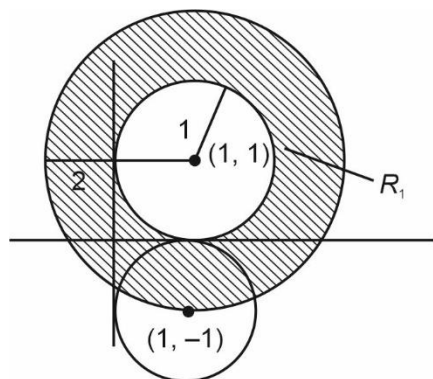# MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $A = \{z \in \mathbf{C} : 1 \le |z - (1 + i)| \le 2\}$ and $B = \{z \in A : |z - (1 - i)| = 1\}$. Then, B :
   (A) Is an empty set
   (B) Contains exactly two elements
   (C) Contains exactly three elements
   (D) Is an infinite set

Answer (D)

Sol.

Set A represents region 1 *i.e.* R_1 and clearly set B has infinite points in it.

2. The remainder when 3^{2022} is divided by 5 is :
   (A) 1   (B) 2
   (C) 3   (D) 4

Answer (D)

Sol. $3^{2022} = (10 - 1)^{1011} = {}^{1011}C_0(10)^{1011} (-1)^0 + {}^{1011}C_1(10)^{1010} (-1)^1 + ... + {}^{1011}C_{1010}(10)^1 (-1)^{1010} + {}^{1011}C_{1011}(10)^0 (-1)^{1011}$

   $= 5k - 1$, where $k \in I$

   So when divided by 5, it leaves remainder 4.

3. The surface area of a balloon of spherical shape being inflated, increases at a constant rate. If initially, the radius of balloon is 3 units and after 5 seconds, it becomes 7 units, then its radius after 9 seconds is :
   (A) 9   (B) 10
   (C) 11   (D) 12

Answer (A)

Sol. $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = \text{constant so} \Rightarrow r \frac{dr}{dt} = k \text{ (Let)}$$

$$r \, dr = k \, dt \Rightarrow \frac{r^2}{2} = kt + C$$

at $t = 0$, $r = 3$

$$\frac{9}{2} = C$$

at $t = 5$,

$$\frac{49}{2} = k \cdot 5 + \frac{9}{2} \Rightarrow k = 4$$
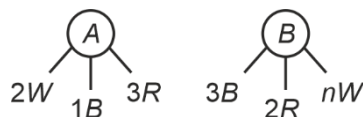
At $t = 9$, $\dfrac{r^2}{2} = \dfrac{81}{2}$

So, $r = 9$

4. Bag A contains 2 white, 1 black and 3 red balls and bas B contains 3 black, 2 red and n white balls. One bag is chosen at random and 2 balls drawn from it at random, are found to be 1 red and 1 black. If the probability that both balls come from Bag A is $\dfrac{6}{11}$, then n is equal to _____.
   (A) 13   (B) 6
   (C) 4   (D) 3

Answer (C)

Sol.

$$P(1R \text{ and } 1B) = P(A) \cdot P\left(\frac{1R\,1B}{A}\right) + P(B) \cdot P\left(\frac{1R\,1B}{B}\right)$$

$$= \frac{1}{2} \cdot \frac{{}^3C_1 \cdot {}^1C_1}{{}^6C_2} + \frac{1}{2} \cdot \frac{{}^2C_1 \cdot {}^3C_1}{{}^{n+5}C_2}$$

$$P\left(\frac{1R\,1B}{A}\right) = \frac{\dfrac{1}{2} \cdot \dfrac{3}{15}}{\dfrac{1}{2} \cdot \dfrac{3}{15} + \dfrac{1}{2} \cdot \dfrac{6 \cdot 2}{(n+5)(n+4)}} = \frac{6}{11}$$

$$\Rightarrow \frac{\dfrac{1}{10}}{\dfrac{1}{10} + \dfrac{6}{(n+5)(n+4)}} = \frac{6}{11}$$

$\Rightarrow \dfrac{11}{10} = \dfrac{6}{10} + \dfrac{36}{(n+5)(n+4)}$

$\Rightarrow \dfrac{5}{10 \times 36} = \dfrac{1}{(n+5)(n+4)}$

$\Rightarrow n^2 + 9n - 52 = 0$

$\Rightarrow n = 4$ is only possible value

5. Let $x^2 + y^2 + Ax + By + C = 0$ be a circle passing through (0, 6) and touching the parabola $y = x^2$ at (2, 4). Then $A + C$ is equal to _____.

(A) 16

(B) $\dfrac{88}{5}$

(C) 72

(D) −8

Answer (A)

Sol. For tangent to parabola $y = x^2$ at (2, 4)

$\left.\dfrac{dy}{dx}\right|_{(2,4)} = 4$

Equation of tangent is

$y - 4 = 4(x - 2)$

$\Rightarrow 4x - y - 4 = 0$

Family of circle can be given by

$(x - 2)^2 + (y - 4)^2 + \lambda(4x - y - 4) = 0$

As it passes through (0, 6)

$2^2 + 2^2 + \lambda(-10) = 0$

$\Rightarrow \lambda = \dfrac{4}{5}$

Equation of circle is

$(x - 2)^2 + (y - 4)^2 + \dfrac{4}{5}(4x - y - 4) = 0$

$\Rightarrow (x^2 + y^2 - 4x - 8y + 20) + \left(\dfrac{16}{5}x - \dfrac{4}{5}y - \dfrac{16}{5}\right) = 0$

$A = -4 + \dfrac{16}{5},\ C = 20 - \dfrac{16}{5}$

So, $A + C = 16$

6. The number of values of α for which the system of equations :

$x + y + z = \alpha$

$\alpha x + 2\alpha y + 3z = -1$

$x + 3\alpha y + 5z = 4$

is inconsistent, is

(A) 0   (B) 1

(C) 2   (D) 3

Answer (B)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 2\alpha & 3 \\ 1 & 3\alpha & 5 \end{vmatrix}$

$= 1(10\alpha - 9\alpha) - 1(5\alpha - 3) + 1(3\alpha^2 - 2\alpha)$

$= \alpha - 5\alpha + 3 + 3\alpha^2 - 2\alpha$

$= 3\alpha^2 - 6\alpha + 3$

For inconsistency $\Delta = 0$ i.e. $\alpha = 1$

Now check for $\alpha = 1$

$x + y + z = 1$ …(i)

$x + 2y + 3z = -1$ …(ii)

$x + 3y + 5z = 4$ …(iii)

By (ii) × 2 − (i) × 1

$x + 3y + 5z = -3$

so equations are

inconsistent for $\alpha = 1$

7. If the sum of the squares of the reciprocals of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)^2$ is equal to :

(A) 18   (B) 24

(C) 36   (D) 96

Answer (B)

Sol. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = 15 \Rightarrow \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = 15$

$\Rightarrow \dfrac{\dfrac{\lambda^2}{9} + \dfrac{2}{3}}{\dfrac{1}{9}} = 15$

$\Rightarrow \dfrac{\lambda^2}{9} = 1 \Rightarrow \lambda^2 = 9$

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= \left(\dfrac{-\lambda}{3}\right)\left(\dfrac{\lambda^2}{9} - 3\left(\dfrac{-1}{3}\right)\right) = \left(\dfrac{-\lambda}{3}\right)\left(\dfrac{\lambda^2}{9} + 1\right) = \dfrac{-2\lambda}{3}$

$6\left(\alpha^3 + \beta^3\right)^2 = 6 \cdot \dfrac{4\lambda^2}{9} = 24$

8. The set of all values of k for which $\left(\tan^{-1}x\right)^3 + \left(\cot^{-1}x\right)^3 = k\pi^3, x \in R$, is the interval:

(A) $\left[\dfrac{1}{32}, \dfrac{7}{8}\right)$

(B) $\left(\dfrac{1}{24}, \dfrac{13}{16}\right)$

(C) $\left[\dfrac{1}{48}, \dfrac{13}{16}\right]$

(D) $\left[\dfrac{1}{32}, \dfrac{9}{8}\right)$

Answer (A)

Sol. Let $\tan^{-1}x = t \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$\cot^{-1}x = \dfrac{\pi}{2} - t$

$f(t) = t^3 + \left(\dfrac{\pi}{2} - t\right)^3 \Rightarrow f'(t) = 3t^2 - 3\left(\dfrac{\pi}{2} - t\right)^2$

$f'(t) = 0$ at $t = \dfrac{\pi}{4}$

$f(t)\big|_{min} = \dfrac{\pi^3}{64} + \dfrac{\pi^3}{64} = \dfrac{\pi^3}{32}$

Max will occur around $t = -\dfrac{\pi}{2}$

Range of $f(t) = \left[\dfrac{\pi^3}{32}, \dfrac{7\pi^3}{8}\right)$

$k \in \left[\dfrac{1}{32}, \dfrac{7}{8}\right)$

9. Let $S = \left\{\sqrt{n} : 1 \le n \le 50 \text{ and } n \text{ is odd}\right\}$.

Let $a \in S$ and $A = \begin{bmatrix} 1 & 0 & a \\ -1 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}$

If $\sum\limits_{a \in S} \det$ (adj A) = 100 λ, then λ is equal to :

(A) 218

(B) 221

(C) 663

(D) 1717

Answer (B)

Sol. $|A| = a^2 + 1$

$|\text{adj } A| = (a^2 + 1)^2$

$S = \left\{1, \sqrt{3}, \sqrt{5}, \sqrt{7}, ..., \sqrt{49}\right\}$

$\sum\limits_{a \in S} \det(\text{adj } A) = (1^2 + 1)^2 + (3 + 1)^2 + (5 + 1)^2 + ... + (49 + 1)^2$

$= 2^2 (1^2 + 2^2 + 3^2 + ... + 25^2)$

$= 4 \cdot \dfrac{25 \cdot 26 \cdot 51}{6} = 100 \cdot 221$
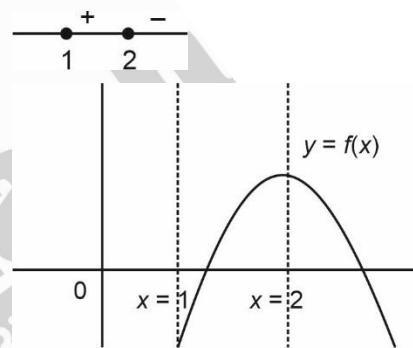
$\lambda = 221$

10. For the function

$f(x) = 4 \log_e(x - 1) - 2x^2 + 4x + 5, x > 1$, which one of the following is NOT correct?

(A) f is increasing in $(1, 2)$ and decreasing in $(2, \infty)$

(B) $f(x) = -1$ has exactly two solutions

(C) $f'(e) - f''(2) < 0$

(D) $f(x) = 0$ has a root in the interval $(e, e + 1)$

Answer (C)

Sol. $f'(x) = \dfrac{4}{x-1} - 4x + 4 = \dfrac{4\left(2x - x^2\right)}{x - 1}$

So maxima occurs at $x = 2$

$f(2) = 4 \cdot 0 - 2 \cdot 2^2 + 4 \cdot 2 + 5 = 5$

so clearly $f(x) = -1$ has

exactly 2 solutions

$f''(x) = \dfrac{4(2 - 2x)(x - 1)}{(x - 1)^2} - \left(2x - x^2\right)$

so $f'(e) - f''(2) > 0$

so option c is not correct

11. If the tangent at the point (x_1, y_1) on the curve $y = x^3 + 3x^2 + 5$ passes through the origin, then (x_1, y_1) does NOT lie on the curve :

(A) $x^2 + \dfrac{y^2}{81} = 2$

(B) $\dfrac{y^2}{9} - x^2 = 8$

(C) $y = 4x^2 + 5$

(D) $\dfrac{x}{3} - y^2 = 2$

Answer (D)

Sol. $m_{op} - m_{Tangent}$

$$\frac{y_1}{x_1} = 3x_1^2 + 6x_1$$

$$\Rightarrow \frac{x_1^3 + 3x_1^2 + 5}{x_1} = 3x_1^2 + 6x_1$$

$$\Rightarrow x_1^3 + 3x_1^2 + 5 = 3x_1^3 + 6x_1^2$$

$$\Rightarrow 2x_1^3 + 3x_1^2 - 5 = 0$$

$$\Rightarrow (x_1 - 1)(2x_1^2 + 5x_1 + 5) = 0$$

So, $(x_1, y_1) = (1, 9)$

12. The sum of absolute maximum and absolute minimum values of the function $f(x) = |2x^2 + 3x - 2| + \sin x \cos x$ in the interval $[0, 1]$ is :

(A) $3 + \dfrac{\sin(1)\cos^2\left(\dfrac{1}{2}\right)}{2}$

(B) $3 + \dfrac{1}{2}(1 + 2\cos(1))\sin(1)$

(C) $5 + \dfrac{1}{2}(\sin(1) + \sin(2))$

(D) $2 + \sin\left(\dfrac{1}{2}\right)\cos\left(\dfrac{1}{2}\right)$

Answer (B)

Sol. $f(x) = |(2x-1)(x+2)| + \dfrac{\sin 2x}{2}$

$\underline{0 \le x < \dfrac{1}{2}} \quad f(x) = (1-2x)(x+2) + \dfrac{\sin 2x}{2}$

$f'(x) = -4x - 3 + \cos 2x < 0$

For $x \ge \dfrac{1}{2}$: $f'(x) = 4x + 3 + \cos 2x > 0$

So, minima occurs at $x = \dfrac{1}{2}$

$f(x)|_{min} = \left| 2\left(\dfrac{1}{2}\right)^2 + \dfrac{3}{2} - 2 \right| + \sin\left(\dfrac{1}{2}\right) \cdot \cos\left(\dfrac{1}{2}\right)$

$= \dfrac{1}{2}\sin 1$

So, maxima is possible at $x = 0$ or $x = 1$

Now checking for $x = 0$ and $x = 1$, we can see it attains its maximum value at $x = 1$

$f(x)|_{max} = |2 + 3 - 2| + \dfrac{\sin 2}{2}$

$= 3 + \dfrac{1}{2}\sin 2$

Sum of absolute maximum and minimum value

$= 3 + \dfrac{1}{2}(\sin 1 + \sin 2)$

13. If $\{a_i\}_{i=1}^n$, where n is an even integer, is an arithmetic progression with common difference 1, and $\displaystyle\sum_{i=1}^n a_i = 192$, $\displaystyle\sum_{i=1}^{n/2} a_{2i} = 120$, then n is equal to :

(A) 48 (B) 96

(C) 92 (D) 104

Answer (B)

Sol. $a_1 + a_2 + \ldots + a_n = 192 \Rightarrow \dfrac{n}{2}(a_1 + a_n) = 192$ …(1)

$a_2 + a_4 + a_6 + \ldots + a_n = 120$

$\Rightarrow \dfrac{n}{4}(a_1 + 1 + a_n) = 120$ …(2)

From (2) & (1)

$\dfrac{480}{n} - \dfrac{384}{n} = 1 \Rightarrow n = 96$

14. If $x = x(y)$ is the solution of the differential equation $y\dfrac{dx}{dy} = 2x + y^3(y+1)e^y$, $x(1) = 0$; then $x(e)$ is equal to :

(A) $e^3(e^e - 1)$ (B) $e^e(e^3 - 1)$

(C) $e^2(e^e + 1)$ (D) $e^e(e^2 - 1)$

Answer (A)

Sol. $\dfrac{dx}{dy} - \dfrac{2x}{y} = y^2(y+1)e^y$

If $= e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = \dfrac{1}{y^2}$

Solution is given by

$x \cdot \dfrac{1}{y^2} = \int y^2(y+1)e^y \cdot \dfrac{1}{y^2}\, dy$

$\Rightarrow \dfrac{x}{y^2} = \int (y+1)e^y\, dy$

$\Rightarrow \dfrac{x}{y^2} = ye^y + c$

$\Rightarrow \quad x = y^2 (ye^y + c)$

at, $y = 1$, $x = 0$

$\Rightarrow \quad 0 = 1(1.e^1 + c) \Rightarrow c = -e$

at $y = e$,

$x = e^2(e.e^e - e)$

15. Let $\lambda x - 2y = \mu$ be a tangent to the hyperbola $a^2x^2 - y^2 = b^2$. Then $\left(\dfrac{\lambda}{a}\right)^2 - \left(\dfrac{\mu}{b}\right)^2$ is equal to :

(A) −2

(B) −4

(C) 2

(D) 4

Answer (D)

Sol. $\dfrac{x^2}{\left(\dfrac{b^2}{a^2}\right)} - \dfrac{y^2}{b^2} = 1$

Tangent in slope form $\Rightarrow y = mx \pm \sqrt{\dfrac{b^2}{a^2}m^2 - b^2}$

i.e., same as $y = \dfrac{\lambda x}{2} - \dfrac{\mu}{2}$

Comparing coefficients,

$m = \dfrac{\lambda}{2}, \dfrac{b^2}{a^2}m^2 - b^2 = \dfrac{\mu^2}{4}$

Eliminating $m, \dfrac{b^2}{a^2} \cdot \dfrac{\lambda^2}{4} - b^2 = \dfrac{\mu^2}{4}$

$\Rightarrow \dfrac{\lambda^2}{a^2} - \dfrac{\mu^2}{b^2} = 4$

16. Let $\hat{a}, \hat{b}$ be unit vectors. If $\vec{c}$ be a vector such that the angle between $\hat{a}$ and $\vec{c}$ is $\dfrac{\pi}{12}$, and $\hat{b} = \vec{c} + 2(\vec{c} \times \hat{a})$, then $\left|6\vec{c}\right|^2$ is equal to:

(A) $6\left(3 - \sqrt{3}\right)$

(B) $3 + \sqrt{3}$

(C) $6\left(3 + \sqrt{3}\right)$

(D) $6\left(\sqrt{3} + 1\right)$

Answer (C)

Sol. $\because \quad \hat{b} = \vec{c} + 2(\vec{c} \times \hat{a})$

$\Rightarrow \quad \hat{b} \cdot \vec{c} = \left|\vec{c}\right|^2$ \qquad ...(i)

$\therefore \quad \hat{b} - \vec{c} = 2(\vec{c} \times \vec{a})$

$\Rightarrow \quad \left|\hat{b}\right|^2 + \left|\vec{c}\right|^2 - 2\,\hat{b}\cdot\vec{c} = 4\left|\vec{c}\right|^2\left|\vec{a}\right|^2 \sin^2\dfrac{\pi}{12}$

$\Rightarrow \quad 1 + \left|\vec{c}\right|^2 - 2\left|\vec{c}\right|^2 = 4\left|\vec{c}\right|^2\left(\dfrac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2$

$\Rightarrow \quad 1 = \left|\vec{c}\right|^2\left(3 - \sqrt{3}\right)$

$\Rightarrow \quad 36\left|\vec{c}\right|^2 = \dfrac{36}{3 - \sqrt{3}} = 6\left(3 + \sqrt{3}\right)$

17. If a random variable X follows the Binomial distribution $B(33, p)$ such that $3P(X = 0) = P(X = 1)$, then the value of $\dfrac{P(X = 15)}{P(X = 18)} - \dfrac{P(X = 16)}{P(X = 17)}$ is equal to:

(A) 1320

(B) 1088

(C) $\dfrac{120}{1331}$

(D) $\dfrac{1088}{1089}$

Answer (A)

Sol. $3P(X = 0) = P(X = 1)$

$3 \cdot {}^nC_0 P^0 (1 - P)^n = {}^nC_1 P^1 (1 - P)^{n-1}$

$\dfrac{3}{n} = \dfrac{P}{1 - P} \Rightarrow \dfrac{1}{11} = \dfrac{P}{1 - P}$

$\Rightarrow 1 - P = 11P$

$\Rightarrow P = \dfrac{1}{12}$

$\dfrac{P(X = 15)}{P(X = 18)} - \dfrac{P(X = 16)}{P(X = 17)}$

$\Rightarrow \dfrac{{}^{33}C_{15} P^{15}(1 - P)^{18}}{{}^{33}C_{18} P^{18}(1 - P)^{15}} - \dfrac{{}^{33}C_{16} P^{16}(1 - P)^{17}}{{}^{33}C_{17} P^{17}(1 - P)^{16}}$

$\Rightarrow \left(\dfrac{1 - P}{P}\right)^3 - \left(\dfrac{1 - P}{P}\right)$

$\Rightarrow 11^3 - 11 = 1320$

18. The domain of the function $f(x) = \dfrac{\cos^{-1}\left(\dfrac{x^2 - 5x + 6}{x^2 - 9}\right)}{\log_e(x^2 - 3x + 2)}$ is:

(A) $(-\infty, 1) \cup (2, \infty)$

(B) $(2, \infty)$

(C) $\left[-\dfrac{1}{2}, 1\right] \cup (2, \infty)$

(D) $\left[-\dfrac{1}{2}, 1\right] \cup (2, \infty) - \left\{\dfrac{3 + \sqrt{5}}{2}, \dfrac{3 - \sqrt{5}}{2}\right\}$

Answer (D)

Sol. $-1 \le \dfrac{x^2 - 5x + 6}{x^2 - 9} \le 1$ and $x^2 - 3x + 2 > 0, \ne 1$

$\dfrac{(x-3)(2x+1)}{x^2 - 9} \ge 0 \ \bigg| \ \dfrac{5(x-3)}{x^2 - 9} \ge 0$

Solution to this inequality is

$x \in \left[\dfrac{-1}{2}, \infty\right) - \{3\}$

for $x^2 - 3x + 2 > 0$ and $\ne 1$

$x \in (-\infty, 1) \cup (2, \infty) - \left\{\dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}\right\}$

Combining the two solution sets (taking intersection)

$x \in \left[-\dfrac{1}{2}, 1\right) \cup (2, \infty) - \left\{\dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}\right\}$

19. Let $S = \left\{\theta \in [-\pi, \pi] - \left\{\pm\dfrac{\pi}{2}\right\} : \sin\theta\tan\theta + \tan\theta = \sin 2\theta\right\}$. If

$T = \displaystyle\sum_{\theta \in S} \cos 2\theta$, then $T + n(S)$ is equal to:

(A) $7 + \sqrt{3}$

(B) 9

(C) $8 + \sqrt{3}$

(D) 10

Answer (B)

Sol. $\tan\theta (\sin\theta + 1) - \sin 2\theta = 0$

$\tan\theta(\sin\theta + 1 - 2\cos^2\theta) = 0$

$\Rightarrow \ \tan\theta = 0$ or $2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \ (2\sin\theta + 1)(\sin\theta - 1) = 0$

$\Rightarrow \ \sin\theta = \dfrac{-1}{2}$ or 1

But, $\sin\theta = 1$ not possible

$\theta = 0, \pi, -\pi, -\dfrac{\pi}{6}, \dfrac{-5\pi}{6}$

$n(S) = 5$

$T = \sum \cos 2\theta = \cos 0° + \cos 2\pi + \cos(-2\pi)$

$+ \cos\left(-\dfrac{5\pi}{3}\right) + \cos\left(-\dfrac{\pi}{3}\right)$

= 4

20. The number of choices for $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$, such that $(p \Delta q) \Rightarrow ((p \Delta \sim q) \vee ((\sim p) \Delta q))$ is a tautology, is

(A) 1

(B) 2

(C) 3

(D) 4

Answer (B)

Sol. Let $x : (p\Delta q) \Rightarrow (p\Delta \sim q) \vee (\sim p\Delta q)$

Case-I

When Δ is same as $\vee$

Then $(p\Delta \sim q) \vee (\sim p\Delta q)$ becomes

$(p \vee \sim q) \vee (\sim p \vee q)$ which is always true, so x becomes a tautology.

Case-II

When Δ is same as $\wedge$

Then $(p \wedge q) \Rightarrow (p\wedge \sim q) \vee (\sim p \wedge q)$

If $p \wedge q$ is T, then $(p \wedge \sim q) \vee (\sim p \wedge q)$ is F

so x cannot be a tautology.

Case-III

When Δ is same as $\Rightarrow$

Then $(p \Rightarrow \sim q) \vee (\sim p \Rightarrow q)$ is same at $(\sim p \vee \sim q) \vee (p \vee q)$, which is always true, so x becomes a tautology.

Case-IV

When Δ is same as $\Leftrightarrow$

Then $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow \sim q) \vee (\sim p \Leftrightarrow q)$

$p \Leftrightarrow q$ is true when p and q have same truth values, then $p \Leftrightarrow \sim q$ and $\sim p \Leftrightarrow q$ both are false. Hence x cannot be a tautology.

So finally x can be $\vee$ or $\Rightarrow$.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, −00.33, −00.30, 30.27, −27.30) using the mouse andw the on-screen virtual numeric keypad in the place designated to enter the answer.

1. The number of one-one functions $f : \{a, b, c, d\} \to \{0, 1, 2, ..., 10\}$ such that $2f(a) - f(b) + 3f(c) + f(d) = 0$ is _____.

Answer (31)

Sol. $\because$ $3f(c) + 2f(a) + f(d) = f(b)$

Value of $f(c)$	Value of $f(a)$	Number of functions
0	1	7
	2	5
	3	3
	4	2
1	0	6
	2	2
	3	1
2	0	3
	1	1
3	0	1
	Total Number of functions =	**31**

2. In an examination, there are 5 multiple choice questions with 3 choices, out of which exactly one is correct. There are 3 marks for each correct answer, –2 marks for each wrong answer and 0 mark if the question is not attempted. Then, the number of ways a student appearing in the examination gets 5 marks is _____.

Answer (40)

Sol. Let student marks x correct answers and y incorrect. So

$3x - 2y = 5$ and $x + y \leq 5$ where $x, y \in W$

Only possible solution is $(x, y) = (3, 2)$

Student can mark correct answer by only one choice but for incorrect answer, there are two choices. So total number of ways of scoring 5 marks = $^5C_3(1)^3 \cdot (2)^2 = 40$

3. Let $A\left(\dfrac{3}{\sqrt{a}}, \sqrt{a}\right), a > 0$, be a fixed point in the xy-plane, The image of A in y-axis be B and the image of B in x-axis be C. If $D(3\cos\theta, a\sin\theta)$ is a point in the fourth quadrant such that the maximum area of $\triangle ACD$ is 12 square units, then a is equal to _____.

Answer (8)

Sol. Clearly B is $\left(-\dfrac{3}{\sqrt{a}}, +\sqrt{a}\right)$ and C is $\left(-\dfrac{3}{\sqrt{a}}, -\sqrt{a}\right)$

Area of $\triangle ACD = \dfrac{1}{2}\begin{vmatrix} \dfrac{3}{\sqrt{a}} & \sqrt{a} & 1 \\ -\dfrac{3}{\sqrt{a}} & -\sqrt{a} & 1 \\ 3\cos\theta & a\sin\theta & 1 \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ -\dfrac{3}{\sqrt{a}} & -\sqrt{a} & 1 \\ 3\cos\theta & a\sin\theta & 1 \end{vmatrix}$

$\Rightarrow \Delta = \left|3\sqrt{a}\sin\theta + 3\sqrt{a}\cos\theta\right| = 3\sqrt{a}\left|\sin\theta + \cos\theta\right|$

$\Rightarrow \Delta_{max} = 3\sqrt{a} \cdot \sqrt{2} = 12 \Rightarrow a = \left(2\sqrt{2}\right)^2 = 8$

4. Let a line having direction ratios 1, –4, 2 intersect the lines $\dfrac{x-7}{3} = \dfrac{y-1}{-1} = \dfrac{z+2}{1}$ and $\dfrac{x}{2} = \dfrac{y-7}{3} = \dfrac{z}{1}$ at the points A and B. Then $(AB)^2$ is equal to _____.

Answer (84)

Sol. Let $A(3\lambda + 7, -\lambda + 1, \lambda - 2)$ and $B(2\mu, 3\mu + 7, \mu)$

So, DR's of $AB \propto 3\lambda - 2\mu + 7, -(\lambda + 3\mu + 6), \lambda - \mu - 2$

Clearly $\dfrac{3\lambda - 2\mu + 7}{1} = \dfrac{\lambda + 3\mu + 6}{4} = \dfrac{\lambda - \mu - 2}{2}$

$\Rightarrow 5\lambda - 3\mu = -16$ ...(i)

And $\lambda - 5\mu = 10$ ...(ii)

From (i) and (ii) we get $\lambda = -5$, $\mu = -3$

So, A is $(-8, 6, -7)$ and B is $(-6, -2, -3)$

$AB = \sqrt{4 + 64 + 16} \Rightarrow (AB)^2 = 84$

5. The number of points where the function

$f(x) = \begin{cases} |2x^2 - 3x - 7| & \text{if} \quad x \leq -1 \\ \left[4x^2 - 1\right] & \text{if} \quad -1 < x < 1 \\ |x+1| + |x-2| & \text{if} \quad x \geq 1, \end{cases}$

$[t]$ denotes the greatest integer $\leq t$, is discontinuous is _____.

Answer (7)

Sol. $\because$ $f(-1) = 2$ and $f(1) = 3$

For $x \in (-1, 1)$, $(4x^2 - 1) \in [-1, 3)$

hence $f(x)$ will be discontinuous at $x = 1$ and also

whenever $4x^2 - 1 = 0$, 1 or 2

$\Rightarrow \quad x = \pm\dfrac{1}{2}, \ \pm\dfrac{1}{\sqrt{2}}$ and $\pm\dfrac{\sqrt{3}}{2}$

So there are total 7 points of discontinuity.

6. Let $f(\theta) = \sin\theta + \displaystyle\int_{-\pi/2}^{\pi/2} (\sin\theta + t\cos\theta) f(t)dt$. Then

the value of $\left|\displaystyle\int_0^{\pi/2} f(\theta)d\theta\right|$ is _____.

Answer (1)

Sol. $f(\theta) = \sin\theta\left(1 + \displaystyle\int_{-\pi/2}^{\pi/2} f(t)\,dt\right) + \cos\theta\left(\displaystyle\int_{-\pi/2}^{\pi/2} tf(t)dt\right)$

Clearly $f(\theta) = a\sin\theta + b\cos\theta$

Where $a = 1 + \displaystyle\int_{-\pi/2}^{\pi/2} (a\sin t + b\cos t)\,dt \Rightarrow a = 1 + 2b$

...(1)

and $b = \displaystyle\int_{-\pi/2}^{\pi/2} (at\sin t + bt\cos t)dt \Rightarrow b = 2a$ ...(2)

from (1) and (2) we get

$a = -\dfrac{1}{3}$ and $b = -\dfrac{2}{3}$

So $f(\theta) = -\dfrac{1}{3}(\sin\theta + 2\cos\theta)$

$\Rightarrow \left|\displaystyle\int_0^{\pi/2} f(\theta)\,d\theta\right| = \dfrac{1}{3}(1 + 2\times 1) = 1$

7. Let $\displaystyle\operatorname*{Max}_{0\le x\le 2}\left\{\dfrac{9 - x^2}{5 - x}\right\} = \alpha$ and $\displaystyle\operatorname*{Min}_{0\le x\le 2}\left\{\dfrac{9 - x^2}{5 - x}\right\} = \beta$.

If $\displaystyle\int_{\beta - \frac{8}{3}}^{2\alpha - 1} \operatorname{Max}\left\{\dfrac{9 - x^2}{5 - x}, x\right\}dx = \alpha_1 + \alpha_2\log_e\left(\dfrac{8}{15}\right)$ then

$\alpha_1 + \alpha_2$ is equal to _____.

Answer (34)

Sol. Let $f(x) = \dfrac{x^2 - 9}{x - 5} \Rightarrow f'(x) = \dfrac{(x-1)(x-9)}{(x-5)^2}$

So, $\alpha = f(1) = 2$ and $\beta = \min(f(0), f(2)) = \dfrac{5}{3}$

Now, $\displaystyle\int_{-1}^{3}\max\left\{\dfrac{x^2-9}{x-5}, x\right\}dx = \displaystyle\int_{-1}^{9/5}\dfrac{x^2-9}{x-5}dx + \displaystyle\int_{9/5}^{3} xdx$

$= \displaystyle\int_{-1}^{9/5}\left(x + 5 + \dfrac{16}{x-5}\right)dx + \dfrac{x^2}{2}\Big|_{9/5}^{3}$

$= \dfrac{28}{25} + 14 + 16\ln\left(\dfrac{8}{15}\right) + \dfrac{72}{25} = 18 + 16\ln\left(\dfrac{8}{15}\right)$

Clearly $\alpha_1 = 18$ and $\alpha_2 = 16$, so $\alpha_1 + \alpha_2 = 34$.

8. If two tangents drawn from a point (α, β) lying on the ellipse $25x^2 + 4y^2 = 1$ to the parabola $y^2 = 4x$ are such that the slope of one tangent is four times the other, then the value of $(10\alpha + 5)^2 + (16\beta^2 + 50)^2$ equals _____.

Answer (2929)

Sol. $\because$ (α, β) lies on the given ellipse, $25\alpha^2 + 4\beta^2 = 1$

...(1)

Tangent to the parabola, $y = mx + \dfrac{1}{m}$ passes

through (α, β). So, $\alpha m^2 - \beta m + 1 = 0$ has roots m_1 and $4m_1$,

$m_1 + 4m_1 = \dfrac{\beta}{\alpha}$ and $m_1 \cdot 4m_1 = \dfrac{1}{\alpha}$

Gives that $4\beta^2 = 25\alpha$ ...(2)

from (1) and (2)

$25(\alpha^2 + \alpha) = 1$ ...(3)

Now, $(10\alpha + 5)^2 + (16\beta^2 + 50)^2$

$= 25(2\alpha + 1)^2 + 2500 (2\alpha + 1)^2$

$= 2525 (4\alpha^2 + 4\alpha + 1)$ from equation (3)
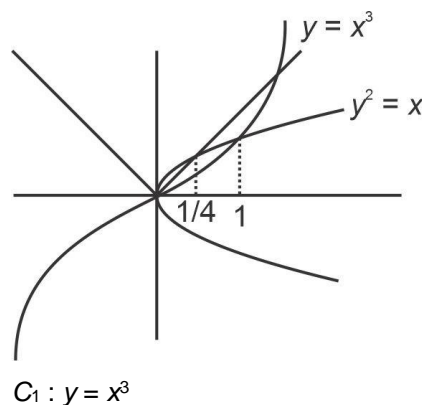
$= 2525\left(\dfrac{4}{25} + 1\right)$

$= 2929$

9. Let S be the region bounded by the curves $y = x^3$ and $y^2 = x$. The curve $y = 2|x|$ divides S into two regions of areas R_1 and R_2.

If $\max\{R_1, R_2\} = R_2$, then $\dfrac{R_2}{R_1}$ is equal to _____.

Answer (19)

Sol.

$C_1 : y = x^3$

$C_2 : y^2 = x$

and $C_3 = y = 2|x|$

C_1 and C_2 intersect at $(1, 1)$

C_2 and C_3 intersect at $\left(\dfrac{1}{4}, \dfrac{1}{2}\right)$

Clearly $R_1 = \displaystyle\int_0^{1/4}\left(\sqrt{x} - 2x\right)dx = \dfrac{2}{3}\left(\dfrac{1}{8}\right) - \dfrac{1}{16} = \dfrac{1}{48}$

and $R_1 + R_2 = \displaystyle\int_0^1\left(\sqrt{x} - x^3\right)dx = \dfrac{2}{3} - \dfrac{1}{4} = \dfrac{5}{12}$

So, $\dfrac{R_1 + R_2}{R_1} = \dfrac{5/12}{1/48} \Rightarrow 1 + \dfrac{R_2}{R_1} = 20$

$\Rightarrow \dfrac{R_2}{R_1} = 19$

10. If the shortest distance between the lines $\vec{r} = \left(-\hat{i} + 3\hat{k}\right) + \lambda\left(\hat{i} - a\hat{j}\right)$ and $\vec{r} = \left(-\hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} - \hat{j} + \hat{k}\right)$ is $\sqrt{\dfrac{2}{3}}$, then the integral value of a is equal to _____.

Answer (2)

Sol. $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -a & 0 \\ 1 & -1 & 1 \end{vmatrix} = -a\hat{i} - \hat{j} + (a-1)\hat{k}$

$\vec{a}_1 - \vec{a}_2 = -\hat{i} + \hat{j} + \hat{k}$

Shortest distance $= \left|\dfrac{\left(\vec{a}_1 - \vec{a}_2\right)\cdot\left(\vec{b}_1 \times \vec{b}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right|$

$\Rightarrow \sqrt{\dfrac{2}{3}} = \dfrac{2(a-1)}{\sqrt{a^2 + 1 + (a-1)^2}}$

$\Rightarrow 6\left(a^2 - 2a + 1\right) = 2a^2 - 2a + 2$

$\Rightarrow (a-2)(2a-1) = 0 \Rightarrow a = 2$ because $a \in z$.

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