

MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $A = \{x \in \mathbb{R} : |x+1| < 2\}$ and $B = \{x \in \mathbb{R} : |x-1| \geq 2\}$. Then which one of the following statements is **NOT** true?
 (A) $A - B = (-1, 1)$ (B) $B - A = \mathbb{R} - (-3, 1)$
 (C) $A \cap B = (-3, -1]$ (D) $A \cup B = \mathbb{R} - [1, 3]$

Answer (B)

Sol. $A = (-3, 1)$ and $B = (-\infty, -1] \cup [3, \infty)$

$$\text{So, } A - B = (-1, 1)$$

$$B - A = (-\infty, -3] \cup [3, \infty) = \mathbb{R} - (-3, 3)$$

$$A \cap B = (-3, -1]$$

$$\text{and } A \cup B = (-\infty, 1) \cup [3, \infty) = \mathbb{R} - [1, 3)$$

2. Let $a, b \in \mathbb{R}$ be such that the equation $ax^2 - 2bx + 15 = 0$ has a repeated root α . If α and β are the roots of the equation $x^2 - 2bx + 21 = 0$, then $\alpha^2 + \beta^2$ is equal to
 (A) 37 (B) 58
 (C) 68 (D) 92

Answer (B)

Sol. $ax^2 - 2bx + 15 = 0$ has repeated root so $b^2 = 15a$

$$\text{and } \alpha = \frac{15}{b}$$

$\therefore \alpha$ is a root of $x^2 - 2bx + 21 = 0$

$$\text{So } \frac{225}{b^2} = 9 \Rightarrow b^2 = 25$$

$$\text{Now } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4b^2 - 42 = 100 - 42 = 58$$

3. Let z_1 and z_2 be two complex numbers such that

$$\overline{z_1} = i\overline{z_2} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \pi. \text{ Then}$$

$$(A) \arg z_2 = \left(\frac{\pi}{4}\right) \quad (B) \arg z_2 = -\frac{3\pi}{4}$$

$$(C) \arg z_1 = \frac{\pi}{4} \quad (D) \arg z_1 = -\frac{3\pi}{4}$$

Answer (C)

$$\text{Sol. } \because \frac{z_1}{z_2} = -i \Rightarrow z_1 = -iz_2$$

$$\Rightarrow \arg(z_1) = -\frac{\pi}{2} + \arg(z_2) \dots (i)$$

$$\text{Also } \arg(z_1) - \arg(\bar{z}_2) = \pi$$

$$\Rightarrow \arg(z_1) + \arg(z_2) = \pi \dots (ii)$$

From (i) and (ii), we get

$$\arg(z_1) = \frac{\pi}{4} \text{ and } \arg(z_2) = \frac{3\pi}{4}$$

4. The system of equations

$$-kx + 3y - 14z = 25$$

$$-15x + 4y - kz = 3$$

$$-4x + y + 3z = 4$$

is consistent for all k in the set

- (A) R (B) $R - \{-11, 13\}$
 (C) $R - \{13\}$ (D) $R - \{-11, 11\}$

Answer (D)

Sol. The system may be inconsistent if

$$\begin{vmatrix} -k & 3 & -14 \\ -15 & 4 & -k \\ -4 & 1 & 3 \end{vmatrix} = 0 \Rightarrow k = \pm 11$$

Hence if system is consistent then $k \in R - \{11, -11\}$

5. $\lim_{x \rightarrow \frac{\pi}{2}} \tan^2 x \left(\left(2 \sin^2 x + 3 \sin x + 4\right)^{\frac{1}{2}} - \left(\sin^2 x + 6 \sin x + 2\right)^{\frac{1}{2}} \right)$ is

equal to

$$(A) \frac{1}{12}$$

$$(B) -\frac{1}{18}$$

$$(C) -\frac{1}{12}$$

$$(D) \frac{1}{6}$$

Answer (A)

Sol.

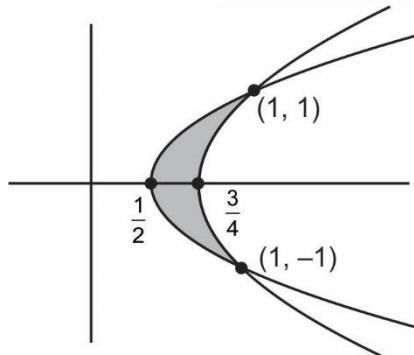
$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \tan^2 x \left\{ \sqrt{2 \sin^2 x + 3 \sin x + 4} - \sqrt{\sin^2 x + 6 \sin x + 2} \right\} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan^2 x (\sin^2 x - 3 \sin x + 2)}{\sqrt{2 \sin^2 x + 3 \sin x + 4} + \sqrt{\sin^2 x + 6 \sin x + 2}} \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)(2 - \sin x)}{\cos^2 x} \cdot \sin^2 x \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(2 - \sin x) \sin^2 x}{1 + \sin x} \\ &= \frac{1}{12} \end{aligned}$$

6. The area of the region enclosed between the parabolas $y^2 = 2x - 1$ and $y^2 = 4x - 3$ is

- (A) $\frac{1}{3}$ (B) $\frac{1}{6}$
 (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Answer (A)

Sol. Area of the shaded region



$$\begin{aligned} &= 2 \int_0^1 \left(\frac{y^2 + 3}{4} - \frac{y^2 + 1}{2} \right) dy \\ &= 2 \int_0^1 \left(\frac{1}{4} - \frac{y^2}{4} \right) dy \\ &= 2 \left[\frac{1}{4} - \frac{1}{12} \right] = \frac{1}{3} \end{aligned}$$

7. The coefficient of x^{101} in the expression $(5 + x)^{500} + x(5 + x)^{499} + x^2(5 + x)^{498} + \dots + x^{500}$, $x > 0$, is

- (A) ${}^{501}C_{101} (5)^{399}$ (B) ${}^{501}C_{101} (5)^{400}$
 (C) ${}^{501}C_{100} (5)^{400}$ (D) ${}^{500}C_{101} (5)^{399}$

Answer (A)

Sol. Coeff. of x^{101} in $\frac{x^{500} \left[\left(\frac{x+5}{x} \right)^{501} - 1 \right]}{x+5 - 1}$

$$\begin{aligned} &= \text{Coeff. of } x^{101} \text{ in } \frac{1}{5} \left[(x+5)^{501} - x^{501} \right] \\ &= \frac{1}{5} {}^{501}C_{101} \cdot 5^{400} \\ &= {}^{501}C_{101} \cdot 5^{399} \end{aligned}$$

8. The sum $1 + 2 \cdot 3 + 3 \cdot 3^2 + \dots + 10 \cdot 3^9$ is equal to

- (A) $\frac{2 \cdot 3^{12} + 10}{4}$ (B) $\frac{19 \cdot 3^{10} + 1}{4}$
 (C) $5 \cdot 3^{10} - 2$ (D) $\frac{9 \cdot 3^{10} + 1}{2}$

Answer (B)

$$\text{Sol. Let } S = 1 \cdot 3^0 + 2 \cdot 3^1 + 3 \cdot 3^2 + \dots + 10 \cdot 3^9$$

$$\begin{aligned} 3S &= 1 \cdot 3^1 + 2 \cdot 3^2 + \dots + 10 \cdot 3^{10} \\ -2S &= (1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^2 + \dots + 1 \cdot 3^9) - 10 \cdot 3^{10} \\ \Rightarrow S &= \frac{1}{2} \left[10 \cdot 3^{10} - \frac{3^{10} - 1}{3 - 1} \right] \\ \Rightarrow S &= \frac{19 \cdot 3^{10} + 1}{4} \end{aligned}$$

9. Let P be the plane passing through the intersection of the planes

$\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$, and the point $(2, 1, -2)$. Let the position vectors of the points X and Y be $\hat{i} - 2\hat{j} + 4\hat{k}$ and $5\hat{i} - \hat{j} + 2\hat{k}$ respectively. Then the points

- (A) X and $X + Y$ are on the same side of P
 (B) Y and $Y - X$ are on the opposite sides of P
 (C) X and Y are on the opposite sides of P
 (D) $X + Y$ and $X - Y$ are on the same side of P

Answer (C)

Sol. Let the equation of required plane

$$\pi : (x + 3y - z - 5) + \lambda(2x - y + z - 3) = 0$$

$$\therefore (2, 1, -2) \text{ lies on it so, } 2 + \lambda(-2) = 0$$

$$\Rightarrow \lambda = 1$$

$$\text{Hence, } \pi : 3x + 2y - 8 = 0$$

$$\therefore \pi_x = -9, \pi_y = 5, \pi_{x+y} = 4$$

$$\pi_{x-y} = -22 \text{ and } \pi_{y-x} = 6$$

Clearly X and Y are on opposite sides of plane π

10. A circle touches both the y-axis and the line $x + y = 0$. Then the locus of its center is
- (A) $y = \sqrt{2}x$ (B) $x = \sqrt{2}y$
 (C) $y^2 - x^2 = 2xy$ (D) $x^2 - y^2 = 2xy$

Answer (D)

Sol. Let the centre be (h, k)

$$\text{So, } |h| = \left| \frac{h+k}{\sqrt{2}} \right|$$

$$\Rightarrow 2h^2 = h^2 + k^2 + 2hk$$

$$\text{Locus will be } x^2 - y^2 = 2xy$$

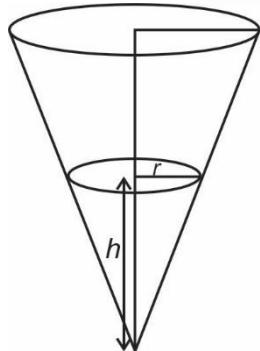
11. Water is being filled at the rate of $1 \text{ cm}^3/\text{sec}$ in a right circular conical vessel (vertex downwards) of height 35 cm and diameter 14 cm. When the height of the water level is 10 cm, the rate (in cm^2/sec) at which the wet conical surface area of the vessel increase, is

- (A) 5 (B) $\frac{\sqrt{21}}{5}$
 (C) $\frac{\sqrt{26}}{5}$ (D) $\frac{\sqrt{26}}{10}$

Answer (C)

$$\text{Sol. } \because V = \frac{1}{3}\pi r^2 h \text{ and } \frac{r}{h} = \frac{7}{35} = \frac{1}{5}$$

$$\Rightarrow V = \frac{1}{75}\pi h^3$$



$$\frac{dV}{dt} = \frac{1}{25}\pi h^2 \frac{dh}{dt} = 1$$

$$\Rightarrow \frac{dh}{dt} = \frac{25}{\pi h^2}$$

$$\text{Now, } S = \pi rl = \pi \left(\frac{h}{5} \right) \sqrt{h^2 + \frac{h^2}{25}} = \frac{\pi}{25} \sqrt{26} h^2$$

$$\Rightarrow \frac{dS}{dt} = \frac{2\sqrt{26}\pi h}{25} \cdot \frac{dh}{dt} = \frac{2\sqrt{26}}{h}$$

$$\Rightarrow \frac{dS}{dt} \Big|_{(h=10)} = \frac{\sqrt{26}}{5}$$

12. If $b_n = \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx}{\sin x} dx, n \in \mathbb{N}$, then
- (A) $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in an A.P. with common difference -2
 (B) $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in an A.P. with common difference 2
 (C) $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in a G.P.
 (D) $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in an A.P. with common difference -2

Answer (D)

$$\text{Sol. } b_n - b_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx - \cos^2(n-1)x}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{-\sin(2n-1)x \cdot \sin x}{\sin x} dx$$

$$= \frac{\cos(2n-1)x}{2n-1} \Big|_0^{\pi/2} = -\frac{1}{2n-1}$$

So, $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in H.P.

$\Rightarrow \frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in A.P. with common difference -2.

13. If $y = y(x)$ is the solution of the differential equation $2x^2 \frac{dy}{dx} - 2xy + 3y^2 = 0$ such that $y(e) = \frac{e}{3}$, then $y(1)$ is equal to

- (A) $\frac{1}{3}$ (B) $\frac{2}{3}$
 (C) $\frac{3}{2}$ (D) 3

Answer (B)

$$\text{Sol. } 2x^2 \frac{dy}{dx} - 2xy + 3y^2 = 0$$

$$\Rightarrow 2x(x dy - y dx) + 3y^2 dx = 0$$

$$\Rightarrow 2 \left(\frac{x dy - y dx}{y^2} \right) + 3 \frac{dx}{x} = 0$$

$$\Rightarrow -\frac{2x}{y} + 3 \ln x = C$$

$$\therefore y(e) = \frac{e}{3} \Rightarrow -6 + 3 = C \Rightarrow C = -3$$

$$\text{Now, at } x = 1, -\frac{2}{y} + 0 = -3$$

$$y = \frac{2}{3}$$

14. If the angle made by the tangent at the point (x_0, y_0) on the curve $x = 12(t + \sin t \cos t)$,

$y = 12(1 + \sin t)^2$, $0 < t < \frac{\pi}{2}$, with the positive x-axis

is $\frac{\pi}{3}$, then y_0 is equal to:

(A) $6(3+2\sqrt{2})$

(B) $3(7+4\sqrt{3})$

(C) 27

(D) 48

Answer (C)

$$\text{Sol.} \because \frac{dy}{dx} = \frac{24(1+\sin t)\cos t}{12(1+\cos 2t)} = \frac{1+\sin t}{\cos t} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

$$\therefore \frac{dy}{dx}_{(x_0, y_0)} = \sqrt{3} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

$$\Rightarrow t = \frac{\pi}{6}$$

$$\text{So, } y_0 \left(\text{at } t = \frac{\pi}{6}\right) = 12 \left(1 + \sin \frac{\pi}{6}\right)^2 = 27$$

15. The value of $2\sin(12^\circ) - \sin(72^\circ)$ is :

(A) $\frac{\sqrt{5}(1-\sqrt{3})}{4}$

(B) $\frac{1-\sqrt{5}}{8}$

(C) $\frac{\sqrt{3}(1-\sqrt{5})}{2}$

(D) $\frac{\sqrt{3}(1-\sqrt{5})}{4}$

Answer (D)

Sol. $2\sin 12^\circ - \sin 72^\circ$

$$= \sin 12^\circ + (-2\cos 42^\circ \cdot \sin 30^\circ)$$

$$= \sin 12^\circ - \cos 42^\circ$$

$$= \sin 12^\circ - \sin 48^\circ$$

$$= 2\sin 18^\circ \cdot \cos 30^\circ$$

$$= -2\left(\frac{\sqrt{5}-1}{4}\right) \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}(1-\sqrt{5})}{4}$$

16. A biased die is marked with numbers 2, 4, 8, 16, 32, 32 on its faces and the probability of getting a face with mark n is $\frac{1}{n}$. If the die is thrown thrice, then the probability, that the sum of the numbers obtained is 48, is :

(A) $\frac{7}{2^{11}}$

(B) $\frac{7}{2^{12}}$

(C) $\frac{3}{2^{10}}$

(D) $\frac{13}{2^{12}}$

Answer (D)

Sol. There are only two ways to get sum 48, which are (32, 8, 8) and (16, 16, 16)

So, required probability

$$= 3\left(\frac{2}{32} \cdot \frac{1}{8} \cdot \frac{1}{8}\right) + \left(\frac{1}{16} \cdot \frac{1}{16} \cdot \frac{1}{16}\right)$$

$$= \frac{3}{2^{10}} + \frac{1}{2^{12}}$$

$$= \frac{13}{2^{12}}$$

17. The negation of the Boolean expression $((\sim q) \wedge p) \Rightarrow ((\sim p) \vee q)$ is logically equivalent to :

(A) $p \Rightarrow q$

(B) $q \Rightarrow p$

(C) $\sim(p \Rightarrow q)$

(D) $\sim(q \Rightarrow p)$

Answer (C)

Sol. Let $S : ((\sim q) \wedge p) \Rightarrow ((\sim p) \vee q)$

$$\Rightarrow S : \sim((\sim q) \wedge p) \vee ((\sim p) \vee q)$$

$$\Rightarrow S : (q \vee (\sim p)) \vee ((\sim p) \vee q)$$

$$\Rightarrow S : (\sim p) \vee q$$

$$\Rightarrow S : p \Rightarrow q$$

So, negation of S will be $\sim(p \Rightarrow q)$

18. If the line $y = 4 + kx$, $k > 0$, is the tangent to the parabola $y = x - x^2$ at the point P and V is the vertex of the parabola, then the slope of the line through P and V is :

(A) $\frac{3}{2}$

(B) $\frac{26}{9}$

(C) $\frac{5}{2}$

(D) $\frac{23}{6}$

Answer (C)

Sol. \because Line $y = kx + 4$ touches the parabola $y = x - x^2$.

So, $kx + 4 = x - x^2 \Rightarrow x^2 + (k-1)x + 4 = 0$ has only one root

$$(k-1)^2 = 16 \Rightarrow k = 5 \text{ or } -3 \text{ but } k > 0$$

$$\text{So, } k = 5.$$

$$\text{And hence } x^2 + 4x + 4 = 0 \Rightarrow x = -2$$

So, $P(-2, -6)$ and V is $\left(\frac{1}{2}, \frac{1}{4}\right)$

$$\text{Slope of } PV = \frac{\frac{1}{2} + 6}{\frac{1}{2} + 2} = \frac{5}{2}$$

19. The value of $\tan^{-1} \left(\frac{\cos\left(\frac{15\pi}{4}\right) - 1}{\sin\left(\frac{\pi}{4}\right)} \right)$ is equal to :

(A) $-\frac{\pi}{4}$

(B) $-\frac{\pi}{8}$

(C) $-\frac{5\pi}{12}$

(D) $-\frac{4\pi}{9}$

Answer (B)

Sol. $\tan^{-1} \left(\frac{\cos\left(\frac{15\pi}{4}\right) - 1}{\sin\left(\frac{\pi}{4}\right)} \right)$

$$= \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}}} \right)$$

$$= \tan^{-1}(1 - \sqrt{2}) = -\tan^{-1}(\sqrt{2} - 1)$$

$$= -\frac{\pi}{8}$$

20. The line $y = x + 1$ meets the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at two points P and Q . If r is the radius of the circle with PQ as diameter then $(3r)^2$ is equal to :

(A) 20

(B) 12

(C) 11

(D) 8

Answer (A)

Sol. Let point $(a, a+1)$ as the point of intersection of line and ellipse.

$$\text{So, } \frac{a^2}{4} + \frac{(a+1)^2}{2} = 1 \Rightarrow a^2 + 2(a^2 + 2a + 1) = 4$$

$$\Rightarrow 3a^2 + 4a - 2 = 0$$

If roots of this equation are α and β .

So, $P(\alpha, \alpha+1)$ and $Q(\beta, \beta+1)$

$$PQ^2 = 4r^2 = (\alpha - \beta)^2 + (\alpha - \beta)^2$$

$$\Rightarrow 9r^2 = \frac{9}{4}(2(\alpha - \beta)^2)$$

$$\begin{aligned} &= \frac{9}{2}[(\alpha + \beta)^2 - 4\alpha\beta] \\ &= \frac{9}{2}\left[\left(-\frac{4}{3}\right)^2 + \frac{8}{3}\right] \\ &= \frac{1}{2}[16 + 24] = 20 \end{aligned}$$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE**. For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$. Then the number of elements in the set $\{(n, m) : n, m \in \{1, 2, \dots, 10\} \text{ and } nA^n + mB^m = I\}$ is _____.

Answer (1)

Sol. $A^2 = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = A \Rightarrow A^K = A, K \in I$$

$$B^2 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} = B$$

So, $B^K = B, K \in I$

$$nA^n + mB^m = nA + mB$$

$$= \begin{bmatrix} 2n - 2n \\ n - n \end{bmatrix} + \begin{bmatrix} -m & 2m \\ -m & 2m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $2n - m = 1, -n + m = 0, 2m - n = 1$

$$\text{So, } (m, n) = (1, 1)$$

2. Let $f(x) = [2x^2 + 1]$ and $g(x) = \begin{cases} 2x - 3, & x < 0 \\ 2x + 3, & x \geq 0 \end{cases}$, where $[t]$ is the greatest integer $\leq t$. Then, in the open interval $(-1, 1)$, the number of points where fog is discontinuous is equal to _____.

Answer (62)

$$\text{Sol. } f(g(x)) = \begin{cases} [2(2x-3)^2] + 1, & x < 0 \\ [2(2x+3)^2] + 1, & x \geq 0 \end{cases}$$

The possible points where $fog(x)$ may be discontinuous are

$$2(2x-3)^2 \in I \text{ & } x \in (-1, 0)$$

$$2(2x+3)^2 \in I \text{ & } x \in [0, 1)$$

$$x \in (-1, 0) \quad x \in [0, 1)$$

$$2x-3 \in (-5, -3) \quad 2x+3 \in [3, 5]$$

$$2(2x-3)^2 \in (18, 50) \quad 2(2x+3)^2 \in [18, 50)$$

So, no. of points = 31
It is discontinuous at all points except $x = 0$ of no. points = 31

So, total = 62

$$3. \text{ The value of } b > 3 \text{ for which } 12 \int_3^b \frac{1}{(x^2-1)(x^2-4)} dx = \log_e\left(\frac{49}{40}\right), \text{ is equal to}$$

Answer (6)

$$\text{Sol. } I = \int \frac{1}{(x^2-1)(x^2-4)} dx = \frac{1}{3} \int \left(\frac{1}{x^2-4} - \frac{1}{x^2-1} \right) dx \\ = \frac{1}{3} \left(\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right) + C$$

$$12I = \ln \left| \frac{x-2}{x+2} \right| - 2 \ln \left| \frac{x-1}{x+1} \right| + C$$

$$12 \int_3^b \frac{dx}{(x^2-4)(x^2-1)}$$

$$= \ln \left(\frac{b-2}{b+2} \right) - 2 \ln \left(\frac{b-1}{b+1} \right) - \left(\ln \left(\frac{1}{5} \right) - 2 \ln \left(\frac{1}{2} \right) \right)$$

$$= \ln \left(\left(\frac{b-2}{b+2} \right) \cdot \frac{(b+1)^2}{(b-1)^2} \right) - \left(\ln \frac{4}{5} \right)$$

$$\text{So, } \frac{49}{40} = \frac{(b-2)}{(b+2)} \cdot \frac{(b+1)^2}{(b-1)^2} \cdot \frac{5}{4}$$

$$\Rightarrow b = 6$$

4. If the sum of the co-efficients of all the positive even powers of x in the binomial expansion of $\left(2x^3 + \frac{3}{x}\right)^{10}$ is $5^{10} - \beta \cdot 3^9$, the β is equal to ____.

Answer (83)

$$\text{Sol. } T_{r+1} = 10C_r (2x^3)^{10-r} \left(\frac{3}{x}\right)^r$$

$$= 10C_r 2^{10-r} 3^r x^{30-4r}$$

$$\text{So, } r \neq 8, 9, 10$$

$$\text{Sum of required Coeff.} = \left(2 \cdot 1^3 + \frac{3}{1} \right)^{10}$$

$$\left({}^{10}C_8 2^2 3^8 + {}^{10}C_9 2^1 3^9 + {}^{10}C_{10} 2^0 3^{10} \right)$$

$$= 5^{10} - 3^9 \left(\frac{{}^{10}C_8 \cdot 2^2}{3} + {}^{10}C_9 \cdot 2^1 + {}^{10}C_{10} \cdot 3 \right)$$

$$\beta = \frac{4}{3} \cdot {}^{10}C_8 + 20 + 3 = 83$$

5. If the mean deviation about the mean of the numbers $1, 2, 3, \dots, n$, where n is odd, is $\frac{5(n+1)}{n}$, then n is equal to _____.

Answer (21)

$$\text{Sol. Mean} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

$$\text{M.D.} = \frac{2 \left(\frac{n-1}{2} + \frac{n-3}{2} + \frac{n-5}{2} + \dots + 0 \right)}{n} = \frac{5(n+1)}{n}$$

$$\Rightarrow ((n-1) + (n-3) + (n-5) + \dots + 0) = 5(n+1)$$

$$\Rightarrow \left(\frac{n+1}{4} \right) \cdot (n-1) = 5(n+1)$$

$$\text{So, } n = 21$$

6. Let $\vec{b} = \hat{i} + \hat{j} + \lambda \hat{k}$, $\lambda \in \mathbb{R}$. If \vec{a} is a vector such that $\vec{a} \times \vec{b} = 13\hat{i} - \hat{j} - 4\hat{k}$ and $\vec{a} \cdot \vec{b} + 21 = 0$, then $(\vec{b} - \vec{a})(\hat{k} - \hat{j}) + (\vec{b} + \vec{a})(\hat{i} - \hat{k})$ is equal to

Answer (14)

Sol. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{So, } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & \lambda \end{vmatrix} = \hat{i}(y-z) + \hat{j}(z-\lambda x) + \hat{k}(x-y)$$

$$\Rightarrow \lambda y - z = 13, z - \lambda x = -1, x - y = -4$$

$$\text{and } x + y + \lambda z = -21$$

$$\Rightarrow \text{clearly, } \lambda = 3, x = -2, y = 2 \text{ and } z = -7$$

So, $\vec{b} - \vec{a} = 3\hat{i} - \hat{j} + 10\hat{k}$
 and $\vec{b} + \vec{a} = -\hat{i} + 3\hat{j} - 4\hat{k}$
 $\Rightarrow (\vec{b} - \vec{a}) \cdot (\hat{k} - \hat{j}) + (\vec{b} + \vec{a}) \cdot (\hat{i} - \hat{k}) = 11 + 3 = 14$

7. The total number of three-digit numbers, with one digit repeated exactly two times, is _____.

Answer (243)

Sol. C-1 : All digits are non-zero

$${}^9C_2 \cdot 2 \cdot \frac{3!}{2} = 216$$

C-2 : One digit is 0

$$0, 0, x \Rightarrow {}^9C_1 \cdot 1 = 9$$

$$0, x, x \Rightarrow {}^9C_1 \cdot 2 = 18$$

$$\text{Total} = 216 + 27 = 243$$

8. Let $f(x) = |(x-1)(x^2-2x-3)| + x - 3, x \in R$. If m and M are respectively the number of points of local minimum and local maximum of f in the interval $(0, 4)$, then $m + M$ is equal to

Answer (3)

Sol. $f(x) = |(x-1)(x+1)(x-3)| + (x-3)$

$$f(x) = \begin{cases} (x-3)(x^2) & 3 \leq x \leq 4 \\ (x-3)(2-x^2) & 1 \leq x < 3 \\ (x-3)(x^2) & 0 < x < 1 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 - 6x & 3 < x < 4 \\ -3x^2 + 6x + 2 & 1 < x < 3 \\ 3x^2 - 6x & 0 < x < 1 \end{cases}$$

$$f'(3^+) > 0 \quad f'(3^-) < 0 \rightarrow \text{Minimum}$$

$$f'(1^+) > 0 \quad f'(1^-) < 0 \rightarrow \text{Minimum}$$

$$x \in (1, 3) \quad f'(x) = 0 \quad \text{at one point} \rightarrow \text{Maximum}$$

$$x \in (3, 4) \quad f'(x) \neq 0$$

$$x \in (0, 1) \quad f'(x) \neq 0$$

So, 3 points

9. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be $\frac{5}{4}$. If the equation of the normal at the point $\left(\frac{8}{\sqrt{5}}, \frac{12}{5}\right)$ on the hyperbola is $8\sqrt{5}x + \beta y = \lambda$, then $\lambda - \beta$ is equal to _____.

Answer (85)

Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \left(e = \frac{5}{4} \right)$
 $\Rightarrow b^2 = a^2 \left(\frac{25}{16} - 1 \right) \Rightarrow b = \frac{3}{4}a$

Also $\left(\frac{8}{\sqrt{5}}, \frac{12}{5}\right)$ lies on the given hyperbola

$$\text{So, } \frac{64}{5a^2} - \frac{144}{25 \left(\frac{9a^2}{16} \right)} = 1 \Rightarrow a = \frac{8}{5} \text{ and } b = \frac{6}{5}$$

Equation of normal

$$\frac{64}{25} \left(\frac{x}{\frac{8}{\sqrt{5}}} \right) + \frac{36}{25} \left(\frac{y}{\frac{12}{5}} \right) = 4$$

$$\Rightarrow \frac{8}{5\sqrt{5}}x + \frac{3}{5}y = 4$$

$$\Rightarrow 8\sqrt{5}x + 15y = 100$$

$$\text{So, } \beta = 15 \text{ and } \lambda = 100$$

$$\text{Gives } \lambda - \beta = 85$$

10. Let l_1 be the line in xy -plane with x and y intercepts $\frac{1}{8}$ and $\frac{1}{4\sqrt{2}}$ respectively and l_2 be the line in zx -plane with x and z intercepts $-\frac{1}{8}$ and $-\frac{1}{6\sqrt{3}}$ respectively. If d is the shortest distance between the line l_1 and l_2 , then d^2 is equal to _____.

Answer (51)

Sol. $\frac{x - \frac{1}{8}}{\frac{1}{8}} = \frac{y}{-\frac{1}{4\sqrt{2}}} = \frac{z}{0} \quad \dots \text{L}_1$

$$\text{or } \frac{x - \frac{1}{8}}{1} = \frac{y}{-\sqrt{2}} = \frac{z}{0} \quad \dots \text{(i)}$$

Equation of L_2

$$\frac{x + \frac{1}{8}}{-6\sqrt{3}} = \frac{y}{0} = \frac{z}{8} \quad \dots \text{(ii)}$$

$$d = \frac{|\vec{c} - \vec{a}| \cdot |\vec{b} \times \vec{d}|}{|\vec{b} \times \vec{d}|}$$

$$= \frac{\left(\frac{1}{4}\hat{i} \right) \cdot (4\sqrt{2}\hat{i} + 4\hat{j} + 3\sqrt{6}\hat{k})}{\sqrt{(4\sqrt{2})^2 + 4^2 + (3\sqrt{6})^2}}$$

$$= \frac{\sqrt{2}}{\sqrt{32 + 16 + 54}} = \frac{1}{\sqrt{51}}$$

$$d^2 = 51$$