R = 3

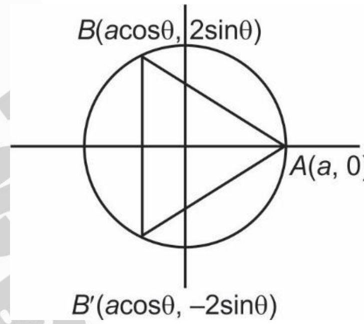
9. Let the maximum area of the triangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1,\ a>2,$ having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be $6\sqrt{3}$. Then the eccentricity of the ellipse is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\frac{\sqrt{3}}{4}$

Answer (A)

Sol. Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1,\ a>2$



$\therefore$ Let $A(\theta)$ be the area of $\Delta ABB'$

Then $A(\theta)=\frac{1}{2}4\sin\theta(a+a\cos\theta)$

$A'(\theta)=a\left(2\cos\theta+2\cos^2\theta\right)$

For maxima $A'(\theta) = 0$

$\Rightarrow\quad \cos\theta=-1,\ \cos\theta=\frac{1}{2}$

But for maximum area $\cos\theta=\frac{1}{2}$

$\therefore\quad A(\theta)=6\sqrt{3}$

$\Rightarrow\quad 2\frac{\sqrt{3}}{2}\left(a+\frac{a}{2}\right)=6\sqrt{3}$

$\Rightarrow\quad a = 4$

$\therefore\quad e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{4}{16}}=\frac{\sqrt{3}}{2}$

10. Let the area of the triangle with vertices $A(1, \alpha)$, $B(\alpha, 0)$ and $C(0, \alpha)$ be 4 sq. units. If the points $(\alpha, -\alpha)$, $(-\alpha, \alpha)$ and (α^2, β) are collinear, then β is equal to

(A) 64 (B) –8

(C) –64 (D) 512

Answer (C)

Sol. $\because$ $A(1, \alpha)$, $B(\alpha, 0)$ and $C(0, \alpha)$ are the vertices of ΔABC and area of $\Delta ABC = 4$

$$\therefore \left|\frac{1}{2}\begin{vmatrix} 1 & \alpha & 1 \\ \alpha & 0 & 1 \\ 0 & \alpha & 1 \end{vmatrix}\right| = 4$$

$$\Rightarrow \left|1(1-\alpha)-\alpha(\alpha)+\alpha^2\right| = 8$$

$$\Rightarrow \alpha = \pm 8$$

Now, $(\alpha, -\alpha)$, $(-\alpha, \alpha)$ and (α^2, β) are collinear

$$\therefore \begin{vmatrix} 8 & -8 & 1 \\ -8 & 8 & 1 \\ 64 & \beta & 1 \end{vmatrix} = 0 = \begin{vmatrix} -8 & 8 & 1 \\ 8 & -8 & 1 \\ 64 & \beta & 1 \end{vmatrix}$$

$$\Rightarrow 8(8-\beta)+8(-8-64)+1(-8\beta-8\times 64)=0$$

$$\Rightarrow 8-\beta-72-\beta-64=0$$

$$\Rightarrow \beta = -64$$

11. The number of distinct real roots of the equation $x^7 - 7x - 2 = 0$ is

(A) 5 (B) 7

(C) 1 (D) 3

Answer (D)

Sol. Given equation $x^7 - 7x - 2 = 0$

Let $f(x) = x^7 - 7x - 2$

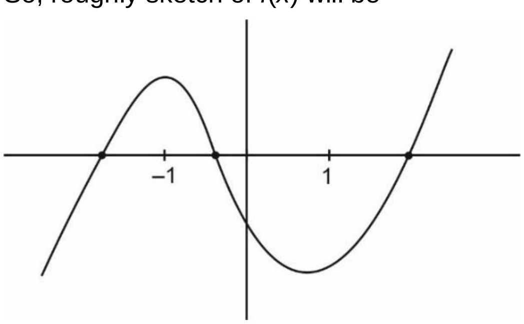
$f'(x) = 7x^6 - 7 = 7(x^6 - 1)$

and $f'(x) = 0 \Rightarrow x = +1$

and $f(-1) = -1 + 7 - 2 = 5 > 0$

$f(1) = 1 - 7 - 2 = -8 < 0$

So, roughly sketch of $f(x)$ will be



So, number of real roots of $f(x) = 0$ and 3

12. A random variable X has the following probability distribution :

X	0	1	2	3	4
$P(X)$	k	$2k$	$4k$	$6k$	$8k$

The value of $P(1 < X < 4 \mid x \le 2)$ is equal to

(A) $\frac{4}{7}$ (B) $\frac{2}{3}$

(C) $\frac{3}{7}$ (D) $\frac{4}{5}$

Answer (A)

Sol. $\because$ x is a random variable

$\therefore$ $k + 2k + 4k + 6k + 8k = 1$

$\therefore$ $k = \frac{1}{21}$

Now, $P(1 < x < 4 \mid x \le 2) = \frac{4k}{7k} = \frac{4}{7}$

13. The number of solutions of the equation $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right) = \frac{1}{4}\cos^2 2x$, $x \in [-3\pi, 3\pi]$ is:

(A) 8 (B) 5

(C) 6 (D) 7

Answer (D)

Sol. $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right) = \frac{1}{4}\cos^2 2x$, $x \in [-3\pi, 3\pi]$

$$\Rightarrow \cos 2x + \cos\frac{2\pi}{3} = \frac{1}{2}\cos^2 2x$$

$$\Rightarrow \cos^2 2x - 2\cos 2x - 1 = 0$$

$\Rightarrow$ $\cos 2x = 1$

$\therefore$ $x = -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$

$\therefore$ Number of solutions = 7

14. If the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{\lambda}$ and $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-5}{5}$ is $\frac{1}{\sqrt{3}}$, then the sum of all possible values of λ is :

(A) 16

(B) 6

(C) 12

(D) 15

Answer (A)

Sol. Let $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{p} = 2\hat{i} + 3\hat{j} + \lambda\hat{k},\ \vec{q} = \hat{i} + 4\hat{j} + 5\hat{k}$

$\therefore\ \vec{p} \times \vec{q} = (15 - 4\lambda)\hat{i} - (10 - \lambda)\hat{j} + 5\hat{k}$

$\vec{a}_2 - \vec{a}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore$ Shortest distance

$$= \left|\frac{(15-4\lambda) - 2(10-\lambda) + 10}{\sqrt{(15-4\lambda)^2 + (10-\lambda)^2 + 25}}\right| = \frac{1}{\sqrt{3}}$$

$\Rightarrow\ 3(5-2\lambda)^2 = (15-4\lambda)^2 + (10-\lambda)^2 + 25$

$\Rightarrow\ 5\lambda^2 - 80\lambda + 275 = 0$

$\therefore$ Sum of values of $\lambda = \frac{80}{5} = 16$

15. Let the points on the plane P be equidistant from the points (–4, 2, 1) and (2, –2, 3). Then the acute angle between the plane P and the plane $2x + y + 3z = 1$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{12}$

Answer (C)

Sol. Let $P(x, y, z)$ be any point on plane P_1

Then $(x+4)^2 + (y-2)^2 + (z-1)^2 = (x-2)^2 + (y+2)^2 + (z-3)^2$

$\Rightarrow\ 12x - 8y + 4z + 4 = 0$

$\Rightarrow\ 3x - 2y + z + 1 = 0$

And $P_2 : 2x + y + 3z = 1$

$\therefore$ angle between P_1 and P_2

$\cos\theta \left|\frac{6-2+3}{14}\right| \Rightarrow \theta = \frac{\pi}{3}$

16. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that $\left|(\hat{a}+\hat{b}) + 2(\hat{a}\times\hat{b})\right| = 2$. If $\theta \in (0, \pi)$ is the angle between $\hat{a}$ and $\hat{b}$, then among the statements:

(S1) : $2\left|\hat{a}\times\hat{b}\right| = \left|\hat{a}-\hat{b}\right|$

(S2) : The projection of $\hat{a}$ on $(\hat{a} + \hat{b})$ is $\frac{1}{2}$

(A) Only (S1) is true

(B) Only (S2) is true

(C) Both (S1) and (S2) are true

(D) Both (S1) and (S2) are false

Answer (C)

Sol. $\because\ \left|\hat{a}+\hat{b}+2(\hat{a}\times\hat{b})\right| = 2,\ \theta \in (0, \pi)$

$\Rightarrow\ |\hat{a}+\hat{b}+2(\hat{a}\times\hat{b})|^2 = 4$.

$\Rightarrow\ |\hat{a}|^2 + |\hat{b}|^2 + 4|\hat{a}\times\hat{b}|^2 + 2\hat{a}\cdot\hat{b} = 4.$

$\therefore\ \cos\theta = \cos2\theta$

$\therefore\ \theta = \frac{2\pi}{3}$

where θ is angle between $\hat{a}$ and $\hat{b}$.

$\therefore\ 2|\hat{a}\times\hat{b}| = \sqrt{3} = |\hat{a}-\hat{b}|$

(S1) is correct

And projection of $\hat{a}$ on $(\hat{a}+\hat{b}) = \left|\frac{\hat{a}\cdot(\hat{a}+\hat{b})}{|\hat{a}+\hat{b}|}\right| = \frac{1}{2}$.

(S2) is correct.

17. If $y = \tan^{-1}\left(\sec x^3 - \tan x^3\right), \frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then

(A) $xy'' + 2y' = 0$ (B) $x^2y'' - 6y + \frac{3\pi}{2} = 0$

(C) $x^2y'' - 6y + 3\pi = 0$ (D) $xy'' - 4y' = 0$

Answer (B)

Sol. Let $x^3 = \theta \Rightarrow \frac{\theta}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\therefore\ y = \tan^{-1}(\sec\theta - \tan\theta)$

$= \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$

$\therefore\ y = \frac{\pi}{4} - \frac{\theta}{2}.$

$y = \frac{\pi}{4} - \frac{x^3}{2}$

$\therefore\ y' = \frac{-3x^2}{2}$

$y'' = -3x$

$\therefore\ x^2y'' - 6y + \frac{3\pi}{2} = 0$.

18. Consider the following statements:

A : Rishi is a judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement “if Rishi is a judge and he is not arrogant, then he is honest” is

(A) $B \rightarrow (A \vee C)$ (B) $(\sim B) \wedge (A \wedge C)$

(C) $B \rightarrow ((\sim A) \vee (\sim C))$ (D) $B \rightarrow (A \wedge C)$

Answer (B)

Sol. $\because$ given statement is

$(A \wedge C) \to B$

Then its negation is

$\sim \{(A \wedge C) \to B\}$

or $\sim \{\sim (A \wedge C) \vee B\}$

$\therefore$ $(A \wedge C) \wedge (\sim B)$

or $(\sim B) \wedge (A \wedge C)$

19. The slope of normal at any point (x, y), $x > 0$, $y > 0$ on the curve $y = y(x)$ is given by $\dfrac{x^2}{xy - x^2y^2 - 1}$. If the curve passes through the point (1, 1), then $e \cdot y(e)$ is equal to

(A) $\dfrac{1-\tan(1)}{1+\tan(1)}$ (B) $\tan(1)$

(C) 1 (D) $\dfrac{1+\tan(1)}{1-\tan(1)}$

Answer (D)

Sol. $\because$ $-\dfrac{dx}{dy} = \dfrac{x^2}{xy - x^2y^2 - 1}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{x^2y^2 - xy + 1}{x^2}$

Let $xy = v \Rightarrow y + x\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore$ $\dfrac{dv}{dx} - y = \dfrac{(v^2 - v + 1)y}{v}$

$\therefore$ $\dfrac{dv}{dx} = \dfrac{v^2 + 1}{x}$

$\because$ $y(1) = 1 \Rightarrow \tan^{-1}(xy) = \ln x + \tan^{-1}(1)$

Put $x = e$ and $y = y(e)$ we get

$\tan^{-1}(e \cdot y(e)) = 1 + \tan^{-1} 1.$

$\tan^{-1}(e \cdot y(e)) - \tan^{-1} 1 = 1$

$\therefore$ $e(y(e)) = \dfrac{1+\tan(1)}{1-\tan(1)}$

20. Let λ^* be the largest value of λ for which the function $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ is increasing for all $x \in \mathbb{R}$. Then $f_{\lambda^*}(1) + f_{\lambda^*}(-1)$ is equal to :

(A) 36 (B) 48

(C) 64 (D) 72

Answer (D)

Sol. $\because$ $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$

$\therefore$ $f'_\lambda(x) = 12(\lambda x^2 - 6\lambda x + 3)$

For $f_\lambda(x)$ increasing : $(6\lambda)^2 - 12\lambda \le 0$

$\therefore$ $\lambda \in \left[0, \dfrac{1}{3}\right]$

$\therefore$ $\lambda^* = \dfrac{1}{3}$

Now, $f^*_\lambda(x) = \dfrac{4}{3}x^3 - 12x^2 + 36x + 48$

$\therefore$ $f^*_\lambda(1) + f^*_\lambda(-1) = 73\dfrac{1}{2} - 1\dfrac{1}{2}$

$= 72.$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse andw the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $S = \{z \in \mathbb{C} : |z - 3| \le 1$ and $z(4 + 3i) + \bar{z}(4 - 3i) \le 24\}$. If $\alpha + i\beta$ is the point in S which is closest to $4i$, then $25(\alpha + \beta)$ is equal to ______.

Answer (80)

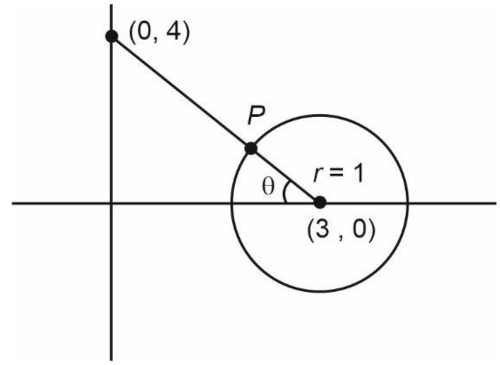
Sol. Here $|z - 3| < 1$

$\Rightarrow$ $(x - 3)^2 + y^2 < 1$

and $z = (4 + 3i) + \bar{z}(4 - 3i) \le 24$

$\Rightarrow$ $4x - 3y \le 12$

$\tan\theta = \dfrac{4}{3}$



$\therefore$ Coordinate of $P = (3 - \cos\theta, \sin\theta)$

$$= \left(3 - \frac{3}{5}, \frac{4}{5}\right)$$

$\therefore$ $\alpha + i\beta = \frac{12}{5} + \frac{4}{5}i$

$\therefore$ $25(\alpha + \beta) = 80$

2. Let $S = \left\{\begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}; a, \ b \in \{1, 2, 3, ... 100\}\right\}$ and let T_n $= \{A \in S : A^{n(n+1)} = I\}$. Then the number of elements in $\bigcap\limits_{n=1}^{100} T_n$ is _____.

Answer (100)

Sol. $S = \left\{\begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix} : a, b \in \{1, 2, 3, ..., 100\}\right\}$

$\because$ $A = \begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}$ then even powers of

A as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, if $b = 1$ and $a \in \{1,....., 100\}$

Here, $n(n + 1)$ is always even.

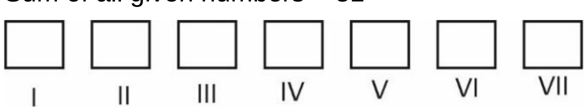
$\therefore$ $T_1, T_2, T_3, ..., T_n$ are all I for $b = 1$ and each value of a.

$\therefore$ $\bigcap\limits_{n=1}^{100} T_n = 100$

3. The number of 7-digit numbers which are multiples of 11 and are formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is _____.

Answer (576)

Sol. Sum of all given numbers = 31



Difference between odd and even positions must be 0, 11 or 22, but 0 and 22 are not possible.

$\therefore$ Only difference 11 is possible

This is possible only when either 1, 2, 3, 4 is filled in odd position in some order and remaining in other order. Similar arrangements of 2, 3, 5 or 7, 2, 1 or 4, 5, 1 at even positions.

$\therefore$ Total possible arrangements = $(4! \times 3!) \times 4$

$= 576$

4. The sum of all the elements of the set $\{\alpha \in \{1, 2, ..., 100\} : \text{HCF}(\alpha, 24) = 1\}$ is

Answer (1633)

Sol. The numbers upto 24 which gives g.c.d. with 24 equals to 1 are 1, 5, 7, 11, 13, 17, 19 and 23.

Sum of these numbers = 96

There are four such blocks and a number 97 is there upto 100.

$\therefore$ Complete sum

$= 96 + (24 \times 8 + 96) + (48 \times 8 + 96) + (72 \times 8 + 96) + 97$

$= 1633$

5. The remainder on dividing $1 + 3 + 3^2 + 3^3 + ... + 3^{2021}$ by 50 _____ is

Answer (4)

Sol. $1 + 3 + 3^2 + ... + 3^{2021} = \frac{3^{2022} - 1}{2}$

$= \frac{1}{2}\left\{(10-1)^{1011} - 1\right\}$

$= \frac{1}{2}\{100k + 10110 - 1 \ - 1\}$

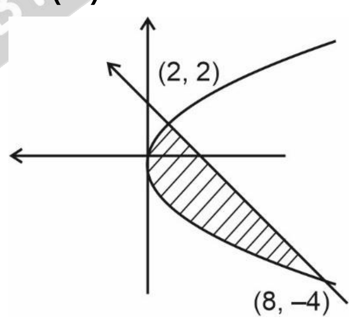
$= 50\,k_1 + 4$

$\therefore$ Remainder = 4

6. The area (in sq. units) of the region enclosed between the parabola $y^2 = 2x$ and the line $x + y = 4$ is _______.

Answer (18)

Sol.



The required area = $\int_{-4}^{2}\left(4 - y - \frac{y^2}{2}\right)dy$

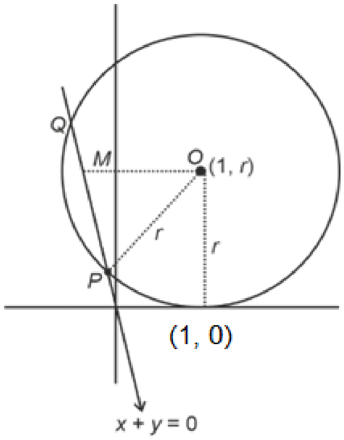
$$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6}\right]_{-4}^{2}$$

= 18 square units

7. Let a circle $C : (x - h)^2 + (y - k)^2 = r^2$, $k > 0$, touch the x-axis at (1, 0). If the line $x + y = 0$ intersects the circle C at P and Q such that the length of the chord PQ is 2, then the value of $h + k + r$ is equal to _____.

Answer (7)

Sol.



Here, $OM^2 = OP^2 - PM^2$

$$\left(\frac{|1+r|}{\sqrt{2}}\right)^2 = r^2 - 1$$

$\therefore$ $r^2 - 2r - 3 = 0$

$\therefore$ $r = 3$

$\therefore$ Equation of circle is

$$(x-1)^2 + (y-3)^2 = 3^2$$

$\therefore$ $h = 1, k = 3, r = 3$

$\therefore$ $h + k + r = 7$

8. In an examination, there are 10 true-false type questions. Out of 10, a student can guess the answer of 4 questions correctly with probability $\frac{3}{4}$ and the remaining 6 questions correctly with probability $\frac{1}{4}$. If the probability that the student guesses the answers of exactly 8 questions correctly out of 10 is $\frac{27k}{4^{10}}$, then k is equal to

Answer (479)

Sol. Student guesses only two wrong. So there are three possibilities

(i) Student guesses both wrong from 1st section

(ii) Student guesses both wrong from 2nd section

(iii) Student guesses two wrong one from each section

Required probabilities = $^4C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^6 +$

$$^6C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^4 + {}^4C_1\cdot{}^6C_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^5$$

$$= \frac{1}{4^{10}}\left[6\times 9 + 15\times 9^4 + 24\times 9^2\right]$$

$$= \frac{27}{4^{10}}\left[2 + 27\times 15 + 72\right]$$

$$= \frac{27\times 479}{4^{10}}$$

9. Let the hyperbola $H: \frac{x^2}{a^2} - y^2 = 1$ and the ellipse $E: 3x^2 + 4y^2 = 12$ be such that the length of latus rectum of H is equal to the length of latus rectum of E. If e_H and e_E are the eccentricities of H and E respectively, then the value of $12\left(e_H^2 + e_E^2\right)$ is equal to ___.

Answer (42)

Sol. $\because$ $H: \frac{x^2}{a^2} - \frac{y^2}{1} = 1$

$\therefore$ Length of latus rectum = $\frac{2}{a}$

$E: \frac{x^2}{4} + \frac{y^2}{3} = 1$

Length of latus rectum = $\frac{6}{2} = 3$

$\because$ $\frac{2}{a} = 3 \Rightarrow a = \frac{2}{3}$

$\therefore$ $12\left(e_H^2 + e_E^2\right) = 12\left(1 + \frac{9}{4}\right) + \left(1 - \frac{3}{4}\right)$ = 42

10. Let P_1 be a parabola with vertex (3, 2) and focus (4, 4) and P_2 be its mirror image with respect to the line $x + 2y = 6$. Then the directrix of P_2 is $x + 2y$ = _______.

Answer (10)

Sol. Focus = (4, 4) and vertex = (3, 2)

$\therefore$ Point of intersection of directrix with axis of parabola = A = (2, 0)

Image of $A(2, 0)$ with respect to line $x + 2y = 6$ is $B(x_2, y_2)$

$\therefore$ $\frac{x_2 - 2}{1} = \frac{y_2 - 0}{2} = \frac{-2(2+0-6)}{5}$

$\therefore$ $B(x_2, y_2) = \left(\frac{18}{5}, \frac{16}{5}\right)$.

Point B is point of intersection of direction with axes of parabola P_2.

$\therefore$ $x + 2y = \lambda$ must have point $\left(\frac{18}{5}, \frac{16}{5}\right)$

$\therefore$ $x + 2y = 10$

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