MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x) = x - 1$ and $g:\mathbb{R}-\{1,-1\}\to\mathbb{R}$ be defined as $g(x)=\frac{x^2}{x^2-1}$.

   Then the function *fog* is:

   (A) One-one but not onto

   (B) Onto but not one-one

   (C) Both one-one and onto

   (D) Neither one-one nor onto

Answer (D)

Sol. $f:\mathbb{R}\to\mathbb{R}$ defined as

$f(x) = x - 1$ and $g:\mathbb{R}\to\{1,-1\}\to\mathbb{R}$, $g(x)=\frac{x^2}{x^2-1}$

Now $fog(x)\ \frac{x^2}{x^2-1}-1=\frac{1}{x^2-1}$

$\therefore$ Domain of $fog(x) = \mathbb{R} - \{-1, 1\}$

And range of $fog(x) = (-\infty, -1] \cup (0, \infty)$

Now, $\frac{d}{dx}(fog(x))=\frac{-1}{(x^2-1)^2}\cdot 2x=\frac{2x}{(1-x^2)^2}$

$\therefore\ \frac{d}{dx}(fog(x))>0$ for $\frac{2x}{((1-x)(1+x))^2}>0$

$\Rightarrow\ \frac{x}{((x-1)(x+1))^2}<0$

$\therefore\ x\in(-\infty,0)$

and $\frac{d}{dx}(fog(x))<0$ for $x\in(0,\infty)$

$\therefore$ *fog*(*x*) is neither one-one nor onto.

2. If the system of equations

   $\alpha x + y + z = 5$, $x + 2y + 3z = 4$, $x + 3y + 5z = \beta$

   has infinitely many solutions, then the ordered pair (α, β) is equal to:

   (A) (1, –3) (B) (–1, 3)

   (C) (1, 3) (D) (–1, –3)

Answer (C)

Sol. Given system of equations

$\alpha x + y + z = 5$

$x + 2y + 3z = 4$, has infinite solution

$x + 3y + 5z = \beta$

$\therefore\ \Delta=\begin{vmatrix}\alpha&1&1\\1&2&3\\1&3&5\end{vmatrix}=0\Rightarrow \alpha(1)-1(2)+1(1)=0$

$\Rightarrow\ \boxed{\alpha=1}$

and $\Delta_1=\begin{vmatrix}5&1&1\\4&2&3\\\beta&3&5\end{vmatrix}=0$

$\Rightarrow\ 5(1) - 1(20 - 3\beta) + 1(12 - 2\beta) = 0$

$\Rightarrow\ \beta = 3$

And $\Delta_2=\begin{vmatrix}1&5&1\\1&4&3\\1&\beta&5\end{vmatrix}=0\Rightarrow(20-3\beta)-5(2)+1(\beta-4)=0$

$\Rightarrow\ -2\beta + 6 = 0$

$\Rightarrow\ \beta = 3$

Similarly,

$\Rightarrow\ \Delta_3=\begin{vmatrix}1&1&5\\1&2&4\\1&3&\beta\end{vmatrix}=0\Rightarrow\beta=3$

$\therefore\ (\alpha, \beta) = (1, 3)$

3. If $A=\sum_{n=1}^{\infty}\frac{1}{\left(3+(-1)^n\right)^n}$ and $B=\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(3+(-1)^n\right)^n}$, then $\frac{A}{B}$ is equal to:

   (A) $\frac{11}{9}$ (B) 1

   (C) $-\frac{11}{9}$ (D) $-\frac{11}{3}$

Answer (C)

Sol. $A=\sum_{n=1}^{\infty}\frac{1}{\left(3+(-1)^n\right)^n}$ and $B=\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(3+(-1)^n\right)^n}$

$A=\frac{1}{2}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+....$

$B=\frac{-1}{2}+\frac{1}{4^2}-\frac{1}{2^3}+\frac{1}{4^4}+....$

$$A = \frac{\frac{1}{2}}{1-\frac{1}{4}} + \frac{\frac{1}{16}}{1-\frac{1}{16}},\ B = \frac{-\frac{1}{2}}{1-\frac{1}{4}} + \frac{\frac{1}{16}}{1-\frac{1}{16}}$$

$$A = \frac{11}{15},\ B = \frac{-9}{15}$$

$$\therefore \frac{A}{B} = \frac{-11}{9}$$

4. $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to:

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{6}$ (D) $\frac{1}{12}$

Answer (C)

Sol. $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4} = \lim_{x\to 0} \frac{2\sin(x+\sin x)\cdot \sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$$= \lim_{x\to 0} 2\cdot\left(\frac{\left(\frac{x+\sin x}{2}\right)\left(\frac{x-\sin x}{2}\right)}{x^4}\right)$$

$$= \lim_{x\to 0} \frac{1}{2}\cdot\left(\frac{\left(x+x-\frac{x^3}{3!}+\frac{x^5}{5!}...\right)\left(x-x+\frac{x^3}{3!}...\right)}{x^4}\right)$$

$$= \lim_{x\to 0} \frac{1}{2}\cdot\left(2-\frac{x^2}{3!}+\frac{x^4}{5!}...\right)\left(\frac{1}{3!}-\frac{x^2}{5!}-1\right)$$

$$= \frac{1}{6}$$

5. Let $f(x) = \min\{1, 1 + x\sin x\}$, $0 \le x \le 2\pi$. If m is the number of points, where f is not differentiable and n is the number of points, where f is not continuous, then the ordered pair (m, n) is equal to

(A) (2, 0) (B) (1, 0)

(C) (1, 1) (D) (2, 1)

Answer (B)

Sol. $f(x) = \min\{1, 1 + x\sin x\}$, $0 \le x \le 2\pi$

$$f(x) = \begin{cases} 1, & 0 \le x < \pi \\ 1 + x\sin x, & \pi \le x \le 2\pi \end{cases}$$

Now at $x = \pi$, $\lim_{x\to\pi^-} f(x) = 1 = \lim_{x\to\pi^+} f(x)$

$\therefore$ $f(x)$ is continuous in $[0, 2\pi]$

Now, at $x = \pi$ L.H.D = $\lim_{h\to 0} \frac{f(\pi-h)-f(\pi)}{-h} = 0$

R.H.D. = $\lim_{h\to 0} \frac{f(\pi+h)-f(\pi)}{h} = 1 - \frac{(\pi+h)\sin h - 1}{h}$

$= -\pi$

$\therefore$ $f(x)$ is not differentiable at $x = \pi$

$\therefore$ $(m, n) = (1, 0)$

6. Consider a cuboid of sides $2x$, $4x$ and $5x$ and a closed hemisphere of radius r. If the sum of their surface areas is a constant k, then the ratio $x : r$, for which the sum of their volumes is maximum, is

(A) 2 : 5 (B) 19 : 45

(C) 3 : 8 (D) 19 : 15

Answer (B)

Sol. $\because$ $s_1 + s_2 = k$

$$76x^2 + 3\pi r^2 = k$$

$$\therefore 152x\frac{dx}{dr} + 6\pi r = 0$$

$$\therefore \frac{dx}{dr} = \frac{-6\pi r}{152x}$$

Now $V = 40x^3 + \frac{2}{3}\pi r^3$

$$\therefore \frac{dv}{dr} = 120x^2\cdot\frac{dx}{dr} + 2\pi r^2 = 0$$

$$\Rightarrow 120x^2\cdot\left(\frac{-6\pi}{152}\frac{r}{x}\right) + 2\pi r^2 = 0$$

$$\Rightarrow 120\left(\frac{x}{r}\right) = 2\pi\left(\frac{152}{6\pi}\right)$$

$$\Rightarrow \left(\frac{x}{r}\right) = \frac{152}{3}\ \frac{1}{120} = \frac{19}{45}$$

7. The area of the region bounded by $y^2 = 8x$ and $y^2 = 16(3 - x)$ is equal to

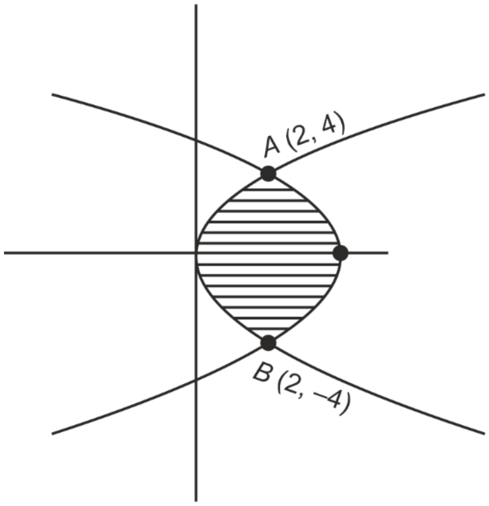
(A) $\frac{32}{3}$ (B) $\frac{40}{3}$

(C) 16 (D) 19

Answer (C)

Sol. $c_1 : y^2 = 8x$

$c_2 : y^2 = 16(3 - x)$



Solving c_1 and c_2

$48 - 16x = 8x$

$\boxed{x = 2}$

$\therefore \quad y = \pm 4$

$\therefore$ Area of shaded region

$$= 2\int_0^4 \left\{\left(\frac{48 - y^2}{16}\right) - \left(\frac{y^2}{8}\right)\right\} dy$$

$$= \frac{1}{8}\left[48y - y^3\right]_0^4 = 16$$

8. If $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g(x) + c,\ g(1) = 0,$ then $g\left(\frac{1}{2}\right)$ is equal to

(A) $\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) + \frac{\pi}{3}$ (B) $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \frac{\pi}{3}$

(C) $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{\pi}{3}$ (D) $\frac{1}{2}\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) - \frac{\pi}{6}$

Answer (A)

Sol. $\because \quad \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g(x) + c$

$$\int_1^{\frac{1}{2}} \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g\left(\frac{1}{2}\right) - g(1)$$

$$\therefore \quad g\left(\frac{1}{2}\right) = \int_1^{\frac{1}{2}} \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$$

$\cot x = \cos 2\theta$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos 2\theta} \cdot \frac{\sin\theta}{\cos\theta}(-2\sin 2\theta)d\theta$$

$$= -\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\cos 2\theta}d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \frac{\left(1 - 2\sin^2\theta\right) - 1}{\cos 2\theta}d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} (1 - \sec 2\theta)d\theta$$

$$= \frac{\pi}{3} - 2 \cdot \frac{1}{2}\left[\ln\left|\sec 2\theta + \tan 2\theta\right|\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} - \left[\ln\left|2 + \sqrt{3}\right| - \ln 1\right]$$

$$= \frac{\pi}{3} + \ln\left(\frac{1}{2 + \sqrt{3}}\right)$$

$$= \frac{\pi}{3} + \ln\left|\frac{\sqrt{3}-1}{\sqrt{3}+1}\right|$$

9. If $y = y(x)$ is the solution of the differential equation $x\frac{dy}{dx} + 2y = xe^x, y(1) = 0$ then the local maximum value of the function $z(x) = x^2y(x) - e^x, x \in R$ is

(A) $1 - e$ (B) 0

(C) $\frac{1}{2}$ (D) $\frac{4}{e} - e$

Answer (D)

Sol. $x\frac{dy}{dx} + 2y = xe^x,\ y(1) = 0$

$$\frac{dy}{dx} + \frac{2}{x}y = e^x, \text{ then } e^{\int \frac{2}{x}dx}dx = x^2$$

$$y \cdot x^2 = \int x^2e^x dx$$

$$yx^2 = x^2e^x - \int 2xe^x dx$$

$$= x^2e^x - 2\left(xe^x - e^x\right) + c$$

$$yx^2 = x^2e^x - 2xe^x + 2e^x + c$$

$$yx^2 = \left(x^2 - 2x + 2\right)e^x + c$$

$$0 = e + c \Rightarrow c = -e$$

$$y(x) \cdot x^2 - e^x = (x-1)^2 e^x - e$$

$$z(x) = (x-1)^2 e^x - e$$

For local maximum $z'(x) = 0$

$\therefore \quad 2(x-1)e^x + (x-1)^2 e^x = 0$

$\therefore \quad x = -1$

And local maximum value $= z(-1)$

$$= \frac{4}{e} - e$$

10. If the solution of the differential equation $\frac{dy}{dx} + e^x (x^2 - 2)y = (x^2 - 2x)(x^2 - 2)e^{2x}$ satisfies $y(0) = 0$, then the value of $y(2)$ is ______.

(A) –1 (B) 1

(C) 0 (D) e

Answer (C)

Sol. $\because \quad \frac{dy}{dx} + e^x (x^2 - 2)y = (x^2 - 2x)(x^2 - 2)e^{2x}$

Here, I.F. $= e^{\int e^x (x^2 - 2)dx}$

$$= e^{(x^2 - 2x)e^x}$$

$\therefore$ Solution of the differential equation is

$$y \cdot e^{(x^2-2x)e^x} = \int (x^2 - 2x)(x^2 - 2)e^{2x} \cdot e^{(x^2-2x)e^x} dx$$

$$= \int (x^2 - 2x)e^x \cdot (x^2 - 2)e^x \cdot e^{(x^2-2x)e^x} dx$$

Let $(x^2 - 2x)e^x = t$

$\therefore \quad (x^2 - 2)e^x dx = dt$

$$y \cdot e^{(x^2-2x)e^x} = \int t \cdot e^t dt$$

$$y \cdot e^{(x^2-2x)e^x} = (x^2 - 2x - 1)e^{(x^2-2x)e^x} + c$$

$\therefore \quad y(0) = 0$

$\therefore \quad c = 1$

$\therefore \quad y = (x^2 - 2x - 1) + e^{(2x - x^2)e^x}$

$\therefore \quad y(2) = -1 + 1$

$= 0$

11. If m is the slope of a common tangent to the curves $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $x^2 + y^2 = 12$, then $12m^2$ is equal to:

(A) 6 (B) 9

(C) 10 (D) 12

Answer (B)

Sol. $C_1 : \frac{x^2}{16} + \frac{y^2}{9} = 1$ and $C_2 : x^2 + y^2 = 12$

Let $y = mx \pm \sqrt{16m^2 + 9}$ be any tangent to C_1 and if this is also tangent to C_2 then

$$\left| \frac{\sqrt{16m^2 + 9}}{\sqrt{m^2 + 1}} \right| = \sqrt{12}$$

$\Rightarrow \quad 16m^2 + 9 = 12m^2 + 12$

$\Rightarrow \quad 4m^2 = 3 \Rightarrow 12m^2 = 9$

12. The locus of the mid-point of the line segment joining the point (4, 3) and the points on the ellipse $x^2 + 2y^2 = 4$ is an ellipse with eccentricity:

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{2}$

Answer (C)

Sol. Let $P(2\cos\theta, \sqrt{2}\sin\theta)$ be any point on ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ and $Q(4, 3)$ and let (h, k) be the mid point of PQ

then $h = \frac{2\cos\theta + 4}{2}$, $k = \frac{\sqrt{2}\sin\theta + 3}{2}$

$\therefore \quad \cos\theta = h - 2, \sin\theta = \frac{2k - 3}{\sqrt{2}}$

$\therefore \quad (h-2)^2 + \left(\frac{2k-3}{\sqrt{2}}\right)^2 = 1$

$$\Rightarrow \quad \frac{(x-2)^2}{1} + \frac{\left(y - \frac{3}{2}\right)^2}{\frac{1}{2}} = 1$$

$\therefore \quad e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$

13. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{9}=1$ at the point $\left(8, 3\sqrt{3}\right)$ on it passes through the point:

(A) $\left(15, -2\sqrt{3}\right)$ (B) $\left(9, 2\sqrt{3}\right)$

(C) $\left(-1, 9\sqrt{3}\right)$ (D) $\left(-1, 6\sqrt{3}\right)$

Answer (C)

Sol. Given hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{9}=1$

$\because$ It passes through $(8, 3\sqrt{3})$

$\because \frac{64}{a^2}-\frac{27}{9}=1 \Rightarrow a^2=16$

Now, equation of normal to hyperbola

$\frac{16x}{8}+\frac{9y}{3\sqrt{3}}=16+9$

$\Rightarrow 2x+\sqrt{3}y=25$ ...(i)

$\left(-1, 9\sqrt{3}\right)$ satisfies (i)

14. If the plane $2x + y - 5z = 0$ is rotated about its line of intersection with the plane $3x - y + 4z - 7 = 0$ by an angle of $\frac{\pi}{2}$, then the plane after the rotation passes through the point:

(A) (2, –2, 0) (B) (–2, 2, 0)

(C) (1, 0, 2) (D) (–1, 0, –2)

Answer (C)

Sol. $P_1 : 2x + y - 52 = 0$, $P_2 : 3x - y + 4z - 7 = 0$

Family of planes P_1 and P_2

$P : P_1 + \lambda P_2$

$\therefore \quad P : (2+3\lambda)x+(1-\lambda)y+(-5+4\lambda)z-7\lambda=0$

$\because \quad P \perp P_1 \therefore 4+6\lambda+1-\lambda+25-20\lambda=0$

$\boxed{\lambda=2}$

$\therefore \quad P : 8x - y + 32 - 14 = 0$

It passes through the point (1, 0, 2)

15. If the lines $\vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(3\hat{j}-\hat{k}\right)$ and $\vec{r}=\left(\alpha\hat{i}-\hat{j}\right)+\mu\left(2\hat{j}-3\hat{k}\right)$ are co-planar, then the distance of the plane containing these two lines from the point $(\alpha, 0, 0)$ is :

(A) $\frac{2}{9}$ (B) $\frac{2}{11}$

(C) $\frac{4}{11}$ (D) 2

Answer (B)

Sol. $\because$ Both lines are coplanar, so

$$\begin{vmatrix} \alpha-1 & 0 & -1 \\ 0 & 3 & -1 \\ 2 & 0 & -3 \end{vmatrix}=0$$

$\Rightarrow \quad \alpha=\frac{5}{3}$

Equation of plane containing both lines

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 0 & 3 & -1 \\ 2 & 0 & -3 \end{vmatrix}=0$$

$\Rightarrow \quad 9x + 2y + 6z = 13$

So, distance of $\left(\frac{5}{3}, 0, 0\right)$ from this plane

$=\frac{2}{\sqrt{81+4+36}}=\frac{2}{11}$

16. Let $\vec{a}=\hat{i}+\hat{j}+2\hat{k}, \vec{b}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+\hat{k}$ be three given vectors. Let $\vec{v}$ be a vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{2}{\sqrt{3}}$. If $\vec{v}\cdot\hat{j}=7$, then $\vec{v}\cdot\left(\hat{i}+\hat{k}\right)$ is equal to :

(A) 6 (B) 7

(C) 8 (D) 9

Answer (D)

Sol. Let $\vec{v}=\lambda_1\vec{a}+\lambda_2\vec{b}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$.

$=(\lambda_1+2\lambda_2)\hat{i}+(\lambda_1-3\lambda_2)\hat{j}+(2\lambda_1+\lambda_2)\hat{k}$

$\because$ Projection of $\vec{v}$ on $\vec{c}$ is $\frac{2}{\sqrt{3}}$

$\therefore \quad \frac{\lambda_1+2\lambda_2-\lambda_1+3\lambda_2+2\lambda_1+\lambda_2}{\sqrt{3}}=\frac{2}{\sqrt{3}}$

$\therefore \quad \lambda_1 + 3\lambda_2 = 1$ ...(i)

and $\vec{v}\cdot\hat{j}=7 \Rightarrow \lambda_1-3\lambda_2=7$ ...(ii)

from equation (i) and (ii)

$\lambda_1 = 4, \lambda_2 = -1$

$\therefore \quad \vec{v}=2\hat{i}+7\hat{j}+7\hat{k}$

$\therefore \quad \vec{v}\cdot(\hat{i}+\hat{k})=2+7$

$= 9$

17. The mean and standard deviation of 50 observations are 15 and 2 respectively. It was found that one incorrect observation was taken such that the sum of correct and incorrect observations is 70. If the correct mean is 16, then the correct variance is equal to :

(A) 10 (B) 36

(C) 43 (D) 60

Answer (C)

Sol. Given $\bar{x} = 15, \sigma = 2 \Rightarrow \sigma^2 = 4$

$\therefore$ $x_2 + x_2 + \ldots\ldots + x_{50} = 15 \times 50 = 750$

$$4 = \frac{x_1^2 + x_2^2 + \ldots. + x_{50}^2}{50} - 225$$

$$\therefore \quad x_1^2 + x_2^2 + \ldots\ldots + x_{50}^2 = 50 \times 229$$

Let a be the correct observation and b is the incorrect observation

then $a + b = 70$

and $16 = \dfrac{750 - b + a}{50}$

$\therefore$ $a - b = 50 \Rightarrow a = 60, b = 10$

$\therefore$ Correct variance = $\dfrac{50 \times 229 + 60^2 - 10^2}{50} - 256$

$= 43$

18. 16 sin(20°) sin(40°) sin(80°) is equal to :

(A) $\sqrt{3}$ (B) $2\sqrt{3}$

(C) 3 (D) $4\sqrt{3}$

Answer (B)

Sol. 16sin20° · sin40° · sin80°

$= 4\sin 60° \{\because 4\sin\theta \cdot \sin(60° - \theta) \cdot \sin(60° + \theta) = \sin 3\theta\}$

$= 2\sqrt{3}$

19. If the inverse trigonometric functions take principal values, then $\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$ is equal to :

(A) 0 (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Answer (C)

Sol. $\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$

$= \cos^{-1}\left(\frac{3}{10} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{4}{5}\right)$

$= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

20. Let $r \in \{p, q, \sim p, \sim q\}$ be such that the logical statement $r \vee (\sim p) \Rightarrow (p \wedge q) \vee r$ is a tautology. Then r is equal to :

(A) p (B) q

(C) $\sim p$ (D) $\sim q$

Answer (C)

Sol. Clearly r must be equal to $\sim p$

$\because$ $\sim p \vee \sim p = \sim p$

and $(p \wedge q) \vee \sim p = p$

$\therefore$ $\sim p \Rightarrow p$ = tautology.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = 2^x f(y) + 4^y f(x)$, $\forall x, y \in \mathbb{R}$. If $f(2) = 3$, then $14 \cdot \dfrac{f'(4)}{f'(2)}$ is equal to ____.

Answer (248)

Sol. $\because$ $f(x + y) = 2^x f(y) + 4^y f(x)$ …(1)

Now, $f(y + x)\ 2^y f(x) + 4^x f(y)$ …(2)

$\therefore$ $2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$

$(4^y - 2^y) f(x) = (4^x - 2^x) f(y)$

$\dfrac{f(x)}{4^x - 2^x} = \dfrac{f(y)}{4^y - 2^y} = k$ (Say)

$\therefore$ $f(x) = k(4^x - 2^x)$

$\because$ $f(2) = 3$ then $k = \dfrac{1}{4}$

$\therefore \quad f(x) = \dfrac{4^x - 2^x}{4}$

$\therefore \quad f'(x) = \dfrac{4^x \ln 4 - 2^x \ln 2}{4}$

$f'(x) = \dfrac{(2.4^x - 2^x)\ ln\ 2}{4}$

$\therefore \quad \dfrac{f'(4)}{f'(2)} = \dfrac{2.256 - 16}{2.16 - 4}$

$\therefore \quad 14\dfrac{f'(4)}{f'(2)} = 248$

2. Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\dfrac{1}{p}+\dfrac{1}{q}\right)^{-2}$ is equal to _______.

Answer (4)

Sol. $\because \quad p + q = 3$ …(i)

and $p^4 + q^4 = 369$ …(ii)

$\{(p + q)^2 - 2pq\}^2 - 2p^2q^2 = 369$

or $(9 - 2pq)^2 - 2(pq)^2 = 369$

or $(pq)^2 - 18pq - 144 = 0$

$\therefore \quad pq = -6$ or 24

But $pq = 24$ is not possible

$\therefore \quad pq = -6$

Hence, $\left(\dfrac{1}{p}+\dfrac{1}{q}\right)^{-2} = \left(\dfrac{pq}{p+q}\right)^2 = (-2)^2 = 4$

3. If $z^2 + z + 1 = 0$, $z \in \mathbb{C}$, then $\left|\sum_{n=1}^{15}\left(z^n + (-1)^n \dfrac{1}{z^n}\right)^2\right|$ is equal to ________.

Answer (2)

Sol. $\because \quad z^2 + z + 1 = 0 \quad \Rightarrow \quad \omega$ or ω^2

$\because \quad \left|\sum_{n=1}^{15}\left(z^n + (-1)^n \dfrac{1}{z^n}\right)^2\right|$

$= \left|\sum_{n=1}^{15} z^{2n} + \sum_{n=1}^{15} z^{-2n} + 2\cdot\sum_{n=1}^{15}(-1)^n\right|$

$= |0 + 0 - 2|$

$= 2$

4. Let $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $Y = \alpha I + \beta X + \gamma X^2$ and

$Z = \alpha^2 I - \alpha\beta X + \left(\beta^2 - \alpha\gamma\right)X^2$, $\alpha, \beta, \gamma \in \mathbb{R}$.

If $Y^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, then $(\alpha - \beta + \gamma)^2$ is equal to ____________.

Answer (100)

Sol. $\because \quad X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \quad X^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \quad Y = \alpha I + \beta X + \gamma X^2 \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix}$

$\because \quad Y \cdot Y^{-1} = I$

$\therefore \quad \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} \frac{\alpha}{5} & \frac{\beta - 2\alpha}{5} & \frac{\alpha - 2\beta + \gamma}{5} \\ 0 & \frac{\alpha}{5} & \frac{\beta - 2\alpha}{5} \\ 0 & 0 & \frac{\alpha}{5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad \alpha = 5, \beta = 10, \gamma = 15$

$\therefore \quad (\alpha - \beta + \gamma)^2 = 100$

5. The total number of 3-digit numbers, whose greatest common divisor with 36 is 2, is _______.

Answer (150)

Sol. $\because$ $x \in [100, 999]$, $x \in N$

Then $\frac{x}{2} \in [50, 499], \frac{x}{2} \in N$

Number whose G.C.D with 18 is 1 in this range have the required condition. There are 6 such number from 18 × 3 to 18 × 4. Similarly from 18 × 4 to 18 × 5….., 26 × 18 to 27 × 18

$\therefore$ Total numbers = 24 × 6 + 6 = 150

The extra numbers are 53, 487, 491, 493, 497 and 499.

6. If $\left({}^{40}C_0\right)+\left({}^{41}C_1\right)+\left({}^{42}C_2\right)+...+\left({}^{60}C_{20}\right)=\frac{m}{n}\,{}^{60}C_{20}$

m and n are coprime, then $m + n$ is equal to ______.

Answer (102)

Sol. ${}^{40}C_0 + {}^{41}C_1 + {}^{42}C_2 + \;......+\; {}^{60}C_{20}$

$= {}^{40}C_{40} + {}^{41}C_{40} + {}^{42}C_{40} + \;......+\; {}^{60}C_{40}$

$= {}^{61}C_{41}$

$\frac{61}{41}\cdot {}^{60}C_{40}$

$\therefore$ m = 61, n = 41

$\therefore$ $m + n = 102$

7. If a_1 (> 0), a_2, a_3, a_4, a_5 are in a G.P., $a_2 + a_4 = 2a_3 + 1$ and $3a_2 + a_3 = 2a_4$, then $a_2 + a_4 + 2a_5$ is equal to ________.

Answer (40)

Sol. Let G.P. be $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$, ……

$\because$ $3a_2 + a_3 = 2a_4$

$\Rightarrow$ $3ar + ar^2 = 2ar^3$

$\Rightarrow$ $2ar^2 - r - 3 = 0$

$\therefore$ $r = -1$ or $\frac{3}{2}$

$\because$ $a_1 = a > 0$ then $r \neq -1$

Now, $a_2 + a_4 = 2a_3 + 1$

$ar + ar^3 = 2ar^2 + 1$

$a\left(\frac{3}{2}+\frac{27}{8}-\frac{9}{2}\right)=1$

$\therefore$ $a = \frac{8}{3}$

$\therefore$ $a_2 + a_4 + 2a_5 = a\left(r + r^3 + 2r^4\right)$

$= \frac{8}{3}\left(\frac{3}{2}+\frac{27}{8}+\frac{81}{8}\right)$

= 40

8. The integral $\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{\left(2-x^2\right)dx}{\left(2+x^2\right)\sqrt{4+x^4}}$ is equal to ________.

Answer (3)

Sol. $I = \frac{24}{\pi}\int_0^{\sqrt{2}}\frac{2-x^2}{\left(2+x^2\right)\sqrt{4+x^4}}dx$

Let $x = \sqrt{2}t \Rightarrow dx = \sqrt{2}dt$

$I = \frac{24}{\pi}\int_0^1\frac{\left(2-2t^2\right)\cdot\sqrt{2}dt}{\left(2+2t^2\right)\sqrt{4+4t^4}}$

$= \frac{12\sqrt{2}}{\pi}\int_0^1\frac{\left(\frac{1}{t^2}-1\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{\left(t+\frac{1}{t}\right)^2-2}}$

Let $t + \frac{1}{t} = u$

$\Rightarrow$ $\left(1-\frac{1}{t^2}\right)dt = du$

$= \frac{12\sqrt{2}}{\pi}\int_\infty^2\frac{-du}{u\sqrt{4^2-2}}$

$= \frac{12\sqrt{2}}{\pi}\int_2^\infty\frac{du}{u^2\sqrt{-\left(\frac{\sqrt{2}}{u}\right)^2}}$

$= \frac{12\sqrt{2}}{\pi}\int_{\frac{1}{\sqrt{2}}}^0\frac{-\frac{1}{\sqrt{2}}dp}{\sqrt{1-p^2}}$

$= \frac{12}{\pi}\left[\sin^{-1}p\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{12}{\pi}\cdot\frac{\pi}{4}$

= 3

9. Let a line L_1 be tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$ and let L_2 be the line passing through the origin and perpendicular to L_1. If the locus of the point of intersection of L_1 and L_2 is $(x^2+y^2)^2=\alpha x^2+\beta y^2$, then $\alpha+\beta$ is equal to___.

Answer (12)

Sol. Equation of L_1 is

$$\frac{x\sec\theta}{4}-\frac{y\tan\theta}{2}=1 \quad \ldots\text{(i)}$$

Equation of line L_2 is

$$\frac{x\tan\theta}{2}+\frac{y\sec\theta}{4}=0 \quad \ldots\text{(ii)}$$

$\because$ Required point of intersection of L_1 and L_2 is (x_1, y_1) then

$$\frac{x_1\sec\theta}{4}-\frac{y_1\tan\theta}{2}-1=0 \quad \ldots\text{(iii)}$$

and $\frac{y_1\sec\theta}{4}+\frac{x_1\tan\theta}{2}=0 \quad \ldots\text{(iv)}$

From equations (iii) and (iv)

$$\sec\theta=\frac{4x_1}{x_1^2+y_1^2} \text{ and } \tan\theta=\frac{-2y_1}{x_1^2+y_1^2}$$

$\therefore$ Required locus of (x_1, y_1) is

$$(x^2+y^2)^2=16x^2-4y^2$$

$\therefore \quad \alpha=16, \beta=-4$

$\therefore \quad \alpha+\beta=12$

10. If the probability that a randomly chosen 6-digit number formed by using digits 1 and 8 only is a multiple of 21 is p, then 96 p is equal to ________ .

Answer (33)

Sol. Total number of numbers from given Condition = $n(s)$ = 2^6.

Every required number is of the form

$$A = 7\cdot(10^{a_1}+10^{a_2}+10^{a_3}+......)+111111$$

Here 111111 is always divisible by 21.

$\therefore$ If A is divisible by 21 then

$10^{a_1}+10^{a_2}+10^{a_3}+......$ must be divisible by 3.

For this we have ${}^6C_0+{}^6C_3+{}^6C_6$ cases are there

$\therefore \quad n(E)={}^6C_0+{}^6C_3+{}^6C_6=22$

$\therefore$ Required probability = $\frac{22}{2^6}=p$

$\therefore \quad \frac{11}{32}=p$

$\therefore \quad 96p = 33$

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