MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let a circle C touch the lines $L_1 : 4x - 3y + K_1 = 0$ and $L_2 : 4x - 3y + K_2 = 0$, $K_1, K_2 \in \mathbf{R}$. If a line passing through the centre of the circle C intersects L_1 at $(-1, 2)$ and L_2 at $(3, -6)$, then the equation of the circle C is :

   (A) $(x - 1)^2 + (y - 2)^2 = 4$

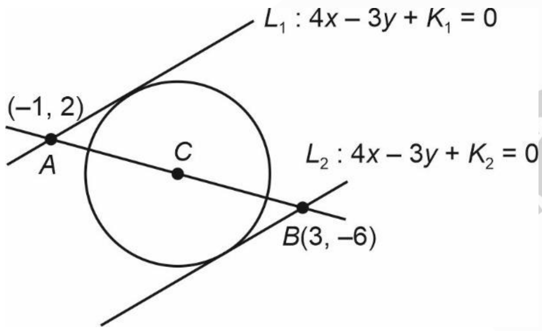
   (B) $(x + 1)^2 + (y - 2)^2 = 4$

   (C) $(x - 1)^2 + (y + 2)^2 = 16$

   (D) $(x - 1)^2 + (y - 2)^2 = 16$

Answer (C)

Sol.



Co-ordinate of centre $C \equiv \left(\frac{3+(-1)}{2}, \frac{-6+2}{2}\right) \equiv (1, -2)$

L_1 is passing through A

$\Rightarrow -4 - 6 + K_1 = 0$

$\Rightarrow K_1 = 10$

L_2 is passing through B

$\Rightarrow 12 + 18 + K_2 = 0$

$\Rightarrow K_2 = -30$

Equation of $L_1 : 4x - 3y + 10 = 0$

Equation of $L_1 : 4x - 3y - 30 = 0$

Diameter of circle $= \left|\frac{10+30}{\sqrt{4^2+(-3)^2}}\right| = 8$

$\Rightarrow$ Radius = 4

Equation of circle $(x - 1)^2 + (y + 2)^2 = 16$

2. The value of $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$ is equal to :

   (A) $\frac{\pi^2}{4}$

   (B) $\frac{\pi^2}{2}$

   (C) $\frac{\pi}{4}$

   (D) $\frac{\pi}{2}$

Answer (C)

Sol. $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$

Let $\cos x = t$

$\sin x\, dx = dt$

$= \int_1^{-1} \frac{-e^t dt}{(1+t^2)(e^t+e^{-t})}$

$I = \int_{-1}^{1} \frac{e^t}{(1+t^2)(e^t+e^{-t})}dt$ ...(i)

$I = \int_{-1}^{1} \frac{e^{-t}}{(1+t^2)(e^{-t}+e^t)}dt$ ...(ii)

Adding (i) and (ii)

$2I = \int_{-1}^{1} \frac{dt}{1+t^2}$

$2I = \tan^{-t}\Big]_{-1}^{1}$

$2I = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$

$2I = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

3. Let a, b and c be the length of sides of a triangle ABC such that $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}$ If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively, then the value of $\frac{R}{r}$ is equal to :

(A) $\frac{5}{2}$ (B) 2

(C) $\frac{3}{2}$ (D) 1

Answer (A)

Sol. $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}=\lambda$

$$a+b=7\lambda$$
$$b+c=8\lambda$$
$$c+a=9\lambda$$
$$a+b+c=12\lambda$$

$\therefore$ $a = 4\lambda$, $b = 3\lambda$, $c = 5\lambda$

$$S=\frac{4\lambda+3\lambda+5\lambda}{2}=6\lambda$$

$$\Delta=\sqrt{S(s-a)(s-b)(s-c)}=\sqrt{(6\lambda)(2\lambda)(3\lambda)(\lambda)}$$
$$=6\lambda^2$$

$$R=\frac{abc}{4\Delta}=\frac{(4\lambda)(3\lambda)(5\lambda)}{4(6\lambda^2)}=\frac{5}{2}\lambda$$

$$r=\frac{\Delta}{s}=\frac{6\lambda^2}{6\lambda}=\lambda$$

$$\frac{R}{r}=\frac{\frac{5}{2}\lambda}{\lambda}=\frac{5}{2}$$

4. Let $f:\mathbf{N}\to\mathbf{R}$ be a function such that $f(x + y) = 2f(x)\,f(y)$ for natural numbers x and y. If $f(1) = 2$, then the value of α for which

$$\sum_{k=1}^{10} f(\alpha+k)=\frac{512}{3}(2^{20}-1)$$

holds, is :

(A) 2

(B) 3

(C) 4

(D) 6

Answer (C)

Sol. $f(x + y) = 2f(x)f(y)$ & $f(1) = 2$

$x = y = 1$

$$\left.\begin{array}{l}\Rightarrow\ f(2)=2^3\\ \quad x=2,\ y=1\\ \Rightarrow\ f(3)=2^5\end{array}\right\} f(x)=2^{(2x-1)}$$

Now,

$$\sum_{k=1}^{10} f(\alpha+k)=\frac{512}{3}(2^{20}-1)$$

$$2\sum_{k=1}^{10} f(\alpha)f(k)=\frac{512}{3}(2^{20}-1)$$

$$2f(\alpha)\left[f(1)+f(2)+.....+f(10)\right]=\frac{512}{3}(2^{20}-1)$$

$$=2f(\alpha)\left[2+2^3+2^5+.....\text{upto 10 terms}\right]=\frac{512}{3}(2^{20}-1)$$

$$=2f(\alpha)\cdot 2\left(\frac{2^{20}-1}{4-1}\right)=\frac{512}{3}(2^{20}-1)$$

$f(\alpha) = 128 = 2^{2\alpha} - 1$

$= 2\alpha - 1 = 7$

$\Rightarrow$ $\alpha = 4$

5. Let A be a 3×3 real matrix such that

$$A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix};\ A\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}\ \text{and}\ A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}.$$

If $X = (x_1, x_2, x_3)^T$ and I is an identity matrix of order 3, then the system $(A-2I)X=\begin{pmatrix}4\\1\\1\end{pmatrix}$ has :

(A) No solution

(B) Infinitely many solutions

(C) Unique solution

(D) Exactly two solutions

Answer (B)

Sol. Let $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$

$$A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}\Rightarrow\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}\Rightarrow\begin{array}{r}a+b=1\\ d+e=1\\ g+h=0\end{array}$$

$$A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix} \Rightarrow \begin{matrix}a+c=-1\\d+f=0\\g+i=1\end{matrix}$$

$$A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix} \Rightarrow \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix} \Rightarrow \begin{matrix}c=1\\f=1\\i=2\end{matrix}$$

Solving will get

$a = -2$, $b = 3$, $c = 1$, $d = -1$, $e = 2$, $f = 1$, $g = -1$, $h = 1$, $i = 2$

$$A=\begin{bmatrix}-2 & 3 & 1\\-1 & 2 & 1\\-1 & 1 & 2\end{bmatrix} \Rightarrow A=2I=\begin{bmatrix}-4 & 3 & 1\\-1 & 0 & 1\\-1 & 1 & 0\end{bmatrix}$$

$$(A-2I)x=\begin{bmatrix}4\\1\\1\end{bmatrix}$$

$\Rightarrow\ -4x_1 + 3x_2 + x_3 = 4$ ...(i)

$-x_1 + x_3 = 1$ ...(ii)

$-x_1 + x_2 = 1$ ...(iii)

So 3(iii) + (ii) = (i)

$\therefore$ Infinite solution

6. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as

$f(x) = x^3 + x - 5$

If $g(x)$ is a function such that $f(g(x)) = x, \forall 'x' \in \mathbf{R}$, then $g'(63)$ is equal to______.

(A) $\frac{1}{49}$ (B) $\frac{3}{49}$

(C) $\frac{43}{49}$ (D) $\frac{91}{49}$

Answer (A)

Sol. $f(x) = 3x^2 + 1$

$f'(x)$ is bijective function

and $f(g(x)) = x \Rightarrow g(x)$ is inverse of $f(x)$

$g(f(x)) = x$

$g'(f(x))\cdot f'(x) = 1$

$$g'(f(x)) = \frac{1}{3x^2+1}$$

Put $x = 4$ we get

$$g'(63) = \frac{1}{49}$$

7. Consider the following two propositions :

$P1 : \sim(p \to \sim q)$

$P2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \to ((\sim p) \vee q)$ is evaluated as FALSE, then :

(A) $P1$ is TRUE and $P2$ is FALSE

(B) $P1$ is FALSE and $P2$ is TRUE

(C) Both $P1$ and $P2$ are FALSE

(D) Both $P1$ and $P2$ are TRUE

Answer (C)

Sol. Given $p \to (\sim p \vee q)$ is false

$\Rightarrow\ \sim p \vee q$ is false and p is true

Now p = True.

$\sim T \vee q = F$

$F \vee q = F \Rightarrow q$ is false

$P1 : \sim(T \to \sim F) \equiv \sim(T \to T) \equiv$ False.

$P2 : (T \wedge \sim F) \wedge (\sim T \vee F) \equiv (T \wedge T) \wedge (F \vee F)$

$\equiv T \wedge F \equiv$ False

8. If $\frac{1}{2\cdot 3^{10}} + \frac{1}{2^2\cdot 3^9} + \ldots + \frac{1}{2^{10}\cdot 3} = \frac{K}{2^{10}\cdot 3^{10}}$, then the remainder when K is divided by 6 is :

(A) 1 (B) 2

(C) 3 (D) 5

Answer (D)

Sol. $\frac{1}{2\cdot 3^{10}} + \frac{1}{2^2\cdot 3^9} + \ldots + \frac{1}{2^{10}\cdot 3} = \frac{K}{2^{10}\cdot 3^{10}}$

$$\Rightarrow \frac{1}{2\cdot 3^{10}}\left[\frac{\left(\frac{3}{2}\right)^{10}-1}{\frac{3}{2}-1}\right] = \frac{K}{2^{10}\cdot 3^{10}}$$

$$= \frac{3^{10}-2^{10}}{2^{10}\cdot 3^{10}} = \frac{K}{2^{10}\cdot 3^{10}} \Rightarrow K = 3^{10}-2^{10}$$

Now $K = (1 + 2)^{10} - 2^{10}$

$= {}^{10}C_0 + {}^{10}C_1\, 2 + {}^{10}C_2\, 2^3 + \ldots + {}^{10}C_{10}\, 2^{10} - 2^{10}$

$= {}^{10}C_0 + {}^{10}C_1\, 2 + 6\lambda + {}^{10}C_9 \cdot 2^9$

$= 1 + 20 + 5120 + 6\lambda$

$= 5136 + 6\lambda + 5$

$= 6\mu + 5$

$\lambda, \mu \in N$

$\therefore$ remainder = 5

9. Let $f(x)$ be a polynomial function such that $f(x) + f'(x) + f''(x) = x^5 + 64$. Then, the value of $\lim\limits_{x\to 1}\frac{f(x)}{x-1}$ is equal to :

(A) –15 (B) –60

(C) 60 (D) 15

Answer (A)

Sol. $\lim\limits_{x\to 1}\frac{f(x)}{x-1}$

$f(x) + f'(x) + f''(x) = x^5 + 64$

Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$

$f'(x) = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 20x^3 + 12ax^2 + 6bx + 2c$

$x^5 + (a + 5)x^4 + (b + 4a + 20)x^3 + (c + 3b + 12a)x^2 + (d + 2c + 6b)x + e + d + 2c = x^5 + 64$

$\Rightarrow a + 5 = 0$

$b + 4a + 20 = 0$

$c + 3b + 12a = 0$

$d + 2c + 6b = 0$

$e + d + 2c = 64$

$\therefore$ $a = -5, b = 0, c = 60, d = -120, e = 64$

$\therefore$ $f(x) = x^5 - 5x^4 + 60x^2 - 120x + 64$

Now, $\lim\limits_{x\to 1}\frac{x^5-5x^4+60x^2-120x+64}{x-1}$ is $\left(\frac{0}{0}\text{ from}\right)$

By L' Hopital rule

$\lim\limits_{x\to 1}\frac{5x^4-20x^3+120x-120}{1}$

$= -15$

10. Let E_1 and E_2 be two events such that the conditional probabilities $P(E_1|E_2)=\frac{1}{2}$, $P(E_2|E_1)=\frac{3}{4}$ and $P(E_1\cap E_2)=\frac{1}{8}$. Then :

(A) $P(E_1\cap E_2)=P(E_1)\cdot P(E_2)$

(B) $P(E_1'\cap E_2')=P(E_1')\cdot P(E_2)$

(C) $P(E_1\cap E_2')=P(E_1)\cdot P(E_2)$

(D) $P(E_1'\cap E_2)=P(E_1)\cdot P(E_2)$

Answer (C)

Sol. $P\left(\frac{E_1}{E_2}\right)=\frac{1}{2}\Rightarrow\frac{P(E_1\cap E_2)}{P(E_2)}=\frac{1}{2}$

$P\left(\frac{E_2}{E_1}\right)=\frac{3}{4}\Rightarrow\frac{P(E_2\cap E_1)}{P(E_1)}=\frac{3}{4}$

$P(E_1\cap E_2)=\frac{1}{8}$

$P(E_2)=\frac{1}{4}, P(E_1)=\frac{1}{6}$

(A) $P(E_1\cap E_2)=\frac{1}{8}$ and $P(E_1).P(E_2)=\frac{1}{24}$

$\Rightarrow P(E_1\cap E_2)\neq P(E_1).P(E_2)$

(B) $P(E_1'\cap E_2')=1-P(E_1\cup E_2)$

$=1-\left[\frac{1}{4}+\frac{1}{6}-\frac{1}{8}\right]=\frac{17}{24}$

$P(E_1')=\frac{3}{4}\Rightarrow P(E_1')P(E_2)=\frac{3}{24}$

$\Rightarrow P(E_1'\cap E_2')\neq P(E_1').P(E_2)$

(C) $P(E_1\cap E_2')=P(E_1)-P(E_1\cap E_2)$

$=\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

$P(E_1).P(E_2)=\frac{1}{24}$

$\Rightarrow P(E_1\cap E_2')=P(E_1).P(E_2)$

(D) $P(E_1'\cap E_2)=P(E_2)-P(E_1\cap E_2)$

$=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}$

$P(E_1)P(E_2)=\frac{1}{24}$

$\Rightarrow P(E_1'\cap E_2)\neq P(E_1).P(E_2)$

11. Let $A=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$. If M and N are two matrices given by $M=\sum\limits_{k=1}^{10}A^{2k}$ and $N=\sum\limits_{k=1}^{10}A^{2k-1}$ then MN^2 is :

(A) a non-identity symmetric matrix

(B) a skew-symmetric matrix

(C) neither symmetric nor skew-symmetric matrix

(D) an identity matrix

Answer (A)

Sol. $A=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$

$$A^2=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}=-4I$$

$M = A^2 + A^4 + A^6 + \dots + A^{20}$

$= -4I + 16I - 64I + \dots$ upto 10 terms

$= -I\,[4 - 16 + 64 \dots +$ upto 10 terms$]$

$$=-I.4\left[\frac{(-4)^{10}-1}{-4-1}\right]=\frac{4}{5}(2^{20}-1)I$$

$N=A^1+A^3+A^5+\dots+A^{19}$

$= A - 4A + 16A + \dots$ upto 10 terms

$$=A\left(\frac{(-4)^{10}-1}{-4-1}\right)=-\left(\frac{2^{20}-1}{5}\right)A$$

$$N^2=\frac{(2^{20}-1)^2}{2^5}A^2=\frac{-4}{25}(2^{20}-1)^2I$$

$$MN^2=\frac{-16}{125}(2^{20}-1)^3I=KI \qquad (K\neq\pm1)$$

$(MN^2)^T = (KI)^T = KI$

$\therefore$ A is correct

12. Let $g:(0,\infty)\to R$ be a differentiable function such that $\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)\left(e^x+1-xe^x\right)}{\left(e^x+1\right)^2}\right)dx$ $=\frac{xg(x)}{e^x+1}+c$, for all $x > 0$, where c is an arbitrary constant. Then :

(A) g is decreasing in $\left(0,\frac{\pi}{4}\right)$

(B) g' is increasing in $\left(0,\frac{\pi}{4}\right)$

(C) $g+g'$ is increasing in $\left(0,\frac{\pi}{2}\right)$

(D) $g-g'$ is increasing in $\left(0,\frac{\pi}{2}\right)$

Answer (D)

Sol. $\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c$

Differentiating on both sides

$$\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x-1)^2}$$

$$=\frac{(e^x+1)\left(g(x)+xg^{\frac{1}{x}}\right)-xg(x)e^x}{(e^x+1)^2}$$

$$=\frac{g(x)[e^x+1-xe^x]}{(e^x+1)^2}+\frac{xg'(x)(e^x+1)}{(e^x+1)^2}$$

$$=\frac{x(\cos x-\sin x)}{e^x+1}=\frac{xg'(x)}{e^x+1}$$

$\Rightarrow\ g'(x)=\cos x-\sin x>0$ in $\left(0,\frac{\pi}{4}\right)$

$\Rightarrow$ g(x) is increasing in $\left(0,\frac{\pi}{4}\right)$ $\Rightarrow$ A is wrong

Now, $g''(x)=-\sin x-\cos x<0$ in $\left(0,\frac{\pi}{4}\right)$

$\Rightarrow$ $g(x)$ is increasing in $\left(0,\frac{\pi}{4}\right)\Rightarrow B$ is wrong

Let $h(x) = g(x) + g'(x)$

$\Rightarrow\ h'(x)=g'(x)+g''(x)=-2\sin x<0$ in $x\in\left(0,\frac{\pi}{2}\right)$

$\Rightarrow$ $g + g'$ is decreasing in $\left(0,\frac{\pi}{2}\right)\Rightarrow$

c is wrong

Let $J(x) = g(x) - g'(x)$

$J'(x)=g'(x)-g''(x)=2\cos x>0$ in $\left(0,\frac{\pi}{2}\right)$

$\Rightarrow$ $g - g'$ is increasing in $\left(0,\frac{\pi}{2}\right)\Rightarrow$ correct

13. Let $f:R\to R$ and $g:R\to R$ be two functions defined by $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ and $g(x)=\frac{1-2e^{2x}}{e^x}$. Then, for which of the following range of α, the inequality $f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\frac{5}{3}\right)\right)$ holds?

(A) (2, 3)
(B) (–2, –1)
(C) (1, 2)
(D) (–1, 1)

Answer (A)

Sol. $f(x) = \log_e(x^2+1) - e^{-x} + 1$

$$f'(x) = \frac{2x}{x^2+1} + e^{-x}$$

$$= \frac{2}{x+\frac{1}{x}} + e^{-x} > 0 \quad \forall x \in R$$

$$g(x) = e^{-x} - 2e^{x}$$

$$g'(x) = -e^{-x} - 2e^{x} < 0 \qquad \forall x \in R.$$

$\Rightarrow$ $f(x)$ is increasing and $g(x)$ is decreasing function.

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$$

$$\Rightarrow \frac{(\alpha-1)^2}{3} < \alpha - \frac{5}{3}$$

$= \alpha^2 - 5\alpha + 6 < 0$

$= (\alpha - 2)(\alpha - 3) < 0$

$= \alpha \in (2, 3)$

14. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ a_i > 0, i = 1, 2, 3$ be a vector which makes equal angles with the coordinate axes OX, OY and OZ. Also, let the projection of $\vec{a}$ on the vector $3\hat{i} + 4\hat{j}$ be 7. Let $\vec{b}$ be a vector obtained by rotating $\vec{a}$ with 90°. If $\vec{a}, \vec{b}$ and x-axis are coplanar, then projection of a vector $\vec{b}$ on $3\hat{i} + 4\hat{j}$ is equal to :

(A) $\sqrt{7}$ (B) $\sqrt{2}$

(C) 2 (D) 7

Answer (B)

Sol. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \cos^2\alpha = \frac{1}{3}$

$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\vec{a} = \frac{\lambda}{3}(\hat{i} + \hat{j} + \hat{k}), \lambda > 0$$

$$\frac{\lambda}{\sqrt{3}} \frac{(\hat{i} + \hat{j} + \hat{k}).(3\hat{i} + 4\hat{j})}{\sqrt{3^2 + 4^2}} = 7$$

$$\Rightarrow \frac{\lambda}{\sqrt{3}}(3 + 4) = 7 \times 5$$

$$\therefore \lambda = 5\sqrt{3}$$

$$\vec{a} = 5(\hat{i} + \hat{j} + \hat{k})$$

Let $\vec{b} = p\hat{i} + q\hat{j} + r\hat{k}$

$\vec{a}.\vec{b} = 0$ and $[\vec{a}\, \vec{b}\, \hat{i}] = 0$

$\Rightarrow p + q + r = 0$ ...(i)

$$\& \begin{vmatrix} p & q & r \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \boxed{q = r} \boxed{p = -2r}$$

$$\vec{b} = -2r\hat{i} + r\hat{j} + r\hat{k}$$

$$\vec{b} = r(-2\hat{i} + \hat{j} + \hat{k})$$

Now $|\vec{a}| = |\vec{b}|$

$$5\sqrt{3} = |r|\sqrt{b} \Rightarrow |r| = \frac{5}{\sqrt{2}}$$

$$\Rightarrow \text{Projection of } \vec{b} \text{ on } 3\hat{i} + 4\hat{j} = \left|\frac{\vec{b}.(3\hat{i} + 4\hat{j})}{\sqrt{3^2 + 4^2}}\right|$$

$$= |r|\frac{(-6+4)}{5} = \left|\frac{-2r}{5}\right|$$

$$\text{Projection} = \frac{2}{5} \times \frac{5}{\sqrt{2}} = \sqrt{2}$$

$\therefore$ B is correct

15. Let $y = y(x)$ be the solution of the differential equation $(x + 1)y' - y = e^{3x}(x + 1)^2$, with $y(0) = \frac{1}{3}$.

Then, the point $x = -\frac{4}{3}$ for the curve $y = y(x)$ is :

(A) not a critical point

(B) a point of local minima

(C) a point of local maxima

(D) a point of inflection

Answer (B)

Sol. $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

$$\text{If } e^{-\int \frac{1}{x+1}x} = e^{-\log(x+1)} = \frac{1}{x+1}$$

$$\therefore y\left(\frac{1}{x+1}\right) = \int \frac{e^{3x}(x+1)}{x+1}dx$$

$$\frac{y}{x+1}=\int e^{3x}dx$$

$$\boxed{\frac{y}{x+1}=\frac{e^{3x}}{3}+c}$$

$$\because\ y(0)=\frac{1}{3}$$

$$\frac{1}{3}=\frac{1}{3}+c$$

$\therefore\ c = 0$

So : $\boxed{y=\frac{e^{3x}}{3}(x+1)}$

$$y'=e^{3x}(x+1)+\frac{e^{3x}}{3}=e^{3x}\left(x+\frac{4}{3}\right)$$

$$y''=3e^{3x}\left(x+\frac{4}{3}\right)+e^{3x}=e^{3x}(3x+5)$$

$$y'=0 \text{ at } x=\frac{-4}{3}\ \&\ y''=e^{-4}(1)>0 \text{ at } x=\frac{-4}{3}$$

$\Rightarrow\ x=\frac{-4}{3}$ is point of local minima

16. If $y = m_1x + c_1$ and $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of circle $x^2 + y^2 = 2$ and parabola $y^2 = x$, then the value of $8|m_1m_2|$ is equal to :

(A) $3+4\sqrt{2}$ (B) $-5+6\sqrt{2}$

(C) $-4+3\sqrt{2}$ (D) $7+6\sqrt{2}$

Answer (C)

Sol. Let tangent to $y^2 = x$ be

$$y=mx+\frac{1}{4m}$$

For it being tangent to circle.

$$\left|\frac{\frac{1}{4}m}{\sqrt{1+m^2}}\right|=\sqrt{2}$$

$\Rightarrow\ 32m^4 + 32m^2 - 1 = 0$

$$\Rightarrow\ m^2=\frac{-32\pm\sqrt{(32)^2+4(32)}}{64}$$

$$\Rightarrow\ 8m_1m_2=-4+3\sqrt{2}$$

17. Let Q be the mirror image of the point $P(1, 0, 1)$ with respect to the plane S: $x + y + z = 5$. If a line L passing through $(1, -1, -1)$, parallel to the line PQ meets the plane S at R, then QR^2 is equal to :

(A) 2 (B) 5

(C) 7 (D) 11

Answer (B)

Sol. As L is parallel to PQ d.r.s of S is <1, 1, 1>

$$\therefore\ L\equiv\frac{x-1}{1}=\frac{y+1}{1}=\frac{z+1}{1}$$

Point of intersection of L and S be λ

$\Rightarrow\ (\lambda + 1) + (\lambda - 1) + (\lambda - 1) = S$

$\Rightarrow\ \lambda = 2$

$\therefore\ R \equiv (3, 1, 1)$

Let $Q(\alpha, \beta, \gamma)$

$$\Rightarrow\ \frac{\alpha-1}{1}=\frac{\beta}{1}=\frac{\gamma-1}{1}=\frac{-2(-3)}{3}$$

$\Rightarrow\ \alpha = 3, \beta = 2, \gamma = 3$

$\Rightarrow\ Q \equiv (3, 2, 3)$

$(QR)^2 = 0^2 + (1)^2 + (2)^2 = 5$

18. If the solution curve $y = y(x)$ of the differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$, which passes through the point (1,1) and intersects the line $y=\sqrt{3}x$ at the point $\left(\alpha, \sqrt{3}\alpha\right)$, then value of $\log_e\left(\sqrt{3}\alpha\right)$ is equal to :

(A) $\frac{\pi}{3}$

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{12}$

(D) $\frac{\pi}{6}$

Answer (C)

Sol. $\frac{dy}{dx}=\frac{y^2}{xy-x^2-y^2}$

Put $y = vx$ we get

$$v+x\frac{dv}{dx}=\frac{v^2}{v-1-v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - v^2 + v + v^3}{v - 1 - v^2}$$

$$\Rightarrow \quad \int \frac{v - 1 - v^2}{v(1+v^2)} dv = \int \frac{dx}{x}$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x}\right) = \ln x + c$$

As it passes through (1, 1)

$$c = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x}\right) = \ln x + \frac{\pi}{4}$$

Put $y = \sqrt{3}x$ we get

$$\Rightarrow \quad \frac{\pi}{3} - \ln\sqrt{3} = \ln x + \frac{\pi}{4}$$

$$\Rightarrow \quad \ln x = \frac{\pi}{12} - \ln\sqrt{3} = \ln\alpha$$

$$\therefore \quad \ln\left(\sqrt{3}\alpha\right) = \ln\sqrt{3} + \ln\alpha$$

$$= \ln\sqrt{3} + \frac{\pi}{12} - \ln\sqrt{3} = \frac{\pi}{12}$$

19. Let $x = 2t$, $y = \frac{t^2}{3}$ be a conic. Let S be the focus and B be the point on the axis of the conic such that $SA \perp BA$, where A is any point on the conic. If k is the ordinate of the centroid of the ΔSAB, then $\lim\limits_{t\to 1} k$ is equal to

(A) $\frac{17}{18}$

(B) $\frac{19}{18}$

(C) $\frac{11}{18}$

(D) $\frac{13}{18}$

Answer (D)

Sol. $x = 2t$, $y = \frac{2}{3}$

$$t \to 1 \quad A \equiv \left(2, \frac{1}{3}\right)$$

Given conic is $x^2 = 12y \Rightarrow S \equiv (0, 3)$

Let $B \equiv (0, \beta)$

Given $SA \perp BA$

$$\left(\frac{\frac{1}{3}}{2-3}\right)\left(\frac{\beta - \frac{1}{3}}{-2}\right) = -1$$

$$\Rightarrow \quad \left(\beta - \frac{1}{3}\right)\frac{1}{3} = -2$$

$$\Rightarrow \quad \beta = \frac{1}{3}\left(\frac{-17}{3}\right)$$

$$\text{Ordinate of centroid} = \underset{(\text{as } t\to 1)}{k} = \frac{\beta + \frac{1}{3} + 3}{3}$$

$$= \frac{\frac{-17}{9} + \frac{10}{3}}{3} = \frac{13}{18}$$

20. Let a circle C in complex plane pass through the points $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z(\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3, then $\arg(z)$ is equal to:

(A) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right) - \pi$

(B) $\tan^{-1}\left(\frac{24}{7}\right) - \pi$

(C) $\tan^{-1}(3) - \pi$

(D) $\tan^{-1}\left(\frac{3}{4}\right) - \pi$

Answer (B)

Sol. $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$

Clearly $C \equiv x^2 + y^2 = 25$

Let $z(x, y)$

$$\Rightarrow \quad \left(\frac{y-4}{x-3}\right)\left(\frac{2}{-4}\right) = -1$$

$\Rightarrow \quad y = 2x - 2 \equiv L$

$\therefore \quad z$ is intersection of C & L

$$\Rightarrow \quad z \equiv \left(\frac{-7}{5}, \frac{-24}{5}\right)$$

$$\therefore \quad \text{Arg}(z) = -\pi + \tan^{-1}\left(\frac{24}{7}\right)$$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let C_r denote the binomial coefficient of x^r in the expansion of $(1 + x)^{10}$. If for $\alpha, \beta \in \mathbf{R}$, $C_1 + 3\cdot 2\, C_2 + 5\cdot 3\, C_3 + \ldots$ upto 10 terms

$$= \frac{\alpha \times 2^{11}}{2^{\beta}-1}\left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots \text{ upto 10 terms}\right) \quad \text{then}$$

the value of $\alpha + \beta$ is equal to ________.

Answer (286*)

Sol. Given that $C_1 + 2\cdot 3C_2 + 5\cdot 3C_3 + \ldots$ 10 terms

$$= \frac{\alpha.2^{11}}{2^{\beta}-1}\left(C_1 + \frac{C_2}{2} + \ldots\right)$$

$$\Rightarrow \sum_{r=1}^{10} r(2r-1)C_r = \frac{\alpha.2^{11}}{2^{\beta}-1}\left(\sum_{r=0}^{10} \frac{C_r}{r}\right)$$

Using $C_1 + 2C_2 + \ldots + nC_n = n.2^{n-1}$,

$1^2C_1 + 2^2C_2 + \ldots + n^2C_n = n.2^{n-1} + n(n-1)2^{n-2}$

and $C_0 + \frac{C_1}{2} + \ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ we get

$$\Rightarrow 2(10.2^9 + 10.9.2^8) - 10.2^9 = \frac{\alpha.2^{11}}{2^{\beta}-1}\frac{(2^{11}-1)}{11}$$

Comparing both side we get

$$2^{11}.25 = \frac{\alpha.2^{11}}{2^{\beta}-1}\frac{(2^{11}-1)}{11}$$

$\Rightarrow \alpha = 25 \times 11 = 275$ & $\beta = 11$

$\Rightarrow \alpha + \beta = 286$

(*RHS shall have II-terms)

2. The number of 3-digit odd numbers, whose sum of digits is a multiple of 7, is ________.

Answer (63)

Sol. For odd number unit place shall be 1, 3, 5, 7 or 9.

$\therefore$ $\underline{x}\ \underline{y}\ \underline{1}$, $\underline{x}\ \underline{y}\ \underline{3}$, $\underline{x}\ \underline{y}$ 5, x y 7, x y 9 are the type of numbers.

If $x\ y$ 1 then

$x + y = 6, 13, 20 \ldots$ Cases are required

i.e., $6 + 6 + 0 + \ldots = 12$ ways

If $x\ y$ 3 then

$x + y = 4, 11, 18, \ldots$. Cases are required

i.e., $4 + 8 + 1 + 0 \ldots = 13$ ways

Similarly for $x\ y$ 5, we have

$x + y = 2, 9, 16, \ldots$

i.e., $2 + 9 + 3 = 14$ ways

for $x\ y$ 7 we have

$x + y = 0, 7, 14, \ldots$.

i.e., $0 + 7 + 5 = 12$ ways

And for $x\ y$ 9 we have

$x + y = 5, 12, 19 \ldots$

i.e., $5 + 7 + 0 \ldots = 12$ ways

$\therefore$ Total 63 ways

3. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 4, |\vec{b}| = 3$ and $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$. Then $\left|\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)\right|^2 + 4\left(\vec{a}\,.\,\vec{b}\right)^2$ is equal to ______.

Answer (576)

Sol. $|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})|^2 + 4(\vec{a}\cdot\vec{b})^2$

$$\Rightarrow |\vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}|^2 + 4(\vec{a}\cdot\vec{b})^2$$

$$\Rightarrow |2(\vec{a}\times\vec{b})|^2 + 4(\vec{a}\cdot\vec{b})^2$$

$$\Rightarrow 4\left(\vec{a}\times\vec{b}\right)^2 + \left(\vec{a}\cdot\vec{b}\right)^2$$

$$\Rightarrow 4|\vec{a}|^2|\vec{b}|^2 = 4.16.9 = 576$$

4. Let the abscissae of the two points P and Q be the roots of $2x^2 - rx + p = 0$ and the ordinates of P and Q be the roots of $x^2 - sx - q = 0$. If the equation of the circle described on PQ as diameter is $2(x^2 + y^2) - 11x - 14y - 22 = 0$, then $2r + s - 2q + p$ is equal to __________.

Answer (7)

Sol. Let $P(x_1, y_1)$ & $Q(x_2, y_2)$

$$\Rightarrow\ 2x^2 - rx + p = 0 \begin{cases} x_1 \\ x_2 \end{cases}$$

$$\&\ \ x^2 - sx - q = 0 \begin{cases} y_1 \\ y_2 \end{cases}$$

$\therefore$ Equation of circle $\equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow\ x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$

$$\Rightarrow\ x^2 - \frac{r}{2}x + \frac{p}{2} + y^2 + sy - q = 0$$

$\Rightarrow\ 2x^2 + 2y^2 - rx + 2sy + p - 2q = 0$

Compare with $2x^2 + 2y^2 - 11x - 14y - 22 = 0$

We get $r = 11$, $s = 7$, $p - 2q = -22$

$\Rightarrow\ 2r + s + p - 2q = 22 + 7 - 22 = 7$

5. The number of values of x in the interval $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$ for which $14\operatorname{cosec}^2x - 2\sin^2x = 21 - 4\cos^2x$ holds, is ______________.

Answer (4)

Sol. $$\frac{14}{\sin^2 x} - 2\sin^2 x = 21 - 4(1 - \sin^2 x)$$

Let $\sin^2x = t$

$\Rightarrow\ 14 - 2t^2 = 21t - 4t + 4t^2$

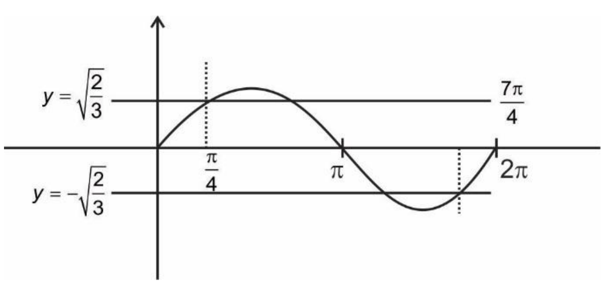
$\Rightarrow\ 6t^2 + 17t - 14 = 0$

$\Rightarrow\ 6t^2 + 21t - 4t - 14 = 0$

$\Rightarrow\ 3t(2t + 7) - 2(2t + 7) = 0$

$$\Rightarrow\ \sin^2 x = \frac{2}{3} \text{ or } -\frac{7}{3} \text{ (rejected)}$$

$$\Rightarrow\ \sin x = \pm\sqrt{\frac{2}{3}}$$



$$\therefore\ \sin x = \pm\sqrt{\frac{2}{3}} \text{ has 4 solutions in } \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$$

6. For a natural number n, let $\alpha_n = 19^n - 12^n$. Then, the value of $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$ is ______________.

Answer (4)

Sol. $\alpha_n = 19^n - 12^n$

Let equation of roots 12 & 19 i.e.

$x^2 - 31x + 228 = 0$

$$\Rightarrow\ (31 - x) = \frac{228}{x} \qquad \text{(where } x \text{ can be 19 or 12)}$$

$$\therefore\ \frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} = \frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57(19^8 - 12^8)}$$

$$= \frac{19^9(31 - 19) - 12^9(31 - 12)}{57(19^8 - 12^8)}$$

$$= \frac{228(19^8 - 12^8)}{57(19^8 - 12^8)} = 4.$$

7. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \left(2\left(1 - \frac{x^{25}}{2}\right)(2 + x^{25})\right)^{\frac{1}{50}}$. If the function $g(x) = f(f(f(x))) + f(f(x))$, then the greatest integer less than or equal to $g(1)$ is ____________.

Answer (2)

Sol. $$f(x) = \left(2\left(\frac{2 - x^{25}}{2}\right)(2 + x^{25})\right)^{\frac{1}{50}}$$

$$= (4 - x^{50})^{\frac{1}{50}}$$

$$f(f(x)) = \left(4 - \left((4 - x^{50})^{\frac{1}{50}}\right)^{50}\right)^{\frac{1}{50}} = x$$

As $f(f(x)) = x$ we have

$g(x) = f(f(f(x))) + f(f(x)) = f(x) + x$

$\Rightarrow\ g(x) = (4 - x^{50})^{1/50} + x$

$\Rightarrow\ g(1) = 3^{1/50} + 1$

$\Rightarrow\ [g(1)] = 2$

8. Let the lines

$L_1 : \vec{r} = \lambda\left(\hat{i} + 2\hat{j} + 3\hat{k}\right), \lambda \in R$

$L_2 : \vec{r} = \left(\hat{i} + 3\hat{j} + \hat{k}\right) + \mu\left(\hat{i} + \hat{j} + 5\hat{k}\right); \mu \in R,$

intersect at the point S. If a plane $ax + by - z + d = 0$ passes through S and is parallel to both the lines L_1 and L_2, then the value of $a + b + d$ is equal to ________.

Answer (5)

Sol. As plane is parallel to both the lines we have d.r's of normal to the plane as < 7, – 2, –1>

$$\left(\text{from} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 7\hat{i} - \hat{j}(2) + \hat{k}(-1)\right)$$

Also point of intersection of lines is $2\hat{i} + 4\hat{j} + 6\hat{k}$

$\therefore$ Equation of plane is

$7(x - 2) - 2(y - 4) - 1(z - 6) = 0$

$\Rightarrow$ $7x - 2y - z = 0$

$a + b + d = 7 - 2 + 0 = 5$

9. Let A be a 3 × 3 matrix having entries from the set {–1, 0, 1}. The number of all such matrices A having sum of all the entries equal to 5, is ____________.

Answer (414)

Sol. Let matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

We need

$a + b + c + d + e + f + g + h + i = 5$

Possible cases	Number of ways
$5 \to$ 1's, $4 \to$ zeroes	$\frac{9!}{5!4!} = 126$
$6 \to$ 1's, $2 \to$ zeroes, $1 \to -1$	$\frac{9!}{6!2!} = 252$
$7 \to$ 1's, $2 \to -1$'s	$\frac{9!}{7!2!} = 36$

Total ways = 126 + 252 + 36 = 414

10. The greatest integer less than or equal to the sum of first 100 terms of the sequence $\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \ldots$ is equal to ______________.

Answer (98)

Sol. $S = \frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \ldots$

$$= \sum_{r=1}^{100}\left(\frac{3^r - 2^r}{3^r}\right)$$

$$= 100 - \frac{2}{3}\,\frac{\left(1 - \left(\frac{2}{3}\right)^{100}\right)}{1/3}$$

$$= 98 + 2\left(\frac{2}{3}\right)^{100}$$

$\therefore$ $[S] = 98$

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